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Hajime Ishihara
Japan Advanced Institute of Science and Technology,
Tatsunokuchi, Ishikawa, Japan

Bakhadyr Khoussainov
Cornell University, Ithaca NY, USA,
The University of Auckland, Auckland,
New Zealand

Anil Nerode
Cornell University, Ithaca NY, USA

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Hajime Ishihara*
Japan Advanced Institute of Science and Technology,
Tatsunokuchi, Ishikawa, 923-12 Japan

Bakhadyr Khoussainov†
Cornell University, Ithaca, NY, 14850, USA,
The University of Auckland, Auckland, New Zealand

Anil Nerode‡
Cornell University, Ithaca NY 14853 USA

1 Motivation

The introduction of computable (alternately, recursive) function theory by Post, Church, Kleene, Godel, Turing, Malcev made it possible to analyse the computability of mathematical notions and constructions within the context of classical mathematics. Quite separately, the constructiveness of algebra was a principal concern of Kronecker in the late nineteenth century, and the constructiveness of analysis was a principle concern of Brouwer in the early twentieth century. Brouwer's work motivated the definition of first order intuitionistic logic as introduced by his disciple Heyting. Kroneckerian field theory was reworked as computable field theory by Fr/"oelich and Shepherdson in the 1950's, [4]. Systematic study of recursive algebra and recursive

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models of classical predicate logic was initiated by Rabin [9] and Malcev[6] in the 1960's. In the 1970's, Ershov's school in Russia and Nerode's school in the United States began the systematic use of the priority method to determine whether or not classical constructions can be made computable throughout mathematics, in such areas as vector spaces, orderings, boolean algebras, abelian groups, fields, rings, and models of classical first order logic.

We refer the reader to Nerode-Remmel [7], Hazarinov [5] and Millar [8] for surveys.

Things are more complicated for the model theory of intuitionistic logic. There are several model theories for intuitionistic logic with quite different flavors. One is lambda calculus models, leading to the work of Girard and of Martin Lof on typed lambda calculi, or, as D. Scott has observed, equivalently leading to closed cartesian categories (untyped lambda calculi). In such models existential quantifiers are interpreted as functionals (lambda terms). A second style of model is Kripke and Beth models. A third is the topological models as introduced by Rasiowa and Sikorski from prior work of Tarski, for their early 1950's proof of completeness of intuitionistic predicate logic within classical mathematics.

All these classes of models are adaquate to give classical proofs of completeness of intuitionistic predicate logic, although the literature is especially opaque when you look for the equivalences and proofs of completenss (see the work of Lauchli and also of D. Scott). There is also a body of work on constructive proofs of completeness of predicate intuitionistic logic. These are based on a very careful choice of definition of model and a very careful formulation of the statement of completeness. These proofs use so-called feeble (in plain English, contradictory) models, see Troelstra and Van Dalen, volume 2 of [12].

What does computable model theory look like for these model theories? Here we look only at Kripke models of intuitionistic predicate logic, leaving the others for other papers.

Classical completeness of standard predicate order logic can be expressed by the assertion that if T is consistent, then T has a classical model. Classical completeness of intuitionistic predicate logic can be expressed by the assertion that if T is consistent in intuitionistic predicate logic, then T has a Kripke model $\mathcal{M}$. Moreover a single Kripke model $\mathcal{M}$ can be chosen so that the statements forced in $\mathcal{M}$ are exactly those intuitionistically provable from T. The standard proof can be thought of as a generalization to Kripke frames of the the Henkin 1949 construction for classical predicate logic, see

[3] or [10] , in which the maximal filters of classical Lindenbaum Boolean algebras have to be replaced by presheaves of prime filters of intuitionistic Lindenbaum distributive lattices.

Straightforward adaptation of that argument gives the result of Gabbay [2] : for any decidable finitely axiomatized intuitionistic theory T and any sentence ϕ not intuitionistically derivable from T , there is a Kripke model of T which does not force ϕ , based on an underlying partially ordered set with a computably enumerable partial ordering, and such that forcing restricted to atomic statements is computably enumerable.

