ON SOME HISTORICAL ASPECTS OF THE THEORY OF RIEMANN ZETA FUNCTION

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Contents

Preface

Briefly speaking, according to the methodological and epistemological considerations explained in (Fromm 1979, 1.), the philosophical *access* to a theoretical system of every thinker is possible only when we have a clear and detailed framework which comprises, above all, the historical contextuality within which it has sprung out as a critical and subversive (i.e., not institutionally recognized or not codified) system with respect to the given *paideia*[1] coeval with the thinker under consideration. All this is, on the other hand, also supported by the well-known Kuhnian epistemological ideas about the unavoidable relationships between internal and external history. This initial state of non-official institutional recognition of a given system of thought is apparently contradictory because, in a given historical period, this system will be in conflict with the so-called *normal science,* for being then gradually considered with even more attention until up when it will be accepted by a given social-cultural community, so joining the new normal science course. Along this historical pathway, which often is characterized by a *paradigmatic change* in the Kuhnian sense, the nets of internal and external history are inextricably intertwined amongst them, above all as concern natural sciences. Nevertheless, as regard mathematics, these epistemological considerations should be considered with a certain caution, because the relationships between internal and external history are quite circumstantialized, like in the case, for instance, of the theory of Riemann zeta function, as we will see later.

Indeed, differently from other celebrated conjectures of mathematics, the so-called *Riemann conjecture,* still resists to every attempt of resolution, notwithstanding its centenarian history which has seen the birth of a wide and variegated knowledge's field grew up just around Riemann zeta function. In a epoch-making communication to the Berlin Academy of Science, dating back the late 1850s, G.F.B. Riemann presented his

[1] *Paideia* is a Greek term, used by Marcus Tullius Cicero and Marcus Terentius Varro (see (Riera Matute 1970)), which refers to the overall cultural formation of the man constrained, in a given historical period, to a truth based on the philosophical knowledge seen as the higher and most worthy knowledge form. The corresponding Latin term is *humanitas.*

unique work on number theory in which he argues on the possible estimates of prime numbers less than a given quantity, on the basis of the previous works mainly made by L. Euler, J.L.F. Bertrand, P.G.L. Dirichlet, A.M. Legendre, P.L. Čebyšev e K.F. Gauss (see (Narkiewicz 2000, Preface) and (Niven et al. 1991, Chapter 8)). To be precise, Riemann commenced with using the famous *Euler relation*

$$(\zeta(s) \doteq) \sum_{n\in\mathbb{N}} \frac{1}{n^s} = \prod_{p\in\mathcal{P}} \frac{1}{1-p^{-s}} \qquad s \in \mathbb{C} \quad \Re(s) > 1,$$

where $\mathcal{P} \doteq \{p; p \in \mathbb{N}, p \text{ prime}, p \geq 2\}$, introducing a new complex function, the $\zeta(s)$ on the left-hand side, which will be explicitly considered, as a complex function, just by Riemann, and later called *Riemann zeta function* or simply *Riemann* ζ. With this remarkable contribution, Euler opened the way to explicitly use the tools and techniques of analysis applied to problems of number theory (see (Karatsuba 1994, Introduction)), along this pathway having then continued Gauss, Legendre and, above all, Riemann in carrying out his only unique work on number theory of the years 1858-59. Indeed, Riemann tried to deduce properties of the distribution of prime numbers by means of the mathematical properties of this new complex function, one of these having given rise to the celebrated *Riemann hypothesis* (in short, RH). Following (Karatsuba 1994, Introduction), the idea expressed by the above Euler's relation proved to be very fruitful and gave great impetus to the development of an important line of investigation in number theory. So with Euler[2], we might identify the origins of analytic number theory which has had mainly to do with two chief problems (see also next chapter 2), the one concerning the distribution of prime numbers (*multiplicative number theory*) and the other one regarding the resolution of equations in integers that was, by Euler, initially approached by means of the so-called *method of generating functions* (*additive number theory*) and that, in turn, has given rise to other new methods like the *Hardy-Littlewood-Ramanujan circle method* and the *Vinogradov method of trigonometric sums.* Finally, another line of investigation in number theory is the theory of transcendental num-

[2]In this regard, see also (Weil 1975).

bers. Our historical sight regards multiplicative number theory because the involvement of the Riemann zeta function is mainly motivated by prime number theory and related distribution's issues: to be precise, we are concerned with some historical moments regarding certain analytic aspects of the Riemann zeta function, laid out within the multiplicative number theory framework, which go through Riemann himself to Hadamard, Poincaré, Pólya, and so forth.

As pointed out by P.B. Borwein, except noteworthy histories of Riemann zeta function and related conjecture given at an informative or popular level, there is neither much specialized literature nor a single monograph devoted to the broad history of the Riemann zeta function realm with its very wide *plethora* of results. The only available historical sources are disseminated into the various technical textbooks and papers on the subject, which therefore are the necessary, and currently the unique available, starting points for every possible attempt to build up an organic and systematic historical framework of Riemann zeta function and related conjecture. Perhaps, this might be due to basically two main reasons. On the one hand, the fact that there exist only failed attempts to prove such a conjecture and this, usually, does not make history because of an usual bias which may be easily expressed by an adage of the Italian historian Niccolò Rodolico (1873-1969), according to which «the history is made by the winners»! On the other hand, in a work devoted to this subject, Alain Connes reports a significant remark, due to his teacher Gustave Choquet (1915-2006), according to which a mathematician is rather negatively remembered for her or his failed attempts to prove RH than for her or his other previous positive achievements[3]. Therefore, not wholly senseless, it would be, for instance, to shed a look at to a possible history of all the attempts made to prove RH, if nothing else from an epistemology of errors standpoint (see (Binanti 2001) for an interesting historical-anthological collectanea of writes on the pedagogy and epistemology of errors and mistakes). Furthermore, following this

[3]To be precise, he textually says that *«According to my first teacher Gustave Choquet one does, by openly facing a well known unsolved problem, run the risk of being remembered more by one's failure than anything else. After reaching a certain age, I realized that waiting "safely" until one reaches the end-point of one's life is an equally self defeating alternative»* (see (Connes 2000)).

historical research program, it would also be possible to have a global vision of what have been the various pathways treated and not. On the other hand, to carry out a similar program, a not brief time would have been necessary to pursue it, so that we have chosen to restrict our attention to a particular part of this wide and ambitious program, consisting in focussing on those historical aspects lying into that non-void ground given by the intersection between the Riemann zeta function theory and the entire function theory, through 19th to 20th century, which have seen involved, amongst others, the names of K. Weierstrass, J. Hadamard and G. Pólya. Finally, because of the unavoidable role that physics has played (and still plays) in mathematics and its development (see also what is said later by K. Maurin), we shall also put attention to some possible applications in physics of what discussed here, and this in coherence with the primeval aim that originally was, on the wake of the previous work of A.L. Cauchy, at the basis of Riemann work on complex functions, that never was disjoined, whenever possible, by the related physical motivations (see (Enriques 1982, Book III, Chapter I, Section 6) and (Klein 1979, Chapter VI)). On the other hand, the history of physics, above all the fundamental physics of 20th century, plainly says us that it cannot leave aside by considering, at the same time, history of complex analysis. Thus, we have considered an historical aspect quite neglected by either history of physics and history of mathematics, concerning the deduction of some notable rigorous results of statistical mechanics, that is to say, the formulation of some important theorems due to T.D. Lee and C.N. Yang of the early 1950s, which started just from some previous results of the 1930s achieved by Pólya upon certain integral representation of the so-called *Riemann ξ function,* and that has seen also involved Hadamard's work on entire function theory.

Hors d'œuvre

Here, we wish textually report some emblematic words of Krzysztof Maurin (see (Maurin 1997, Foreword, pp. xiii-xxii)) from which it is also possible to descry the motif of his notable work (Maurin 1997) on *Riemann oeuvre,* and that might make a brief but meaningful epistemological synoptical framework within which to lay out our work. Exactly, around great Riemann's figure, Maurin argues as follows

«The study of the ways in which great mathematical ideas are born, develop and die out (i.e., their so-called 'history' [of ideas]*) is undoubtedly one of the most fascinating branches of history. However, it requires an extensive and profound knowledge of contemporary mathematics. Being involved with the life of great mathematical ideas is fruitful not only for mathematics and physics but also for the person involved. It enables him (or her) to come into contact with and participate in the life of the world of ideas (the 'cosmos noethos' of the Platonists). For nowhere can we see more concretely, one is tempted to say almost palpably, the enormous spiritual energy which, although acting in people, still lacks clear contours and desires 'to be mounded' and developed by people - people called mathematicians. Plato, being strongly influenced by the Pythagoreans, was aware of this. So was Eudoxos, one of the greatest mathematicians of antiquity active in Plato's Academy.* [...] *It was Riemann, who probably more than anyone else, enriched mathematics with new ideas. These ideas display an unusual degree of vitality and impulse the whole mathematics as well as many branches of physics. The world of ideas is 'one' - i.e., it is a cohesive living organism in which all 'parts' interact and where even slight stimulations propagate producing echoes in the (seemingly) distant organs*[4] *which may be called theories or 'branches of mathematics'. Similarly like in Weierstrass-Riemann principle of 'an-*

[4]This is just one of the possible definition of Philosophy, centered around the notion of rational *referring* within a given philosophical system or between different philosophical systems, thanks to which the historical unity of knowledge is attainable in such a manner to overcome the various specialistic picket fences, in search for the possible common or fixed points obtained by means of a general comparative method.

alytic continuation', a change of a (meromorphic) function even within a very small domain (environment) affects through analytic continuation the whole of Riemann surface, or analytic manifold. Riemann was a master in applying this principle and also the first who noticed and emphasized that a meromorphic function is determined by its 'singularities'. Therefore, he is rightly regarded as the father of the huge 'theory of singularities' which is developing so quickly and whose importance (also for physics) can hardly be overestimated. [...] *As we have seen, the most fascinating phenomena in mathematics are those which link seemingly disparate branches of the discipline: analysis and geometry, analysis and arithmetic, geometry and arithmetic, local and global. The last pair is probably the 'hermetic' relation between micro- and macrocosm. Mathematics and physics make up*[5] *one organism - man's task is to actualize this unity of the world of ideas. Riemann was deeply aware of this: he thought of himself as mathematician and physicist. Constantly repeated, puzzled questions about 'the mysterious and incomprehensible congruity between mathematics and physics'* [d'après E.P. Wigner] *have as their source the unconscious and stubborn inclination to dissect this one (and the same) organism of mathematics-physics. Thus, two different, artificially created, entities come into being which are in fact organs of one great reality. Riemann outward life was brief, punctuated by the deaths of those he loved. It did not abound in any great worldly adventures. But his true life was devoted to the enrichment of the world of mathematical ideas, which was only natural as he was a profound philosopher, a discipline of G.T. Fechner and his 'Zend Avesta'. Creator lives in his creations!»*

With these few words, Maurin has efficaciously summarized the main lines of Riemann general philosophy. Our work would want to be coherent with what Maurin has said above. Furthermore, reconnecting us to the first part of this Maurin foreword, we also quote, *en passant*, what has been just said by Chen Ning Yang in (Yang 1961, Preface) about some methodological aspects of the history of science; precisely, he affirms that

[5]As concern unitary character of mathematics, see also (Loria 1946, Appendice, § IV).

«Obviously, a concept, and especially a scientific one, has no full meaning if it is not defined with respect to that context of knowledge from which it derived and has developed».

All these yet brief methodological recalls may find a coherent and homogeneous setting into the modern historiographical theories as, for instance, those exposed in (Kragh 1990), which is a fundamental work and an unavoidable reference for everyone has to do with history of science because it contains the minimal prerequisites to be known. Likewise, we will also talk about one of the last works of Gino Loria in history of mathematics, namely (Loria 1946), in which interesting historiography of mathematics remarks and hints are exposed. In any way, as concern the importance of the history of mathematics oriented toward primary historical sources, it is just enough to recall a single case study, that is to say, the truly notable work made by Carl Ludwig Siegel (1896-1981) - who, inter alia, was also a scholar of history of mathematics - on the 1850s Riemann unpublished manuscripts, the so-called *Riemann's Nachlaß,* from which he deduced, in 1932, a fundamental result of the theory of Riemann zeta-function, which he wanted to call *Riemann-Siegel formula* to highlight his obligation to Riemann himself (see (Siegel 1932) and (Neuenschwander 1988)). Also André Weil (1906-1998), besides to be a great mathematician, was too a scholar in history of mathematics (see, for instance, (Weil 1975; 1978; 1984)), who has stressed some crucial points concerning historiography of mathematics as done, for example, in (Weil 1978). His historiographical considerations[6] will constitute the main methodological lines that we shall follow in pursuing our historical research's work.

[6]Obviously, there are many other trends of historiography of mathematics besides the one considered by Weil (see, for instance, (Dauben 1994)), but we have chosen to follow it.

1. A few notes on the historiographical method

As has just been said above, Weil, in (Weil 1978), briefly outlines the main lines of what a history of mathematics should be, its study's object and related methods. The clear and dry form with which Weil, in a few pages, delineates the essence of the historiography of mathematics according his viewpoint, will be in accordance with our point of view. We focus on some central points of Weil's paper, precisely on those just centered on historiography of mathematics. At first he begins distinguishing between two main methodological approaches to a scientific subject-matter, which he respectively calls *tactic* and *strategic*. In this regard, we report textual words with which Weil explains their meaning and difference. Weil states that

«[...] *one has to make clear the distinction, in scientific matters, between tactics and strategy. By tactics I understand the day-to-day handling of the tools at the disposal of the scientist or scholar at a given moment; this is best learnt from a competent teacher and the study of contemporary work. For the mathematician it may include the use of differential calculus at one time, of homological algebra at another. For the historian of mathematics, tactics have much in common with those of the general historian. He must seek his documentation at its source, or as close to it as practicable; second-hand information is of small value. In some areas of research one must learn to hunt for and read manuscripts; in others one may be content with published texts, but then the question of their reliability or lack of it must always be kept in mind. An indispensable requirement is an adequate knowledge of the language of the sources; it is a basic and sound principle of all historical research that a translation can never replace the original when the latter is available. Luckily the history of Western mathematics after the* XV^{th} *century seldom requires any linguistic knowledge besides Latin and the modern Western European languages; for many purposes French, German and sometimes English might even be enough.*

In contrast with this, strategy means the art of recognizing the main problems, attacking them at their weak points, setting up future lines of *advance. Mathematical strategy is concerned with long-range objectives; it requires a deep understanding of broad trends and of the evolution of ideas over long periods. This is almost indistinguishable from what Gustav Eneström used to describe as the main object of mathematical history, viz., "the mathematical ideas, considered historically", or, as Paul Tannery put it, "the filiation of ideas and the concatenation of discoveries". There we have the core of the discipline we are discussing, and it is a fortunate fact that the aspect towards which, according to Eneström and Tannery, the mathematical historian has chiefly to direct his attention is also the one of greatest value for any mathematician who wants to look beyond the everyday practice of his craft.* [...] *However that may be,* [...] *we have agreed that mathematical ideas are the true object of mathematical history.* [...] *It is obvious that the ability to recognize mathematical ideas in obscure or inchoate form, and to trace them under the many disguises which they are apt to assume before coming out in full daylight, is most likely to be coupled with a better than average mathematical talent. More than that, it is an essential component of such talent, since in large part the art of discovery consists in getting a firm grasp on the vague ideas which are "in the air", some of them flying all around us, some (to quote Plato) floating around in our own minds*».

