Proof Theory for Admissible Rules

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Abstract

The admissible rules of a logic are the rules under which the set of theorems of the logic is closed. In this paper a Gentzen-style framework is introduced for defining analytic proof systems that derive the admissible rules of various non-classical logics. Just as Gentzen systems for derivability treat sequents as basic objects, for admissibility, sequent rules are basic. Proof systems are defined here for the admissible rules of classes of both modal logics, including $\mathsf{K4}$, $\mathsf{S4}$, and GL, and intermediate logics, including Intuitionistic logic IPC, De Morgan (or Jankov) logic KC, and logics $\mathsf{BC_n}$ $(n = 1, 2, \ldots)$ with bounded cardinality Kripke models. With minor restrictions, proof search in these systems terminates, giving decision procedures for admissibility in the corresponding logics.

Key words: Admissible Rules, Proof Theory, Gentzen Systems, Intuitionistic Logic, Modal Logics, Intermediate Logics.

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1 Introduction

Investigation of logical systems usually concentrates on the derivability of theorems. However, it is also interesting to "move up a level" and consider which rules are admissible for the system; that is, to investigate under which rules the set of theorems is closed. Algebraically, this corresponds to the study of quasi-varieties generated by free algebras, while from a computational perspective the investigation is significant since adding further (admissible) rules to a system may improve proof search. In Classical logic CPC, such questions are trivial: admissible rules are also derivable; that is, CPC is structurally complete. However, in Intuitionistic logic IPC, and many other non-classical (e.g. modal and intermediate) logics this is no longer the case. It is therefore an interesting and significant task to provide characterizations of admissibility for these logics.

In recent years, one successful approach to characterizing admissible rules has been via *bases*, which may be viewed (roughly speaking) as axiomatizations for sets of rules. More precisely, a basis for admissible rules in a logic is a set of admissible rules such that adding these to the logic allows all the admissible rules to be derived. That the set of admissible rules of IPC has no finite basis but is nevertheless decidable was proved by Rybakov [14], answering a problem originally posed by Friedman [3]. It has also been shown independently by Iemhoff [7] and Rozière [13] (confirming a conjecture of de Jongh and Visser) that such a basis is provided by the following so-called "Visser rules":

$$(V_n) \;\; (C \to (A_{n+1} \vee A_{n+2})) \vee D \;/\; (\bigvee_{j=1}^{n+2} C \to A_j) \vee D$$

for $n = 1, 2, \ldots$, where $C = \bigwedge_{i=1}^{n}(A_i \to B_i)$. Related characterizations have since been obtained for intermediate logics by Iemhoff [10] and for modal logics by Jerábek [11]; proofs being based on Ghilardi's work on unification and projective approximations [4, 5]. We mention also that the basis of Visser rules given above has been used to define a first basic provability logic for IPC [17, 8].

Although decision procedures for admissibility are described (or implicit) in the works of Rybakov and Ghilardi, a systematic presentation of analytic proof systems for deriving admissible rules has been lacking. Such a presentation is important not only for developing systems that reason directly about rules, but also for investigating relationships between admissibility in different logics, and meta-logical properties such as complexity and interpolation. A first step in this direction was taken by Iemhoff in [9] where an analytic proof system is defined for deriving admissible rules of IPC based partly on an algorithm by Ghilardi for projectivity [6]. However, this system makes use of a number of inelegant syntactic divisions and semantic checks, and is unsuitable for generalization to other logics.

In this work we develop a general framework for defining Gentzen-style proof systems for admissible rules. The key idea is to give a uniform proof-theoretic characterization of admissibility by generalizing proof calculi at the theorem level. For derivability, the basic objects are typically sequents, not formulas. Similarly, for admissibility, we take the basic objects of our systems to be not rules, but *sequent rules*. Rules (now one level up) of these systems thus have sequent rules as premises, and a sequent rule as the conclusion. Each logical connective is characterized by four rules: the connective can occur either on the left or the right of a sequent, and the sequent itself can occur either as a premise or a conclusion of a sequent rule. Our systems also include weakening and contraction rules, and rules that allow sequents to interact: an anti-cut rule corresponding to the admissibility of cut for the logic, a projection rule reflecting the fact that derivability implies admissibility, and one or two so-called "Visser Rules" capturing key facts of admissibility in the logic.

We begin, following the work of Jerábek [11] and Ghilardi [5], by considering a wide class of (so-called extensible) *modal logics* extending $\mathsf{K4}$, treating as particular case studies $\mathsf{K4}$, $\mathsf{S4}$, and Gödel-Löb logic GL. More precisely, we obtain analytic systems for admissibility in these logics as uniform extensions of systems for derivability. The extra "Visser rules" depend on whether the logic can be characterized as transitive or intransitive. We then provide a system for the fundamental (and historically most studied) case of IPC, making essential use of theorems by Ghilardi [4]. We extend this approach to a class of intermediate logics, including De Morgan (or Jankov) logic KC and logics with bounded cardinality Kripke models $\mathsf{BC_n}$ ($n = 1, 2, \ldots$), by treating rules dealing with hypersequents, a natural extension of sequents introduced by Avron [1]. With minor modifications, all these systems terminate, and hence provide the basis for decision procedures for deriving admissible rules in these logics.

2 Preliminaries

We treat logics as consequence relations based upon propositional languages with binary connectives $\wedge$, $\vee$, $\rightarrow$, a constant $\bot$, and sometimes also a modal connective $\Box$. Other connectives are then defined as:

$$\neg A =_{def} A \rightarrow \bot \qquad A \leftrightarrow B =_{def} (A \rightarrow B) \wedge (B \rightarrow A)$$
$$\top =_{def} \neg\bot \qquad \boxdot A =_{def} \Box A \wedge A$$

We denote propositional variables by $p, q, \ldots$, formulas by $A, B, \ldots$, and sets of formulas by $\Gamma, \Pi, \Sigma, \Delta, \Theta, \Psi$. Propositional variables and constants are called *atoms* and denoted by $a, b, \ldots$, and formulas $a \rightarrow b$ and $\Box a$ are called *atomic implications* and *boxed atoms* respectively. We adopt the convention of writing $\bigvee \Gamma$ and $\bigwedge \Gamma$ where $\bigvee \emptyset = \bot$ and $\bigwedge \emptyset = \top$ for iterated disjunctions and conjunctions of formulas in a finite set Γ. We also write $\Box\Gamma$ and $\boxdot\Gamma$ for $\{\Box A : A \in \Gamma\}$ and $\Gamma \cup \Box\Gamma$, respectively. Finally, for brevity we write $\{x\}_{x\in\Gamma}$ for the set $\{x : x \in \Gamma\}$, reserving ordinary brackets (and) for clarification.

2.1 Generalized Rules and Admissibility

Rules are usually asymmetric, having many premises but just one conclusion. However, it is convenient when considering admissibility to treat "generalized" rules having also many conclusions.

Definition 1 *A* generalized rule *is an ordered pair of finite sets of formulas:*

$$A_1, \ldots, A_n \;/\; B_1, \ldots, B_m$$

Intuitively, a generalized rule is admissible for a logic L if whenever a substitution makes all the premises theorems of L, then it makes one of the conclusions a theorem. More precisely:

Definition 2 *Let L be a logic. An* L-unifier *for a formula A is a substitution σ such that $\vdash_{\mathsf{L}} \sigma A$.*

Definition 3 *Let L be a logic. A generalized rule $\Gamma \;/\; \Delta$ is* L-admissible*, written $\Gamma \mathrel{|\!\!\sim}_{\mathsf{L}} \Delta$, if each L-unifier for all $A \in \Gamma$, is an L-unifier for some $B \in \Delta$.*

In developing proof systems for derivability in a logic it is helpful to consider sequents, which in this context, we define and interpret as follows:

Definition 4 *A* sequent *S is an ordered pair of finite sets of formulas:*

$$\Gamma \Rightarrow \Delta$$

S is L-derivable for a logic L, written $\vdash_{\mathsf{L}} S$, iff $\vdash_{\mathsf{L}} I(S)$ where:

$$I(\Gamma \Rightarrow \Delta) =_{def} \bigwedge \Gamma \to \bigvee \Delta$$

Similarly, for admissibility, rather than deal with rules involving only formulas, we consider *sequent rules*. These are represented as "implications" between multisets of sequents using the symbol $\rhd$ as follows:

Definition 5 *A* generalized sequent rule *(gs-rule for short) $\mathcal{R}$ is an ordered pair of multisets of sequents:*

$$\{\Gamma_i \Rightarrow \Delta_i\}_{i=1}^{n} \rhd \{\Pi_j \Rightarrow \Sigma_j\}_{j=1}^{m}$$

$\mathcal{R}$ *is* L-admissible *for a logic* L*, written* $\mathrel{|\!\sim}_{\mathsf{L}} \mathcal{R}$*, iff:*

$$\{I(\Gamma_i \Rightarrow \Delta_i)\}_{i=1}^{n} \mathrel{|\!\sim}_{\mathsf{L}} \{I(\Pi_j \Rightarrow \Sigma_j)\}_{j=1}^{m}$$

$\mathcal{R}$ *is* L-derivable *for a logic* L*, written* $\vdash_{\mathsf{L}} \mathcal{R}$*, iff:*

$$\bigwedge_{i=1}^{n} I(\Gamma_i \Rightarrow \Delta_i) \vdash_{\mathsf{L}} \bigvee_{j=1}^{m} I(\Pi_j \Rightarrow \Sigma_j)$$

Note that unlike generalized rules, gs-rules consist of multisets, not sets, of premises and conclusions, denoted by the variables $\mathcal{G}$, $\mathcal{H}$. However, crucially:

$$\mathrel{|\!\sim}_{\mathsf{L}} A_1, \ldots, A_n \;/\; B_1, \ldots, B_m \;\text{ iff }\; \mathrel{|\!\sim}_{\mathsf{L}} (\Rightarrow A_1), \ldots, (\Rightarrow A_n) \rhd (\Rightarrow B_1), \ldots, (\Rightarrow B_m)$$

Hence a proof system for the admissibility of gs-rules is also a proof system for the admissibility of generalized rules.

Rules (now at the next level up) for gs-rules consist of a set of premises $\mathcal{R}_1, \ldots, \mathcal{R}_n$, which we often write as $[\mathcal{R}_i]_{i=1}^{n}$, and a conclusion $\mathcal{R}$; rules with no premises being called *initial gs-rules*. Such rules are *sound* with respect to a logic L if whenever $\mathrel{|\!\sim}_{\mathsf{L}} \mathcal{R}_i$ for $i = 1 \ldots n$, then $\mathrel{|\!\sim}_{\mathsf{L}} \mathcal{R}$, and *invertible*, if whenever $\mathrel{|\!\sim}_{\mathsf{L}} \mathcal{R}$, then $\mathrel{|\!\sim}_{\mathsf{L}} \mathcal{R}_i$ for $i = 1 \ldots n$.

Example 6 *As an illustration of these ideas, consider the disjunction property, which can be written as the generalized rule* $p \vee q \;/\; p, q$*. Clearly, this rule is* L*-admissible iff the gs-rule:*

$$(\Rightarrow p \vee q) \rhd (\Rightarrow p), (\Rightarrow q)$$

is L*-admissible. Observe now that if* σ *is an* IPC*-unifier for* $p \vee q$*, i.e.* $\vdash_{\mathsf{IPC}} (\sigma p) \vee (\sigma q)$*, then* σ *must be an* IPC*-unifier for* p *or* q*, i.e. either* $\vdash_{\mathsf{IPC}} (\sigma p)$ *or* $\vdash_{\mathsf{IPC}} (\sigma q)$*. However, this does not hold for* CPC*. For example, let* $\sigma(p) = p$ *and* $\sigma(q) = \neg p$*; plainly* $\vdash_{\mathsf{CPC}} p \vee \neg p$*, but* $\nvdash_{\mathsf{CPC}} p$ *and* $\nvdash_{\mathsf{CPC}} \neg p$*.*

2.2 Projectivity and the Extension Property

Admissibility and derivability do not coincide in general for non-classical logics. However, Ghilardi [4, 5] has identified classes of so-called “projective” formulas where if A is projective, then the relationship “$A \mathrel{|\!\sim}_{\mathsf{L}} B$ iff $A \vdash_{\mathsf{L}} B$” holds for all B.

Definition 7 *Let* L *be a logic and* A *a formula.* A *is* L-projective *if there exists a substitution* σ*, called an* L-projective unifier *for* A*, such that:*

$$\vdash_{\mathsf{L}} \sigma A, \; \textit{and for all atoms } a \; (A \vdash_{\mathsf{L}} \sigma(a) \leftrightarrow a).$$

Lemma 8 *Let* L *be an intermediate logic or a normal modal logic:* [3]

(a) If a formula A *is* L*-projective, then for all sets of formulas* Δ*,* $A \mathrel{|\!\sim}_{\mathsf{L}} \Delta$ *iff* $A \vdash_{\mathsf{L}} B$ *for some* $B \in \Delta$*.*
(b) If L *has the disjunction property, then for any* L*-projective formula* A *and sets of formulas* Δ*,* $A \mathrel{|\!\sim}_{\mathsf{L}} \bigvee \Delta$ *iff* $A \vdash_{\mathsf{L}} B$ *for some* $B \in \Delta$*.*
(c) If $A_1, \ldots, A_n$ *are* L*-projective formulas, then for all formulas* B*,* $\bigvee_{i=1}^{n} A_i \mathrel{|\!\sim}_{\mathsf{L}} B$ *iff* $\bigvee_{i=1}^{n} A_i \vdash_{\mathsf{L}} B$*.*
(d) If L *is an intermediate logic and* $A_1, \ldots, A_n$ *are* IPC*-projective formulas, then for all formulas* B*,* $\bigvee_{i=1}^{n} A_i \mathrel{|\!\sim}_{\mathsf{L}} B$ *iff* $\bigvee_{i=1}^{n} A_i \vdash_{\mathsf{L}} B$*.*

Proof.

(a) The right-to-left direction is immediate. For the other direction, let $A \mathrel{|\!\sim}_{\mathsf{L}} \Delta$. Since A is L-projective, there exists an L-projective unifier σ of A, such that $\vdash_{\mathsf{L}} \sigma B$ for some $B \in \Delta$. But using the Leibniz property for L and the fact that σ is an L-projective unifier, $A \vdash_{\mathsf{L}} \sigma B \to B$. Hence, by modus ponens, $A \vdash_{\mathsf{L}} B$.
(b) Since L has the disjunction property, $A \mathrel{|\!\sim}_{\mathsf{L}} \bigvee \Delta$ iff $A \mathrel{|\!\sim}_{\mathsf{L}} \Delta$ and the result follows directly from (a).
(c) Again, the right-to-left direction is immediate. For the other direction, suppose that $\bigvee_{i=1}^{n} A_i \mathrel{|\!\sim}_{\mathsf{L}} B$. Clearly $A_i \mathrel{|\!\sim}_{\mathsf{L}} B$ for $i = 1 \ldots n$. Hence by (a), $A_i \vdash_{\mathsf{L}} B$ for $i = 1 \ldots n$. So easily $\bigvee_{i=1}^{n} A_i \vdash_{\mathsf{L}} B$ as required.
(d) All IPC-projective formulas are L-projective (since each L extends IPC), so the result follows from (c). □

What makes projective formulas particularly interesting (and useful) is the fact that for certain logics they can also be characterized in terms of *Kripke models.* [4]

Definition 9 *For Kripke models* $K_1, \ldots, K_n$*, let* $(\sum_i K_i)'$ *denote the Kripke model obtained by attaching one new node below all nodes in* $K_1, \ldots, K_n$ *where no propositional variables are forced.*

Definition 10 *Two Kripke models* K, K' *are* (modal) variants *of each other when they have the same set of nodes and order (or accessibility in the case of modal variants) relation, and their forcing relations agree on all nodes except possibly the root.*

Definition 11 *A class of Kripke models* $\mathcal{K}$ *has the* extension property *if for every finite family of models* $K_1, \ldots, K_n \in \mathcal{K}$*, there is a variant of* $(\sum_i K_i)'$ *in* $\mathcal{K}$*. An intermediate logic is* extensible *if its class of models has the extension property.*

[3] In fact all we require here is a list of basic conditions on the logic such as the Leibniz property and closure under modus ponens.
[4] For basic definitions regarding Kripke models refer to [**?**].