Here our main result is that by using a more refined argument we get that any decidable intuitionistic theory T has a Kripke model $\mathcal{M}$ with decidable forcing such that for all sentences ϕ, ϕ is an intuitionistic consequence of T if and only if $\mathcal{M}$ forces ϕ. This generalizes the theorem in classical computable model theory that a decidable theory has a decidable model.

Examination of the proof of that older theorem shows that the crucial observation is that a computably enumerable maximal filter in a recursively presented Boolean algebra is in fact recursive. This is not the case for prime filters in computably presented distibutive lattices such as the Lindenbaum algebra of a decidable theory in intuitionistic logic. There are generally lots of prime filters that are computably enumerable but not computable (recursive.) So what we do is to locate a subclass of prime filters for which, in recursively presented distributive lattices, computably enumerable implies computable. These are the prime filters P for which there is a non-zero element $\psi \notin P$ such that every element not less than or equal to ψ is in P.

A function is *computable* if there is a Turing machine which computes it. We denote the set of all natural numbers by ω. A subset of natural numbers is *computable* if its characteristic function is computable. A set of natural numbers is *computably enumerable* (c.e.) if it is the range of a computable function. We refer to Soare [11] for the basic computability theory. We fix an effective enumeration $\Phi_0, \Phi_1, \ldots$ of all computable partial functions. We call number x an *index* of Φ_x. We also use the λ–notation for functions.

2 Decidable Kripke Models

Let $L =< P_0^{n_0}, \ldots, P_k^{n_k} \ldots, c_0, c_1, \ldots >$ be a countable first order language without any function symbols. We suppose that the language L is computable, that is the set of constants $C = \{c_0, c_1, \ldots\}$ and the function $n \to n_k$

are computable. We denote the set of all sentences of L by $Sn(L)$.

Definition 2.1 *A* **theory** *is a consistent set of sentences closed under the deduction rules of intuitionistic logic.*

Here is the Kripke model semantics.

A **frame** is a triple $F = (W, \leq, D)$ consisting of a non-empty set W, ("**states of knowledge**" or "**forcing conditions**"), a partial order $\leq$ of W, and a map D on W to a power set such that $v \leq w$ implies $D(v) \subseteq D(w)$. D is called the **domain function**.The partially ordered set $(W, \leq)$ is called the **base** of the frame. Let $L(w)$ be the extension of predicate logic language L obtained by adding to L a constant (name) c_a for each element $a \in D(w)$.

We suppose given a mapping V, called a **valuation ,** which assigns to each pair consisting of a w and an n–ary predicate symbol P (resp. constant c) from L , a n–ary relation on $D(w)$ (resp. element of $D(w)$). Let $A(w)$ be the set of all atomic statements of language $L(w)$ (classically) true in $D(w)$ under the valuation V. Suppose that for all $v \leq w$ the set of all atomic sentences from $A(v)$ is a subset of $A(w)$. Then the 4–tuple $\mathcal{M} = (W, \leq, D, V)$ is called a **Kripke model (over frame** F**)**.

Definition 2.2 *Let $(W, \leq, D, V)$ be a Kripke model of language L, w be in W and ϕ be a sentence from $L(w)$. We give a definition of w* **forces**ϕ *by induction on the complexity of ϕ.*

1. *For atomic senetences ϕ, w forces ϕ iff $\phi \in A(w)$.*

2. *w forces $\phi \to \psi$ iff for all $v \geq w$, v forces ϕ implies w forces ψ.*

3. *w forces $\neg\phi$ iff for all $v \geq w$, q does not force ϕ.*

4. *w forces $\forall x\phi$ iff for all $v \geq w$ and all constants $c \in L(v)$, v forces $\phi(c)$.*

5. *w forces $\exists x\phi$ iff for some $c \in L(w)$, w forces $\phi(c)$.*

6. *w forces $\phi \vee \psi$ iff w forces ϕ or w forces ψ.*

7. *w forces $\phi \& \psi$ iff w forces ϕ and w forces ψ.*

We say that $\mathcal{M}$ **forces** a sentence ϕ of language L if every $w \in W$ forces ϕ. By induction on the length of sentences $\phi \in L(w)$, we can prove that if w forces ϕ and $v \geq w$, then v forces ϕ. Following the lines of Henkin's proof for classical logic, one can prove the classical completeness of intuitionistic logic.