Due to their fundamental importance from an historiographical standpoint, we are particularly interested in the last words of Weil, namely, at the cost of repeating them again, when he says that

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Indeed, Weil highlights what is peculiar of a process of mathematical creation in coming off a mathematical idea (or object), that is to say, to *synchronously expliciting* what is *diachronically implicit* in the past work of mathematics[7], and, in this regard, we refer to (Kragh 1990, Chapter 9) for the notions of synchronic and diachronic and their role in history of science, as well as for a general overview of what means doing history of science[8]. Of course, this main process of the history of mathematics centered on the evolution of a mathematical idea or object, which belongs to the so-called *internal history* and is mainly pursued over primary literature and sources, is surrounded too by all the other historiographical procedures which make complete a general historical recognition, that is to say, to analyze possible correspondences, to examine objective biographical data, to sift secondary literature and sources, all procedures, these, belonging to the so-called *external history.* Likewise, following (Loria 1946, Libro II, Capitolo I), every investigation on the historical evolution of an arbitrary product of human thought relies on the twice consideration of both the author of this outcome (external history point of view) and her of his works (internal history point of view). A paradigmatic instance of doing history of mathematics according to what has just been said above, is the notable work on history of complex function theory recently accomplished by U. Bottazzini and J. Gray in their treatise (Bottazzini & Gray 2013) which has been one of the main references of our research, above all as regard canonical historical aspects of the route followed in this work. Anyhow, we will adhere to remarkable Weil's historiographical considerations in pursuing

[7]A concrete example related to a specific historical case study of mathematics regarding the crucial passage from *implicit* to *explicit* in exact sciences, is expounded in (Iurato 2012).

[8]Following (Kragh 1990, Chapter 9), according to *synchronic* viewpoint of history, the past should be studied in the light of present knowledge we have, according to the perspective to comprehend its further developments which have could led to the current epoch. It is allowed, if not even necessary, that historian analyzes past with knowledge he or she has today. On the other hand, there are no other means to avoid this, because how is it possible for an historian of her or his own time to do otherwise? Every human being cannot wholly avoid from her or his own knowledge's heritage with whom to evaluate every object of thinking through her of his unavoidable mental grids. Instead, the *diachronic* viewpoint tries to study past only in the light of real situations and conceptions of the time under examinations, neglecting every other thing may have had some relationship with the given historical fact. In this regard, the historical re-enactments advocated by R.G. Collingwood belong to the diachronic viewpoint.

our historical research herein exposed. Furthermore, following (Neuenschwander 1994), the use of primary sources play an important role in history of mathematics, since they provide the means for any critical study in the field. As can be seen from numerous examples, historical developments are seldom as straightforward and logical as many theoretically oriented historians and philosophers of science like to pretend. The precise historical development can rarely be reconstructed by purely intellectual means, since influences from outside the field, and even mere accidents, sometimes play a certain unavoidable role. In order to determine the significant and decisive factors and to avoid false conclusions, it is therefore essential to undertake an extensive evaluation of primary and secondary sources. The usefulness and the importance of the study of primary sources in the history of mathematics have been demonstrated by researches and studies achieved by E. Neuenschwander in regard to three mathematicians from three different countries, namely J. Liouville, B. Riemann and F. Casorati. His investigations revealed the richness of the estates of mathematicians of the 19th century, as nearly all the mathematical notes of these three mathematicians have survived, and in the case of Liouville and Riemann, for example, even their school and university reports are still to be found. On the other hand, it is possible to reconstruct the discovery of mathematical theorems down to the smallest detail by the use of primary sources, as was shown by a survey of the history of Casorati-Weierstrass theorem (see (Neuenschwander 1978)), which is another exemplary instance of history of a mathematical idea or object according to Weil. On the other hand, according to (Peri 1971, Parte I^a, Capitolo 2), the historical research has often a sectorial nature, above all as regard history of science. Indeed, in our case, mathematics does not develop ever only through great systems, but very often also through specific contributions and particular problems, so that we have either an historical examination brought with *intensive* (or *comprehensive*) *method* and an historical recognition carried out with *extensive method.* Here, we are just interested in a particular contribution to mathematics falling into the intersection between Riemann zeta function theory and entire function theory, and conducted with intensive method. But, notwithstanding that one brings forward a comprehensive history upon a particular problem or specific subject-matter, it is a basic role of historiography to consider the various possible historical connections which regard it, in such a manner to give an as more coherent and harmonic possible framework centered around the original historical issue.

Following (Kragh 1990, Chapter 2), roughly speaking, historiography is right manner of the writing of history, especially the writing of history based on the critical examination of sources, the selection of particular details from the authentic materials in those sources, and the synthesis of those details into a narrative story that stands the test of critical examination. The term historiography also refers to the theory and history of historical writing. It is customary to distinguish between two main different levels or meanings of the term 'history'. History (hereafter H_1) can describe the actual phenomena or events that occurred in the past, that is to say, objective history. In such expressions, history has to be understood as 'the past' or the phenomena that actually occurred in the past. But since we only have, and only ever will have, a limited knowledge of the reality of the past, most of what actually took place in the past will forever be beyond our grasp. The part of history (H_1) that we do know is not just limited in extent but is also the product of a research process that includes the selections, interpretations and hypotheses of the historian. We do not have direct access to H_1, but only to parts of H_1 which have been transmitted via various sources. The term history (hereafter H_2) is also used to refer to the analysis of historical actuality (H_1), that is to say, the historical research and its results. The object of history (H_2) is thus history (H_1) in the same way as the object of natural science is nature. Just as our (scientific) knowledge of nature is limited to the research results of science that are not nature but a theoretical interpretation of it, so our knowledge of the events of the past is limited to the results of history (H_2) that are not the past but a theoretical interpretation of it. Radically positivist philosophers have maintained that the existence of an objective nature is a meaningless fiction and that it is impossible to distinguish between nature and our knowledge of it. In the same way, some idealist historians maintained that the distinction between H_1 and H_2 is a fiction that serves no useful purpose; that there is no actual history apart from that which the historian constructs from his sources. There is no need, however, in the present context, for us to take this idealist view of history seriously. The term *historiography* is often used to refer to H_2, meaning writings about history. In practice, historiography can have two meanings. It can simply mean (professional) writing about history, that is, accounts of the events of the past as written by historians, as well as it can mean theory or philosophy of history, that is, theoretical reflections on the nature of history (H_2). In its latter meaning, historiography is, therefore, a meta-discipline, whose object is H_2; purely descriptive history will not itself be historiography but it can be the object of an historiographical analysis. According to the historiographical theory associated with positivism (positivist historiography), history is a description of the past, based on a series of well-documented facts. Positivist historiography is based on the following assumptions:

1. History (i.e., the past, H_1) is an objective reality that is the unchangeable object of interest to the historian.

2. It is an historian's task to try to reconstruct the past as it actually was, i.e., give a true description of the course of events of the past. But it is not her or his task to interpret or evaluate the occurrences of the past or to draw conclusions about the present or future on the basis of history. The study of history is the study of the past as the past in itself.

3. It is, in fact, possible to write history 'wie es eigentlich gewesen', i.e. to attain an objective knowledge of parts of the historical past. This epistemological objectivity implies, among other things, that the subject (i.e., the historian) can be separated from the object (i.e., the historic events) that can be viewed impartially, to be seen 'from without' (ideal of impartiality).

4. History can be viewed as an organized sum of simple, particular facts that can be discovered through the study of documents from the past, using methods that are the critical of sources. It is the most exalted task of the historian to uncover these facts. Interpretations and conclusions can only be made and drawn when all the

relevant facts have been collected. The historian G. Sarton compares the historian of science with the entomologist: in the one case, it is insects that are collected and arranged, in the other, it is scientific ideas.

Following (Kragh 1990, Chapter 7), because of their placement in the past, historical occurrences cannot be re-created or manipulated. For this reason hypothetical or contrary-to-fact statements are often regarded as unacceptable in historical works. A contrary-to-fact statement is a statement based on an assumption that is known to be factually false, in other words, that cannot be reconciled with the known facts. Such statements are also called *counterfactual* statements. They contain the conditional 'if ...' followed by the false statement P. The hypotheses are normally statements whose truth value is not known, but which are used heuristically in order to deduce testable statements that will then support or weaken the hypothesis. Counterfactual history seems to presuppose that individual historical occurrences can be taken out of their context without disturbing anything more than a few other occurrences. According to many historians with a 'holistic' view, this presupposition is fundamentally unjustified since all historical occurrences are connected to each other. The assumption that an actual occurrence had not taken place, would have changed all subsequent occurrences in a totally unpredictable way. In spite of these objections and in spite of the fact that we can never determine the truth value of counterfactual historical situations with certainty, they are of value in history. In practice, counterfactual questions are not infrequent in the history of science. According to M. Bernal, «we ought to demand not only how was this discovery made, but why was it not made before then and what would have been the course if history had gone differently». Questions of why occurrences took place as they did are of course an important part of history. Such factual questions can, however, also be formulated counterfactually, especially when they are attempts to make causal connections between occurrences.

Following (Kragh 1990, Chapter 8), the structural framework of the historian includes, among other things, divisions into historical periods. Obviously, periodization is the work of the historian, not of history.

There not exist any objective or natural way to periodize which is intrinsic of the historical course of events. This does not mean, however, that all ways of organizing the historical materials are equally good. In the historiography of modern science, a tradition has arisen for working with chronological periods that follow the century in question: e.g., science in the 20th century, in the 19th, 18th and 17th centuries, and so on. The division is obviously arbitrary, in the sense that it does not reflect any internal tendency in the development of science. One way of organizing history of science is to divide it into 'horizontal' and 'vertical' sections. The *horizontal* history of science is understood here to mean the study of the development through time of a given, narrow topic, like a scientific speciality, a problem area or an intellectual theme[9]. In some cases, it is possible to identify the origin (t_α) and the end (t_ω) of the topic in which cases the time boundaries are given. In other cases, the upper boundary is the present day[10] (t_ν). This case appears frequently since the reason for tracing a particular topic backwards in time is often tied up with the present importance of that particular topic. Horizontal history is typically discipline history or history of a sub-discipline. Instead, *vertical* history is an alternative way of organizing history of science materials. The vertically inclined historian starts out from a perspective that is more interdisciplinary in nature where the science that is in focus is seen as merely one element in the cultural and social context of a given period. An element that cannot be isolated from other elements of the period and which, together with these, characterizes the 'spirit of the age' (*Zeitgeist*) that constitutes the real field of this type of history of science. While horizontal history is a film of a narrow part of science, vertical history is a snapshot of the overall situation. In horizontally organized history, the historian isolates a particular discipline or problem from other, contemporary disciplines. This approach involves the danger of falling into anachronisms relying as it does on an assumption of disciplinary continuity. If the historian applies a narrow, horizontal

[9] Like in the historical case study treated in this our research's work.

[10] Following the New Testament biblical tradition, we have deliberately chosen to use the first letter (α), the mid letter (ν) and the last letter (ω) of the Greek alphabet as indexes to denote such particular instants.

perspective, the dependency on problems that lie outside the specialist subject may not be revealed. Disciplinary, horizontal history tends to become a bloodless recapitulation, a record of the origin, development and decay of the internal aspects of the discipline. As such, it will not only be relatively uninteresting but also artificially confined. The historian of mathematics who studies the development of geometry, cannot allow himself to study pure geometry alone; he must be prepared to study the histories of art, architecture, philosophy, cartography, physics and perhaps several other fields. In spite of the criticism that can be raised against horizontally organized histories of disciplines, it would be wrong to follow those who repudiate this approach completely. At least in some cases, it is possible to identify disciplines and specialist themes in earlier periods without committing sins of anachronism, like in history of mathematics. The only problem is that these themes will only rarely be identical to modern themes and only rarely be unchanged throughout long periods of time. The risk one runs in cutting oneself off from important vertically integrated connections depends on the period and discipline under consideration. An increasing disciplinary isolation is characteristic of the kind of highly organized, specialized science that has developed since the turn of the century. As far as modern science is concerned, it is, therefore, less problematic to organize history horizontally. Whether one needs to adopt a vertical, cross-disciplinary approach is not a matter of principle but of historical contingency. While vertically organized historiography avoids the problems connected with identifying a stable discipline throughout a longer period of time, it lays itself open to other problems. The historian who follows the advice given by who investigates the science in a short period of time, including its integration with social-cultural context in general, will perhaps cut herself or himself off from acquiring knowledge about possible historical causes of the situation being analyzed. The degree of arbitrariness in the choice of period or of the complex discipline, will often be no less than the degree of arbitrariness to be found in the horizontally inclined historian who has to mark out her or his field.

A special kind of organization of history that contains both horizontal and vertical traits is connected with the thesis of *invariant his-*

torical themes, or the *invariance thesis,* for short. This is the thesis that history can be viewed as a variation on a relatively small number of constant themes or *unit-ideas* that manifest themselves at different times in all-important branches of culture. According to A. Lovejoy, who was an important spokesman for the invariance thesis in the history of ideas, unit-ideas can be compared with atoms of elements: just as the hundreds of thousands of chemical compounds can be understood to be combinations of a few kinds of atoms, the complex and extremely varied forms in the history of ideas can be conceived as combinations of a few unit-ideas. Since it attempts to integrate different elements that make up culture and to simultaneously follow these through time, the thesis can be regarded as an attempt to circumvent the conflict between horizontal and vertical historiography. Lovejoy describes the thesis as follows

«The postulate [...] *is that the working of a given conception, of an explicit or tacit presupposition*[11]*, of a type of mental habit, or of a specific thesis or argument, needs, if its nature and its historic role are to be fully understood, to be traced connectedly through all the phases of men's reflective life in which those workings manifest themselves, or through as many of them as the historian's resources permit. It is inspired by the belief that there is a great deal more that is common to more than one of these provinces than is usually recognized, that the same idea often appears, sometimes considerably disguised, in the most diverse regions of the intellectual world».*

Ever since Lovejoy, the thesis of invariant unit-ideas has been developed by many authors amongst whom is M. Sachs, a physicist and philosopher, who writes

«It is my thesis that the actual truths sought by the philosopher and the scientist about the real world emerge in the form of abstract, invariant relations that are independent of the domain of understanding to which they may be applied, whether in the arts, the sciences, the phi-

[11] Notice here the reference to implicit/explicit duality.

losophy of religion, or any other intellectual discipline, and that these relations are invariant with respect to the different periods of history during which they may be expressed. In the language of theoretical physics, I am contending that the principle of relativity - the assertion that the laws of nature are independent of the frame of reference in which they may be expressed - applies equally to the relations that govern the evolution of human understanding, i.e. the history of ideas, as it does to the natural phenomena of the inanimate world of the stars, planets and elementary particles».