Theorem 12 (Ghilardi [4]) *A formula is* IPC*-projective iff its class of Kripke models has the extension property.*

Example 13 *Using this definition it is not difficult to see that the formulas* p, $\neg p$, $\neg p \to (q \wedge r)$ *or* $p \to A$ *are* IPC*-projective (e.g. for* p *and* $\neg p$ *the constant substitutions* $\top$ *and* $\bot$ *are* IPC*-projective unifiers), while* $\neg p \to (q \vee r)$ *and* $\neg p \vee \neg\neg p$ *are not. In particular, the antecedents of the atomic version of the Visser rules:*

$$((p \to q) \to (r_{n+1} \vee r_{n+2})) \vee s \;/\; (\bigvee_{j=1}^{n+2} (p \to q) \to r_j) \vee s,$$

are not IPC*-projective, while every one of the disjuncts of the conclusion is.*

Note moreover that De Jongh and Bezhanishvili have showed that for any finite set of propositional variables there are only finitely many IPC*-projective formulas containing only atoms in the given set, and have given a characterization of the* IPC*-projective formulas in one (De Jongh in [**?**]) and two variables.*

Ghilardi [5] has also extended this characterization to a wide range of modal logics (we follow here the terminology of [11]).

Definition 14 *Let* K_k *denote the Kripke model* K *restricted to the domain* $\{l : kRl \text{ or } k = l\}$*. The root of* K *is the cluster* $\{k : \forall l \neq k(kRl)\}$*. An* L*-frame is a frame such that every model on that frame is a model of* L*. An* L*-model is a model based on an* L*-frame.*

Definition 15 *For frames* $F_1, \ldots, F_n$*, denote by* $(\Sigma F_j)^i$ *and* $(\Sigma F_j)^r$*, the frames obtained by adding, respectively, one irreflexive and one reflexive node below all nodes in the frame.*

Definition 16 *A normal modal logic* L *has the* finite model property *if every refutable formula is refutable on a finite* L*-frame.* L *is* extensible *if it is a normal extension of* K4 *with the finite model property such that for all finite sets of* L*-frames* $F_1, \ldots, F_n$ *the frame* $(\Sigma F_j)^i$ *is an* L*-frame unless* L *is reflexive, and* $(\Sigma F_j)^r$ *is an* L*-frame unless* L *is irreflexive.*

Definition 17 *A class of finite modal models* $\mathcal{K}$ *has the* modal extension property *if for every model* K*, if* $K_k \in \mathcal{K}$ *for all* k *not in the root of* K*, then there is a variant of* K *in* $\mathcal{K}$*.*

Theorem 18 (Ghilardi [5]) *For every normal extension* L *of* K4 *with the finite model property a formula is* L*-projective iff its class of* L*-models has the modal extension property.*

Example 19 *Using this theorem it is not difficult to see that for each extensible modal logic* L*, the formulas* $\boxdot p$*,* $\neg p$*,* $p \to A$ *are* L*-projective, while e.g.* $\boxdot p \to$

$(q \lor r)$ is not. In particular, the antecedents of the atomic versions of the Visser rules for the modal case:

$$(A^{\bullet}) \qquad \Box p \to \bigvee_{i=1}^{n} \Box q_i \;/\; \{\boxdot p \to q_i\}_{i=1}^{n}$$

$$(A^{\circ}) \qquad \bigwedge_{j=1}^{m}(p_j \leftrightarrow \Box p_j) \to \bigvee_{i=1}^{n} \Box q_i \;/\; \{\bigwedge_{j=1}^{m} \boxdot p_j \to q_i\}_{i=1}^{n},$$

are not L*-projective, while every element of the conclusion is.*

3 Modal Logics

In this section we define uniform Gentzen-style calculi for deriving admissible gs-rules of extensible modal logics. We begin by introducing systems for the paradigmatic cases K4, S4, and GL. We then use these systems to show that any calculus for derivability in an extensible modal logic can be extended to a proof system for admissibility in that logic.

3.1 Proof Systems

We construct calculi for admissibility in much the same way as for derivability: we give rules for connectives on the left and right of sequents. The difference here is that the sequents themselves occur either on the left or the right; that is, as premises or conclusions of a gs-rule, doubling the number of rules required. For sequents occurring on the right, we adapt rules from calculi for derivability by adding variables $\mathcal{G}$ and $\mathcal{H}$ standing for arbitrary multisets of sequents. For sequents occurring on the left, we make use of invertibility properties of the rules on the right. Calculi are then completed by adding structural rules and various rules that allow sequents to interact.

We define the following core set of rules for extensible modal logics:

Definition 20 (Core Modal Rules)

Initial GS-Rules

$$\frac{}{\mathcal{G} \rhd (\Gamma, A \Rightarrow A, \Delta), \mathcal{H}}\ (ID) \qquad\qquad \frac{}{\mathcal{G} \rhd (\Gamma, \bot \Rightarrow \Delta), \mathcal{H}}\ (\bot)$$

Right Logical Rules

$$\frac{\mathcal{G} \rhd (\Gamma \Rightarrow A, \Delta), \mathcal{H} \quad \mathcal{G} \rhd (\Gamma \Rightarrow B, \Delta), \mathcal{H}}{\mathcal{G} \rhd (\Gamma \Rightarrow A \wedge B, \Delta), \mathcal{H}} \rhd(\Rightarrow\wedge) \qquad \frac{\mathcal{G} \rhd (\Gamma, A, B \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \rhd (\Gamma, A \wedge B \Rightarrow \Delta), \mathcal{H}} \rhd(\wedge\Rightarrow)$$

$$\frac{\mathcal{G} \rhd (\Gamma, A \Rightarrow \Delta), \mathcal{H} \quad \mathcal{G} \rhd (\Gamma, B \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \rhd (\Gamma, A \vee B \Rightarrow \Delta), \mathcal{H}} \rhd(\vee\Rightarrow) \qquad \frac{\mathcal{G} \rhd (\Gamma \Rightarrow A, B, \Delta), \mathcal{H}}{\mathcal{G} \rhd (\Gamma \Rightarrow A \vee B, \Delta), \mathcal{H}} \rhd(\Rightarrow\vee)$$

$$\frac{\mathcal{G} \rhd (\Gamma \Rightarrow A, \Delta), \mathcal{H} \quad \mathcal{G} \rhd (\Gamma, B \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \rhd (\Gamma, A \to B \Rightarrow \Delta), \mathcal{H}} \rhd(\to\Rightarrow) \qquad \frac{\mathcal{G} \rhd (\Gamma, A \Rightarrow B, \Delta), \mathcal{H}}{\mathcal{G} \rhd (\Gamma \Rightarrow A \to B, \Delta), \mathcal{H}} \rhd(\Rightarrow\to)$$

Left Logical Rules

$$\frac{\mathcal{G}, (\Gamma, A, B \Rightarrow \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, A \wedge B \Rightarrow \Delta) \rhd \mathcal{H}} (\wedge\Rightarrow)\rhd \qquad \frac{\mathcal{G}, (\Gamma \Rightarrow A, \Delta), (\Gamma \Rightarrow B, \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow A \wedge B, \Delta) \rhd \mathcal{H}} (\Rightarrow\wedge)\rhd$$

$$\frac{\mathcal{G}, (\Gamma \Rightarrow A, B, \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow A \vee B, \Delta) \rhd \mathcal{H}} (\Rightarrow\vee)\rhd \qquad \frac{\mathcal{G}, (\Gamma, A \Rightarrow \Delta), (\Gamma, B \Rightarrow \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, A \vee B \Rightarrow \Delta) \rhd \mathcal{H}} (\vee\Rightarrow)\rhd$$

$$\frac{\mathcal{G}, (\Gamma, B \Rightarrow \Delta), (\Gamma \Rightarrow A, \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, A \to B \Rightarrow \Delta) \rhd \mathcal{H}} (\to\Rightarrow)\rhd \qquad \frac{\mathcal{G}, (\Gamma, A \Rightarrow B, \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow A \to B, \Delta) \rhd \mathcal{H}} (\Rightarrow\to)\rhd$$

$$\frac{\mathcal{G}, (\Gamma, \Box p \Rightarrow \Delta), (A \Rightarrow p) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, \Box A \Rightarrow \Delta) \rhd \mathcal{H}} (\Box\Rightarrow)\rhd \qquad \frac{\mathcal{G}, (\Gamma \Rightarrow \Box p, \Delta), (p \Rightarrow A) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow \Box A, \Delta) \rhd \mathcal{H}} (\Rightarrow\Box)\rhd$$

where in $(\Box\Rightarrow)\rhd$ *and* $(\Rightarrow\Box)\rhd$*,* A *is non-atomic and* p *does not occur in* $\mathcal{G}$*,* $\mathcal{H}$*,* Γ*,* Δ*,* A*.*

Structural Rules

$$\frac{\mathcal{G} \rhd \mathcal{H}}{\mathcal{G}, S \rhd \mathcal{H}} (W)\rhd \qquad \frac{\mathcal{G} \rhd \mathcal{H}}{\mathcal{G} \rhd S, \mathcal{H}} \rhd(W) \qquad \frac{\mathcal{G}, S, S, \rhd \mathcal{H}}{\mathcal{G}, S \rhd \mathcal{H}} (C)\rhd \qquad \frac{\mathcal{G} \rhd S, S, \mathcal{H}}{\mathcal{G} \rhd S, \mathcal{H}} \rhd(C)$$

Anti-Cut and Projection Rules

$$\frac{\mathcal{G}, (\Gamma, \Pi \Rightarrow \Sigma, \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, A \Rightarrow \Delta), (\Pi \Rightarrow A, \Sigma) \rhd \mathcal{H}} (AC) \qquad \frac{\mathcal{G} \rhd (\Gamma, \boxdot I(S) \Rightarrow \Delta), \mathcal{H}}{\mathcal{G}, S \rhd \mathcal{H}} (PJ)$$

where $(\Gamma \Rightarrow \Delta) \in \mathcal{H} \cup \{\Rightarrow\}$

We now extend this core set to obtain proof systems for admissibility in the paradigmatic cases of K4, S4, and GL.

Definition 21 GAK4 *consists of the core modal rules plus:*

$$\frac{\mathcal{G} \rhd (\boxdot\Gamma \Rightarrow A), \mathcal{H}}{\mathcal{G} \rhd (\Box\Gamma, \Pi \Rightarrow \Box A, \Delta), \mathcal{H}} \rhd(\Box)_{\mathsf{K4}}$$

and the Visser Rules*:*

$$\frac{[\mathcal{G}, (\boxdot\Gamma \Rightarrow A) \rhd \mathcal{H}]_{A\in\Delta}}{\mathcal{G}, (\Box\Gamma \Rightarrow \Box\Delta) \rhd \mathcal{H}} (V^i) \qquad \frac{[\mathcal{G}, \boxdot(\Gamma \cup \Pi) \Rightarrow A) \rhd \mathcal{H}]_{A\in\Delta}}{\mathcal{G}, (\Gamma \equiv \Box\Pi \Rightarrow \Box\Delta) \rhd \mathcal{H}} (V^r)$$

where $(\Gamma \equiv \Box\Pi \Rightarrow \Delta)$ *denotes any set of sequents* X *such that:*

$$\forall \boxdot Z \subseteq (\Gamma \cup \Box\Pi) \exists \Sigma \subseteq \boxdot Z \exists \Theta \subseteq ((\Gamma \cup \Box\Pi) - \boxdot Z) \exists \Psi \subseteq \Delta (\Sigma \Rightarrow \Theta, \Psi) \in X)$$

Note that in particular, taking $\Sigma = \boxdot Z$ and $\Psi = \Delta$ in (V^r), we obtain the rule:

$$\frac{[\mathcal{G}, (\boxdot(\Gamma \cup \Pi) \Rightarrow A) \ \triangleright \ \mathcal{H}]_{A \in \Delta}}{\mathcal{G}, \{\boxdot Z \Rightarrow ((\Gamma \cup \Box\Pi) - \boxdot Z), \Box\Delta \mid \boxdot Z \subseteq \Gamma \cup \Box\Pi\} \ \triangleright \ \mathcal{H}}$$

Definition 22 $\mathbf{GAS4}$ *consists of the core modal rules plus (V^r), $\triangleright(\Box)_{\mathsf{K4}}$, and:*

$$\frac{\mathcal{G} \ \triangleright \ (\boxdot\Gamma, \Pi \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \ \triangleright \ (\Box\Gamma, \Pi \Rightarrow \Delta), \mathcal{H}} \ \triangleright(\Box)_{\mathsf{S4}}$$

Definition 23 $\mathbf{GAGL}$ *consists of the core modal rules plus (V^i) and:*

$$\frac{\mathcal{G} \ \triangleright \ (\boxdot\Gamma, \Box A \Rightarrow A), \mathcal{H}}{\mathcal{G} \ \triangleright \ (\Box\Gamma, \Pi \Rightarrow \Box A, \Delta), \mathcal{H}} \ \triangleright(\Box)_{\mathsf{GL}}$$

Some explanation is required. First note that weakening and contraction are "built in" to the right logical rules (taken from [15]). This is not strictly necessary. In fact, any calculus for derivability in the logic can be used as a template for the right logical rules. However, for $\wedge$, $\vee$, and $\to$, the rules given here are easily "inverted" to obtain corresponding rules on the left; that is, by replacing the conclusion of the original sequent rule with the premises of that rule. This approach fails in the case of the (non-invertible) modal rules. Instead the rules $(\Box \Rightarrow)\triangleright$ and $(\Rightarrow \Box)\triangleright$ decompose modal formulas on the left by replacing the formula A in $\Box A$ by a new propositional variable p. The soundness of these rules follows from the fact that any substitution for the conclusion can be extended (since p does not occur there) by subsituting A for p.

The structural rules permit weakening and contraction of sequents occurring as premises and conclusions of sequent rules. The "Projection Rule" (PJ) allows sequents on the left to be used as modal implications on the right, corresponding to the fact that derivability implies admissibility. [5]

Example 24 *It is easy to see that any gs-rule containing the same sequent on both sides (i.e. as a premise and as a conclusion) is derivable using (PJ). Indeed, generalizing a little, the following gs-rule may be taken as a useful derived initial gs-rule:*

$$\frac{}{\mathcal{G}, (\Gamma \Rightarrow \Delta) \ \triangleright \ (\Gamma, \Pi \Rightarrow \Sigma, \Delta), \mathcal{H}} \ (SID)$$

Just observe that the following gs-rule:

$$\mathcal{G} \ \triangleright \ (\Gamma, \Pi, (\bigwedge \Gamma \to \bigvee \Delta) \Rightarrow \Sigma, \Delta), \mathcal{H}$$

is derivable using the initial gs-rules and right logical rules, and hence that (SID) is derivable using (PJ).

[5] In the particular cases of $\mathbf{GAK4}$, $\mathbf{GAS4}$, and $\mathbf{GAL}$, $\boxdot I(S)$ in (PJ) can be replaced with $I(S)$, allowing sequents on the left to be used directly as implications on the right.

The "Anti-Cut Rule" (AC) corresponds directly to the fact that the usual cut rule is admissible in the logic. Observe however that, unlike cut, (AC), and indeed all the rules except $(\Rightarrow\Box)\triangleright$ and $(\Box\Rightarrow)\triangleright$, have the subformula property. That is, every formula occurring in a premise of such a rule occurs as a subformula of a formula in the conclusion. Note, moreover, that a suitable cut rule for admissibility would be of the form:

$$\frac{\mathcal{G}, S \triangleright \mathcal{H} \quad \mathcal{G}' \triangleright S, \mathcal{H}'}{\mathcal{G}, \mathcal{G}' \triangleright \mathcal{H}', \mathcal{H}}\ (CUT)$$

However, rather than eliminate (CUT) syntactically, here we obtain the admissibility of the rule indirectly via a (semantic) completeness proof.

Example 25 *Consider the following cut rule:*

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Pi \Rightarrow A, \Sigma}{\Gamma, \Pi \Rightarrow \Sigma, \Delta}$$

The gs-rule version is derivable as follows:

$$\dfrac{\dfrac{}{(\Gamma, \Pi \Rightarrow \Sigma, \Delta) \triangleright (\Gamma, \Pi \Rightarrow \Sigma, \Delta)}\ (SID)}{(\Gamma, A \Rightarrow \Delta), (\Pi, A \Rightarrow \Sigma) \triangleright (\Gamma, \Pi \Rightarrow \Sigma, \Delta)}\ (AC)$$

The "Visser Rules" (V^i) and (V^r) are a little harder to understand, corresponding to the rules $(A^\bullet)$ and (A°), respectively, given by Jerabek in [11] (see Example 19).