Theorem 2.1 *For any intuitionistically consistent theory T of language L , there exists a Kripke model $\mathcal{M}$ such that for all ϕ, $\mathcal{M}$ forces ϕ if and only if ϕ is deducible from T.*

To motivate our next notions we need to expand on the classical proof of this theorem. Exact proofs can be found in [3] or [10]. The proof is based on constructing so-called "prime theories" containing T. These are theories in languages obtained by adding to the original L infinitely many new constant symbols. Informally, these theories are the "prime filters with witnesses" of the distributive lattice which is the Lindenbaum algebra defined by intuitionistic deducibility in T. The **base** of the desired Kripke model is the set of all such prime theories. The ordering is set–theoretic inclusion. Thus, one can say that under an appropriate coding of all formulas of the expansion of language L by constants, the elements of the base are subsets of natural numbers. Thus we are lead to the following definition.

Definition 2.3 *A* **frame** *is a triple $(S, \subseteq, D)$ with the following properties:*

1. *S is a family of subsets of ω, that is $S \subset 2^{\omega}$.*
2. *$\subseteq$ is the set–theoretical inclusion between subsets of ω.*
3. *D is a function assigning to any $p \in S$ a subset $D(p)$ of the natural numbers such that $D(p) \subset D(q0$ if $p \subset q$ for all $p, q \in S$.*

Now we can define the notion of computable frame. Informally, a frame $(S, \subseteq, D)$ is computable if the sets S and $\{D(p)|p \in S\}$ are uniformly computably enumerable. Here is an exact definition.

Definition 2.4 *A* **computable frame** *is 5–tuple $(S, \subseteq, D, f, g)$ such that $(S, \subseteq, D)$ is a frame, and f, g are computable functions with the following properties:*

1. *The set S coincides with $\{range(\lambda y f(x, y)|x \in \omega\}$.*
2. *For all $x \in \omega$, if $p = range(\lambda y f(x, y)$, then $D(p) = range(\lambda y g(x, y))$.*

If $(S, \subseteq, D, f, g)$ is a computable frame, we abuse notation by omitting mention of the functions f and g and simply say that $(S, \subseteq, D)$ is a computable frame. We are interested in those Kripke models for which forcing is a decidable relation. In other words, informally, a decidable frame is one for which there is a procedure which applied to any state of knowledge p from the frame and any statement ϕ from $L(p)$ decides whether p forces ϕ. Here is a formal definition.

Definition 2.5 *A Kripke model $\mathcal{M}$ over a computable frame $(S, \subseteq, D, f, g)$ is* **decidable** *if the set*

$$\{(i, \phi) | i \in \omega,\ \phi \in L(range(\lambda y(f(i, y)))) \text{ and } range(\lambda y(f(i, y))) \text{ forces } \phi\}$$

is a computable set.

Now we are ready to state our main theorem. But first we need a basic definition. A theory T is **computable** if there is procedure which applied to any sentence ϕ answers if ϕ is intuitionistically deducible from T or not.

Theorem 2.2 *For every computable theory T of the language L there is a decidable Kripke model $\mathcal{M}$ such that for all $\phi \in L$, $\mathcal{M}$ forces ϕ if and only if ϕ is deducible from T.*

Proof. The proof is based on effectivizing the proof of the completeness theorem. First, we show that computable prime theories withg witnesses containing T exist. The classical proofs in [3] or [10] do not contruct computable prime theories containing T . We give a slightly different construction of a prime theory containing T with an additional property that guarantee computability. Here is the crucial new definition.