Also S. Sambursky concludes that «the inner logic of scientific patterns of thought has remained unchanged by the passage of centuries and the coming and going of civilizations». In the same way, several historians have fixed on what they consider to be striking similarities between concepts in classical natural philosophy and in modern science. The thesis of invariance has been developed into a so-called *analysis* by G. Holton, according to whom one can profitably interpret pioneering scientific work as being based on underlying, possibly unconscious, concepts, methods and commitments that act as 'private' motives or restraints during the process of research. These themata are non-scientific in the sense that they are often not acknowledged by the scientist and rarely appear in official scientific discourse. The themata to which a scientist is committed do not necessarily stem from science. They can have been formed in early years or be the result of any sort of influence. Like other forms of invariant ideas, themata do not have the status of theories. Their validity cannot be tested empirically or established by means of rational argumentation. Holton's use of thematic analysis differs from the Lovejoy version of the invariance thesis in that it focuses on a short period of time and on individual scientists. In other words, it is used vertically rather than horizontally. However, Holton believes that there are only a few themata in the history of science and that it is only very rarely that new themata arise. No matter how radical the advances will seem in the near future, they will with high probability still be fashioned chiefly in terms of currently used themata. The themata considered by Holton typically appear as opposing pairs of thesis-antithesis, such as evolution/devolution, plenum/vacuum, hierarchy/unity, reductionism/holism and symmetry/asymmetry. However inspiring and interesting the invariance thesis may be, it should be used with caution, not as an infallible framework for organizing history but rather as a heuristic principle. In most cases, it is problematic to talk of actually invariant unit-ideas as independent historical quantities. Unit-ideas are the result of a comparative analysis made by the historians, which are 'labels' implying that different works are analogous or belong to the same category. The selection of the historian and her or his interest in historical constancy may result in unit-ideas whose constancy in time is an illusion, since the actual historical context in which they appear is disregarded. Concepts and ideas are rarely or never quite the same over a long period of time. Although the names given to them by historians might turn out to be unchanged, fundamental concepts often develop beyond recognition through the historical process. The problem with using the invariance thesis over long periods of time, is that it tends to press modern concepts and forms of thought down on earlier science instead of studying the latter in terms of its own premises.

Following (Kragh 1990, Chapter 9), according to the *synchronic viewpoint,* the science of the past ought to be studied in the light of the knowledge that we have today, and with a view to understand this later development, especially how it leads until to the present. It is considered legitimate, if not necessary, that the historian should 'intervene' in the past with the knowledge that he possesses by virtue of his placement later in time. Synchronic historiography, in the sense is used here, involves a certain type of anachronism, but it is not necessarily anachronistic in the usual, derogatory sense. Today, synchronic history of science is rarely a conscious historiographical strategy. On the contrary, there is a broad agreement about praising a non-synchronic ideal. Even so, in practice, synchronic history of science is widespread and difficult to avoid. The doctrine is connected with the *presentist* view of history which may be seen as a theoretical justification of synchronic historiography. Furthermore, this perspective is legitimate from the points of view that regard the goal of history of science as primarily bound up with the present situation. If one believes that it is the task of the historian of science to understand the technical contents of older science and to pass this understanding to the scientists of today, then a way of presentation that is synchronic in tendency will be natural. A text will then be taken to have been understood if its true contents, in the current sense, can be represented with modern formalism and using modern knowledge. The *diachronic* ideal instead is to study the science of the past in the light of the situation and the views that actually existed in the past, in other words to disregard all later occurrences that could not have had any influence on the period in question. Occurrences that took place before, but which were actually unknown at the time, have to be regarded as non-existent as well. So, ideally, in the diachronic perspective one imagines oneself to be an observer in the past, not just of the past. This fictitious journey backwards in time, has the result that the memory of the historian-observer is cleansed of all knowledge that comes from later periods. The diachronic historian is therefore not interested in evaluating the extent to which historical agents behaved rationally or whether they produced true knowledge in an absolute or modern sense. The only thing that matters is how far the actions of the agent were judged to be rational and true by the agent's own time. In this sense, one may say that there is a relativistic element in diachronic historiography. In many ways, as has been said above, Collingwood's view of history is in accordance with the diachronic ideal. In synchronic historiography the subject-matter of history of science is the same as the subject-matter of science. Scientific facts and theories are regarded as having a permanent, almost transcendental existence even in periods when they were not recognized. In the words of G. Buchdal, synchronic historiography is based on «the misleading presupposition that "science" (as against *scientia*) is a quasi-object latently existing in all ages, signs or symptoms of which may be discerned to appear during any stages of world history». Accordingly, science becomes a phenomenon that is bound to make progress in the direction of truth. It is then the task of the historian to elucidate this development towards true knowledge as it takes place through successive experiments and theories. The philosophy of science that lies behind synchronic historiography leads to the temptation to write history backwards, to teleological history of science. This is an approach that has been badly shaken by the criticisms put forward by T.S. Kuhn and other post-positivistic philosophers of science.

A fundamental aspect of synchronic ideal, is the so-called *anticipation,* which regards too our research's work. There is a long tradition in the history of science of taking an interest in which persons or theories were the forerunners of a particular later theory. This interest has recently been criticized by many authors and historians. The point is partly that assertions about anticipation necessarily involve speculative interpretations directed by next knowledge. And partly that scientific discoveries ought to be judged with respect to their actual historical significance: discoveries can only be regarded as effective if they have achieved a widespread acceptance. We notice, however, this entails a non-diachronic view according to which earlier science ought to be judged according to the same criteria as modern science, that is to say, the unchangeable rules of scientific discussion. By its very nature, the idea of anticipation involves a synchronic perspective. In itself, this may not be problematic, but it becomes so if 'clairvoyant' abilities are ascribed to predecessors and if later theories are projected upon the works of predecessors. If these pitfalls are not avoided, the result is a pure anachronism. The problem about the concept of anticipation is that, largely, it is the historian's interpretation of the forerunner that decides to what extent there is an historical connection between the alleged forerunner and the successive doctrine. This is an unavoidable element in anticipation historiography. As has been pointed out, anticipation is a context-dependent concept that will often be evaluated differently by scientists and historians. Anticipation historiography is closely connected with the thesis of invariance and, in general, with continuity of the history of science. If scientific development is seen as a continuous and conservative process, then the search for direct predecessors becomes a central task for the historian[12]. This method, in which a development is presented as a sequence of small changes and in which it does not, therefore, have any clear beginning, has been called the *emergence technique* by P. Duhem, who also defined the so-called *emergence chains.* Can one conclude from

[12] And this is the point of view adopted by us in the follows.

the critics of the so-called *Whig historiography*[13] that all synchronic elements ought to be avoided and that history of science should dealt with a purely diachronic perspective? The answer is no, because a totally diachronic history of science would not be able to live up to the demands that are normally made on historical expositions. It might perhaps give a true representation of the past, but it would also be antiquarian and inaccessible to all except a few specialists. Diachronic historiography can only be an ideal. The historian cannot liberate himself from his own age and cannot completely avoid the use of contemporary standards. During the preliminary study of a specific period, one cannot use the period's own standards for evaluation and selection, since these standards form part of a period that has not yet been studied and they will only gradually be revealed. In order to have a whole view of a subject-matter, one has to wear glasses, and these glasses must, unavoidably, be the glasses of the present. The historian cannot purely rely on criteria of significance accepted in the past. Only in a few cases there will be an undisputed consensus on the priorities in the past. Usually the establishment of consensus will involve selection and hence imply the historian's intervention and her or his will. In many cases, it will be the obvious thing to do to use modern knowledge in the analysis of a historical event, and, by so doing, one may be led to interesting questions that could not be formulated on a purely diachronic basis. Similarly, it is only in a retrospect standpoint that many important connections manifest themselves. It is only if one allows a synchronic perspective that it can be seen that, in fact, different instances of the 'same' discovery taken place. We conclude that in practice the historian has not to be confronted with a choice between a diachronic or a synchronic perspective. Usually both elements should be present, their relative weights depending on the particular subject being investigated and the purpose of the investigation. The historian of science has to be a person with a kind of the 'two-faced head of a Janus' who, at the same time, is able to respect the conflicting diachronic and synchronic points of view.

[13] Roughly speaking, it is an historiographic point of view which focuses on the successful chain of theories and experiments that led to present-day science, ignoring failed and dead theories.

2. Moments of the history of number theory

Loosely speaking, number theory deals with $\mathbb{Z}$, the set of integral numbers, with related properties and possible structures. Number theory is the oldest trend of mathematics, dating back to Sumerians, Akkadians, Egyptians, Babylonians, Chineses, and Greeks, who faced the problem of counting and computing starting from numerical evidences and empirical relationships between numbers. Notwithstanding that, number theory is so wide to touch every other mathematical sector, it is however possible to identify a few fundamental arguments which have been the pivotal routes along which number theory historically developed, which are

1. *Diophantine equations.* These are polynomial equations in more variables whose solutions are in $\mathbb{Z}$, one of the most celebrated being the Pythagorean one $x^2 + y^2 = z^2$. Such equations date back to Diophantus of Alexandria (lived ca. between 200 and 300 A.C.), in his famous treatise *Arithmetica* there being the first systematic study of some equations of this type, even if equations of the simplest type $ax+by=c$ with $a, b, c \in \mathbb{Z}$ had already been considered by Euclid in his *Elements.* But maybe such equations date back as early as thousand years ago. For a modern history of Pythagorean theorem, see the recent monograph (Maor 2007).

2. *Distribution of prime numbers.* Prime numbers are distributed in a very irregular manner along numerical line, and ever since ancient times many efforts were accomplished to determine the laws ruling such a distribution. The *analytic number theory* is the modern trend of number theory which deals with this problematic.

3. *Algebraic number theory.* Such a trend of number theory deals with factorization of integral numbers into prime numbers, starting from the so-called *fundamental theorem of arithmetic* (see later).

4. *Congruences.* An equations of the type $x \equiv y \bmod n$ means find integer solutions to $x - y = pn$, for a certain $p \in \mathbb{N}$. Such a

types of equations, said to be a *congruence,* formalizes divisibility properties of integral numbers and their factorizations into prime numbers. The modern theory of congruences is therefore a chapter of analytic number theory, and basically originated from C.F. Gauss work on number theory (see (Berzolari et al. 1930-1951, Volume I, Parte 1^a)); furthermore, Gauss work put the foundations from which started the construction of most modern number theory of 19th and 20th century.

According to (Stopple, 2003), number theory is a subject which is so old that nobody can say when it started, this also making hard to describe what it is. More or less, it is the study of interesting properties of integers, even if, of course, what is interesting depends on the personal taste of every individual. In the study of right triangles in geometry, one encounters triples of integers x, y, z such that $x^2 + y^2 = z^2$, as, for example, $3^2 + 4^2 = 5^2$. These are called *Pythagorean triples,* but their study predates even Pythagoras. In fact, there is a Babylonian cuneiform tablet, considered to be oldest historical source of the first arithmetical fact, and designated Plimpton 322 in the archives of Columbia University from the nineteenth century B.C., that lists fifteen very large Pythagorean triples; for example, $12709^2 + 13500^2 = 18541^2$. The Babylonians seem to have known the theorem that such triples can be generated as $x = 2st, y = s^2 - t^2, z = s^2 + t^2$ for integers s, t. This, therefore, is the oldest theorem in mathematics. Pythagoras and his followers were fascinated by mystical properties of numbers, believing that numbers constitute the nature of all things. The Pythagorean school of mathematics also noted this interesting example with sums of cubes $3^3 + 4^3 + 5^3 = 216 = 6^3$. This number, 216, is the Geometrical Number in Plato's Republic. The other important tradition in number theory is based on the *Arithmetica* of Diophantus. More or less, his subject was the study of integer solutions of equations. Diophantus' work was lost to the Western world for more than a thousand years. Although many results of number theory were already known to ancients, the history of this mathematical section wants to begin with he great French mathematician Pierre de Fermat, who was a lawman interested in number theory as an hobby. The problems raised by him, were quite modern for his contemporaries, and were the starting points for almost all the number theory researches from 18th century on. Fermat worked out many of his assertions and statements in letters sent to his contemporaries, amongst whom are M. Mersenne and C. Huygens, but rarely given a complete proof of them. A large part of number theory that developed from this period onward, was just devoted to prove Fermat's arguments. Fermat was reading Diophantus' comments on the Pythagorean theorem, mentioned above, when he conjectured that for an exponent $n > 2$, the equation $x^n + y^n = z^n$ has no integer solutions x, y, z (other than the trivial solution when one of the integers is zero). This was called "Fermat Last Theorem", although he gave no proof of it, except the case $n = 4$ by means of a new method of proof, called *method of descent,* and claiming that the margin of the book was too small to be impossible to fit the complete proof's steps for the general case. For more than 350 years, Fermat Last Theorem was considered the hardest open question in mathematics, until it was brilliantly solved by Andrew Wiles in 1994. This, then, is the most recent major breakthrough in mathematics. Fermat introduced the so-called *two-square* and *four-square theorems,* the first proved by him, the second completely proved, for the first time, by J.L. Lagrange around 1770 after some unfruitful previous attempts due to L. Euler.