Example 26 *For non-reflexive logics the gs-rule versions of $(A^\bullet)$ are derived using (V^i) as follows:*

$$\dfrac{\dfrac{}{[(\boxdot A \Rightarrow B) \triangleright \{\boxdot A \Rightarrow B\}_{B\in\Delta}]_{B\in\Delta}}\ (SID)}{(\Box A \Rightarrow \Box\Delta) \triangleright \{\boxdot A \Rightarrow B\}_{B\in\Delta}}\ (V^i)$$

For non-irreflexive logics, we can use (V^r) to show:

$$\dfrac{\dfrac{}{[(\boxdot\Gamma \Rightarrow A) \triangleright \{\boxdot\Gamma \Rightarrow A\}_{A\in\Delta}]_{A\in\Delta}}\ (SID)}{(\Gamma \equiv \Box\Gamma \Rightarrow \Box\Delta) \triangleright \{\boxdot\Gamma \Rightarrow A\}_{A\in\Delta}}\ (V^r)$$

Let $\Gamma \leftrightarrow \Box\Gamma$ stand for the set $\{A \leftrightarrow \Box A \mid A \in \Gamma\}$. To derive $(\Gamma \leftrightarrow \Box\Gamma \Rightarrow \Box\Delta) \triangleright \{\boxdot\Gamma \Rightarrow A\}_{A\in\Delta}$, the gs-rule version of (A°), it is hence sufficient that for any $\mathcal{H}$, Δ, and Γ, we can derive $(\Gamma \leftrightarrow \Box\Gamma \Rightarrow \Delta) \triangleright \mathcal{H}$ from $(\Gamma \equiv \Box\Gamma \Rightarrow \Delta) \triangleright \mathcal{H}$. For example, if $(A, \Box A \Rightarrow \Delta), (\Rightarrow A, \Box A, \Delta) \triangleright \mathcal{H}$ is derivable, then we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(A, \Box A \Rightarrow \Delta), (\Rightarrow A, \Box A, \Delta) \triangleright \mathcal{H}}{(A \Rightarrow A, \Delta), (A, \Box A \Rightarrow \Delta), (\Rightarrow A, \Box A, \Delta) \triangleright \mathcal{H}}\ (W)\triangleright}{(A \Rightarrow A, \Delta), (A, \Box A \Rightarrow \Delta), (\Box A \Rightarrow \Box A, \Delta), (\Rightarrow A, \Box A, \Delta) \triangleright \mathcal{H}}\ (W)\triangleright}{(A \to \Box A, A \Rightarrow \Delta), (\Box A \Rightarrow \Box A, \Delta), (\Rightarrow A, \Box A, \Delta) \triangleright \mathcal{H}}\ (\to\Rightarrow)\triangleright}{(A \to \Box A, A \Rightarrow \Delta), (A \to \Box A \Rightarrow \Box A, \Delta) \triangleright \mathcal{H}}\ (\to\Rightarrow)\triangleright}{(A \to \Box A, \Box A \to A \Rightarrow \Delta) \triangleright \mathcal{H}}\ (\to\Rightarrow)\triangleright$$

Extensible modal logics possessing a natural sequent calculus for derivability provide the most elegant examples of our systems for admissibility. However, all that we really require for the rules on the right is that they provide a sound and complete method for establishing derivability in the logic at hand. We can then expand this calculus with the core modal rules, plus (V^i) if the logic is not reflexive, and (V^r) if the logic is not irreflexive. The result is a calculus which, as we show in the next section, is sound and complete for admissibility in the logic.

Definition 27 *Let* L *be an extensible modal logic. A calculus* $\mathbf{GAL}$ *is* L*-fitting if:*

(1) $\mathbf{GAL}$ *extends the core modal rules.*
(2) If L *is not reflexive, then* (V^i) *is a rule of* $\mathbf{GAL}$.
(3) If L *is not irreflexive, then* (V^r) *is a rule of* $\mathbf{GAL}$.
(4) If $\vdash_{\mathsf{L}} S$*, then* $\vdash_{\mathbf{GAL}} \rhd S$.
(5) If $\vdash_{\mathbf{GAL}} \mathcal{R}$*, then* $\mathrel{|\!\sim}_{\mathsf{L}} \mathcal{R}$.

3.2 Soundness and Completeness

We first show that the core modal rules and (where appropriate) the Visser rules are sound for extensible modal logics.

Proposition 28 *Let* L *be an extensible modal logic.*

(a) All the core modal rules are L*-sound.*
(b) If L *is not reflexive, then* (V^i) *is* L*-sound.*
(c) If L *is not irreflexive, then* (V^r) *is* L*-sound.*

Proof. (a) The initial gs-rules and right logical rules (taken from a calculus for CPC in [15]) are clearly L-sound. For the left logical rules for $\wedge$, $\vee$, and $\to$, soundness follows directly from the CPC-invertibility of the rules on the right. For $(\Box\Rightarrow)\rhd$, suppose that the premise is L-admissible and let σ be an L-unifier for $I(S)$ for all $S \in \mathcal{G}$ and $I(\Gamma, \Box A \Rightarrow \Delta)$. Since p does not occur in $\mathcal{G}, \mathcal{H}, \Gamma, \Delta, A$ we can extend σ by mapping p to A. It follows that σ is an L-unifier for $I(\Gamma, \Box p \Rightarrow \Delta)$ and $I(A \Rightarrow p)$. Hence, by the admissibility of the premise, σ is an L-unifier for some $S \in \mathcal{H}$ as required. The argument for $(\Box\Rightarrow)\rhd$ is very similar.

It is easy to see that the structural rules are L-sound. For (AC), suppose that the premise is L-admissible. Let σ be an L-unifier for $I(S)$ for all $S \in \mathcal{G}$, $I(\Gamma, A \Rightarrow \Delta)$, and $I(\Pi \Rightarrow A, \Sigma)$. By the L-admissibility of the cut rule for L, we get that σ is an L-unifier for $I(\Gamma, \Pi \Rightarrow \Sigma, \Delta)$ and the result follows using the L-admissibility of the premise. For (PJ), suppose that the premise is L-admissible and that σ is an L-unifier for $I(S')$ for all $S' \in \mathcal{G}$ and $I(S)$. It follows that σ is an L-unifier either for $I(\Gamma, \boxdot I(S) \Rightarrow \Delta)$ or for $I(S')$ for some $S' \in \mathcal{H}$. In the latter case we are done. In the former case, since σ is an L-unifier for $I(S)$ it is an L-unifier for $\boxdot I(S)$, and hence also for $I(\Gamma \Rightarrow \Delta)$. Since there is no L-unifier for the empty sequent $\Rightarrow$, we have $(\Gamma \Rightarrow \Delta) \in \mathcal{H}$ and we are done.

(b) For (V^i), suppose that all the premises are L-admissible. Let σ be an L-unifier for $I(S)$ for all $S \in \mathcal{G}$ and $I(\Box\Gamma \Rightarrow \Box\Delta)$. If σ is a L-unifier for $I(\boxdot\Gamma \Rightarrow A)$ for some $A \in \Delta$, then we are done using the admissibility of the premises. Otherwise let K_A be a L-model refuting $\sigma(I(\boxdot\Gamma \Rightarrow A))$ for each $A \in \Delta$. They exist because L has the finite model property. The fact that L is extensible and not reflexive implies that $(\Sigma_{A\in\Delta}K_A)^i$ is also an L-model. But $\sigma(I(\Box\Gamma \Rightarrow \Box\Delta))$ is refuted at the root of this model, a contradiction.

(c) For (V^r), assume

$$\forall A \in \Delta : \ \mathrel{|\!\sim} \mathcal{G}, (\boxdot(\Gamma \cup \Pi) \Rightarrow A) \rhd \mathcal{H}.$$

Let σ be an L-unifier for $\mathcal{G}$ and $\Gamma \equiv \Box\Pi \Rightarrow \Box\Delta$, recalling that the latter denotes a set of sequents X such that:

$$\forall \boxdot Z \subseteq \Gamma \cup \Box\Pi \exists \Sigma \subseteq \boxdot Z \exists \Theta \subseteq ((\Gamma \cup \Box\Pi) - \boxdot Z) \exists \Psi \subseteq \Delta((\Sigma \Rightarrow \Theta, \Psi) \in X)$$

Arguing by contradiction, suppose that σ is not an L-unifier for $\mathcal{H}$. Hence σ is not a unifier for $\boxdot(\Gamma \cup \Pi) \Rightarrow A$ for all $A \in \Delta$. Let K_A be the L-models that are counter models in which $\sigma(\boxdot(\Gamma \cup \Pi))$ is forced and $\sigma(A)$ is not, which exist by the finite model property of L. Since L is extensible, $(\Sigma_{A\in\Delta}K_A)^r$ is an L-model with a reflexive root r. Consider $\Sigma = \{A \in \Gamma \cup \Box\Pi \mid r \Vdash \sigma(A)\}$. Because of the reflexivity of r and since each K_A forces $\sigma(\boxdot\Gamma)$, we have $r \Vdash \bigwedge\sigma(\Sigma)$ but $r \nVdash \sigma(\bigvee((\Gamma \cup \Box\Pi) - \Sigma) \bigvee \Box\Delta)$, contradicting the fact that $\vdash \sigma(X)$. □

In particular, using the fact that the rules on the right for **GAK4**, **GAS4**, and **GAL** are sound and complete for K4, S4, and GL, respectively (see e.g. [**?**] for references), we obtain:

Corollary 29 **GAL** *is* L*-fitting for* $\mathsf{L} \in \{\mathsf{K4}, \mathsf{S4}, \mathsf{GL}\}$.

Our completeness proof consists of several stages. First we show completeness for a restricted class of gs-rules: L-derivable gs-rules with at most one sequent on the right. The idea being (to look ahead a little) to show eventually that all L-admissible gs-rules are **GAL**-derivable from gs-rules in this class.

Lemma 30 *Let* L *be an extensible modal logic and let* **GAL** *be* L*-fitting. If* $\vdash_{\mathsf{L}} \mathcal{G} \rhd \mathcal{H}$ *where* $|\mathcal{H}| \leq 1$*, then* $\vdash_{\mathbf{GAL}} \mathcal{G} \rhd \mathcal{H}$*.*

Proof. Suppose that $\vdash_{\mathsf{L}} \mathcal{G} \rhd \mathcal{H}$ where $|\mathcal{H}| \leq 1$. If $\mathcal{H} = \{\Gamma \Rightarrow \Delta\}$, or letting $\Gamma = \Delta = \emptyset$ if $\mathcal{H} = \emptyset$, then:

$$\{I(S)\}_{S\in\mathcal{G}} \vdash_{\mathsf{L}} I(\Gamma \Rightarrow \Delta)$$

But then using the modal deduction theorem (see e.g. [**?**] for details):

$$\vdash_{\mathsf{L}} \bigwedge_{S \in \mathcal{G}} \boxdot I(S) \rightarrow I(\Gamma \Rightarrow \Delta)$$

Hence $\vdash_{\mathsf{L}} \Gamma, \{\boxdot I(S)\}_{S \in \mathcal{G}} \Rightarrow \Delta$, and since $\mathbf{GAL}$ is L-fitting:

$$\vdash_{\mathbf{GAL}} \rhd (\Gamma, \{\boxdot I(S)\}_{S \in \mathcal{G}} \Rightarrow \Delta)$$

So by repeated applications of (PJ), $\vdash_{\mathbf{GAL}} \mathcal{G} \rhd \mathcal{H}$ as required. □

The next step is to show that the left logical rules are invertible. Since, each of these rules has fewer connectives in its premise than its conclusion, it then follows that the question of the admissibility of gs-rules can be restricted to a particular subclass.

Lemma 31 *Let* L *be an extensible modal logic. The left logical rules are* L*-invertible.*

Proof. The cases for $\wedge$, $\vee$, and $\rightarrow$ follow from the L-soundness of the left logical rules. As an example, we consider $(\Rightarrow \wedge)\rhd$. Suppose that the conclusion is L-admissible and let σ be a unifier for $I(\Gamma \Rightarrow A, \Delta)$, $I(\Gamma \Rightarrow B, \Delta)$, and $I(S)$ for all $S \in \mathcal{G}$. In particular, $\vdash_{\mathsf{L}} \sigma(I(\Gamma \Rightarrow A, \Delta))$ and $\vdash_{\mathsf{L}} \sigma(I(\Gamma \Rightarrow B, \Delta))$. Hence $\vdash_{\mathsf{L}} \sigma((I(\Gamma \Rightarrow A \wedge B, \Delta))$, i.e. σ is a unifier for $I(\Gamma \Rightarrow A \wedge B, \Delta)$. It follows therefore by the admissibility of the conclusion, that σ is a unifier for $I(S)$ for some $S \in \mathcal{H}$. For $(\Box \Rightarrow)\rhd$, suppose that the conclusion is L-admissible and let σ be an L-unifier for $I(\Gamma, \Box p \Rightarrow \Delta)$, $I(A \Rightarrow p)$, and $I(S)$ for all $S \in \mathcal{G}$. Since L is a normal modal logic, σ is an L-unifier for $I(\Box A \Rightarrow \Box p)$. Hence by the L-admissibility of cut, σ is an L-unifier for $I(\Gamma, \Box A \Rightarrow \Delta)$ and the result follows using the L-admissibility of the conclusion. The case of $(\Rightarrow \Box)\rhd$ is very similar. □

Definition 32 *A gs-rule* $\mathcal{G} \rhd \mathcal{H}$ *is* modal-irreducible *if the sequents in* $\mathcal{G}$ *contain only atoms and boxed atoms.*

Definition 33 *We define the following measures:*

- $c(q) = 1$ *for all propositional variables* q, *and* $c(\#(A_1, \ldots, A_n)) = c(A_1) + \ldots + c(A_n) + 1$ *for formulas* $A_1, \ldots, A_n$, *and each connective* $\#$ *with arity* n.
- $mc(\Gamma \Rightarrow \Delta)$ *is the multiset* $\{c(A) \ : \ A \in \Gamma \cup \Delta\}$ *for a sequent* $\Gamma \Rightarrow \Delta$.
- $mmc(\mathcal{G} \rhd \mathcal{H})$ *is the multiset* $\{mc(S) \ : \ S \in \mathcal{G} \cup \mathcal{H}\}$ *for a gs-rule* $\mathcal{G} \rhd \mathcal{H}$.

Definition 34 *For multisets* α, β *of integers:* $<_m$ *is the transitive closure of* $<_1$, *where* $\alpha <_1 \beta$ *if* α *is obtained by replacing an element* n *of* β *by finitely many (possibly* 0*) copies of* m *for some* $m < n$. *Similarly, for multisets* ϕ, ψ *of multisets of integers,* $<_{mm}$ *is the transitive closure of* $<_2$, *where* $\phi <_2 \psi$ *if* ϕ *is obtained by replacing an element* α *of* ψ *by finitely many (possibly* 0*) copies of* β *for some* $\beta <_m \alpha$.

Theorem 35 (Dershowitz and Manna [12]) *$<_m$ and $<_{mm}$ are well-orderings.*

Lemma 36 *Let L be an extensible modal logic. Every L-admissible gs-rule is derivable from an L-admissible modal irreducible gs-rule using the left logical rules.*

Proof. Since $<_{mm}$ is a well-ordering, we can prove the lemma by induction on $mmc(\mathcal{R})$ where $\mathcal{R}$ is an L-admissible gs-rule. If $\mathcal{R}$ is modal-irreducible, then we are done. Otherwise there is an instance of a left logical rule $\mathcal{R}' \;/\; \mathcal{R}$ such that $mmc(\mathcal{R}') < mmc(\mathcal{R})$, where by L-invertibility $\mathcal{R}'$ is an L-admissible gs-rule. Hence, by the induction hypothesis, $\mathcal{R}'$ is $\mathbf{GAL}$-derivable from an L-admissible modal irreducible gs-rule, and then clearly so also is $\mathcal{R}$. $\Box$

As a consequence of the previous lemma, it is sufficient to establish completeness for modal irreducible L-admissible gs-rules. Intuitively, we do this (working upwards) by applying the anti-cut rule (AC) and the Visser rules (V^i) and (V^r) exhaustively, using structural rules at each step to ensure that sequents occurring in the conclusion occur also in the premises. Since these rules have the subformula property, the number of possible sequents obtained in this way is finite and the process terminates with gs-rules which we call L-modal full.