Definition 2.6 *Let ψ_0 be a sentence. A theory Γ of a language L is L–* **maximal with respect to** ψ_0 *if*

1. *Γ does not contain ψ_0 (and therefore is consistent).*
2. *For all ϕ, if ϕ is intuitionistically deducible from Γ, then $\phi \in \Gamma$.*
3. *For all ϕ and ψ if $\phi \vee \psi \in \Gamma$, then either $\phi \in \Gamma$ or $\psi \in \Gamma$.*
4. *For all formulas $\phi(x)$ of one free variable x, if $\exists x \phi(x) \in \Gamma$, then $\phi(c) \in \Gamma$ for some constant $c \in L$.*
5. *For all ϕ, if ψ_0 can not be deduced from $\Gamma \bigcup \{\phi\}$, then $\phi \in \Gamma$.*

The last condition is the new one [3] [10].

Lemma 2.1 *Suppose that T is a computable theory in language L . Suppose that ψ_0 is not intuitionistically deducible from T. Let $\bar{L} = L \bigcup C$ where C is an infinite computable set of constants such that $L \bigcap C = \emptyset$. Then there is a $\bar{L}$–maximal computable theory Γ with respect to ψ_0 such that $T \subset \Gamma$ and ψ_0 is not deducible from Γ.*

Proof. Let ϕ_0, ϕ_1, ϕ_2, ... be a computable sequence of all sentences of the language $\bar{L}$ in which every sentence appears infinitely many times. We construct Γ by stages. At stage $t+1$ we define Γ_{t+1} such that $\Gamma_t \subseteq \Gamma_{t+1}$. At the end we put $\Gamma = \bigcup_t \Gamma_t$. At each stage $t+1$ we treat the sentence ϕ_t. If we do not put ϕ_t into Γ_{t+1}, then ϕ will not belong to Γ. Since the procedure is effective, Γ will be computable. We note the following simple fact. If T is computable and Δ is a finite set, then by deduction theorem the closure of $T \bigcup \Delta$ with respect to intuitionistic deduction is also computable.

Stage 0. $\Gamma_0 = T$.

Stage $t+1$. Suppose that Γ_t has been constructed. Take ϕ_t. We have three cases.

Case 1. ϕ_t is $A \vee B$. If ψ_0 is not intuitionistically deducible from $\Gamma_t \bigcup \{A\}$, then we define Γ_{t+1} as the closure of $\Gamma_t \bigcup \{A\}$ under intuitionistic deduction. Suppose that ψ_0 is intuitionistically deducible from $\Gamma_t \bigcup \{A\}$. Then if ψ_0 is not intuitionistically deducible from $\Gamma_t \bigcup \{B\}$, we define Γ_{t+1} as be the closure of $\Gamma_t \bigcup \{B\}$ under intuitionistic deduction. Otherwise, we define $\Gamma_{t+1} = \Gamma_t$.

Case 2. ϕ_t is $\exists x \phi(x)$. If ψ_0 is not deducible from $\Gamma_t \bigcup \{\phi_t\}$, then we define Γ_{t+1} as the closure of $\Gamma_t \bigcup \{\phi_t, \phi(c)\}$ under intuitionistic deduction, where c is the first constant not belonging to Γ_t. Otherwise, we define $\Gamma_{t+1} = \Gamma_t$.

Case 3. Suppose that the previous neither of the previous cases holds. If ψ_0 is not deducible from $\Gamma_t \bigcup \{\phi_t\}$, then we define Γ_{t+1} as the closure of $\Gamma_t \bigcup \{\phi_t\}$ under intuitionistic deduction. Otherwise, we define $\Gamma_{t+1} = \Gamma_t$.

This ends the construction.

Define Γ to be $\bigcup_t \Gamma_t$. We prove that Γ is a $\bar{L}$–maximal theory with respect to ψ_0.

First, we show that ψ_0 is not intuitionistically deducible from Γ. Suppose otherwise. Then there exists a t such that ψ_0 is intuitionistically deducible from Γ_{t+1}. We prove by induction on k thatψ_0 is not intuitionistically deducible from Γ_k . Clearly, ψ_0 is not intuitionistically deducible from Γ_0 . Suppose that ψ_0 is not intuitionistically deducible fromΓ_t .