Following (Weil, 1984), (Scharlau & Opolca 1985) and (Watkins 2014) to outline the main moments of the modern era of analytic number theory until up Riemann's work, we may start saying that after more than a thousand years of general stagnation and decay, the rejuvenation and revitalization of western mathematics - particularly algebra and number theory - starts with Leonardo of Pisa, known as Fibonacci (about 1180-1250). Occasionally, the formula $(a^2 + b^2)(c^2 + d^2) - (ac - bd)^2 + (ad + be)^2$ is ascribed to him: if two numbers are the sums of two squares, their product is a sum of two squares as well. This development was continued by the Italian renaissance mathematicians Scipione dal Ferro (about 1465-1526), Nicolò Fontana, known as Tartaglia (about 1500-1557), Gerolamo Cardano (1501-1576), and Ludovico Ferrari (1522-1565). Their solution of algebraic equations of the third and fourth degree marks the first real progress over ancient mathematics. Next in this line is François Viète (1530-1603) who introduced the use of letters in mathematics. With Viète, we enter into the seventeenth century mathematics, and, from that time on, mathematics enjoys an uninterrupted, continuous and exponentially accelerating development. This new era, the era of modern mathematics, starts with four great French mathematicians, namely Girard Desargues (1591-1661), René Descartes (1596-1650), Fermat (1601-1665), and Blaise Pascal (1623-1662). Fermat was the most important one, considered to be the father of modern number theory, deriving much of his inspiration from Diophantus' works. He was a royal councillor at the Parliament of Toulouse, a position that, in today's terms, can be described as a high-level administrator. Fermat's profession apparently provided him with all the leisure he needed to occupy himself with mathematics. His style of work was slow, his letters, which contain all his important number-theoretical results, are laconic and dry. The majority of these were directed to Mersenne. Several of these correspondents were important in the development of number theory, among them B. Frénicle de Bessy, Pascal and P. de Carcavi. In these letters, Fermat formulated number-theoretical problems, but there are also several definitive statements and discussions of special numerical examples. Fermat never gave proofs and only once did he indicate his method of proof. This makes it difficult to determine what Fermat really proved as opposed to what he conjectured on the basis of partial results or numerical evidence. Many of his theorems cannot be proved easily, and first-rate mathematicians, such as Euler, had great trouble proving them. Nevertheless, there can be no doubt that Fermat knew how to prove many if not most of his theorems completely. His letters indicate that at about 1635, inspired by Mersenne, Fermat began to occupy himself with number-theoretical questions. His first interests were perfect numbers, amicable numbers, and similar arithmetical brain-teasers. He describes several ways to construct such numbers, but far more remarkable is that - showing more insight than any of his contemporaries - he succeeded in proving an important theorem in this still very barren area, the so-called *Fermat little theorem,* $a^p \equiv 1 \bmod p$ for every prime number p and every number a prime with p, today this theorem being proved with basic notions of group theory. Fermat's most important number-theoretical heritage is a letter to Carcavi in August 1650 that himself considers as his testament, a fact which he expresses in the following words: "Voila sommairement le compte de mes reveries sur le sujet des nombres". At the beginning of this letter, one finds the passage where he describes a certain method of proof which he himself discovered and used with great success. He then formulates a number of theorems all of which were contained in earlier letters or papers, but it is obvious that he wanted to compile what he himself considered his most beautiful and important results. Among the Fermat's main outcomes in arithmetic besides the above mentioned Fermat's little theorem, we recall the following ones. If n and m are coprime, then $n^2 + m^2$ is not divisible by any prime congruent to -1 modulo 4, while every prime congruent to 1 modulo 4 can be written in the form $n^2 + m^2$, these two statements dating back to 1640 and proved in 1659, with his *method of infinite descent,* already mentioned above, which is an argument by contradiction that roughly reads as follows: if a given natural number n, with assigned properties, implies that there exists at least a smaller one with the same properties, then there exist too infinitely many of such numbers, which it is impossible. As said above, Fermat started to consider Diophantine equations ever since 1850s, through the method of infinite descent, but often without giving a correct and complete proof, but developing methods to find points on some elementary algebraic curves. In particular, he claimed $x^4 + y^4 = z^4$ does not have non-trivial integer solutions, while $x^3 + y^3 = z^3$ does not have non-trivial solutions, stating that this could be proven via the method of infinite descent, but the first correct proofs of these questions were due to Euler around 1753. Moreover, Fermat also affirmed to have prove there are no solutions to the equation $x^n + y^n = z^n$ for all $n \geq 3$ (*Fermat last theorem*). With Diophantus and, above all, Fermat's works, we have the dawn of *algebraic number theory* (see also (Maurin 1997) and (Goldstein et al. 2007)). Anyway, Fermat was the first to discover really deep properties of the integers.

After 1650, number theory stood virtually still for a hundred years. This period saw the development of analysis in the work of Isaac Newton (1643-1727), Gottfried Wilhelm Leibniz (1646-1716), the Bernoullis

(Jacob, 1655-1705: Johann I, 1667-1748; Nicholas II, 1687-1759; Daniel 1700-1792), and Euler (1707-1783). Analytic methods have played an important role in number theory ever since Dirichlet's work. This interplay between analysis and number theory has its early origins in the work of Euler, which, therefore, marked the emergence of analytic number theory. Euler, whose life is opposed to the Leibnitz's one, made meaningful contributions in every field of mathematics in which he worked. Doubtlessly, his most important achievements are in analysis (infinite series, theory of functions, differential and integral calculus, differential equations, calculus of variations, and so on). Euler's application of infinite series to different number theoretical problems was of principal importance. The study of the series $\sum n^{-2k}$ leads to series of the form $\sum n^{-s},\ s \in \mathbb{N}$. The case where $s = 2k+1$ runs into major difficulties, and even today, no explicit formula is known for the corresponding series. Only later H. Minkowski discovered interesting and very different interpretations for these expressions. Euler was probably the first to see that these series can be applied to number theory. He was in correspondence with C. Goldbach and J.L Lagrange just on number theory questions. His proof of the existence of infinitely many primes uses the divergence of the harmonic series $\sum n^{-1}$ and using the above fundamental theorem of arithmetic which says that every natural number can uniquely be written as a product of powers of primes. Afterwards, P.L.G. Dirichlet (1805-1859) systematically introduced analytical methods in number theory. Among other things, he investigated the series $\sum n^{-s}$ for real s, while B. Riemann (1826-1866) allowed complex s. Another typical example for Euler's way of thinking is the following attempt to prove the four-square theorem on the wake of the previous Fermat's work, for example proving Fermat little theorem, some special cases of Fermat last theorem for $n = 3, 4$, the statement that every integer number is the sum of four squares, and considering the question of which prime numbers can be expressed in the form $x^2 + ky^2$, where k is a given integer, hence, by numerical evidence, identifying first forms (together similar Legendre's attempts) of the so-called *law of quadratic reciprocity,* according to which, in Gauss notations, the congruence $x^2 \equiv p \bmod q$ is solvable if and only if $x^2 \equiv (-1)^{(q-1)/2} q \bmod p$ is solvable, but not giving a rigorous proof of it; the first rigorous proof will be provided by Gauss[14]; see (Frei 1994) for a more detailed historical account of this law. The problem of four-square theorem fascinated Euler over several decades but he never found a complete proof, improving and simplifying the first proof of the theorem, due to Lagrange; as said above, he gave only a proof of the two-square theorem. Euler also approached some Diophantine equations, amongst which the equation $x^2 - dy^2 = 1$, where d is a non-squared natural number[15], and introduced and studied the so-called *numeri idonei,* that is to say, the numbers $d = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 16, 18, 211320, 1365, 1848$ (altogether 65) which have the following property: if $ab = d$ and if a number can be uniquely written in the form $ax^2 + by^2$ with ax, by relatively prime, then this number is of the form $p, 2p$ or 2^k, where p is a prime number. Specifically, any odd number greater than 1 which can be written uniquely in this fashion, is prime. Euler calls these numbers 'numeri idonei' because they can be used for tests of primality. However, number theory, in a way, did not exist when Euler began his work, since Fermat had not left any proof. Initially, Euler was quite isolated, and only later Lagrange joined him as a versatile and knowledgeable partner. It is difficult to realize today what kind of obstacles Euler faced, obstacles which we can overcome easily today with the help of simple algebraic concepts such as those provided by group theory.

Lagrange was the first to give rigorous proofs to many Fermat's and Euler's statements. Lagrange's number-theoretical papers belong to the Berlin era, mainly dating back to the years 1766-1777. Lagrange's main inspiration seems to have been Euler's work which he read very carefully; there also was an extensive correspondence between Euler and Lagrange, notwithstanding that they never met. As already said above, Euler was not really successful in treating Fermat's problems, and, in spite of the great efforts made by him, he gave a complete proof, after several unsuccessful attempts, only of the two-square theorem. Euler's contributions to the four-square theorem, or to the theory of the equations $x^3 = y^2 + 2$

[14]The law of quadratic reciprocity anticipated more general results amongst which the modern E. Artin reciprocity law of field class theory.

[15]As has been already said above, this equation was first considered by Fermat, but inexplicably Euler called it *Pell's equation.* Euler proved such an equation have an infinite number of solutions.

or $x^3 + y^3 = z^3$, were almost pursued with success, but serious gaps yet remained, Euler's real achievements having been the presentation of many examples and the use of analytical methods. Lagrange is Fermat's true successor in number theory. He was the first to give proofs for a series of Fermat's propositions and did so without leaving the realm of arithmetic; many of these techniques were his own. Three of Lagrange's works in number theory are particularly important, namely the "Solution d'un problème d'arithmétique" of 1768, the Lagrange treatment of the equation $x^2 - ky^2 = 1$ in "Demonstration d'un théorème d'arithmétique" of 1770, and the paper which contains the first proof of the four-square theorem, i.e., "Recherches d'arithmétique" of 1773 where, moreover, Lagrange developed the theory of binary quadratic forms and derived from the general theory, amongst other things, Fermat's theorems about the representation of prime numbers by $x^2 + 2y^2$ and $x^2 + 3y^2$. We are particularly interested in the latter paper because it is the first work to systematically develop and in a coherent manner a complete arithmetical theory, going much further than the individual problems which are discussed by Fermat and Euler. The importance of this step cannot be overestimated for the further development of number theory and algebra. About 25 years later, Gauss considerably expanded the theory of binary quadratic forms, starting to study congruences[16] of the type $a_2x^2 + a_1x + a_0 \equiv 0 \bmod p$ (for proving the law of quadratic reciprocity), and providing the first elements for a general theory of quadratic binary forms starting from his studies on quadratic Diophantine equations of the type $ax^2+bxy+cy^2 = n$. Lagrange studied quadratic forms of the general type $q(x, y) = ax^2 + bxy + cy^2$ on the basis of previous studies on certain quadratic forms of the type x^2+y^2, x^2+2y^2, x^2+3y^2, x^2-dy^2, which have already been treated by Fermat. Lagrange used the so-called continued fraction algorithm for the solution of Fermat's equation $x^2 - dy^2 = 1$, already called *Pell's equation* by Euler, proving it has a non-trivial integral

[16] With Gauss, the theory of congruences became an autonomous chapter of number theory, with a specific notation which is the current one (see (Berzolari et al. 1930-1951, Volume I, Parte 1^a)). Gauss theory of binary quadratic forms also led to modern algebraic number theory, while his work on congruences has provided first forms of Riemann hypothesis for curves. Finally, as has already been said above, Gauss work on prime number theory were prolegomena of analytic number theory.

solution for any non-squared natural number d. After many individual results and more or less accidentally discovered connections, Euler and, even more so, Lagrange developed the theory of continued fractions in a systematic way, which he substantially extended for this end. In particular, the set of solutions to this Fermat's equation can be interpreted as a group in a natural way, and in this one should identify the very early origins of the algebraic structure of group even before E. Galois work. Lagrange also proved that n is a prime number if and only if it is a solution to $(n-1)! \equiv 1 \mod n$. Euler is even more a member of the "naive" period of discovery, calculation and heuristic methods. But modem mathematics with its rigorous proofs, systematic procedures, and clear descriptions and delineations of the problems begins with Lagrange. A decisive change took place in the development of number theory between Euler and Lagrange, with the dawning of Legendre and Gauss works which were gathered into the first treatises on number theory respectively drawn up in 1798 and 1801, the latter being the celebrated *Disquisitiones Arithmeticæ* of Gauss, considered his greatest work. See (Merzbach 1981) for a detailed historical analysis of Gauss' work on number theory prior to 1799; see also (Goldstein et al. 2007).

After Lagrange, the vestibule of the founder heroes of number theory comprises Legendre, Gauss, Dirichlet, Riemann, and so forth. Nevertheless, for our historical purposes which are mainly oriented towards the central chapter of analytic number theory, that is to say, that concerning the distribution of prime numbers, we shall devote a few historical words to Legendre, Gauss and Dirichlet, till to touch Riemann work, narrowing to the analytic number theory context that just starts with some works of these authors, in turn restricting us to those historical information concerning only distribution of prime numbers. Following (Goldstein 1973) and (Maz'ya & Shaposhnikova 1998, Chapter 10), the sequence of prime numbers, that is to say, $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, ...$ has always fascinated mathematicians, both professionals and amateurs alike. It was already known to Euclid that the number of primes is infinite. The ancient Greeks also knew that all the prime numbers could be obtained using the algorithm known as the *Eratosthenes' sieve.* Up to 18th century, no regularity in the sequence of prime numbers had been found. Euler wrote, in 1747, that

«Until now, mathematicians have tried in vain to discover any order in the sequence of prime numbers, and therefore they believed that it is a mystery which the human mind will never be able to penetrate. In order to convince oneself, one only needs to look at the table of primes, which many mathematicians made great efforts to extend beyond 100,000. From this table, one can see that there is no law governing them».

Let $\pi(x)$ denote the number of primes lower than $[x]$ (= integer part of x). Euclid's theorem simply states that $\pi(x) \to \infty$ as $x \to \infty$, but what can one say about the behavior of $\pi(x)$ as $x \to \infty$? This problem hardly occupied many famous mathematicians from Euclid onwards, till to the pioneering works, simultaneously and independently achieved by J. Hadamard and C. de la Vallée-Poussin in 1896, on the principal term in the asymptotic law of $\pi(x)$. Whereupon, other mathematicians contributed to study such an asymptotic behavior with many other methods, like Tauberian theorems, harmonic analysis, Dirichlet L-function theory, and so on. In a rough form, the basic theorem of the theory of prime number is known as the *prime number theorem* (in short PNT) and allows one to predict, at least roughly, the way in which the primes are distributed. Let x be a positive real number, and let $\pi(x) =$ the number of primes lesser than x. Then the prime number theorem asserts that

$$(*_0) \qquad \lim_{x\to\infty} \pi(x)/(x/\ln x) = 1.$$

In other words, the prime number theorem asserts that

$$(*_1) \qquad \pi(x) = \frac{x}{\ln x} + o\Big(\frac{x}{\ln x}\Big) \qquad (x \to \infty).$$

Actually, for reasons which will become clear later, it is much better to replace $(*_1)$ by the following equivalent assertion

$$(*_2) \qquad \pi(x) = \int_2^x \frac{dy}{\ln y} + o\Big(\frac{x}{\ln x}\Big).$$

The advantage of the version $(*_2)$ is that the function $\mathrm{li}(x) = \int_2^x \frac{dy}{\ln y}$, called the *logarithmic integral,* provides a much closer numerical approximation to $\pi(x)$ than $x/\ln x$. In (Goldstein 1973), the author explores the history of the ideas which led up to the prime number theorem and to its proof, which was not supplied until up some 100 years after the first conjecture was made. The history of the prime number theorem provides a beautiful example of the way in which great mathematical ideas develop and interrelate, feeding upon one another ultimately to constructively yield a coherent theory which rather completely explains observed phenomena.