Definition 37 *Let $\mathcal{R}$ be a rule with premises $\mathcal{G}_i \rhd \mathcal{H}_i$ for $i = 1 \ldots n$, and conclusion $\mathcal{G} \rhd \mathcal{H}$. An application of $\mathcal{R}$ is* non-looping *if for each $i = 1 \ldots n$ there exists either $S \in \mathcal{G}_i$ such that $S \notin \mathcal{G}$ or $S \in \mathcal{H}_i$ such that $S \notin \mathcal{H}$.*

Definition 38 *A gs-rule $\mathcal{R}$ is* full *with respect to a set of rules X if there is no non-looping application of a rule in X to $\mathcal{R}$. For an extensible modal logic L we will say that a gs-rule is* L-modal full *if it is modal-irreducible and full with respect to (AC) plus (V^i) if L is not reflexive, and (AC) plus (V^r) if L is not irreflexive.*

Observe that L-modal fullness is a property that only depends on the left hand side $\mathcal{G}$ of a gs-rule $\mathcal{G} \;\rhd\; \mathcal{H}$.

Lemma 39 *Let L be an extensible modal logic. Every L-admissible gs-rule is derivable from a set of L-modal full L-admissible gs-rules.*

Proof. By Lemma 36 we can restrict our attention to L-admissible modal irreducible gs-rules $\mathcal{R} = \mathcal{G} \rhd \mathcal{H}$, assuming without loss of generality that $\mathcal{G}$ and $\mathcal{H}$ contain no repeated elements. We proceed by induction on $n(\mathcal{R}) = 2M - (|\mathcal{G}| + |\mathcal{H}|)$ where M is the number of different sequents possible containing subformulas of formulas occurring in $\mathcal{R}$. If $n(\mathcal{R}) = 0$, then $\mathcal{R}$ must be L-modal full since any application of a rule will be looping. If $\mathcal{R}$ is not L-modal full, then there exists an instance of (AC), (V^i), or (V^r) with conclusion $\mathcal{G} \rhd \mathcal{H}$, such that for each premise $\mathcal{G}' \rhd \mathcal{H}'$, $\mathcal{G}' \not\subseteq \mathcal{G}$, or $\mathcal{H}' \not\subseteq \mathcal{H}$. Clearly $\mathcal{G} \rhd \mathcal{H}$ is $\mathbf{GAL}$-derivable from L-admissible premises of the form $\mathcal{R}' = \mathcal{G}, \mathcal{G}' \rhd \mathcal{H}', \mathcal{H}$ using also $(C)\rhd$. But $n(\mathcal{R}') < n(\mathcal{R})$.

Hence by the induction hypothesis, each $\mathcal{R}'$ is derivable from a set of L-modal full L-admissible gs-rules, and the result follows. $\square$

We use these lemmas to show that if an L-admissible full gs-rule $\mathcal{G} \rhd \mathcal{H}$ is inconsistent or the formula $\bigwedge_{S \in \mathcal{G}} I(S)$ is L-projective, then is $\mathcal{G} \rhd \mathcal{H}$ is $\mathbf{GAL}$-derivable. We then deal with the case where A is consistent and not L-projective and show, using Ghilardi's characterization of L-projective formulas, that this case cannot occur by deriving a contradiction.

Theorem 40 *Let* L *be an extensible modal logic, and let* $\mathbf{GAL}$ *be* L*-fitting. Then*

$$\mid\!\sim_{\mathsf{L}} \mathcal{R} \quad \textit{iff} \ \vdash_{\mathbf{GAL}} \mathcal{R}$$

Proof. The right-to-left direction follows from the definition of L-fitting. For the other direction, by Lemma 39 it is sufficient to assume that $\mathcal{R} = \mathcal{G} \rhd \mathcal{H}$ is a *modal-full* L-admissible gs-rule. Let $C = \bigwedge_{S \in \mathcal{G}} I(S)$. If C is inconsistent, then $\vdash_{\mathsf{L}} \mathcal{G} \rhd$, and if C is L-projective, then using Lemma 8 (a), $\vdash_{\mathsf{L}} \mathcal{G} \rhd S$ for some $S \in \mathcal{H}$. In both cases, by Lemma 30 and $\rhd(W)$, $\vdash_{\mathbf{GAL}} \mathcal{G} \rhd \mathcal{H}$.

Hence assume that C is consistent and not L-projective. We use Theorem 18 of Ghilardi, which tells us that C does not have the L-extension property, to obtain a non-empty L-model K such that $K_k \Vdash C$ for all k not in the root of K, and such that every variant of K refutes C. We write $K' \Vdash A$ or $A \in K'$ if $K_k \Vdash A$ for all k not in the root of K. Let $M_1, \ldots, M_k$ be the variants of K, and let $S_1, \ldots, S_k$ be the sequents in $\mathcal{G}$ for which $M_i \nVdash S_i$, where $S_i = (\Gamma_i \Rightarrow \Delta_i)$ and $C_i = I(S_i)$. Thus $M_i \Vdash \bigwedge \Gamma_i$ and $M_i \nVdash \bigvee \Delta_i$. Note that Γ_i and Δ_i contain only atoms and boxed atoms by the fullness of $\mathcal{G} \rhd \mathcal{H}$. We distinguish by cases according to whether K is reflexive or not, recalling that in some logics this is possible and others not.

Irreflexive case. First, suppose K is irreflexive. Observe that in this case

$$\Box A \in \Gamma_i \Rightarrow \boxdot A \in K' \quad \text{and} \quad \Box A \in \Delta_i \Rightarrow A \notin K'. \tag{1}$$

Let:

$$A_i =_{def} \bigwedge_{p \in \Gamma_i} p \wedge \bigwedge_{p \in \Delta_i} \neg p \quad \text{and} \quad A =_{def} \bigvee A_i.$$

We show that A is a tautology: consider a valuation v on atoms occuring in C, and define a variant M of K by defining:

$$M \Vdash p \Leftrightarrow v(p) = 1.$$

Suppose that M is the variant M_i. It is not difficult to see that $v(p) = 1$ for $p \in \Gamma_i$ and $v(p) = 0$ for $p \in \Delta_i$; i.e. $v(A_i) = 1$. So A is a tautology, and the formula corresponding to the negation of A, and swapping literals (i.e. p goes to $\neg p$ and vice versa):

$$\bigwedge_i \Big(\bigvee_{p \in \Gamma_i} p \vee \bigvee_{p \in \Delta_i} \neg p \Big),$$

is inconsistent (swapping literals is not necessary but simplifies the reasoning that follows). Therefore, there exists a resolution refutation starting with the clauses:

$$\{p : p \in \Gamma_j\} \cup \{\neg p : p \in \Delta_j\} \text{ for } j = 1 \dots m$$

that ends in the empty clause $\emptyset$. Let $\Theta \cup \Psi'$ be any clause in the refutation, where Θ contains only atoms and Ψ' contains only negated atoms. Define $\Psi = \{p : \neg p \in \Psi'\}$. Then, since $\mathcal{R}$ is full, we can show inductively, using (AC) and (1) for the base case, that there exists $(\Box\Gamma, \Theta \Rightarrow \Psi, \Box\Delta) \in \mathcal{G}$ such that:

$$K' \Vdash \bigwedge \boxdot\Gamma \quad \text{and} \quad \Box A \in \Box\Delta \Rightarrow K' \nVdash A. \tag{2}$$

Now consider the empty clause $\emptyset$, and its corresponding sequent in $\mathcal{G}$ of the form $\Box\Gamma \Rightarrow \Box\Delta$. The rule (V^i) implies that $\boxdot\Gamma \Rightarrow q \in \mathcal{G}$ for some $q \in \Delta$, and hence it follows, as $K' \Vdash C$, that $K' \Vdash \bigwedge \boxdot\Gamma \rightarrow q$. But by (2) we have $K' \Vdash \bigwedge \boxdot\Gamma$ and $K' \nVdash q$. However, $K' \Vdash \bigwedge \boxdot\Gamma \rightarrow q$ implies $K' \Vdash q$, a contradiction.

Reflexive case. Instead of (V^i), the rule (V^r) plays a crucial role here. Recall that we denote the set $\{A \mid K' \Vdash A\}$ by K' and that

$$M_i \Vdash \bigwedge \Gamma_i \quad \text{and} \quad M_i \nVdash \bigvee \Delta_i.$$

Define $K'_c = \{A \mid K' \nVdash A\}$. Observe that $\forall p \in K'$:

$$\begin{aligned} p \in \Gamma_i \Rightarrow M_i \Vdash \boxdot p \qquad & \Box p \in \Gamma_i \Rightarrow M_i \Vdash p \\ p \in \Delta_i \Rightarrow M_i \Vdash \neg p \wedge \neg\Box p \qquad & \Box p \in \Delta_i \Rightarrow M_i \Vdash \neg p \wedge \neg\Box p \end{aligned} \tag{3}$$

In order to apply resolution refutations as in the irreflexive case above, we associate, for $p \in K'$, new atoms l_p with expressions $(p \wedge \Box p)$. Define A_i to be the formula:

$$\bigwedge_{p \in \Gamma_i \setminus K'} p \wedge \bigwedge_{p \in \Gamma_i \cap K'} l_p \wedge \bigwedge_{\Box p \in \Gamma_i, p \in K'} l_p \wedge \bigwedge_{p \in \Delta_i \setminus K'} \neg p \wedge \bigwedge_{p \in \Delta_i \cap K'} \neg l_p \wedge \bigwedge_{\Box p \in \Delta_i, p \in K'} \neg l_p.$$

We show that $A = \bigvee A_i$ is a tautology. Consider a valuation v and define a variant M of K via:

$$\forall p \notin K' : \ M \Vdash p \Leftrightarrow v(p) = 1 \qquad \forall p \in K' : \ M \Vdash p \Leftrightarrow v(l_p) = 1.$$

Suppose M is the variant M_i. This implies that for $p \notin K'$, $p \in \Gamma_i$ implies $v(p) = 1$, and $p \in \Delta_i$ implies $v(p) = 0$. Observe the following relation between atoms l_p and expressions $(p \wedge \Box p)$:

$$v(l_p) = 1 \Leftrightarrow M_i \Vdash p \wedge \Box p \qquad v(l_p) = 0 \Leftrightarrow M_i \Vdash \neg p \wedge \neg\Box p. \tag{4}$$

By (3) we have that for $p \in K'$, $p \in \Gamma_i$ or $\Box p \in \Gamma_i$ implies $v(l_p) = 1$ and $p \in \Delta_i$ or $\Box p \in \Delta_i$ implies $v(l_p) = 0$. Therefore, $v(A_i) = 1$, and thus A is a tautology.

Hence the formula:

$$\bigwedge_{i=1}^{m}\Big(\bigvee_{p\in\Gamma_i\backslash K'} p\vee\bigvee_{p\in\Gamma_i\cap K'} l_p\vee\bigvee_{\Box p\in\Gamma_i, p\in K'} l_p\vee\bigvee_{p\in\Delta_i\backslash K'}\neg p\vee\bigvee_{p\in\Delta_i\cap K'}\neg l_p\vee\bigvee_{\Box p\in\Delta_i, p\in K'}\neg l_p\Big)$$

equivalent to $\neg A$, swapping literals, is inconsistent. So there exists a resolution refutation of:

$$\{p \mid p\in\Gamma_i\backslash K'\}\cup\{l_p \mid p\in K',\ \text{and}\ p\in\Gamma_i\ \text{or}\ \Box p\in\Gamma_i\ \}\cup$$
$$\{\neg p \mid p\in\Delta_i\backslash K'\}\cup\{\neg l_p \mid p\in K',\ \text{and}\ p\in\Delta_i\ \text{or}\ \Box p\in\Delta_i\ \}\ \text{for}\ i\leq m$$

that ends in the empty clause $\emptyset$. Let $\Theta\Pi\cup\Psi\Sigma$ be any clause in the refutation, where Θ contains only atoms not in K', Π contains only atoms of the form l_q, Ψ contains only negated atoms not in K', and Σ contains only negated atoms of the form $\neg l_q$. Observe that no clause contains both p and l_p. Also, the existence of an atom l_p implies $p\in K'$, and $p\notin K'$ implies that there is no l_p. Observe that for the input clauses no variable appears both in the succedent and the antecedent of a sequent. As usual, we assume that no clause in the refutation contains both an atom and its negation. First, some definitions:

$$\Psi_0 =_{def} \{p : \neg p\in\Psi\}\quad \Pi_0 =_{def} \{p, \Box p \mid l_p\in\Pi\}\quad \Sigma_0 =_{def} \{p, \Box p \mid \neg l_p\in\Sigma\}.$$

For a clause R, let:

$$L_R^+ =_{def} \{l_p \mid l_p\in R\} \qquad P_R^+ =_{def} \{\Box p, p \mid l_p\in R\}\cup\{p \mid p\in R\}$$
$$L_R^- =_{def} \{l_p \mid \neg l_p\in R\} \qquad P_R^- =_{def} \{\Box p, p \mid \neg l_p\in R\}\cup\{p \mid \neg p\in R\}$$
$$L_R^c =_{def} \{l_p \mid l_p\notin R, \neg l_p\notin R\} \qquad P_R^c =_{def} \{\Box p, p \mid l_p\in L_R^c\}$$

First we sketch the idea of the proof by an example. The reasoning is similar to the irreflexive case, but more complicated. The idea is to associate with every clause a set of sequents in such a way that for the empty clause the associated set includes $\Gamma\equiv\Box\Gamma\Rightarrow\Box\Delta$ to which we can apply (V^r) and thereby obtain a contradiction. The exact argument in this last step will be given at the end of the proof. Note the similarity with the irreflexive case. What makes this part of the proof more complicated is that we cannot, as in that case, associate sequents with clauses, but rather sets of sequents with clauses.

Suppose that the input clauses are:

$$\{p, l_q, \neg l_r\},\ \{\neg p, l_q, \neg l_r\},\ \{\neg l_q, \neg l_r\},\ \{l_r\}.$$

Moreover, suppose that in this example, if $q\in\Gamma_i\cap K'$, then $\Box q\in\Gamma_i$, and vice versa, and similarly for Δ_i. This is not a necessary assumption, but just facilitates the reasoning below. Thus the initial sequents are

$$p, \boxdot q\Rightarrow r, \Box r \qquad \boxdot\, q\Rightarrow p, r, \Box r \qquad \Rightarrow q, \Box q, r, \Box r \qquad \boxdot\, r\Rightarrow .$$

Observe that this implies that $p \in K'_c$. Following the resolution refutation, we see that the cut on p can be mimicked at the sequent level using the rule (AC), since it implies that $\boxdot q \Rightarrow r, \Box r \in \mathcal{G}$. However, the cuts on l_q and l_r cannot be mimicked at the sequent level. Instead, we keep track of the set P^c_R of all atoms of the form l_x or $\neg l_x$ that do not occur in R (because they have been cut away already, or were never there), and note that for each $\boxdot Z \subseteq P^c_R$ there are $\Gamma \subseteq \boxdot Z$, $\Theta \subseteq P^c_R - \boxdot Z$, $\Pi \subseteq P^+_R$, $\Sigma \subseteq P^-_R$, and $\Delta \subseteq K'_c$ such that $\Gamma, \Pi \Rightarrow \Theta, \Sigma, \Box\Delta \in \mathcal{G}$. This property will be denoted by $V(R, \boxdot Z, \mathcal{G})$.

In our example the fact that this property holds can be shown as follows. For the input clauses R with associated sequents $\Gamma \Rightarrow \Delta$ this is immediate since $\Gamma \subseteq P^+_R$ and $\Delta \subseteq P^-_R$. For the other clauses we choose the sequents as follows.