Suppose that Case 1 of stage $t+1$ holds. Then Γ_{t+1} properly extends Γ_t since by inductive hypothesis ψ_0 is not intuitionistically deducible from Γ_t. It follows that $\phi_t = A \vee B$ and that either Γ_{t+1} is the closure of $\Gamma_t \bigcup \{A\}$ or Γ_{t+1} is the closure of $\Gamma_t \bigcup \{B\}$. If Γ_{t+1} is the closure of $\Gamma_t \bigcup \{A\}$, then by the definition of Γ_{t+1}, ψ_0 is not intuitionistically deducible from$\Gamma_{i+1} \bigcup \{A\}$. Similarly, if Γ_{t+1} is the closure of $\Gamma_t \bigcup \{B\}$, then ψ_0 is not intuitionistically deducible from Γ_{t+1}. This is a contradiction.

Suppose that Case 2 holds. Then Γ_{t+1} is the closure of $\Gamma_t \bigcup \{\phi_t, \phi(c)\}$. Then ψ_0 is intuitionistically deducible from $\Gamma_t \bigcup \{\phi_t, \phi(c)\}$, hence ψ_0 is intuitionistically deducible from $\Gamma_t \bigcup \{\phi_t\}$.

Suppose that Case 3 holds, then $\psi_0$is not intuitionistically dediucible from Γ_{t+1} .

It follows that ψ_0 is not intuitionistically deducible from Γ.

We need to show that Γ is closed under intuitionistic deduction. Suppose that ϕ is intuitionistically deducible from Γ. There is a t such that $\phi = \phi_t$. It follows thatψ_0 is not an intuitionistic consequence of $\Gamma_t \bigcup \{\phi\}$. So, by the definition of Γ_{t+1}, ϕ belongs to Γ_{t+1}.

Suppose that $\phi \vee \psi \in \Gamma$. There is a t such that $\phi \vee \psi \in \Gamma_{t+1}$. Since every sentence ϕ appears infinitely many time in the sequence $\phi_0, \phi_1, \ldots$, we see that there is a $k > t$ such that $\phi_k = \phi \vee \psi$. We need to show that at stage $k+1$ either ϕ or ψ enters Γ. If $\Gamma_k \bigcup \{\phi\}$ and ψ_0 is an intuitionistic consequence of $\Gamma_k \bigcup \{\psi\}$. In this case ψ_0 is an intuitionistic consequence of $\Gamma_k \bigcup \{\phi_k\}$, which is contraduction. Hence at stage $k+1$ either ϕ or ψ enters Γ.

Suppose that $\exists x \phi(x) \in \Gamma$. There is a k such that $\phi_k = \exists x \phi(x)$. At stage $k+1$, $\phi(c)$ enters Γ for some c by the definition of the stage.

Now we prove that if$\psi_0 \Gamma \bigcup \{\phi\}$ is not an intuitionistic consequence of ψ_0, then $\phi \in \Gamma$. There is a t such that $\phi_t = \phi \vee \phi$. Then at stage $t+1$, ϕ enters Γ.

Now we prove that Γ is a computable theory. Take a sentence ϕ. Find a t such that $\phi_t = \phi$. Then $\phi \in \Gamma$ if and only if $\phi_t \in \Gamma_{t+1}$. Hence Γ is computable. The lemma is proved.

To state the next corollary we need some more notation. Let C_0, C_1, ... be an infinite sequence of pairwise disjoint infinite computable sets of constants. We put $L_0 = L$, $L_{i+1} = L_i \bigcup C_i$.

Corollary 2.1 *There is an effective procedure P which, given $i \in \omega$, $\psi \in Sn(L_i)$, and a finite $\Delta \subset Sn(L_i)$ such that : if ψ is not intuitionistically deducible from T, Δ , then $P(i, \psi, \Delta)$ is an index of a computable function which computes a L_i–maximal theory with respect to ψ containing T.*

Proof. The corollary follows from the observation that the lemma above can be proved effectively when ψ and Δ are given.