The real conception of a prime number goes back to ancient time, although it is not possible to precisely say when the concept was explicitly and clearly formulated for the first time. However, a number of elementary facts concerning the primes were known as early as Greek mathematicians. Goldstein cites three examples, all of which appear ever since Euclid's work, namely

1. the *Fundamental Theorem of Arithmetic,* which states that every positive integer n can be written as a product of powers of primes. Moreover, this expression of n is unique up to a rearrangement of the factors;

2. there exist infinitely many primes;

3. the primes may be effectively listed using the so-called *Eratosthenes' sieve.*

There exists a proof of 2. which is quite different from original Euclid's well-known proof and which is very significant for the history of the prime number theorem: indeed, this proof is due to Euler and dates back to the middle of the 18th century, which links together the Fundamental Theorem of Arithmetic with the infinitude of primes, as well as it uses an analytic fact, namely the divergence of the harmonic series, to conclude an arithmetic result. It was just this latter feature to become the cornerstone upon which much of 19th century number theory was erected.

In Euler's enormous legacy is the following identity, which he obtained in 1737

$$\sum_{n\in\mathbb{N}} \frac{1}{n^s} = \prod_{i\in\mathbb{N}} \Big(1 - \frac{1}{p_i^s}\Big)^{-1}, \qquad s > 1.$$

The first published statement which came close to the prime number theorem was due to Legendre in 1798, while analyzing the table of prime numbers. He asserted that $\pi(x)$ is of the form $x/(A\ln x + B)$ for constants A and B. On the basis of numerical works, Legendre refined his conjecture in 1808, asserting that

$$(*_3) \qquad \pi(x) = \frac{x}{\ln x + A(x)}$$

where $A(x)$ is "approximately 1.08366 ... ". Presumably, by this latter statement, Legendre meant that $\lim_{x\to\infty} A(x) = 1.08366$. Nevertheless, it is precisely in regard to $A(x)$ where Legendre was in error. In his memoir *Essai sur la théorie des nombres* of 1808, Legendre formulated another famous conjecture, namely the following one. Let k and 1 be integers which are relatively prime to one another. Then Legendre asserted that there exist infinitely many primes of the form $1 + kn(n = 0, 1, 2, 3, ...)$. In other words, if $\pi_{k,l}(x)$ denotes the number of primes p of the form $1+kn$ for which $p < x$, then Legendre conjectured that

$$(*_4) \qquad \pi_{k,l}(x) \to \infty \qquad \text{as} \quad x \to \infty,$$

whose proof by P.L. Dirichlet in 1837 provided several crucial ideas on how to approach the prime number theorem. Although Legendre was the first to publish a conjectural form of the prime number theorem, Gauss had already done extensive work on the theory of primes in the years 1792-93. Gauss was interested in the asymptotic law all throughout his life from when he was a youth. In his old age, he said he liked to spend a quarter of an hour each day to thinking about this issue. He never published any result on it, but in a letter to the astronomer J.F. Encke, dated December 24, 1848, where he wrote that, while considering the table of prime numbers during 1792-93, he obtained, from numerical evidence, the following approximate formula

$$(*_5) \qquad \pi(x) \approx \mathrm{li}(x) \doteq \int_2^x \frac{dt}{\ln t}$$

which implies $(*_0)$. Evidently Gauss considered the tabulation of primes as some sort of pastime and amused himself by compiling extensive tables on how the primes distribute themselves in various intervals of length 1000. The first table of Gauss, covers the primes from 1 to 50,000, where each entry in the table represents an interval of length 1000. Thus, for example, there are 168 primes from 1 to 1000; 135 primes from 1001 to 2000; 127 primes from 3001 to 4000; and so forth. Gauss suspected that the density with which primes occurred in the neighborhood of the integer n was $1/\ln n$, so that the number of primes in the interval $[a, b[$ should be approximately equal to $\int_a^b dx/\ln x$. In the second set of tables, Gauss investigates the distribution of primes up to 3,000,000 and compares the number of primes found with the above integral, the agreement turning out to be striking. Nevertheless, Gauss never published his investigations on the distribution of primes, even if there is a little reason to doubt Gauss' revendication that he first accomplished this his work in 1792-93, well before the memoir of Legendre was written. Indeed, there are several other known examples of results of the first rank which Gauss proved, but never communicated to anyone until years after the original work had been done. This was the case, for example, with the elliptic functions, where Gauss preceded C.G.J. Jacobi, and with Riemannian geometry, where Gauss anticipated B. Riemann. The only information beyond Gauss' tables concerning Gauss' work on the distribution of primes is contained, as already said above, in a 1849 letter to Encke. In his letter, Gauss describes his numerical experiments and his conjecture concerning $\pi(x)$, while, on the second page of the letter, he compares his approximation to $\pi(x)$, namely $\mathrm{li}(x)$, with Legendre formula. The results are tabulated at the top of the second page and Gauss' formula yields a much larger numerical error. In a very prescient statement, Gauss defends his formula by noting that although Legendre's formula yields a smaller error, the rate of increase of Legendre's error term is much greater than his own. Anyway, as we shall see later, in a certain sense Gauss anticipated what is today known as the "Riemann hypothesis". Another feature of Gauss' letter is that he casts doubt on Legendre's assertion about $A(x)$. He asserts that the numerical evidence does not support any conjecture about the limiting value of $A(x)$. Gauss' calculations are very imposing to contemplate, since they were achieved long before the days of high-speed computers. Gauss' persistence is most impressive. However, Gauss' tables are not error-free: for instance, E. Korn has checked Gauss' tables using an electronic computer and has found a number of errors, but, in spite of these (remarkably few) errors, Gauss' calculations still provide overwhelming evidence in favor of the prime number theorem, and modern students of mathematics should take note of the great care with which data was compiled by such giants as Gauss. Conjectures in those days were rarely guesses of inactivity, since they were usually supported by piles of laboriously gathered evidence.

The next step towards a proof of the prime number theorem was a step in a completely different direction, and was taken by Dirichlet in 1837. In a beautiful memoir, Dirichlet proved Legendre's conjecture $\pi_{k,l}(x) \to \infty$ as $x \to \infty$, concerning the infinitude of primes in an arithmetic progression. Simply, Dirichlet observed that linear polynomials of the type $ax + b$ may provide infinite prime numbers when x runs through all the positive integers and a, b were positive integers with no prime factor in common (like $2x + 1, 4x + 1, 4x + 3$ and so on); it also follows that there exist infinite primes in the arithmetical progression $k, k + l, k + 2l, k + 3l, \ldots$. Dirichlet, in proving this his outcome, went outside the realm of integers, introducing tools of analysis to pursue this. Furthermore, Dirichlet's work contained two radically new ideas: to be precise, Dirichlet's ideas gave birth to the modern theory of duality on locally compact Abelian groups, while Dirichiet's second great idea opened the way to the well-known *Dirichlet L-function theory,* as it nowadays is called, thanks to which he proved a notable theorem on primes in arithmetic progressions, which states that for any two positive coprime integers n and m (that is to say, if the only positive integer that equally divides both of them, is 1), there are infinitely many primes of the form $n + lm$ where l is a non-negative integer, which was one of the major achievements of 19th century mathematics, because it introduced a fertile new idea into number theory, that is to say, that analytic methods (in this case the study of the Dirichlet L-series) could be fruitfully applied to arithmetic problems (in this case the problem of primes in arithmetic progressions). To be noted the following main fact about $L(s, \chi) = \sum_{n\in\mathbb{N}}(\chi(n)/n^s)$ ($\chi(n)$ is a numerical function called *Dirichlet's character modulo* l), to be precise, $L(s, \chi)$ has a product formula of the form

$$L(s, \chi) = \prod_p \Big(1 - \frac{\chi(p)}{p^s}\Big)^{-1}$$

where the product is taken over all primes p, the proof of this formula being very similar to the argument given in Euler's proof of the infinity of prime numbers via his product formula. Dirichlet's idea in proving the infinitude of primes in the arithmetic progression $a, a+n, a+2n, ..., (a, n) = 1$, was to imitate, somehow, Euler's proof of the infinitude of primes, by studying the function $L(s, \chi)$ for s near 1. To the novice, such an application of analysis to number theory would seem to be a waste of time. After all, number theory is the study of the discrete, whereas analysis is the study of the continuous; and what should one have to do with the other! However, Dirichlet's 1837 paper was, together Euler's work, the beginning of a revolution in number-theoretic thought, the substance of which was to apply analysis to number theory. At first, undoubtedly, mathematicians were very uncomfortable with Dirichlet's ideas. They regarded them as very clever devices, which would eventually be supplanted by completely arithmetic ideas. For although analysis might be useful in proving results about the integers, surely the analytic tools were not intrinsic. Rather, they entered into the theory of the integers in an inessential way and could be eliminated by the use of suitably sophisticated arithmetic. However, the history of number theory in the 19th century shows that this idea was eventually repudiated and the rightful connection between analysis and number theory came to be recognized. But, the first major progress towards a proof of the prime number theorem after Dirichlet, was due to the Russian mathematician

P.L. Tchebycheff in two memoirs written in 1851 and 1852. Both Gauss and Legendre conjectures were analyzed by Tchebycheff in his memoir presented at the St. Petersburg Academy of Science in 1848, his first result stating that, for each natural number n, the sum

$$\sum_{n=2}^{\infty}\big(\pi(x+1)-\pi(x)-\frac{1}{\ln x}\big)\frac{(\ln x)^n}{x^{1+\rho}}$$

tends to a finite limit as $\rho\to 0^+$. From this result, Tchebycheff deduced that, if it exists, the limit of the ratio $\pi(x)/\mathrm{li}(x)$, as $x\to\infty$, should be equal to 1. He therefore concludes that the expression $x/\pi(x)-\ln x$ can only tend to -1, so disproving Legendre conjecture. Later, Tchebycheff introduced two functions of a real variable x, namely

$$\theta(x)=\sum_{p\leq x}\ln p, \qquad \psi(x)=\sum_{p^m\leq x}\ln p$$

where p runs over primes and m over positive integers. Tchebycheff proved that the prime number theorem in the first form we have given above, is equivalent to either of the two statement

$$(*_6) \qquad \lim_{x\to\infty}\frac{\theta(x)}{x}=\lim_{x\to\infty}\frac{\psi(x)}{x}=1.$$

These results will play a significant role in the subsequent proofs of prime number theorem. Accordingly, the asymptotic law is often written in the form $\psi(x)\sim x$. Tchebycheff also gave a proof of the so-called *Bertrand's postulate,* which claims that there exists a prime number into any interval of the type $]x,2x[$ for each $x\geq 2$. In 1881, J.J. Sylvester proved that, for sufficiently large x, the value $\pi(x)$ lies into a smaller interval, but the question of making this interval as small as possible remained open, until the next works of Riemann and Hadamard. However, Tchebycheff's methods were of an elementary, combinatorial nature, and as such were not powerful enough to prove the prime number theorem. The first giant strides toward a proof of the prime number theory were taken by B. Riemann in a memoir written in 1859. Riemann followed Dirichlet in connecting problems of an arithmetic nature with the properties of a function of a continuous variable. However, where Dirichlet considered L-functions $L(s, \chi)$ as functions of a real variable s, Riemann took the decisive step in connecting arithmetic with the theory of functions of a complex variable introducing the function $\zeta(s) = \sum_{n \in \mathbb{N}} (1/n^s)$ which has come to be known as the celebrated *Riemann zeta function* in relation to the conjecture about the location of its non-trivial zeros. Very roughly speaking, such a conjecture, the *Riemann Hypothesis* (RH), is an outgrowth of the Pythagorean tradition in number theory. It determines how the prime numbers are distributed among all the integers, raising the possibility that there is a hidden regularity amid the apparent randomness. The key question turns out to be the location of the zeros of a certain function, that is to say, the Riemann zeta function.

3. Outlines of history of complex function theory

Strangely enough, besides a few hints inorganically spread into the various chapters of the many histories of infinitesimal calculus, a very few organic, complete and systematic works on the history of complex function theory there exist so far. Indeed, as a witness of this fact, it is enough to recall what say the authors in[17] (Bottazzini & Gray 2013, Introduction), namely

«This book is the first to be devoted to the history of analytic function theory since Brill and Noether published their Bericht über die Entwicklung der Theorie der algebraischen Functionen in älterer und neuerer Zeit in the Jahresbericht der Deutschen Mathematiker Vereinigung *in 1894. Indeed, because that work leaves out many topics that belong to the theory of analytic functions but not algebraic functions, it can reasonably be argued that our book is the first ever to be written exclusively on this subject. This is rather strange given the importance of analytic function theory within mathematics and the attention that historians of mathematics have paid to the development of the theory of real functions in the nineteenth century. It is indeed surprising that the rise of complex or analytic function theory in the nineteenth century from almost nothing to one of the dominant fields of mathematics has not been told before, because it is a story worth telling and analyzing in its own right. The theory of functions, as it was generally referred to throughout the later half of the century, was much more concerned with complex than with real variables and functions. To tell this story is to redress the balance and to restore a family of overlapping perspectives on the mathematics of the day.*

In fact, we provide here the first full treatment of the work of several major mathematicians in the context of complex function theory. Gauss's work has not been treated in this way since Schlesinger contributed his essay [...] *to the Gauss Werke, although Cox* [...] *has written*

[17] Which will be the main reference quasi verbatim followed in this section.

a good account of Gauss's work on elliptic functions. Ours is the first thorough analysis of Cauchy's contributions. They were described by Smithies [...], *but he stopped his account in 1831, after which the reader must consult Belhostes biography* [...]*; Grabiner* [...]*confined her attention to Cauchy's work on real analysis. Riemann's work was discussed quite fully by Laugwitz* [...], *but he only skirted the central topic of Abelian functions. Weierstrass has also been studied only selectively by Dugac, Manning, and Ullrich; again, our chapter is the first to discuss all his work on complex analysis. Although there are* [...] *good accounts of Kovalevskaya's life and work* [...], *there is nothing on Schwarz, although the English translation of Arild Stubhaug's biography of Mittag-Leffler appeared in 2010. Our account of Poincaré's contribution is fuller than most but still only partial. Hadamard's work in this area has been written about very helpfully by Maz'ya and Shaposhnikova, but there is very little on the other French mathematicians of note, Borel and Montel».*

Bottazzini and Gray have not tried just to make only a history of ideas, but they have instead tried both to describe the rise of complex analysis and to explain it also with respect to the general social-cultural inherent context. What is a major and novel aspect of their work is the emphasis placed on elliptic functions as one of the principal impetuses for a theory of complex functions. Therefore, one of the main results achieved by Bottazzini and Gray, is just to have pointed out that elliptic function theory has been the main source from which complex function theory sprung out. The rising, about 1830, of elliptic functions by N.H. Abel and C.G.J. Jacobi works caused tremendous excitement, Bottazzini and Gray history starting just with this event and its immediate consequences just because of the fact that elliptic functions were a strong incentive to the development of complex function theory[18]. They