	P^c_R	$\boxdot Z$		$\boxdot Z$	
$\{l_q, \neg l_r\}$	$\emptyset$	$\emptyset$	$(\boxdot q \Rightarrow r, \Box r)$		
$\{\neg l_q, \neg l_r\}$	$\emptyset$	$\emptyset$	$(\Rightarrow q, \Box q, r, \Box r)$		
$\{l_q\}$	$\{r, \Box r\}$	$\emptyset$	$(\boxdot q \Rightarrow r, \Box r)$	$\{r, \Box r\}$	$(\boxdot r \Rightarrow)$
$\{l_r\}$	$\{q, \Box q\}$	$\emptyset$	$(\boxdot r \Rightarrow)$	$\{q, \Box q\}$	$(\boxdot r \Rightarrow)$
$\{\neg l_r\}$	$\{q, \Box q\}$	$\emptyset$	$(\Rightarrow q, \Box q, r, \Box r)$	$\{q, \Box q\}$	$(\boxdot q \Rightarrow r, \Box r)$

We leave it to the reader to verify that the associated sequents have the desired form and are elements of $\mathcal{G}$. For example, for $R = \{l_q\}$ and $\boxdot Z = \emptyset$, the sequent $\boxdot q \Rightarrow r, \Box r$ has the desired form since $q, \Box q \in R^+$ and $r, \Box r \in P^c_R - \boxdot Z$. That it is an element of $\mathcal{G}$ follows from the fact that $\mathcal{G}$ is full and contains the sequents $p, \boxdot q \Rightarrow r, \Box r$ and $\boxdot q \Rightarrow p, r, \Box r$. Thus by (AC) it also contains $\boxdot q \Rightarrow r, \Box r$.

As the example shows, there is not one particular sequent corresponding to a clause. Instead, for each of the subsets $\boxdot Z$ of P^c_R, we have a sequent in $\mathcal{G}$ which is a witness of $V(R, \boxdot Z, \mathcal{G})$. The argument showing that the fact that $V(\emptyset, \boxdot Z, \mathcal{G})$ holds for all $\boxdot Z \subseteq P^c_\emptyset$ leads to a contradiction, will be given at the end of the proof.

Define:

$$V(R, X, \mathcal{G}) =_{def} \exists \Gamma \subseteq X \exists \Theta \subseteq (P^c_R - X) \exists \Pi \subseteq P^+_R \exists \Sigma \subseteq P^-_R \exists \Delta \subseteq K'_c \quad (\Gamma, \Pi \Rightarrow \Theta, \Sigma, \Box\Delta) \in \mathcal{G}$$

$$U(R, \mathcal{G}) =_{def} \forall \boxdot Z \subseteq P^c_R \, V(R, \boxdot Z, \mathcal{G}).$$

Claim 1 *For every clause R in the resolution refutation $U(R, \mathcal{G})$ holds.*

Proof of the Claim. For the initial clauses $R = \Theta\Pi \cup \Psi\Sigma$ this is straightforward as they can be divided as $\Gamma_i \Rightarrow \Sigma_i, \Box\Theta_i$, where $\Gamma_i \subseteq P^+_{R_i}$, $\Sigma_i \subseteq P^-_{R_i}$ and $\Theta_i \subseteq K'_c$.

Cuts on p. For the induction step, first consider a cut on an atom p, with input clauses $R \cup \{p\}$ and $R' \cup \{\neg p\}$, and conclusion $R \cup R'$. Therefore, consider $\boxdot Z \subseteq P^c_{R\cup R'} = P^c_R \cap P^c_{R'}$. Denote $\boxdot Z$ by X. We show that $V(R \cup R', \boxdot Z, \mathcal{G})$, i.e. that there exists $S = \Gamma'', \Pi'' \Rightarrow \Theta'', \Sigma'', \Box\Delta''$ in $\mathcal{G}$ such that

$$\Gamma'' \subseteq X \quad \Theta'' \subseteq P^c_{R\cup R'} - X \quad \Pi'' \subseteq P^+_{R\cup R'} \quad \Sigma'' \subseteq P^-_{R\cup R'} \quad \Delta'' \subseteq K'_c. \tag{5}$$

We will leave out all the Δ's in the argument as they play no role in it. Consider

$$Y = X \cup \{q, \Box q \mid l_q \in L^c_R \cap R'\} \subseteq P^c_R,$$

$$Y' = X \cup \{q, \Box q \mid l_q \in L^c_{R'} \cap R\} \subseteq P^c_{R'}.$$

observe that Y and Y' are of the form $\boxdot W$. Therefore, by the induction hypothesis there are sequents $\Gamma, \Pi, p \Rightarrow \Theta, \Sigma$ and $\Gamma', \Pi' \Rightarrow p, \Theta', \Sigma'$ such that $\Gamma \subseteq Y$, $\Pi \subseteq P^+_R$, $\Sigma \subseteq P^-_R$, $\Theta \subseteq P^c_R - Y$, and similarly for the second sequent. The case that p does not occur in one or both of the sequents can be treated in the same way. By (AC) the sequent $\Gamma, \Gamma', \Pi, \Pi' \Rightarrow \Theta, \Theta', \Sigma, \Sigma' \in \mathcal{G}$. We will take for S this sequent, and show how we have to partition it in order to obtain (5). Define

$$\Gamma'' = (\Gamma \cap X) \cup (\Gamma' \cap X) \quad \Pi'' = \Pi \cup \Pi' \cup (\Gamma - X) \cup (\Gamma' - X),$$

$$\Theta'' = (\Theta \cap P^c_{R'}) \cup (\Theta' \cap P^c_R) \quad \Sigma'' = \Sigma \cup (P^-_{R'} \cap \Theta) \cup \Sigma' \cup (P^-_R \cap \Theta').$$

We have to show that they satisfy (5), and that $\Gamma\Gamma'\Pi\Pi' = \Gamma''\Pi''$ and $\Theta\Theta'\Sigma\Sigma' = \Sigma''\Sigma''$. For the first part, note that no clause in the refutation contains both an atom and its negation, which implies that $\Gamma''\Pi''$ and $\Theta''\Sigma''$ do not contain p. Therefore, $(\Gamma - X) \subseteq (Y - X) \subseteq P^+_{R'} \subseteq P^+_{R\cup R'}$, and $(\Gamma' - X) \subseteq (Y' - X) \subseteq P^+_R \subseteq P^+_{R\cup R'}$. This proves the first part. For the last part, that $\Gamma\Gamma'\Pi\Pi' = \Gamma''\Pi''$ is easy to see. For $\Theta\Theta'\Sigma\Sigma' = \Theta''\Sigma''$, observe that $\Theta = \Theta \cap P^+_{R'} \cup \Theta \cap P^-_{R'} \cup \Theta \cap P^c_{R'}$, and that $\Theta \cap P^+_{R'} = \emptyset$ by the definition of Y. And similarly for Θ'.

Cuts on l_p. For a cut on l_p the input clauses are

$$R \cup \{l_p\} \quad R' \cup \{\neg l_p\}$$

and the conclusion is $R \cup R'$. We have to show that $U(R \cup R', \mathcal{G})$. Therefore, consider $\boxdot Z \subseteq P^c_{R\cup R'} = P^c_R \cap P^c_{R'} \cup \{p, \Box p\}$. Denote $\boxdot Z$ by X. We have to show that there is a sequent $\Gamma, \Pi \Rightarrow \Theta, \Sigma, \Box\Delta \in \mathcal{G}$ such that

$$\Gamma \subseteq X \quad \Theta \subseteq P^c_{R\cup R'} - X, \quad \Pi \subseteq P^+_{R\cup R'} \quad \Sigma \subseteq P^-_{R\cup R'} \quad \Delta \subseteq K'_c. \tag{6}$$

We distinguish the two cases $p, \Box p \in X$, and $p, \Box p \notin X$. Observe that since X is of the form $\boxdot Z$ these are the only two cases that can occur. We treat the first case, the second case is similar. Consider

$$Y = (X - \{p, \Box p\}) \cup \{q, \Box q \mid l_q \in L^c_R \cap R'\} \subseteq P^c_R \cap ((X - \{p, \Box p\}) \cup P^+_{R'}).$$

Note that Y is of the form $\boxdot W$. Therefore, by the induction hypothesis there is a sequent $S = \Gamma', \Pi' \Rightarrow \Theta', \Sigma', \Box\Delta' \in \mathcal{G}$ such that

$$\Gamma' \subseteq Y \quad \Theta' \subseteq P^c_R - Y, \quad \Pi' \subseteq P^+_R \quad \Sigma' \subseteq P^-_R \subseteq P^-_{R\cup R'} \quad \Delta' \subseteq K'_c.$$

Consider the following partition of S: $\Gamma = (\Gamma' \cap X) \cup (\Pi' \cap \{p, \Box p\})$, $\Pi = (\Pi' - \{\Box p, p\}) \cup (\Gamma' \cap (Y - X))$, $\Theta = \Theta' \cap P^c_{R'}$, $\Sigma = \Sigma' \cup (P^-_{R'} \cap \Theta')$, and $\Delta' = \Delta$. We have to show that

$$\Gamma\Pi\Theta\Sigma \text{ satisfy (6).} \tag{7}$$

and that it is indeed a partition of S, i.e.

$$\Gamma\Pi = \Gamma'\Pi' \quad \Theta\Sigma = \Theta'\Sigma'. \tag{8}$$

For (7), that $\Gamma \subseteq X$ is clear. For $\Pi \subseteq P^+_{R\cup R'}$ observe that $(\Pi' - \{\Box p, p\})$ is contained in $(P^+_R - \{p, \Box p\}) \subseteq P^+_{R\cup R'}$, and that $(Y - X) \subseteq P^+_{R'} \subseteq P^+_{R\cup R'}$. That $\Theta \subseteq (P^c_{R\cup R'} - X)$ is clear. That $\Sigma \subseteq P^-_{R\cup R'}$ follows from the fact that $\Sigma' \subseteq P^-_R \subseteq P^-_{R\cup R'}$ and that $P^c_{R'} \cap \Theta' \subseteq P^-_{R\cup R'}$ because $\Box p, p \notin \Theta' \subseteq P^c_R$. This finishes the proof of (7). For (8), that $\Gamma\Pi = \Gamma'\Pi'$ is easy to see. For $\Theta\Sigma = \Theta'\Sigma'$, observe that $\Theta \cap P^+_{R'} = \emptyset$ by the definition of Y. This proves (8), and thereby the claim. **End of Claim proof**.

To finish the proof of the theorem, consider the empty clause $\emptyset$, and observe that $P^c_\emptyset = \{p, \Box p \mid p \in K'\} \subseteq K'$. Let $\Gamma = \{p \mid p \in K'\}$. The fact that $U(\emptyset, \mathcal{G})$ implies that $V(\emptyset, \boxdot Z, \mathcal{G})$ for all $\boxdot Z \subseteq P^c_\emptyset$. Since $P^+_\emptyset = P^-_\emptyset = \emptyset$, it follows that $\Gamma \equiv \Box\Gamma \Rightarrow \Box\Delta \in \mathcal{G}$. Thus by the rule (V^r), $\boxdot\Gamma \Rightarrow q \in \mathcal{G}$ for some $q \in \Delta$. Since $K' \Vdash C$, it follows that $K' \Vdash \bigwedge \boxdot\Gamma \to q$. Since also $K' \Vdash \bigwedge \boxdot\Gamma$, $K' \Vdash q$ follows, contradicting $K' \nVdash q$, which follows from $\Delta \subseteq K'_c$. This finishes the proof of the theorem. $\Box$

In combination with Corollary, we obtain in particular soundness and completeness results for $\mathbf{GAK4}$, $\mathbf{GAS4}$, and $\mathbf{GAL}$.

Corollary 41 $\mathrel{|\!\!\sim}_{\mathsf{L}} \mathcal{R}$ *iff* $\vdash_{\mathbf{GAL}} \mathcal{R}$ *for* $\mathsf{L} \in \{\mathsf{K4}, \mathsf{S4}, \mathsf{GL}\}$.

4 Intuitionistic Logic

In this section we turn our attention to the historically most significant case of Intuitionistic Logic IPC, proceeding in much the same way as for modal logics. Namely, we start with a calculus for derivability (taken from [15]) for the right logical rules, and use invertibility properties to obtain left logical rules.

Definition 42 ($\mathbf{GAI}$)

Initial Generalized Sequent Rules

$$\frac{}{\mathcal{G} \,\triangleright\, (\Gamma, A \Rightarrow A, \Delta), \mathcal{H}}\ (ID) \qquad \frac{}{\mathcal{G} \,\triangleright\, (\Gamma, \bot \Rightarrow \Delta), \mathcal{H}}\ (\bot)$$

Right Logical Rules

$$\frac{\mathcal{G} \,\triangleright\, (\Gamma \Rightarrow A, \Delta), \mathcal{H} \quad \mathcal{G} \,\triangleright\, (\Gamma \Rightarrow B, \Delta), \mathcal{H}}{\mathcal{G} \,\triangleright\, (\Gamma \Rightarrow A \wedge B, \Delta), \mathcal{H}}\ \triangleright(\Rightarrow\wedge) \qquad \frac{\mathcal{G} \,\triangleright\, (\Gamma, A, B \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \,\triangleright\, (\Gamma, A \wedge B \Rightarrow \Delta), \mathcal{H}}\ \triangleright(\wedge\Rightarrow)$$

$$\frac{\mathcal{G} \,\triangleright\, (\Gamma, A \Rightarrow \Delta), \mathcal{H} \quad \mathcal{G} \,\triangleright\, (\Gamma, B \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \,\triangleright\, (\Gamma, A \vee B \Rightarrow \Delta), \mathcal{H}}\ \triangleright(\vee\Rightarrow) \qquad \frac{\mathcal{G} \,\triangleright\, (\Gamma \Rightarrow A, B, \Delta), \mathcal{H}}{\mathcal{G} \,\triangleright\, (\Gamma \Rightarrow A \vee B, \Delta), \mathcal{H}}\ \triangleright(\Rightarrow\vee)$$

$$\frac{\mathcal{G} \,\triangleright\, (\Gamma, A \to B \Rightarrow A, \Delta), \mathcal{H} \quad \mathcal{G} \,\triangleright\, (\Gamma, B \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \,\triangleright\, (\Gamma, A \to B \Rightarrow \Delta), \mathcal{H}}\ \triangleright(\to\Rightarrow)^i \qquad \frac{\mathcal{G} \,\triangleright\, (\Gamma, A \Rightarrow B), \mathcal{H}}{\mathcal{G} \,\triangleright\, (\Gamma \Rightarrow A \to B, \Delta), \mathcal{H}}\ \triangleright(\Rightarrow\to)^i$$

Left Logical Rules

$$\frac{\mathcal{G}, (\Gamma, A, B \Rightarrow \Delta) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma, A \wedge B \Rightarrow \Delta) \,\triangleright\, \mathcal{H}}\ (\wedge\Rightarrow)\triangleright \qquad \frac{\mathcal{G}, (\Gamma \Rightarrow A, \Delta), (\Gamma \Rightarrow B, \Delta) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow A \wedge B, \Delta) \,\triangleright\, \mathcal{H}}\ (\Rightarrow\wedge)\triangleright$$

$$\frac{\mathcal{G}, (\Gamma \Rightarrow A, B, \Delta) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow A \vee B, \Delta) \,\triangleright\, \mathcal{H}}\ (\Rightarrow\vee)\triangleright \qquad \frac{\mathcal{G}, (\Gamma, A \Rightarrow \Delta), (\Gamma, B \Rightarrow \Delta) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma, A \vee B \Rightarrow \Delta) \,\triangleright\, \mathcal{H}}\ (\vee\Rightarrow)\triangleright$$

$$\frac{\mathcal{G}, (\Gamma \Rightarrow p, \Delta), (p, A \Rightarrow B) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow A \to B, \Delta) \,\triangleright\, \mathcal{H}}\ (\Rightarrow\to)\triangleright^i \qquad \frac{\mathcal{G}, (\Gamma, p \to q \Rightarrow \Delta), (p \Rightarrow A), (B \Rightarrow q) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma, A \to B \Rightarrow \Delta) \,\triangleright\, \mathcal{H}}\ (\to\Rightarrow)\triangleright^i$$

where in $(\to\Rightarrow)\triangleright^i$, $(\Rightarrow\to)\triangleright^i$, p, q do not occur in $\mathcal{G}$, $\mathcal{H}$, Γ, Δ, A, B, and in $(\to\Rightarrow)\triangleright^i$, A, B are non-atomic

Structural Rules

$$\frac{\mathcal{G} \,\triangleright\, \mathcal{H}}{\mathcal{G}, S \,\triangleright\, \mathcal{H}}\ (W)\triangleright \qquad \frac{\mathcal{G} \,\triangleright\, \mathcal{H}}{\mathcal{G} \,\triangleright\, S, \mathcal{H}}\ \triangleright(W) \qquad \frac{\mathcal{G}, S, S, \,\triangleright \mathcal{H}}{\mathcal{G}, S \,\triangleright\, \mathcal{H}}\ (C)\triangleright \qquad \frac{\mathcal{G} \,\triangleright\, S, S, \mathcal{H}}{\mathcal{G} \,\triangleright\, S, \mathcal{H}}\ \triangleright(C)$$

Anti-Cut and Projection Rules

$$\frac{\mathcal{G}, (\Gamma, \Pi \Rightarrow \Sigma, \Delta) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma, A \Rightarrow \Delta), (\Pi \Rightarrow A, \Sigma) \,\triangleright\, \mathcal{H}}\ (AC) \qquad \frac{\mathcal{G} \,\triangleright\, (\Gamma, I(S) \Rightarrow \Delta), \mathcal{H}}{\mathcal{G}, S \,\triangleright\, \mathcal{H}}\ (PJ)$$

where $(\Gamma \Rightarrow \Delta) \in \mathcal{H} \cup \{\Rightarrow\}$

Implication Rule

$$\frac{\mathcal{G}, (\Gamma, B \Rightarrow \Delta), (\Gamma, A \to B \Rightarrow A, \Delta) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma, A \to B \Rightarrow \Delta) \,\triangleright\, \mathcal{H}}\ (\to)\triangleright$$

Visser Rule

$$\frac{[\mathcal{G}, (\Gamma \Rightarrow A) \,\triangleright\, \mathcal{H}]_{A \in \Delta} \quad [\mathcal{G} \,\triangleright\, (\Gamma^{\Pi}, \Pi \Rightarrow \Delta), \mathcal{H}]_{\Pi \subseteq \Gamma_{\Delta}}}{\mathcal{G}, (\Gamma \Rightarrow \Delta) \,\triangleright\, \mathcal{H}}\ (V)$$

where Γ contains only implications, and:

(1) $\Gamma^{\Pi} = \{A \to B \in \Gamma : A \notin \Pi\}$.