Now introduce the following frame F

$$(\{P(i, \psi, \Delta) | i \in \omega, \psi \in L_i, \Delta \subset Sn(L_i), \quad T \bigcup \Delta \text{ does not deduce } \psi\}, \subseteq, D),$$

where $D(P(i, \psi, \Delta))$ is the set of all constants from L_i. By the corollary above, this frame is computable.

Now we need to define a Kripke model $\mathcal{M}$ over frame F. Consider the state of knowledge $D(P(i, \psi, \Delta))$ from F. For every predicate symbol $R \in L$ we put $R(c_1, \ldots, c_n)$ is (classicaly) true iff $R(c_1, \ldots, c_n)$ belongs to $P(i, \psi, \Delta)$.

Lemma 2.2 *Let $P(i, \psi, \Delta)$ be a state of knowledge from Kripke model $\mathcal{M}$. Let ϕ be a sentence of the language L_i. Then the following properties hold:*

1. *$\phi \rightarrow \phi' \in P(i, \psi, \Delta)$ if, and only if, for all Γ containing $P(i, \psi, \Delta)$ the condition $\phi \in \Gamma$ implies $\phi' \in \Gamma$.*

2. *$\neg\phi \in P(i, \psi, \Delta)$ if, and only if, for all Γ containing $P(i, \psi, \Delta)$ we have $\phi \notin \Gamma$.*

3. *$\phi = \forall x \phi' \in P(i, \psi, \Delta)$ if, and only if, for all Γ containing $P(i, \psi, \Delta)$ and $c \in D(P(i, \psi, \Delta))$ we have $\phi'(c) \in \Gamma$.*

4. *$\phi \& \phi' \in P(i, \psi, \Delta)$ if, and only if, ϕ and ϕ' belong to $P(i, \psi, \Delta)$.*

5. *$\phi \vee \phi' \in P(i, \psi, \Delta)$ if, and only if, either ϕ or ϕ' belong to $P(i, \psi, \Delta)$.*

Proof. We prove the lemma by induction on the length of sentence ϕ. If ϕ is atomic, then by definition $P(i, \psi, \Delta)$ forces ϕ if and only if $\phi \in P(i, \psi, \Delta)$.

To prove parts 4 and 5 note that if ϕ is $\phi' \& \phi''$ or $\phi' \vee \phi''$, then the proof of the lemma follows from the facts that $P(i, \psi, \Delta)$ is closed under deduction and is prime theory.

We now prove part 1. If $\phi \to \phi' \in P(i,\psi,\Delta)$, $\phi \in P(j,\psi',\Delta')$ and $P(i,\psi,\Delta) \subset P(j,\psi',\Delta')$, then since $P(j,\psi',\Delta')$ is a theory we obtain that $\phi' \in P(j,\psi',\Delta')$. Suppose that $\phi \to \phi' \notin P(i,\psi,\Delta)$. It follows that ϕ' is not intuitionistically deducible from $P(i,\psi,\Delta) \bigcup \{\phi\}$.. Hence by Corollary 2.1 there is a computable strongly $L(i+1)$–prime theory Γ containing $P(i,\psi,\Delta)$ such that $\phi' \notin \Gamma$. This proves Part 1.

Part 2 as well as part 3 can be proved in a similar way. So the lemma is proved.

¿From this lemma we obtain that in frame $\mathcal{M}$, the state of knowledge $P(i,\psi,\Delta)$ forces a sentence ϕ if and only if ϕ belongs to $P(i,\psi,\Delta)$. By Corollary 2.1, we get that the forcing in $\mathcal{M}$ is a computable set. Hence the frame is decidable. Moreover, by the previous lemma we see that for any $\phi \in Sn(L)$, ϕ is deducible from T if and only if ϕ is forced in frame $\mathcal{M}$. Hence the theorem is proved.

Definition 2.7 *We say that a theory T is* **complete** *for a class K of of Kripke models if for any ϕ not intuitionistically deducible from T, there is a Kripke model $\mathcal{M}$ from K such that $\mathcal{M}$ is a model of T but not ϕ.*

The next result directly follows from the theorem.

Corollary 2.2 *Every computable intuitionistic theory T is complete for the class of decidable Kripke models.*

□

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