[18]Moreover, if we look at the few people who were doing anything genuinely using complex numbers prior to 1850 (e.g., prior to Riemann's memoir), such as Cauchy, Abel, Jacobi, and Eisenstein, we do not see an aggressive use of complex numbers in a way so different from Euler's intuitive sense of "number". It is true that Abel and Jacobi's treatment of elliptic integrals, elliptic functions and more general "abelian" functions looked at the behavior of functions in $\mathbb{C}$ and $\mathbb{C}^n$, but did not use terribly subtle properties of complex functions, especially not of entire functions (by a private communication with Professor Paul Garrett).

essentially were complex functions, so to say *in nuce,* even if their basic occurrence in the inversion of an integral containing a two valued integrand was soon felt to be either inadequate and mysterious at first sight. But Gauss immediately recognized this fact and took considerable steps to create a properly complex theory of doubly periodic functions that did not start from an integral, whilst Jacobi turned his attention toward the theory of theta functions. But these were not the only motifs: indeed, A.L. Cauchy was attracted by the subject from the analytic standpoint, who had taken profound knowledge and practice towards a rigorization process of the calculus as well as knew very well the way which complex issues had become entangled with problems in evaluating integrals. Cauchy, however, was not so strongly drawn to the theory of elliptic functions, and his insights into the integration of complex functions took over 20 years to suggest a rich and organized theory to him. This was left to others, amongst whom J. Liouville and C. Hermite, to find a way to bring enough analysis in such a way to re-found elliptic functions as doubly periodic complex functions on the wake of Gauss' work. As the time went on, complex function theory gradually became even more to be involved with other branches of mathematics. Bottazzini and Gray argue that these applications or interventions of complex function theory in both pure and applied aspects of mathematics were amongst the main reasons why the theory became more and more highly regarded and appreciated, because it turned out to be of fundamental importance in number theory, in mechanics, in the theory of linear differential equations, and even in geometry, and it developed a most fruitful interaction with potential theory and the theory of harmonic functions. And these, in turn, greatly helped in the advancement of complex function theory itself. Instead, a different situation occurred with the subject of topology, because it was not a preexisting discipline to be enriched by new ideas, for instance coming from applied sciences. Rather, it was a new subject, created in part to meet the increasing need for a rigor in complex function theory itself. From Riemann intuitive vague and yet profound ideas of what is today known as a "Riemann" surface till to the work of A. Harnack on connectedness and path-connectedness, complex function theory generated a number of basic concepts in both geometric and point set topology.

The rise of complex function theory cannot, however, be understood exclusively as an history of ideas, so that Bottazzini and Gray have not only traced such a story as a sequence of ideas, theorems, notions, concepts, and theories, but they have also paid attention to the general social-cultural historical context, the biographies of the main protagonists, the different reactions and developments in various countries depending on the national traditions, and so on. One would expect that the balance would shift from France to Germany, from Paris to Berlin, because that is true of so many aspects of the nineteenth century cultural context, especially regarding scientific disciplines. It took more-or-less than sixty years for the nation whose citizens filled the ranks of Napoleon's Army and watched Moscow burn to see a Prussian army at the gates of Paris, and nobody since then has doubted that this situation has profoundly influenced about almost every aspect of life in those two countries, albeit it is very hard to make definitive clarity in this regard. In the context of the history of mathematics, it is true that those young ambitious persons who sought best education in the 1820s, went to Paris and in the 1870s went to Berlin. But single individuals in small fields can invert trends, and to this end it is enough to mention a few names, like Cauchy, Riemann and Weierstrass. We see a strong presence of those Italian mathematicians who historians of mathematics have considered to form that revitalized Italian community soon arose after unification of 1860s, while the poor presence of British mathematicians for most of the nineteenth century is a further evidence of the narrowly utilitarian views of the British society of then. In any history of ideas, the historian mainly seeks to show how things once thought about in one way then became worked out in another fashion, as well as how complex function theory developed many ideas which were first introduced naively and only then slowly refined, or coming from other fields, like physics, and gradually formalized and rightly contextualized. Definitions were lacking or, when provided, were sometimes inadequate by later standards. Furthermore, even when this latter became available, it could yet be misleading, since mathematicians on occasion may offer a clear definition with very few ideas about its deepest implications, like the example of continuity in real analysis shows. Sometimes these problems can be compared directly, as with the definition of an analytic function, but more often one has to overcome a long period of vagueness and gestation. To mention a few of some specific issues, Cauchy, for example, often used the phrases "continuous" and "finite and continuous" very loosely to mean something like "complex analytic". Similar problems occur with counting roots according to their multiplicities, with lim versus lim sup, as well as with points of infinity and poles. Moreover, mathematicians throughout the eighteenth and nineteenth centuries often spoke freely of many-valued functions (the simplest example being $\sqrt{z}$). It is just in this spirit that often it is more proper to use the nineteenth century term "complex function" for example, where a modern mathematician would use "analytic function". In this regard, it is helpful here to recall the following A. Weil's remark

«The mathematicians of the eighteenth century were in the habit of speaking of "the metaphysics of the infinitesimal calculus" and the "metaphysics of the theory of equations". They understood by this a set of vague analogies, difficult to grasp and difficult to formulate but which nonetheless seemed to play an important role at a given moment in research and mathematical discovery. [...] Nothing is more fertile, all mathematicians know this, than these obscures analogies, these cloudy reflections of one theory on another, these furtive caresses, these inexplicable misunderstandings; nothing gives more pleasure to the researcher».

Afterwards, Bottazzini and Gray going on making some historiographical considerations. To be precise, they state that there is no truly satisfactory manner to represent the original ideas of mathematicians when they are in this state. On the other hand, saying nothing is to produce confusion. On the other hand, from a synchronous point of view, to silently bring them into line with modern standards not only introduces anachronisms but also brings in historical falsehoods and nullifies the purpose of a history. Furthermore, in correcting them in the light of modern sights in more than the most egregious cases would entail unrealistic and undue advances with the simultaneous admission of genuine blunders made by who established such results, and with a consequent belittling of the work of major mathematicians. According to Bottazzini and Gray, the best historiographical policy, therefore, should be read on with a spirit of ideal dialogue with the earlier authors, aware, as one better might do, of the limitations and false implications that their original papers and books may imply, and waiting to see when, if at all in the period, a better light was shone on the subject. In this way, one can grapple with more of the complexity of the past. Thus, going back to the subject of this section, one may affirm that the emergence of elliptic functions should be considered the first step with which to begin a history of complex function theory. As already said above, these functions have their early origins in the study of elliptic integrals, and they, as their name suggests, date back to researches by I. Newton and later workers on the integration of the equations of motion of the planets. Likewise many other matters of mathematical importance, elliptic integrals were taken up and studied in depth by Euler, but it was Legendre's lengthy account of them that may be considered as the first official historical account on the subject. Legendre computed tables of values for them, found differential equations for the so-called 'complete elliptic integrals' as functions of the parameter on which they depend, and described in detail possible applications in mechanics where these integrals could be found, and from which really arise. It was Legendre, modelling himself deliberately on Euler, who made elliptic integrals into a definite mathematical topic, his elliptic integrals providing one of the first new functions to enrich mathematics since Euler's times. Legendre's elliptic integrals were firmly real functions of their upper endpoint, and they involved only a real parameter. Everything changed with the nearly simultaneous realizations by Abel and Jacobi, so realizing that it is much more productive to invert the integral and to let the variables be complex. This step turned the elliptic integrals into elliptic functions strikingly analogous to the familiar trigonometric functions, explained some puzzling features of the previous studies on elliptic integrals, and opened a wide door to new research. Their work was almost immediately recognized as significant with the joint award of the prestigious Paris Académie prize of 1830 to Abel (but posthumously assigned) and Jacobi. Their work provided the first examples, other than the trigonometric functions, of complex-valued functions of a complex variable. But Abel and Jacobi were not the first ones to take this precious momentous step. Unknown to them both, Gauss had been confiding results about elliptic functions to his notebooks since 1797. He had indeed gone further than them, notoriously he having said, about Abel's work, that Abel had "gone one third of the way". Gauss had also connected their study to the hypergeometric differential equation, and in various other ways fully embraced the idea that the proper domain for the theory of functions was the complex domain. He had ideas about the nature of complex integration when Abel and Jacobi were still proceeding formally on several proofs of the fundamental theorem of algebra.

It is well-known that Gauss did not publish much of his work, preferring, as is known, the motto "Pauca, sed matura (i.e., few but ripe)", a practice he only broke when it devoted to astronomy. His French peer in the next generation, Cauchy, had no such inhibitions, but whereas Cauchy's Cours d'analyse of 1821 and its Résumé of 1823, did so much to put real analysis on the map, nevertheless unlucky circumstances hindered his contemporaries to be perfectly aware of the relevance of such an account of complex analysis due to Cauchy. What eluded Cauchy for over 20 years was a good way to make precise the perception that, within the growing collection of facts about maps from $\mathbb{R}^2$ to $\mathbb{R}^2$, implicitly there was a coherent collection of ideas that would make a theory of maps from $\mathbb{C}$ to $\mathbb{C}$. Cauchy's work in the 1810s and 1820s is rooted in eighteenth-century methods for evaluating integrals, specifically those that used what was called the passage from the real to the imaginary. This involved mathematicians such as A.C. Clairaut, J.L. d'Alembert, and Euler in the study of complete differentials and their integrals, and here the famous *Cauchy-Riemann equations* naturally make their first (one might say, unannounced) appearance. In a purely formal way, complex terms appeared in many places when one factorizes expressions or makes substitutions. Cauchy built up this work, and to explain some of its paradoxical conclusions made a detailed study of the introduction of imaginaries in a memoir of 1814, but first published in 1827. There, he showed that the reduction of a double integral to a repeated integral can be affected by the presence of singular integrals, an insight that was eventually to be the key to the theory of integrating a complex function on a closed path. At one time or another in the 1820s, 1830s and early 1840s, Cauchy had a theory of complex integration that included the calculus of residues (a term he introduced in 1826) and a theory of power series expansions of 'complex' functions. But he was forced to be exiled for political reasons in 1830 and was away from France for almost all of the 1830s. This nevertheless did not stop him to publish but deprived him of an opportunity to write up his ideas in a systematic way, so robbing his numerous articles of some of their impact. By the time that he was securely went back in Paris in the late 1840s, he was reaching 60 years old, and this, as well as the rivalry of the younger generations, may have stimulated him to publish, so that only now Cauchy gathered his ideas together and promoted them. In a fast-flowing stream of papers, he responded to the work of others, often to claim priority, to draw out new and forgotten conclusions from his published and unpublished work, and to find new ideas (for example, the logarithmic counter dates from this period). For the first time, he found the right general setting for his ideas and settled on the geometric description of complex numbers as points in the plane, which he had hitherto resisted.

As for elliptic functions, which however were never a topic belonging to Cauchy's interests, from the 1840s onwards, Liouville and Hermite, independently of Cauchy, had begun to lay out them into a theory of complex functions. In doing so, Liouville realized that an analytic function, which is defined and bounded in the entire complex plane, reduces to a constant (such a result being today known as the Liouville's theorem), which can be taken as one of the first major landmarks on the way to recognizing analytic functions as a distinctive class of functions. From all that, it turned out that only now the early origins of analytic function theory could be clearly identified in the elliptic functions. However, Cauchy's uncertainties about geometric approach to function theory implied, excepts V. Puiseux memoir of 1851, that there was still little that could be function-theoretically said about algebraic functions with certainty and rigor, beyond the single case of elliptic functions. In this regard, remarkable was the Jacobi's reformulation of the theory of elliptic functions. Throughout the 1830s, mathematicians had tried to face the unfortunate fact that an elliptic function was defined by inverting an elliptic integral which nevertheless, when treated in the complex setting, has a two-valued integrand. One successful way to cope this fact, was to restart their story with what had been one of its crowning successes, that is to say, the representation of elliptic functions as quotients of theta functions. Jacobi saw that it was possible to reverse the argument, hence he started with theta functions, and derived elliptic function theory from them, although, as Weierstrass pointed out, he did not fully extend the theory from the real to the complex setting in any respect. Other mathematicians, amongst whom is Jacobi, looked at a graver problem still, extending the theory of elliptic integrals and elliptic functions to the situation where the integrand involves not the square root of a polynomial of degree 4 but one of degree 5 or more, what became known as the hyperelliptic case. Even Jacobi stumbled here and seems to have suggested that it could not be done directly, but only by passing to the study of theta functions in two or more variables. This was done later by A. Göpel and G. Rosenhain, hence extended by Hermite and Carl Neumann. In this way, the hyperelliptic integral and related arguments, became established as central problems in the emerging theory of complex functions. So complex functions and related applications, were beginning to find uses and fruitful applications. Jacobi had shown that J.V. Poncelet closure theorem could be tackled by means of them, and Gauss' work showed that they turn up naturally in number theory and in the study of quadratic (and higher) reciprocity. Also Dirichlet used elementary ideas about complex variables to establish the remarkable result that every arithmetic progression without common factors contains infinitely many primes (as we have seen in the previous section).