(2) $\Gamma_{\Delta} = \{A \notin \Delta : \exists B\, (A \to B) \in \Gamma\}$.

$\mathbf{GAI}$ is evidently very similar to the calculi for extensible modal logics, the main differences being changes in the rules for implication, a different Visser Rule, and the addition of the implication rule.

Example 43 *Consider the Kreisel-Putnam rule $\neg A \to (B \lor C) \;/\; (\neg A \to B) \lor (\neg A \to C)$ which is* IPC*-admissible but not* IPC*-derivable. This can be written in gs-rule format as:*

$$(\neg A \Rightarrow B, C) \;\rhd\; (\Rightarrow \neg A \to B, \neg A \to C)$$

We will construct a proof of this in $\mathbf{GAI}$*. First, proceeding backwards, we apply* $(\to)\rhd$*,* $(W)\rhd$*, and* $(C)\rhd$ *to get:*

$$(\neg A \Rightarrow A, B, C), (\neg A \Rightarrow A, B, C) \;\rhd\; (\Rightarrow \neg A \to B, \neg A \to C)$$

We now apply the Visser rule (V) *to the first sequent on the left. Observe that* Γ *here contains just one implication* $\neg A$*, and* Γ_Δ *is in this case just* $\emptyset$ *with* $\Gamma^\emptyset = \neg A$*. Hence we get the following premises to be proved:*

(1) $(\neg A \Rightarrow A), (\neg A \Rightarrow A, B, C) \;\rhd\; (\Rightarrow \neg A \to B, \neg A \to C)$
(2) $(\neg A \Rightarrow B), (\neg A \Rightarrow A, B, C) \;\rhd\; (\Rightarrow \neg A \to B, \neg A \to C)$
(3) $(\neg A \Rightarrow C), (\neg A \Rightarrow A, B, C) \;\rhd\; (\Rightarrow \neg A \to B, \neg A \to C)$
(4) $(\neg A \Rightarrow A, B, C) \;\rhd\; (\neg A \Rightarrow A, B, C), (\Rightarrow \neg A \to B, \neg A \to C)$

(1) *is derivable as follows:*

$$\dfrac{\dfrac{}{\rhd\,(\neg A, \neg A \to A \Rightarrow \neg A, B)}\,(SID) \qquad \dfrac{\dfrac{}{\rhd\,(\bot, A \Rightarrow B)}\,(\bot) \qquad \dfrac{}{\rhd\,(\neg A, A \Rightarrow A, B)}\,(SID)}{\rhd\,(\neg A, A \Rightarrow B)}\,\rhd(\to\Rightarrow)}{\dfrac{\dfrac{\dfrac{\rhd\,(\neg A, \neg A \to A \Rightarrow B)}{(\neg A \Rightarrow A) \;\rhd\; (\neg A \Rightarrow B)}\,(PJ)}{(\neg A \Rightarrow A) \;\rhd\; (\Rightarrow \neg A \to B, \neg A \to C)}\,\rhd(\Rightarrow\to)}{(\neg A \Rightarrow A), (\neg A \Rightarrow A, B, C) \;\rhd\; (\Rightarrow \neg A \to B, \neg A \to C)}\,(W)\rhd}\,\rhd(\to\Rightarrow)$$

(2) *and* (3) *are derivable simply by applying* $\rhd(\Rightarrow\to)$ *to the appropriate disjunct and* (SID)*, while* (4) *is just an instance of* (SID)*.*

The proof of soundness for $\mathbf{GAI}$ with respect to IPC-admissibility proceeds similarly to Lemma 28.

Theorem 44 *If* $\vdash_{\mathbf{GAI}} \mathcal{R}$*, then* $\mathrel{|\!\sim}_{\mathsf{IPC}} \mathcal{R}$*.*

Proof. It is sufficient to show that each rule of $\mathbf{GAI}$ is IPC-sound, concentrating just on those cases different to Lemma 28. For $(\to\Rightarrow)\rhd^i$, let σ be a unifier for $I(S)$ for all $S \in \mathcal{G}$ and $I(\Gamma, A \to B \Rightarrow \Delta)$. Since p and q do not occur in the conclusion of the rule, we can extend σ with $\sigma(p) = \sigma(A)$ and $\sigma(q) = \sigma(B)$. It follows immediately that σ is a unifier for $I(\Gamma, p \to q \Rightarrow \Delta)$, $I(p \Rightarrow A)$, and

$I(B \Rightarrow q)$. Hence, if the premise is IPC-admissible, then σ is a unifier for $I(S)$ for some $S \in \mathcal{H}$ as required. The case of the rule $(\Rightarrow\rightarrow)\triangleright$ follows a similar pattern.

For (V), suppose that σ is a unifier for $I(S)$ for all $S \in \mathcal{G}$ and $I(\Gamma \Rightarrow \Delta)$, and let $\Delta = \{A_1, \ldots, A_n\}$. Using the right set of premises, σ is either a unifier for some $S \in \mathcal{H}$ or for $I(\Gamma^\Pi, \Pi \Rightarrow \Delta)$ for all $\Pi \subseteq \Gamma_\Delta$. In the first case we are done, so assume the latter. It suffices now by the left set of premises to show that σ is a unifier for $I(\Gamma \Rightarrow A_i)$ for some i, $1 \leq i \leq n$. Suppose, arguing contrapositively, that this is not the case. Then there exist countermodels $K_1, \ldots, K_n$ such that $K_i \Vdash \sigma(\bigwedge \Gamma)$ and $K_i \nVdash \sigma(A_i)$ for $i = 1 \ldots n$. Consider the model $K = (\Sigma_{i=1}^n K_i)'$. Let $\Pi = \{D \in \Gamma_\Delta : K \Vdash \sigma(D)\}$. Observe that for all $B \rightarrow C \in \Gamma$ such that $B \notin \Pi$, either $B \in \Delta$ or $K \nVdash \sigma(B)$. Note also that $B \in \Delta$ implies $K \nVdash \sigma(B)$. Hence for all $B \notin \Pi$ it follows that $K \nVdash \sigma(B)$, and so $K \Vdash \sigma(B \rightarrow C)$. It follows that $K \Vdash \sigma(\bigwedge(\Gamma^\Pi \cup \Pi))$. Thus $K \Vdash \sigma(\bigvee \Delta)$, a contradiction. □

We now need a series of lemmas corresponding to those used in the modal case. First we have, exactly as in Lemma 30 (except replacing the application of the modal deduction theorem with the usual deduction theorem), that IPC-derivable gs-rules with at most one sequent on the right are **GAI**-derivable.

Lemma 45 *If* $\vdash_{\mathsf{IPC}} \mathcal{G} \triangleright \mathcal{H}$ *where* $|\mathcal{H}| \leq 1$, *then* $\vdash_{\mathbf{GAI}} \mathcal{G} \triangleright \mathcal{H}$.

We then establish that IPC-admissible rules are **GAI**-derivable from IPC-admissible rules that are full (recalling Definition 38 for the definition of fullness) with respect to (V), $(\rightarrow)\triangleright$, and (AC). Below, we prove just the invertibility step, proofs for the second and third lemmas proceeding in exactly the same way as Lemmas 36 and 39.

Lemma 46 *The left logical rules of* **GAI** *are invertible.*

Proof. The cases for $\wedge$ and $\vee$ are straightforward. For $(\rightarrow\Rightarrow)\triangleright^i$, assume that σ is a unifier for $I(S)$ for all $S \in \mathcal{G}$, $I(\Gamma, p \rightarrow q \Rightarrow \Delta)$, $I(p \Rightarrow A)$, and $I(B \Rightarrow q)$. Since $\vdash_{\mathsf{IPC}} I(A, A \rightarrow B \Rightarrow B)$ it follows that σ is a unifier for $I(p, A \rightarrow B \Rightarrow q)$, and hence for $I(A \rightarrow B \Rightarrow p \rightarrow q)$. So by cut-admissibility for IPC, σ is a unifier for $I(\Gamma, A \rightarrow B \Rightarrow \Delta)$, and hence, by the IPC-admissibility of the conclusion, for $I(S)$ for some $S \in \mathcal{H}$. The case of $(\Rightarrow\rightarrow)\triangleright^i$ is similar. □

Definition 47 *A gs-rule* $\mathcal{G} \triangleright \mathcal{H}$ *is* implication-irreducible *if every sequent in* $\mathcal{G}$ *contains only atoms on the right and atoms and atomic implications on the left.*

Lemma 48 *Every* IPC*-admissible gs-rule is* **GAI***-derivable from an* IPC*-admissible implication-irreducible gs-rule.*

Lemma 49 *Admissible gs-rules are* **GAI***-derivable from* IPC*-admissible gs-rules that are full with respect to* (V), $(\rightarrow)\triangleright$, *and* (AC).

We now use Ghilardi's characterization of IPC-projective formulas to establish completeness for $\mathbf{GAI}$. First, however, we give a technical lemma showing that a crucial property of gs-rules is preserved from premise to conclusion by the rule (AC). For convenience, we define the following conditions for multisets of gs-rules $\mathcal{G}$ and sets of formulas I:

Definition 50 *Recall that*

$$\Gamma^{\Pi} = \{A \to B \in \Gamma : A \notin \Pi\}.$$
$$\Gamma_{\Delta} = \{A \notin \Delta : \exists B\, (A \to B) \in \Gamma\}.$$

Let $\mathcal{G}$ be a multiset of sequents, $\Gamma \Rightarrow \Delta$ a sequent, and I a set of formulas:

(1) $U((\Gamma \Rightarrow \Delta), I, \mathcal{G})$ iff $\exists \Gamma' \subseteq \Gamma^I \cup I\ \exists \Delta' \subseteq \Delta\, (\Gamma' \Rightarrow \Delta' \in \mathcal{G})$
(2) $V((\Gamma \Rightarrow \Delta), \mathcal{G})$ iff $\forall I \subseteq \Gamma_{\Delta} : U((\Gamma \Rightarrow \Delta), I, \mathcal{G})$

Lemma 51 *Let $\mathcal{G}, (\Gamma, p \Rightarrow \Delta), (\Pi \Rightarrow p, \Sigma) \rhd \mathcal{H}$ be a gs-rule full with respect to (AC). If $V((\Gamma, p \Rightarrow \Delta), \mathcal{G})$ and $V((\Pi \Rightarrow p, \Sigma), \mathcal{G})$, then $V((\Gamma, \Pi \Rightarrow \Sigma, \Delta), \mathcal{G})$.*

Proof. Consider $I \subseteq (\Gamma \cup \Pi)_{\Delta \cup \Sigma}$. We show that $U((\Gamma, \Pi \Rightarrow \Sigma, \Delta), I, \mathcal{G})$, i.e. that there are $\Gamma' \subseteq \Pi^I \cup \Gamma^I \cup I$ and $\Delta' \subseteq \Delta \cup \Sigma$ such that $\Gamma' \Rightarrow \Delta' \in \mathcal{G}$. Observe that

$$I \cap \Gamma_{\Delta} \subseteq \Gamma_{\Delta}, \quad I \cap \Pi_{\Sigma} \subseteq \Pi_{\Sigma}, \quad \Gamma^{I \cap \Gamma_{\Delta}} = \Gamma^I, \text{ and } \quad \Pi^{I \cap \Pi_{\Sigma}} = \Pi^I \qquad (9)$$

First, assume $p \notin I$. By the hypothesis and (9), there are $\Gamma' \subseteq \Gamma^I \cup I$, $\Pi' \subseteq \Pi^I \cup I$, $\Delta' \subseteq \Delta$, and $\Sigma' \subseteq \Sigma$ such that $(\Gamma', p \Rightarrow \Delta') \in \mathcal{G}$ and $(\Pi' \Rightarrow p, \Sigma') \in \mathcal{G}$. Whence also $(\Gamma', \Pi' \Rightarrow \Sigma', \Delta') \in \mathcal{G}$ by fullness, and we are done.

Second, assume $p \in I$. By the hypothesis and (9), we have $\Gamma' \subseteq \Gamma^I \cup I$ and $\Delta' \subseteq \Delta$ such that $(\Gamma' \Rightarrow \Delta') \in \mathcal{G}$, and we are done. $\square$

Theorem 52 $\mathrel{|\!\sim}_{\mathsf{IPC}} \mathcal{R}$ *iff* $\vdash_{\mathbf{GAI}} \mathcal{R}$.

Proof. The right to left direction has been proved above. For the other direction, it is sufficient using Lemma 49 to assume that $\mathcal{R} = \mathcal{G} \rhd \mathcal{H}$ is an IPC-admissible gs-rule full with respect to (AC), $(\to)\rhd$, and (V). Let $C = \bigwedge_{S \in \mathcal{G}} I(S)$. If C is inconsistent, then $\vdash_{\mathsf{IPC}} \mathcal{G}\rhd$, and if C is IPC-projective, then using Lemma 8 (a), $\vdash_{\mathsf{IPC}} \mathcal{G} \rhd S$ for some $S \in \mathcal{H}$. Hence in both cases, by Lemma 45 and $\rhd(W)$, $\vdash_{\mathbf{GAI}} \mathcal{G} \rhd \mathcal{H}$.

Hence assume that C is consistent and not IPC-projective. We use Ghilardi's key result, Theorem 12, which tells us that C does not have the extension property, to show that $\mathcal{R}$ is $\mathbf{GAI}$-derivable. Unpacking the definition of the extension property, we consider the set of Kripke models $\mathcal{K}$ for C, and have a model $K \in \mathcal{K}$ such that $K \Vdash C$ and every variant of K' refutes C. We can assume that K has at least one node as otherwise K' would have one node, and, since C is consistent, we know that for such a classical model there is a variant that forces C. Let $M_1, \ldots, M_k$ be all the possible variants of K' and let $C_1, \ldots, C_k$ be the sequents of $\mathcal{G}$ such that $M_i \not\Vdash C_i$. Observe that we can assume for each i that:

$$(p \to q) \in \Gamma_i \text{ implies } p \in \Delta_i \tag{10}$$

For suppose that this is not possible. Then $p \to q \in \Gamma_i$ and $p \notin \Delta_i$. But since $M_i \Vdash \bigwedge \Gamma_i$ and $M_i \not\Vdash \bigvee \Delta_i$, it follows that $M_i \Vdash p \to q$, and hence either $M_i \not\Vdash p$ or $M_i \Vdash q$. This means that either $M_i \not\Vdash I(\Gamma_i \Rightarrow \Delta_i, p)$ or $M_i \not\Vdash I(\Gamma_i - \{p \to q\}, q \Rightarrow \Delta_i)$. But since $\mathcal{R}$ is full with respect to $(\to)\triangleright$, both of these sequents are in $\mathcal{G}$, and can replace $\Gamma_i \Rightarrow \Delta_i$, a contradiction.