Afterwards, it enters into the scene the unique and original insights of Riemann with his distinctive intuitive manner of doing mathematics. The fragmentary discovery of complex function theory as a theory of analytic functions defined on the complex plane, was disrupted so completely by Riemann even if a long time needed before the implications of his ideas were fully understood, although - contrary to the impression sometimes given - they were promptly and energetically accepted and studied. In his doctoral thesis of 1851, Riemann gave the definition of complex differentiability and its consequence, that is, the celebrated *Cauchy-Riemann equations,* so central to his definition of the relevant functions. He deduced from this that a complex function can be defined on any two-dimensional patch, thus opening the way to the study of complex functions upon non-simply connected domains and, in particular, the study of elliptic functions as functions on a torus. He proposed the theorem that any two simply connected domains with boundaries - this ruling out the entire plane - are equivalent for the purposes of complex function theory (this outcome being today known as the Riemann mapping theorem). In 1857, he pushed these ideas through to a resolution of the outstanding problem of the integrals of algebraic functions, Weierstrass having essentially done the hyperelliptic case between 1854 and 1856. Riemann gave finally the first theory of algebraic functions on a complex curve and amplified it with a thoroughgoing theory of theta functions in any number of variables. It was a profound success for his geometric, and indeed topological, way of thinking, quite disliked by Weierstrass. He followed it with his account of the hypergeometric equation in the complex setting, with consequences for the construction of minimal surfaces and, famously in the twentieth century, his treatment of the zeta function as a complex function, with its deep implications for the distributions of the prime numbers (see previous section). Riemann visionary presentation left very much for his successors to do, even if some of his bolder claims collapsed under a careful criticism. For a generation, his use of Dirichlet principle, which he had tried to prove, was held in great suspicion. His hope that his geometric analysis captured exactly what an approach based on infinite arithmetical expressions would capture, was shown to be unfounded. But his elementary insight that the Cauchy-Riemann equations were the place to start, as well as his use of geometric and topological methods to tackle advanced problems in complex function theory, were widely regarded as decisive, albeit very difficult. From the 1850s onwards, complex function theory was increasingly well established in France because of the work of Cauchy and in Germany through the work of Riemann. The next major figure with a vision for the subject was K. Weierstrass, whose lifelong ambition was to create a theory of *Abelian functions,* as those functions obtained by inverting an arbitrary algebraic integral were called in honor of Abel. They would be functions of several complex variables, and so, whenever possible, Weierstrass preferred methods that worked in any number of variables and avoided methods that worked in just one variable. By the time, he arrived in Berlin in his glory as the conqueror of the hyperelliptic integral, he had renounced the Cauchy integral theorem, and he distrusted of the Cauchy-Riemann equations too, which neither work satisfactorily when more than one variable is involved. So, as his famous letter to H.A. Schwarz attests, everything came down more and more to the use of algebraic methods and convergence arguments. In his own way, however, Weierstrass thought as deeply about complex function theory as Riemann had done. He was not the first to observe the crucial distinction between finite poles and essential singularities - even if F. Casorati in Italy and Y.V. Sokhotskii in Russia had noticed this independently - but he was the first to begin to understand it and to make real use of this distinction and thereby to clarify obscure features about the way a function 'becomes infinite'. His representation theorem clarified completely the question of what the zero set of a complex function can be, and his disciple R. Mittag-Leffler then did the same service for the polar set. His theory of elliptic functions evolved until up the 1870s, it was based on his famous $\mathcal{P}$ function, and rooted in a significant argument about what functions can satisfy an algebraic addition theorem. Ironically, it was only in the study of Abelian functions that Weierstrass made any serious mistakes and could not get the deep results he wanted, but that only showed how difficult that subject was.

One way thanks to which the theory of complex functions advanced, was because of the fruitful connections found for it with other, better established, fields of mathematics and of applied sciences. Weierstrass had, of course, disdained the intimate connection that Riemann had exploited between complex function theory and harmonic function theory, but others found it worth to explore. The first difficulty here, however, was that the fundamental theorems of potential theory were themselves in trouble. With Dirichlet principle in disgrace, the Dirichlet problem had to be solved some other way, and among the first to do this was Schwarz, the Weierstrass' most loyal and ambitious former student. Schwarz gave a rigorous account of how Dirichlet problem can be solved for a large but obscure class of boundaries, and he was followed by C. Neumann, A. Harnack and H.J. Poincaré. Schwarz in fact performed the function for Weierstrass of recapturing several of Riemann's theorems in a way that was more acceptable (and indeed more rigorous) than Riemann first presentation of them. He solved special cases of Riemann mapping theorem in this spirit, as did E.B. Christoffel independently, and likewise engaged in the study of minimal surfaces. He also followed L.I. Fuchs, another Berlin graduate much influenced by Weierstrass, in a study of the complex hypergeometric equation and investigated when all its solutions are algebraic functions. Other problems in applied mathematics were enriched by the use of complex methods in surprising ways. Elliptic integrals had been identified by Legendre as important in the motion of the top, and the corresponding elliptic functions studied by Jacobi. In the 1870s, the motion of the top was studied again by S. Kovalevskaya and later on by F. Klein and A. Sommerfeld. The three-body problem, although decisively reformulated by Poincaré in the late 1880s, still concealed answers about collisions that only yielded to K.F. Sundman complex methods at the end of the 19th century. The hypergeometric equation and the related study of the so-called *special functions* of mathematical physics were clarified and deepened by being made complex, as the discovery of G. Stokes' sectors and the theory of confluent differential equations show. Finally, the conformal character of a complex analytic map proved its worth in the reformulation of the theory of minimal surfaces, in conformal transformations of problems in two-dimensional fluid flow, and in the elucidation of a complex structure on a (Riemann) surface. Moreover, two complementary developments in the theory of complex functions came about in the period from 1880 to around 1910. Geometric function theory is the use of geometric, chiefly Riemannian, ideas. It received a considerable boost when R. Dedekind used it, in 1877, to illuminate the theory of modular functions, an important offshoot of the theory of elliptic functions. It was immediately taken up by Klein and, a year or so later, independently by Poincaré, who made it into the first major application of non-Euclidean geometry.

Also in 1879, C.E. Picard published his two theorems about the behavior of a function with an essential singularity, which led to a great many attempts in France and Germany to exploit and deepen his unexpected discovery.

The work of Poincaré and Klein strongly suggested that every Riemann surface is the quotient of the appropriate simply connected surface by the action of a discrete group (this being the content of the nowadays called uniformisation theorem). But any real and rigorous proof of it wasn't provided, first proofs having been found only later by P. Koebe and Poincaré independently in 1907, after Hilbert had made the uniformization theorem the subject of one of his famous 1910 Paris problems. Rigorous proofs of the Riemann mapping theorem also date from this time, when for the first time topological methods could be developed to deal with general boundaries. In the same three decades, French mathematicians (amongst whom are E.N. Laguerre, Poincaré himself, J. Hadamard, E.C. Picard, and E. Borel) had been at work investigating questions that sought to describe properties of a complex function from properties of its power series expansion. They also took up the theory of entire functions and the properties of genre and order of an entire function. Along with the Italian mathematicians C. Arzelá and G. Ascoli and the American mathematician W.F. Osgood, they looked at questions about sequences of functions, and out of this network of ideas, which includes Picard theorems, came Paul Montel work on normal families. At the very end of our period, a remarkable amount of complex function theory was brought together by P. Fatou and G. Julia in their theory of the iteration of rational functions. In passing, it is noteworthy to notice the uncomfortable fact that such a rapid progress was not to be made in the theory of complex functions of more than one variable. Where the direction of influence had run from elliptic functions to complex functions, it was to run from complex functions of several variables to Abelian functions, and then only slowly. The zero set and the singular set of a function of several variables had to be elucidated, and even Weierstrass made mistakes. The theory of Abelian functions and their connection with the general theory of theta functions was again difficult: in fact the central problem, the so-called Schottky problem, which asks for a characterization of the theta functions that arise from Abelian functions, was not to be solved until the 1980s. Some insight came with P. Cousin's generalization of the Mittag-Leffler theorem to the several variables case, and gradually Poincaré, Picard, and P. Appell produced the first rough but complete theory of the subject. Then, in the early years of the twentieth century, F. Hartogs and E.E. Levi were the first to explore the crucial novelties concerning the possible domains of functions of several variables, and Poincaré and K.A. Reinhardt showed how Riemann mapping theorem could not be generalized to higher dimensions. From a wide and detailed pedagogical final analysis of the many texts and treatises on the subject, Bottazzini and Gray have finally identified some unsurprising national features, but also a growing recognition that complex differentiability and the Cauchy-Riemann equations are the place to start, that a transition to the Weierstrass' power series methods should be made quite quickly, especially if the important subject of elliptic functions is to be taught, and that Riemannian methods are likely to be the right way to tackle deeper problems. A consensus could only emerge, however, when the Weierstrassian school was played out, and it emerges, interestingly enough, in the lecture courses that Hilbert gave in Göttingen as well as the well-known textbooks by L. Bieberbach and K. Knopp. These are among the classic texts that Serge Lang hoped everyone would continue to consult.

Cauchy's presentation of his ideas was however frankly poor until the last few years of his life. It has none of the focus of his two major accounts of real analysis, his Cours d'analyse with his next Résumé, and consequently it had a lesser impact. It was scattered between a number of journals and irregular publications, a problem that his voluntary exile from Paris for most of the 1830s inevitably entailed and exacerbated. Only around the 1850s, when he attempted to pull together his many and varied presentations, he gave to French mathematicians good accounts of his work, by then many of his ideas having been independently rediscovered and used. Riemann ideas, on the other hand, drew a much better response than is commonly realized. The lonely and obscure genius of Riemann - who, in H. Ahlfors accurate phrase, wrote "cryptic messages to the future" - must be highly regarded as one of the greatest ingenious men of mathematics. He did indeed leave a body of work that needed for a major rewriting to become rigorous, but over twenty mathematicians contributed to its further development in less than a decade after his early death in 1866. Only his remarkable paper on the zeta function and the distribution of primes lay fallow for thirty years. His great rival, Weierstrass, paid him the compliment of suggesting that his own students take up aspects of Riemann work and re-think as well as re-derive Riemann results via better methods, like the Weierstrassian one[19]. Weierstrass himself had however less success in promoting his own approach, despite having the immeasurable advantage to teach in Berlin University, i.e., the leading place in the world in his lifetime for the education of the mathematical elite. But Weierstrass put all his energies into the presentation of his 2-year, 4-semester cycle of lectures on complex functions, elliptic functions, Abelian functions, and their applications. He published very little of his own work in any journal, even some of his lectures to the Berlin Academy of Science were only read but not printed, and the seven volumes of his collected works are largely full of accounts that were being published for the first time. His present and former students had access to lithographed sets of notes, G.H. Halphen recorded that he saw such sets on every mathematician's desk when he went to Germany, but not to books, reprints, or preprints. Instead, Weierstrass relied on the power of his lectures to spread his ideas, which he revised between each cycle and the next one, and by giving few references to existing literature he inevitably took steps to deprive his students from other possible alternative versions.

The result was that naturally the French relied on a version of the theory that was largely that of Cauchy as described first by C. Briot and J. Bouquet in their celebrated textbook of 1859 (see (Briot & Bouquet, 1859)) and then by a succession of authors, while the German mathematicians turned either to more-or-less Riemannian accounts such as H. Durège provided in 1864 (see (Durège 1864)) or to Weierstrassian approaches derived from lecture notes. Gradually, the geometric versions

[19]As pointed out by Professor Paul Garrett (private communication), the possible relationships between Riemann works and Weierstrass ones are very little clear and would deserve a further deeper historical examination.

that relied on the theory of the complex integral distilled into a Cauchy-style approach and a more advanced Riemannian one, the distinction being drawn according to the extent that the concept of Riemann surface was found to be necessary. Those attracted by a Weierstrassian approach valued it for its pedagogic clarity, its rigor, its numerous insights, and its coherence; but even here a crucial distinction emerged. Weierstrass himself disliked the integral and was at pains to avoid it and to rely on power series methods instead. English and American authors, for example, found this too extreme, and blended Cauchy and Weierstrass' methods as they saw fit. Also German authors felt more free to pick up and mix them, as is often suggested by the historical literature which wants a well-known dichotomy. Even if the elements of complex function theory were almost fully established by the triumvirate of Cauchy, Riemann and Weierstrass, as the literature would often have it, much more was done before it was clear that the foundations had been laid. There was the vigorous series of investigations that form the body of geometric function theory, associated with the names of Carl Neumann, Dedekind, Klein and Poincaré. There was the equally important body of work on understanding functions defined by power series that one might have expected to have been done by Weierstrass or his students, but which was largely a French creation of the 1880s. But there were also significant gaps in the theory that were filled by other mathematicians, such as P. Laurent theorem, E. Rouché theorem, G. Morera theorem and the Schwarz reflection principle (so named by C. Carathéodory in 1913). Curiously, Weierstrass' Berlin may be the only example of a school of function theorists in the modern period. The concept of a school was originally introduced into the history of science and after its loose use in the history of mathematics, for a number of years it was recently refined by K.H. Parshall (see (Parshall 2004)). On Parshall's definition, a school involves a leader who actively pursues research, who has a characteristic approach to her or his subject, who trains students who then develop this approach further, and who collectively publish the results of their research, thereby demonstrating that it has external validity. If the research is particularly successful, the school's focus may produce a new subdiscipline of mathematics, into which the school dissolves. This concept of a school may be particularly well suited to the later nineteenth century and beyond, which is interesting in itself as an observation about the structure of the profession of mathematician (it was, of course, constructed upon the example of E.H. Moore at Chicago). This fits the situation with Weierstrass in Berlin very well, for Weierstrass had a particular approach to the fundamentals of complex function theory, and he taught it with considerable success to many of his students. Riemann, by contrast, was a major influence, but he did not create a school; L. Kronecker was also influential in Berlin, but he did not build up a school around him. In French, there was the group around É Borel, even if it was more like a collection of 'equals of France' than other[20]. These differences are reflected in the way the subject spread out. The school around Weierstrass looked inward to the writings of the master and kept new members away from alternative treatments. Everyone else in France, Italy or further afield, had much more independence, less support, and could be much more eclectic. By the time our story ends, a generation after the death of Weierstrass, it is clear that orthodoxy was no longer a virtue, and a new consensus was in the making, one that has largely survived to the present day.

In what follows, for a sake of completeness, we shall briefly give only those notions of complex analysis which will be need for understanding part of the content of next sections. Following, for example, (Bernardini et al. 1993, Chapter 1, Section 1.6), holomorphic[21] functions are roughly speaking complex functions defined on a non-void open subset of the complex plane, say A, which are complex-differentiable according to *Cauchy-Riemann equations* defined in A. Let $f(z)$ be a monodromic function which is holomorphic into a domain of the complex plane, say D, but eventually in an internal point of it, say z_0. Then, three possible

[20]It is known the snobbish behavior of Borel in regard to H. Lebesgue just due to the notable differences of social status between them, which maybe reflected too on their mathematical work with, for example, the many opposition between their corresponding integration theories (see (Hoare & Lord, 2002)).