Now we define the set of atoms: $P = \{p : p \text{ occurs in } C \text{ and } K \Vdash p\}$. Let $at(\Gamma)$ denote the set of atoms that are elements of Γ. Note that $at(\Gamma_i) \subseteq P$ for all $i = 1 \ldots k$. Define for $i = 1 \ldots k$:

$$A_i =_{def} \bigwedge_{p \in \Gamma_i \cap P} p \wedge \bigwedge_{p \in \Delta_i \cap P} \neg p.$$

We show that $A =_{def} \bigvee_{i=1}^{k} A_i$ is a classical tautology. Since $K \Vdash p$ for all $p \in P$, given a classical valuation v on P, we can consider the variant of K' defined at the root by:

$$M \Vdash p \Leftrightarrow v(p) = 1.$$

where M is M_j for some j, $1 \leq j \leq k$. Observe that $M \Vdash p$ and hence $v(p) = 1$ for all $p \in at(\Gamma_j)$, and $M \not\Vdash p$ and hence $v(p) = 0$ for all $p \in \Delta_j$. Thus $v(A_j) = 1$. Hence A is a tautology and the following formula, equivalent to the negation of A, is inconsistent:

$$\bigwedge_{j=1}^{m} \Big(\bigvee_{p \in \Gamma_j \cap P} p \vee \bigvee_{p \in \Delta_j \cap P} \neg p \Big)$$

Hence there exists a resolution refutation starting with the clauses:

$$\{p : p \in \Gamma_j \cap P\} \cup \{\neg p : p \in \Delta_j \cap P\} \text{ for } j = 1 \ldots m$$

that ends in the empty clause $\emptyset$.

Let $\Theta \cup \Psi'$ be a clause in the refutation, where Θ contains only atoms and Ψ' contains only negated atoms. Define $\Psi = \{p : \neg p \in \Psi'\}$. Then, since $\mathcal{R}$ is full with respect to (AC), there exists $(\Gamma, \Theta \Rightarrow \Psi, \Delta) \in \mathcal{G}$ such that $\Delta \cap P = \emptyset, \Gamma \cap P = \emptyset$, and $K \Vdash \bigwedge \Gamma$. Moreover, since it holds by (10) for $\Gamma_i \Rightarrow \Delta_i$ for $i = 1 \ldots k$, inductively by multiple applications of Lemma 51:

$$\forall I \subseteq (\Gamma \cup \Pi)_{\Delta \cup \Sigma} : U((\Gamma, \Pi \Rightarrow \Sigma, \Delta), I, \mathcal{G}). \tag{11}$$

Hence in particular for the empty clause $\emptyset$: $(\Gamma \Rightarrow \Delta) \in \mathcal{G}$ where Δ contains only atoms not in P, Γ contains only implicational formulas, and $K \Vdash \bigwedge \Gamma$. Since $\mathcal{R}$ is full with respect to (V), either $(\Gamma \Rightarrow q) \in \mathcal{G}$ for some $q \in \Delta$, or $(\Gamma^{\Pi}, \Pi \Rightarrow \Delta) \in \mathcal{H}$ for some $\Pi \subseteq \Gamma_{\Delta}$. In the first case, we get that $K \Vdash \bigwedge \Gamma \to q$, since $K \Vdash C$. But

$K \Vdash \bigwedge \Gamma$ so it follows that $K \Vdash q$, which implies $q \in P$, a contradiction. In the second case, using (11), there exists $(\Gamma' \Rightarrow \Delta') \in \mathcal{G}$ for some $\Gamma' \subseteq \Gamma^{\Pi} \cup \Pi$ and some $\Delta' \subseteq \Delta$. Since $\vdash_{\mathsf{IPC}} (\Gamma' \Rightarrow \Delta') \rhd (\Gamma^{\Pi}, \Pi \Rightarrow \Delta)$, it follows by Lemma 45 that $\vdash_{\mathbf{GAI}} (\Gamma' \Rightarrow \Delta') \rhd (\Gamma^{\Pi}, \Pi \Rightarrow \Delta)$ and hence by $(W)\rhd$ and $\rhd(W)$ that $\vdash_{\mathbf{GAI}} \mathcal{R}$ as required. $\square$

5 Intermediate Logics

In this section we consider intermediate logics, recalling the result of [10] that if the Visser rules are admissible for an intermediate logic, then they form a basis for the admissible rules of that logic. In some cases, such as Gödel-Dummett logic, the Visser rules (and hence all admissible rules) are derivable. Here we consider some logics where this does not happen: de Morgan (or Jankov) logic KC, axiomatized by adding the axiom $\neg A \vee \neg\neg A$ to IPC, and the family of logics with Kripke models of bounded cardinality $\mathsf{BC_n}$ for $n = 1, 2, \ldots$ (noting that for the cases $n = 1, 2$, the Visser rules are in fact derivable). Our treatment of these logics gives a nice illustration of the flexibility of the approach, since to define a calculus for admissibility (and indeed even derivability) in such cases, we require rules dealing with more complicated structures. In particular, we use *hypersequents*, a natural generalization of sequents introduced by Avron in [1].

Definition 53 *A* hypersequent G *is a finite multiset of sequents, written* $S_1 \mid \ldots \mid S_n$*, and* $\vdash_{\mathsf{L}} G$ *iff* $\vdash_{\mathsf{L}} I(G)$ *where* $I(G) = \bigvee_{i=1}^{n} I(S_i)$*.*

Generalized hypersequent rules are defined in exactly the same way as sequent rules. However, here we will deal only with *single-conclusion* generalized hypersequent rules:

Definition 54 *A* generalized hypersequent rule *(gh-rule for short)* $\mathcal{R}$ *is an ordered pair of multisets of hypersequents:*

$$(G_1), \ldots, (G_n) \rhd (H_1), \ldots, (H_m)$$

If $m \leq 1$*, then* $\mathcal{R}$ *is called a* single-conclusion *gh-rule (sgh-rule for short).*

$\mathcal{R}$ *is* L-admissible*, written* $\mathrel{|\!\sim}_{\mathsf{L}} \mathcal{R}$*, iff:*

$$\{I(G_i)\}_{i=1}^{n} \mathrel{|\!\sim}_{\mathsf{L}} \{I(H_j)\}_{j=1}^{m}$$

$\mathcal{R}$ *is* L-derivable*, written* $\vdash_{\mathsf{L}} \mathcal{R}$*, iff:*

$$\bigwedge_{i=1}^{n} I(G_i) \vdash_{\mathsf{L}} \bigvee_{j=1}^{m} I(H_j)$$

We obtain a core set of rules for extensible intermediate logics by taking the single-conclusion versions of rules of **GAI** for IPC and adding context variables G, H standing for arbitrary context hypersequents, the only significant change being in the Visser rule:

Definition 55 (Core Intermediate Rules)

Initial GH-Rules

$$\frac{}{\mathcal{G} \triangleright (G \mid \Gamma, A \Rightarrow A, \Delta)}\ (ID) \qquad \frac{}{\mathcal{G} \triangleright (G \mid \Gamma, \bot \Rightarrow \Delta)}\ (\bot)$$

Right Logical Rules

$$\frac{\mathcal{G} \triangleright (G \mid \Gamma \Rightarrow A, \Delta) \quad \mathcal{G} \triangleright (G \mid \Gamma \Rightarrow B, \Delta)}{\mathcal{G} \triangleright (G \mid \Gamma \Rightarrow A \wedge B, \Delta)}\ \triangleright(\Rightarrow\wedge) \qquad \frac{\mathcal{G} \triangleright (G \mid \Gamma, A, B \Rightarrow \Delta)}{\mathcal{G} \triangleright (G \mid \Gamma, A \wedge B \Rightarrow \Delta)}\ \triangleright(\wedge\Rightarrow)$$

$$\frac{\mathcal{G} \triangleright (G \mid \Gamma, A \Rightarrow \Delta) \quad \mathcal{G} \triangleright (G \mid \Gamma, B \Rightarrow \Delta)}{\mathcal{G} \triangleright (G \mid \Gamma, A \vee B \Rightarrow \Delta)}\ \triangleright(\vee\Rightarrow) \qquad \frac{\mathcal{G} \triangleright (G \mid \Gamma \Rightarrow A, B, \Delta)}{\mathcal{G} \triangleright (G \mid \Gamma \Rightarrow A \vee B, \Delta)}\ \triangleright(\Rightarrow\vee)$$

$$\frac{\mathcal{G} \triangleright (G \mid \Gamma, A \to B \Rightarrow A, \Delta) \quad \mathcal{G} \triangleright (G \mid \Gamma, B \Rightarrow \Delta)}{\mathcal{G} \triangleright (G \mid \Gamma, A \to B \Rightarrow \Delta)}\ \triangleright(\to\Rightarrow)^i \qquad \frac{\mathcal{G} \triangleright (G \mid \Gamma, A \Rightarrow B)}{\mathcal{G} \triangleright (G \mid \Gamma \Rightarrow A \to B, \Delta)}\ \triangleright(\Rightarrow\to)^i$$

Left Logical Rules

$$\frac{\mathcal{G}, (G \mid \Gamma, A, B \Rightarrow \Delta) \triangleright \mathcal{H}}{\mathcal{G}, (G \mid \Gamma, A \wedge B \Rightarrow \Delta) \triangleright \mathcal{H}}\ (\wedge\Rightarrow)\triangleright \qquad \frac{\mathcal{G}, (G \mid \Gamma \Rightarrow A, \Delta), (G \mid \Gamma \Rightarrow B, \Delta) \triangleright \mathcal{H}}{\mathcal{G}, (G \mid \Gamma \Rightarrow A \wedge B, \Delta) \triangleright \mathcal{H}}$$

$$\frac{\mathcal{G}, (G \mid \Gamma \Rightarrow A, B, \Delta) \triangleright \mathcal{H}}{\mathcal{G}, (G \mid \Gamma \Rightarrow A \vee B, \Delta) \triangleright \mathcal{H}}\ (\Rightarrow\vee)\triangleright \qquad \frac{\mathcal{G}, (G \mid \Gamma, A \Rightarrow \Delta), (G \mid \Gamma, B \Rightarrow \Delta) \triangleright \mathcal{H}}{\mathcal{G}, (G \mid \Gamma, A \vee B \Rightarrow \Delta) \triangleright \mathcal{H}}\ (\vee\Rightarrow)\triangleright$$

$$\frac{\mathcal{G}, (G \mid \Gamma \Rightarrow p, \Delta), (p, A \Rightarrow B) \triangleright \mathcal{H}}{\mathcal{G}, (G \mid \Gamma \Rightarrow A \to B, \Delta) \triangleright \mathcal{H}}\ (\Rightarrow\to)\triangleright^i \qquad \frac{\mathcal{G}, (G \mid \Gamma, p \to q \Rightarrow \Delta), (p \Rightarrow A), (B \Rightarrow q) \triangleright \mathcal{H}}{\mathcal{G}, (G \mid \Gamma, A \to B \Rightarrow \Delta) \triangleright \mathcal{H}}\ (\to\Rightarrow)\triangleright^i$$

where in $(\to\Rightarrow)\triangleright^i$, $(\Rightarrow\to)\triangleright^i$, p, q *do not occur in* $\mathcal{G}$, $\mathcal{H}$, Γ, Δ, A, B, *and in* $(\to\Rightarrow)\triangleright^i$, A, B *are non-atomic*

Structural Rules

$$\frac{\mathcal{G} \triangleright \mathcal{H}}{\mathcal{G}, G \triangleright \mathcal{H}}\ (W)\triangleright \qquad \frac{\mathcal{G} \triangleright}{\mathcal{G} \triangleright G}\ \triangleright(W) \qquad \frac{\mathcal{G}, G, G, \triangleright \mathcal{H}}{\mathcal{G}, G \triangleright \mathcal{H}}\ (C)\triangleright$$

Anti-Cut and Projection Rules

$$\frac{\mathcal{G}, (G \mid H \mid \Gamma, \Pi \Rightarrow \Sigma, \Delta) \triangleright \mathcal{H}}{\mathcal{G}, (G \mid \Gamma, A \Rightarrow \Delta), (H \mid \Pi \Rightarrow A, \Sigma) \triangleright \mathcal{H}}\ (IAC) \qquad \frac{\mathcal{G} \triangleright (H \mid \Gamma, I(G) \Rightarrow \Delta)}{\mathcal{G}, (G) \triangleright \mathcal{H}}\ (IPJ)$$

where $(H \mid \Gamma \Rightarrow \Delta) \in \mathcal{H} \cup \{\Rightarrow\}$

Implication Rule

$$\frac{\mathcal{G}, (G \mid \Gamma, B \Rightarrow \Delta), (G \mid \Gamma, A \to B \Rightarrow A, \Delta) \triangleright \mathcal{H}}{\mathcal{G}, (G \mid \Gamma, A \to B \Rightarrow \Delta) \triangleright \mathcal{H}}\ (\to)\triangleright$$

Intermediate Visser Rule

$$\frac{\mathcal{G}, (G \mid \{\Gamma \Rightarrow A\}_{A \in \Delta}) \triangleright \mathcal{H} \quad [\mathcal{G} \triangleright (\Gamma^{\Pi}, \Pi \Rightarrow \Delta)]_{\Pi \subseteq \Gamma_{\Delta}}}{\mathcal{G}, (G \mid \Gamma \Rightarrow \Delta) \triangleright \mathcal{H}}\ (IV)$$

where Γ *contains only implications, and:*

(1) $\Gamma^{\Pi} = \{A \to B \in \Gamma : A \notin \Pi\}$.
(2) $\Gamma_{\Delta} = \{A \notin \Delta : \exists B\, (A \to B) \in \Gamma\}$.

The difference in the Visser rules for IPC and for intermediate logics is due to the fact that IPC has the disjunction property, while intermediate logics in general do not. This is reflected in the fact that by Lemma 8 (c), for IPC the following stronger versions of the Visser rules are admissible:

$$(C \to (A_{n+1} \vee A_{n+2})) \vee D \;/\; \{C \to A_j\}_{j=1}^{n+2} \cup \{D\}.$$

for $n = 1, 2, \ldots$ where $C = \bigwedge_{i=1}^{n} A_i \to B_i$.

We can extend the core set of rules given above to obtain proof systems for admissibility in extensible intermediate logics. In particular, we can make use of hypersequent calculi provided for KC and $\mathsf{BC_n}$ ($n = 1, 2, \ldots$) in [2], to obtain the following systems:

Definition 56 $\mathbf{GAKC}$ *consists of the core intermediate rules plus:*

$$\frac{\mathcal{G} \rhd (G \mid S \mid S)}{\mathcal{G} \rhd (G \mid S)}\ (SC) \qquad \frac{\mathcal{G} \rhd (G \mid \Gamma_1, \Gamma_2 \Rightarrow)}{\mathcal{G} \rhd (G \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2)}\ (J)$$

Definition 57 $\mathbf{GABC_n}$ *for* $n = 1, 2, \ldots$ *consists of the core intermediate rules plus* (SC) *and:*

$$\frac{[\mathcal{G} \rhd (G \mid \Gamma_i, \Gamma_j \Rightarrow \Delta_i)]_{1 \le i < j \le n+1}}{\mathcal{G} \rhd (G \mid \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_{n+1} \Rightarrow \Delta_{n+1})}\ (BC_n)$$

More generally, we can define (as in the modal case) the notion of a calculus being L-fitting for an intermediate logic L.

Definition 58 *Let* L *be an intermediate logic. A calculus* $\mathbf{GAL}$ *is* L*-fitting if:*

(1) $\mathbf{GAL}$ *extends the core intermediate rules.*
(2) If $\vdash_{\mathsf{L}} S$*, then* $\vdash_{\mathbf{GAL}} \rhd S$ *for any sequent* S.
(3) If $\vdash_{\mathbf{GAL}} \mathcal{R}$*, then* $\mathrel{|\!\sim}_{\mathsf{L}} \mathcal{R}$.

Soundness of the core intermediate rules for extensible intermediate logics is established in exactly the same way as for IPC.

Lemma 59 *The core intermediate rules are* L*-sound for every extensible intermediate logic* L*.*

Proof. Let L be an extensible intermediate logic. We just consider the revised Visser rule. Suppose that σ is an L-unifier for $I(H)$ for all $H \in \mathcal{G}$ and $I(G \mid \Gamma \Rightarrow \Delta)$, where $\Delta = \{A_1, \ldots, A_n\}$. Using the right set of premises, σ is an L-unifier for $I(\Gamma^\Pi, \Pi \Rightarrow \Delta)$ for all $\Pi \subseteq \Gamma_\Delta$. It suffices now to show that σ is an L-unifier for $I(G \mid \{\Gamma \Rightarrow A\}_{A \in \Delta})$. Suppose, arguing contrapositively, that this is not the case. Then there exists a countermodel of L for $I(\sigma(G)) \vee \bigvee_{A \in \Delta} I(\sigma(\Gamma) \Rightarrow \sigma(A))$. This implies that for every $A \in \Delta$ there are countermodels K_A such that K_A is a model of L, $K_A \Vdash \sigma(\bigwedge \Gamma)$, and $K_A \nVdash \sigma(A)$. Because L is extensible there is a variant K of $(\Sigma_{A \in \Delta} K_A)'$ that is a model of L. Let $\Pi = \{D \in \Gamma_\Delta : K \Vdash \sigma(D)\}$. Observe that for all $B \to C \in \Gamma$ such that $B \notin \Pi$, either $B \in \Delta$ or $K \nVdash \sigma(B)$. Note also that $B \in \Delta$ implies $K \nVdash \sigma(B)$. Hence for all $B \notin \Pi$ it follows that $K \nVdash \sigma(B)$, and so $K \Vdash \sigma(B \to C)$. It follows that $K \Vdash \sigma(\bigwedge(\Gamma^\Pi \cup \Pi))$. Thus $K \Vdash \sigma(\bigvee \Delta)$, a contradiction. □

In particular, using results from [2], we obtain:

Corollary 60 $\mathbf{GAKC}$ *is* KC*-fitting and* $\mathbf{GABC_n}$ *is* $\mathsf{BC_n}$*-fitting for* $n = 1, 2, \ldots$.

To prove completeness for L-fitting systems $\mathbf{GAL}$ for extensible intermediate logics L, we proceed as for IPC and extensible modal logics. First we can show, exactly as in Lemma 30 (except replacing the application of the modal deduction theorem with the usual deduction theorem), that L-derivable sgh-rules of a certain form are also $\mathbf{GAL}$-derivable.

Lemma 61 *Let* L *be an extensible intermediate logic and let* $\mathbf{GAL}$ *be* L*-fitting. If* $\vdash_{\mathsf{L}} \mathcal{G} \rhd \mathcal{H}$ *where* $\mathcal{H} = \emptyset$ *or* $\mathcal{H}$ *is a sequent, then* $\vdash_{\mathbf{GAL}} \mathcal{G} \rhd \mathcal{H}$.

The completeness theorem is then established similarly to the proof for IPC, the main complication being that we now have to take care of all the different disjuncts occurring in hypersequents on the left.

Theorem 62 *Let* L *be an extensible intermediate logic and let* $\mathbf{GAL}$ *be* L*-fitting. Then if* $\mathcal{H} = \emptyset$ *or* $\mathcal{H}$ *is a sequent:*

$$\mathrel{|\!\sim}_{\mathsf{L}} \mathcal{G} \rhd \mathcal{H} \;\; \textit{iff} \;\; \vdash_{\mathbf{GAL}} \mathcal{G} \rhd \mathcal{H}$$

Proof. The right-to-left direction follows directly from Lemma 59. For the left-to-right direction, it is sufficient to assume (proceeding exactly as in the IPC-case) that $\mathcal{R} = \mathcal{G} \rhd \mathcal{H}$ is an L-admissible implication-irreducible gs-rule that is full with respect to (AC), $(\to)\rhd$, and (IV). Let $\mathcal{G} = G_1, \ldots, G_n$ and $G_i = S^i_1 \mid \ldots \mid S^i_{m_i}$ where $S^i_j = \Gamma^i_j \Rightarrow \Delta^i_j$. Define $C_i =_{def} \bigvee_{j=1}^{m_i} I(S^i_j)$ and $C =_{def} \bigwedge_{i=1}^n C_i$. If C is inconsistent, then $\vdash_{\mathbf{GAL}} \mathcal{G} \rhd \mathcal{H}$ follows immediately by Lemma 61. Define:

$$C_{j_i,\ldots,j_n} =_{def} I(S^1_{j_1}) \wedge \ldots \wedge I(S^n_{j_n}).$$

and observe that by distributivity:

$$\vdash_{\mathsf{IPC}} \Big(\bigvee_{j_1 \leq m_1, \ldots, j_n \leq m_n} C_{j_i, \ldots, j_n} \Big) \leftrightarrow C.$$

By Lemma 8 (d), if each $C_{j_i,\ldots,j_n}$ is IPC-projective, then $\vdash_{\mathsf{L}} \mathcal{G} \rhd \mathcal{H}$. Hence, by Lemma 61, $\vdash_{\mathbf{GAL}} \mathcal{G} \rhd \mathcal{H}$. Note that the fact that $\mathcal{H}$ consists of at most one sequent plays a crucial role here.

It therefore remains to show that each consistent $C_{j_i,\ldots,j_n}$ is IPC-projective. Let:

$$eq(C_1 \wedge \ldots \wedge C_n) =_{def} \{C_i \mid i \leq n, \exists j \leq n(C_j = C_i \wedge j \neq i)\},$$

$$set(C_1 \wedge \ldots \wedge C_n) =_{def} \{C_1, \ldots, C_n\}.$$

It suffices in fact to establish the IPC-projectivity only of the disjuncts $D = C_{j_i,\ldots,j_n}$ for which there is no disjunct D' of C such that $set(D') \subseteq set(D)$ and $eq(D') < eq(D)$. For suppose there is such a D' that is IPC-projective. Then (reasoning as above) $\vdash_{\mathbf{GAL}} D' \rhd \mathcal{H}$, and, since $set(D') \subseteq set(D)$, $\vdash_{\mathbf{GAL}} D \rhd \mathcal{H}$.

Thus, it suffices to show that every disjunct $D = C_{j_i,\ldots,j_n}$ with the desired property is IPC-projective. In proving this we proceed in the same way as in the intuitionistic case, the only difference being that where we used $S \in \mathcal{G}$ there, e.g. in the defintion of the U property, we replace this here by "$I(S)$ is a conjunct of D", which we will denote by $I(S) \in D$. Thus in this setting we define

(1) $U((\Gamma \Rightarrow \Delta), I, D)$ iff $\exists \Gamma' \subseteq \Gamma^I \cup I \; \exists \Delta' \subseteq \Delta \, (\Gamma' \Rightarrow \Delta' \in D)$
(2) $V((\Gamma \Rightarrow \Delta), \mathcal{G})$ iff $\forall I \subseteq \Gamma_\Delta : U((\Gamma \Rightarrow \Delta), I, D)$

Now we want to prove a lemma equivalent to Lemma 51:

Claim 2 *For every disjunct D of C, if $\Gamma, p \Rightarrow \Delta$ and $\Pi \Rightarrow p, \Sigma$ are conjuncts of D, then so is $\Gamma, \Pi \Rightarrow \Delta, \Sigma$, or there is a disjunct D' of C such that $eq(D') < eq(D)$ and $set(D') \subseteq set(D)$.*

Proof. Suppose that a disjunct D of C has conjuncts $\Gamma, p \Rightarrow \Delta$ and $\Pi \Rightarrow p, \Sigma$. W.l.o.g. we can assume that D is of the form $D_1 \wedge \ldots \wedge D_k \wedge E_1 \wedge \ldots \wedge E_l \wedge C_1 \wedge \ldots \wedge C_m$, where the D_i are all the conjuncts in D of the form $\Gamma, p \Rightarrow \Delta$, and the E_i are all the conjuncts in D of the form $\Pi \Rightarrow p, \Sigma$. Thus there are G_i and H_j such that $G_i \mid \Gamma, p \Rightarrow \Delta \in \mathcal{G}$ for all $i \leq k$, and $H_j \mid \Pi \Rightarrow p, \Sigma \in \mathcal{G}$ for all $j \leq l$: namely, hypersequents in $\mathcal{G}$ corresponding to the conjuncts D_i and E_j of D. Note that since the sequents have to come from different hypersequents in $\mathcal{G}$, we have $i \neq j$ for all $i \leq k$ and $j \leq l$. By the (IAC) rule, $G_i \mid H_j \mid \Gamma, \Pi \Rightarrow \Delta, \Sigma \in \mathcal{G}$. If $\Gamma, \Pi \Rightarrow \Delta, \Sigma$ is a conjunct of D we are done. Therefore, suppose this is not the case. This implies that for all i, j there are conjuncts A_{ij} of $C_1 \wedge \ldots \wedge C_m$ that appear in $G_i \mid H_j$, i.e. $G_i \mid H_j = A_{ij} \mid I$ for some I. If A_{ij} appears in H_j we replace E_j by A_{ij}. If A_{ij}

appears in G_i we replace D_i by A_{ij}. In this way we obtain a disjunct D' of C such that $eq(D') < eq(D)$ and $set(D') \subseteq set(D)$. This proves the claim. $\square$

Now consider a disjunct $D = C_{j_i,\ldots,j_n}$ of C for which there is no disjunct D' of C such that $set(D') \subseteq set(D)$ and $eq(D') < eq(D)$. We have to show that it is IPC-projective. Arguing contrapositively, assume it is not. Then in the same way as in the intuitionistic case, using the claim where we there used Lemma 51, we end up with a sequent $\Gamma \Rightarrow \Delta$ of $C_{j_i,\ldots,j_n}$ such that $U(\Gamma \Rightarrow \Delta, D)$, where $K \Vdash \bigwedge \Gamma$ and $\Delta \cap P = \emptyset$ for $K = \{p \mid K \Vdash p\}$. Now in the intuitionistic case we conclude that $\Gamma \Rightarrow A \in \mathcal{G}$ for some $A \in \Delta$, or $\Gamma^\Pi, \Pi \Rightarrow \Delta \in \mathcal{H}$ for some $\Pi \subseteq \Gamma_\Delta$. Here we want to conclude that either $\Gamma \Rightarrow A$ is a conjunct of D for some $A \in \Delta$, or $H = (\Gamma^\Pi, \Pi \Rightarrow \Delta)$ for some $\Pi \subseteq \Gamma_\Delta$. The rest of the argument will then be similar to the intuitionistic case. Therefore, we have proved the theorem once we have proved the following claim.

Claim 3 *For every disjunct D of C, if $\Gamma \Rightarrow \Delta$ is a conjunct of D, where Γ consists of implications only, then $\Gamma \Rightarrow A$ is a conjunct of D, or $H = (\Gamma^\Pi, \Pi \Rightarrow \Delta)$ for some $\Pi \subseteq \Gamma_\Delta$, or there is a disjunct D' of C such that $eq(D') < eq(D)$ and $set(D') \subseteq set(D)$.*

We prove the claim as follows. Suppose that D is of the form $D_1 \wedge \ldots \wedge D_k \wedge C_1 \wedge \ldots \wedge C_m$, where the D_i are all the conjuncts in D of the form $\Gamma \Rightarrow \Delta$. Thus there are G_i such that $(G_i \mid \Gamma \Rightarrow \Delta) \in \mathcal{G}$ for all $i \leq k$. Namely, the hypersequents in $\mathcal{G}$ that correspond to the conjuncts D_i of D. Hence by the Visser rule, for all i either $(G_i \mid \Gamma \Rightarrow A \mid_{A\in\Delta}) \in \mathcal{G}$ or $H = (\Gamma^\Pi, \Pi \Rightarrow \Delta)$ for some $\Pi \subseteq \Gamma_\Delta$. Now if $\Gamma \Rightarrow A$ is a conjunct of D or $H = (\Gamma^\Pi, \Pi \Rightarrow \Delta)$, we are done. If not, there are conjuncts A_i of $C_1 \wedge \ldots \wedge C_m$ that appear in G_i. Therefore, replacing the D_i by A_i results in a disjunct D' of C such that $set(D') \subseteq set(D)$ and $eq(D') < eq(D)$. This proves the claim. $\square$

Corollary 63 *For $\mathsf{L} \in \{\mathsf{KC}, \mathsf{BC}_1, \mathsf{BC}_2, \ldots\}$ and $\mathcal{H} = \emptyset$ or $\mathcal{H}$ is a sequent: $\mid\!\sim_{\mathsf{L}} \mathcal{G} \rhd \mathcal{H}$ iff $\vdash_{\mathbf{GAL}} \mathcal{G} \rhd \mathcal{H}$.*

6 Termination

Our final task will be to show that by adding some control to the application of rules, we obtain calculi for admissibility in our logics that are *terminating* in the sense that applying the rules backwards to any gs-rule or gh-rule terminates. Known decidability results for admissibility in modal and intermediate logics are obtained as corollaries. The basic idea is to apply the invertible left logical rules as much as possible to obtain atomic or implication-irreducible gs-rules or gh-rules, then to apply the remaining non-structural rules with loop-checking.

Definition 64 *For* $\mathcal{R} = \frac{\mathcal{G}_1 \rhd \mathcal{H}_1 \ \dots \ \mathcal{G}_n \rhd \mathcal{H}_n}{\mathcal{G} \rhd \mathcal{H}}$, *let* $\mathcal{R}^+ = \frac{\mathcal{G}_1, \mathcal{G} \rhd \mathcal{H}, \mathcal{H}_1 \ \dots \ \mathcal{G}_n, \mathcal{G} \rhd \mathcal{H}, \mathcal{H}_n}{\mathcal{G} \rhd \mathcal{H}}$.

Definition 65 *Let* $\mathbf{GAL_t}$ *be* $\mathbf{GAL}$ *where every non-left-logical-rule* R *is replaced by* R^+ *restricted to non-looping applications with an irreducible conclusion.*

Intuitively, we are simply ensuring that the left logical rules of the calculus are applied first, and that the other rules add sequents or hypersequents without removing them (this can be achieved by first applying contraction rules). Indeed it is easy to check that these extra restrictions do not interfere with our completeness proofs.

Lemma 66 *Let* L *be an extensible modal logic or an extensible intermediate logic, and let* $\mathbf{GAL}$ *be an* L*-fitting calculus or* $\mathbf{GAI}$ *if* L *is* IPC*. Then:*

$$\mathrel{|\!\sim}_{\mathsf{L}} \mathcal{G} \rhd \mathcal{H} \ \textit{ iff } \ \vdash_{\mathbf{GAL_t}} \mathcal{G} \rhd \mathcal{H}$$

where if L *is an extensible intermediate logic but not* IPC*,* $\mathcal{H} = \emptyset$ *or* $\mathcal{H}$ *is a sequent.*

Theorem 67 *If* $\mathbf{GAL}$ *is* L*-fitting or* $\mathbf{GAI}$ *and every non-left-logical-rule has the subformula property, then* $\mathbf{GAL_t}$ *is terminating.*

Proof. If $\mathcal{R}$ is not irreducible, then by invertibility, left logical rules can be applied, terminating with irreducible gs-rules or hs-rules. From this point onwards only non-looping applications of rules with the subformula property are applied. Observe that there is only a finite number of sequents and non-repetitive hypersequents that can be constructed from subformulas of a given gs-rule or gh-rule $\mathcal{R}$. However, since every rule is non-looping and expansive, each application of a rule adds at least one new sequent or non-repetitive hypersequent. By the preceding observation, this process must terminate. □

In particular, we have terminating systems for our paradigmatic cases of modal and intermediate logics.

Corollary 68 $\mathbf{GAL^t}$ *is terminating for* $\mathsf{L} \in \{\mathsf{K4}, \mathsf{S4}, \mathsf{GL}, \mathsf{IPC}, \mathsf{KC.BC_1}, \mathsf{BC_2}, \dots\}$.

Moreover, we know that decidability for derivability implies decidability for admissibility in these cases.

Corollary 69 *Admissibility is decidable for any decidable extensible modal logic or extensible intermediate logic.*

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