[21]This term was introduced by two Cauchy's alumni C. Briot and J-C. Bouquet in (Briot & Bouquet 1859), the first treatise that organically and systematically collected Cauchy's theory of complex functions previously disseminated into a realm of numerous articles and papers of their schoolmaster (see (Neuenschwander 1981)).

types of singularity of f may exist in such a point. To be precise, z_0 is said to be a *removable singularity* if there exists finite $\lim_{z\to z_0} f(z) = c$, in such a case letting $f(z_0) \doteq c$ in order to make f analytic in the whole of D; z_0 is said to be a *pole* of f if $\lim_{z\to z_0} |f(z)| = \infty$, while z_0 is said to be an *essential singularity* of f if there not exist $\lim_{z\to z_0} f(z)$. Given a pole z_0 of f, the smaller natural number n such that there exists finite and non-zero the limit $\lim_{z\to z_0}(z - z_0)^n f(z)$, is said to be the *order* of such a pole. If $z_0 \in D$, f is analytic in D and $f(z_0) = 0$, then z_0 is said to be a *zero* of *order* n if n is the smaller natural number such that $f(z) = (z - z_0)^n h(z)$ with $h(z)$ analytic function in D and such that $h(z_0) \neq 0$. There exists a close connection between the poles of an analytic function f and the zeros of its inverse $g = 1/f$: namely, a necessary and sufficient condition for f have a pole of order n in z_0, is that g has a zero of order n in it. The zeros of an analytic function are isolated points of it. An *entire function* is a monodromic analytic function which have not finite singularities, whilst a *meromorphic function* is a monodromic analytic function whose only possible finite singularities are poles. Following (Itô, 1993, Article 429), an entire function (or integral function) $f(z)$ is a complex-valued function of a complex variable z that is holomorphic in the finite z-plane, $z \neq \infty$. If $f(z)$ has a pole at ∞, then $f(z)$ is a polynomial in z. A polynomial is called a *rational entire function.* If an entire function is bounded, it is constant (Liouville's theorem). A *transcendental entire function* is an entire function that is not a polynomial, for example, $\exp z, \sin z, \cos z$. An entire function can be developed in a power series $\sum_{n\in\mathbb{N}} a_n z^n$ with infinite radius of convergence. If $f(z)$ is a transcendental entire function, this is actually an infinite series. If a transcendental entire function $f(z)$ has a zero of order $m(m > 0)$ at $z = 0$ and other zeros at $\alpha_1, \alpha_2, ..., \alpha_n, ...$ with $|\alpha_1| \leq |\alpha_2| \leq ... \leq |\alpha_n| \leq ... \to \infty$, multiple zeros being repeated, then $f(z)$ can be written in the form

$$f(z) = e^{g(z)} z^m \prod_{k=1}^{\infty} \Big(1 - \frac{z}{\alpha_k}\Big) e^{g_k(z)}$$

where $g(z)$ is an entire function, $g_k(z) = (z/\alpha_k) + (1/2)(z^2/\alpha_k^2) + ... +$

$(1/p_k)(z^{p_k}/\alpha_k^{p_k})$, and $p_1, p_2, ..., p_k, ...$ are non-zero integers with the property that $\sum_{k=1}^{\infty} |z_k/\alpha_k|^{p_k+1}$ converges for all z (*K. Weierstrass' canonical product* dating back to the 1870s). Later, in the 1880s, E.N. Laguerre introduced the concept of the genus of a transcendental entire function $f(z)$. Assume that there exists an integer p for which $\sum_{k=1}^{\infty}(\alpha_k)^{-(p+1)}$ converges, and take the smallest such p. Assume further that in the representation for $f(z)$ in the previous one, when $p_1 = p_2 = ... = p$, the function $g(z)$ reduces to a polynomial of degree q; then $\max\{p, q\}$ is called the *genus* of $f(z)$. For transcendental entire functions, however, the order is more essential than the genus. The order p of a transcendental entire function $f(z)$ is defined by $p = \limsup(\log\log M(r)/\log r)$, where $M(r)$ is the maximum value of $|f(z)|$ on $|z| = r$. By using the coefficients of $f(z) = \sum_n a_n z^n$, we can write $p = \limsup(n \log n/\log(1/|a_n|))$. The entire functions of order 0, which were studied by G. Valiron and others in the 1910s, have properties similar to polynomials, and the entire functions of order less than 1/2 satisfy $\lim_{r_n\to\infty}(\min_{|z|=r_n}|f(z)|) = \infty$ for some increasing sequence $r_n \uparrow \infty$ (*A. Wiman's theorem* of the early 1900s - see (Wiman 1905)). Hence, entire functions of order less than 1/2 cannot be bounded in any domain extending to infinity. Among the functions of order greater than 1/2, there exist functions bounded in a given angular domain $D : \alpha < arg z < \alpha + \pi/\mu$. If $|f(z)| < \exp r^{\rho}(\rho < \mu)$ and $f(z)$ is bounded on the boundary of D, then $f(z)$ is bounded in the angular domain. In particular, if the order ρ of $f(z)$ is an integer p, then it is equal to the genus, and $g(z)$ reduces to a polynomial of degree $\leq p$ (*J. Hadamard's theorem* of 1893). As we will see later, these theorems originated in the study of the zeros of the Riemann zeta function and constitute the beginning of the theory of entire functions. Just upon all these common aspects relying on that non-void intersection area between transcendental entire functions theory and Riemann zeta function theory, our historical work will be centered around.

4. Outlines of history of entire function factorization theorems

To sum up following (Remmert 1998, Chapter 1), infinite products first appeared in 1579 in the works of F. Vieta, whilst in 1655 J. Wallis gave the famous product for $\pi/2$ in his *Arithmetica Infinitorum.* But L. Euler was the first to systematically work with infinite products and to formulate important product expansions in his 1748 *Introductio in Analysin Infinitorum.* The first convergence criterion is due to Cauchy in his 1821 *Cours d'analyse*[22], whilst the first comprehensive treatment of the convergence theory of infinite products was given by A. Pringsheim in 1889 (see (Pringsheim 1889)). Infinite products found then their permanent place in analysis by 1854 at the latest, through the works of Weierstrass and others. In this section, we wish to deeply outline some historical aspects and moments regarding the dawning of infinite product techniques in complex analysis, with a view towards some of their main applications.

4.1 On the Weierstrass' contribution. Roughly, the history of entire function theory starts with the theorems of factorization of a certain class of complex functions, later called *entire transcendental functions* by Weierstrass (see (Loria 1950, Chapter XLIV, Section 741) and (Klein 1979, Chapter VI)), which made their explicit appearance around the mid-1800s, within the wider realm of complex function theory which had its paroxysmal moment just in the 19th-century. But, if one wished to identify, with a more precision, that chapter of complex function theory which was the crucible of such a theory, then the history would lead to elliptic function theory and related factorization theorems for doubly periodic elliptic functions, these latter being meant as a generalization of trigonometric functions. Following (Vesentini 1984, Chapter VII), the theory of elliptic integrals was the first main historical motif from which elliptic function theory sprung out, whilst the polydromy of such integrals finds its natural environment of development in the geometry of algebraic curves (see (Enriques & Chisini 1985, Volume 1) and (Dieudonné

[22]See its English annotated translation (Bradley & Sandifer 2009).

& Grothendieck 1971)). Again, following (Stillwell 2002, Chapter 12, Section 12.6), the early idea which was as at the basis of the origin of elliptic functions as obtained by inversion of elliptic integrals, is mainly due to Legendre, Gauss, Abel and Jacobi (together two pupils of this last, namely G.A. Göpel (1812-1847) and J.A. Rosenhain (1816-1887) (see (Hermite 1873, pp. 296-297)). Following (Fasano & Marmi 2002, Appendix 2), the elliptic functions was introduced for the first time by J. Wallis in 1655 in computing the arc length of an ellipse whose infinitesimal element is not equal to the differential of an already known elementary function. In their most general form, they are given by $\int R(x,y)dx$ where $R(x,y)$ is a rational function of its arguments and $y = \sqrt{P(x)}$ with $P(x)$ a fourth degree polynomial. Legendre, in 1793, proved that a general elliptic integral is given by a linear combination of elementary functions and three basic elliptic integrals said to be integrals of the first, second and third kind. Gauss had the idea of inversion of elliptic integrals in the late 1790s but didn't publish it; Abel had the same idea in 1823 and published it in 1827, independently of Gauss. Jacobi's independence instead is not quite so clear. He seems to have been approaching the idea of inversion in 1827, but he was only spurred by the appearance of Abel's paper. At any rate, his ideas subsequently developed at an explosive rate, up until he published the first and major book on elliptic functions, the celebrated *Fundamenta nova theoriæ functionum ellipticarum* in 1829. Following (Enriques 1982, Book III, Chapter I, Section 6), on the legacy left, amongst others, by J.L. Lagrange, N.H. Abel, C.G.J. Jacobi, A.L. Cauchy and L. Euler, Riemann and Weierstrass quickly became the outstanding figures of the 19th-century mathematics. Agreeing with Poincaré in his 1908 *Science et méthode,* Riemann was an extremely brilliant intuitive mathematician, whereas Weierstrass was primarily a logician, both personifying, therefore, those two complementary and opposite typical aspects characterizing the mathematical work. Beyond what had been made by Cauchy, they created the main body of the new complex function theory in the period from about 1850 to 1880 (see (Klein 1979, Chapter VI)). Both received a strong impulse from Jacobi's work. The first elements of the theory of functions according to

Weierstrass date back to a period which roughly goes on from 1842-43 to 1854; in the meanwhile, Riemann published, in the early 1850s, his first works on the foundations of complex analysis, followed by the celebrated works on Abelian functions (which are elliptic functions so named by Jacobi) of the years 1856-57, which dismayed Weierstrass himself, influencing his next research program. This last point should be taken with a certain consideration. Instead, following (Klein 1979, Chapter VI), in the period from 1830s to the early 1840s, Weierstrass began to self-taughtly study Jacobi's *Fundamenta nova theoriæ functionum ellipticarum,* hence attended Christoph Gudermann (1798-1852) lectures on elliptic functions (see also (Manning 1975)). He wrote his first paper in 1841 on modular functions, followed by some other papers wrote between 1842 and 1849 on general function theory and differential equations. His first relevant papers were written in the years 1854-56 on hyperelliptic or Abelian functions, which engaged him very much. Afterwards, in the wake of his previous work on analytic, elliptic and Abelian functions, Weierstrass was led to consider the so-called *natural boundaries* (that is to say, curves or points - later called *essential singularities* - in which the function is not regularly defined) of an analytic function to which Riemann put little attention. The first and rigorous treatment of these questions was given by Weierstrass in his masterful 1876 paper entitled *Zur Theorie der eindeutigen anatytischen Funktionen,* where many new results were achieved, amongst which is the well-known *Casorati-Weierstrass theorem* (as we today know it) and the product factorization theorem. Klein (1979, Chapter VI) states that the content of this seminal paper surely dated back to an earlier period, and was chiefly motivated by his research interests in elliptic functions. As pointed out in (Hancock 1910, Introduction), nevertheless, it is quite difficult to discern the right contribution to the elliptic function theory due to Weierstrass from other previous mathematicians, because of the objective fact that Weierstrass started to publish his lessons and researches only after the mid-1860s.

Weierstrass' theory of entire functions and their product decompositions, according to Klein, has found its most brilliant application in the (Weierstrass) theory of elliptic functions, to be precise, in the construction of the basic Weierstrassian σ-function $\sigma(u)$; perhaps - Klein says -

Weierstrass' theory of entire functions even originated from his theory of elliptic functions (see also (Bottazzini & Gray 2013, Chapter 6, Section 6.6.3)). Nevertheless, already Gauss and Abel were gone very close to this σ-function and its properties. Again Klein says that he wishes to conclude his discussion of Weierstrass' theory of complex functions, adding only some remarks on the history, referring to R. Fricke distinguished review on elliptic functions for more information (see (Burkhardt et al. 1899-1927, Zweiter Teil, B.3, pp. 177-348)). If we now ask - again Klein says - from where Weierstrass got the impulse to represent his functions by infinite products, we find his principal forerunner in G. Eisenstein (1823-1852), a student of Gauss, who was also a friend of Riemann with whom often talked about mathematical questions and who very likely stimulated, according to André Weil (see (Narkiewicz 2000, Chapter 4, Section 4.1, Number 2) and references therein), the interest towards prime number theory in Riemann. Following textually (Weil 1975, Second Lecture, p. 21),

«[...] *the case of Riemann is more curious. Of all the great mathematicians of the last century, he is outstanding for many things, but also, strangely enough, for his complete lack of interest for number theory and algebra. This is really striking, when one reflects how close he was, as a student, to Dirichlet and Eisenstein, and, at a later period, also to Gauss and to Dedekind who became his most intimate friend. During Riemann's student days in Berlin, Eisenstein tried (not without some success, he fancied) to attract him to number-theory. ln 1855, Dedekind was lecturing in Gottingen on Galois theory, and one might think that Riemann, interested as he was in algebraic functions, might have paid some attention. But there is not the slightest indication that he ever gave any serious thought to such matters. It is clearly as an analyst that he took up the zeta-function. Perhaps his attention had been drawn to the papers of Schlömilch and Malmquist in 1849, and of Clausen in 1858. Anyway, to him the analytic continuation of the zeta-function and its functional equation may weIl have seemed a matter of routine; what really interested him wa8 the connection with the prime number theorem, and those aspects which we now classify as "analytic number-theory",* *which to me, as 1 have told you, is not number theory at all. Nevertheless, there are two aspects of his famous 1859 paper on the zeta-function which are of vital importance to us here* [i.e., his functional equation for ζ function and the famous Riemann hypothesis]*».*

In his long-paper (see (Eisenstein 1847)), Eisenstein did not attain the fully symmetric normal form, because he still lacked the exponential factors attached to the individual prime factors that will be then introduced by Weierstrass for inducing the product to converge in an absolutely manner. As he himself declared, Weierstrass got this idea from Gauss, who had proceeded in a similar way with his product expansion of the gamma function in 1812 (see the paper on the hypergeometric series in Weierstrass' *Mathematische Werke,* Band II; see also Weierstrass' works on elliptic and other special functions included in Band V). It therefore turns out clear that elliptic and hyperelliptic function theory exerted a notable role in preparing the *humus* in which grew up the Weierstrass work on factorization theorem, and not only this: in general, it exerted a great influence on Riemann and Weierstrass work (see (Bottazzini & Gray 2013, Chapter 4, Section 4.5)). Following (Burkhardt et al. 1899-1927, Zweiter Teil, B.3, Nr. 15-17, 25, 45, 55), amongst others, Abel[23], Euler, Jacobi, Cayley and Gauss (see (Bottazzini & Gray 2013, Chapter 1, Section 1.5.1.1; Chapter 4, Section 4.2.3.1-2)) had already provided product expansions of certain elliptic functions, but it was Eisenstein (see (Eisenstein 1847)), with his infinite product expansion *ansatz,* the closest forerunner of the Weierstrass work on his σ function, in turn based on the previous work made by Jacobi and Gauss (see (Weil 1976)). Following (Remmert 1998, Chapter 1), in his 1847 long-forgotten paper, had already systematically used infinite products, also using conditionally convergent products and series as well as carefully discuss the problems,

[23]In (Greenhill 1892, Chapter IX, Section 258), the author states that the well-known expressions for the circular and hyperbolic functions in the form of finite and infinite products, have their analogues for the elliptic functions as laid down by Abel in some his researches on elliptic functions published in the celebrated *Crelle's Journal*, Issues 2 and 3, years 1827 and 1828. Following (Hancock 1910, Chapter V, Article 89), Abel showed, in the 1820s, that elliptic functions, considered as the inverse of the elliptic integrals, could be expressed as the quotient of infinite products, then systematically reconsidered in a deeper manner by Jacobi.

then barely recognized, of conditional and absolute convergence, but he does not deal with questions of compact convergence. Thus logarithms of infinite products are taken without hesitation, and series are casually differentiated term by term, and this carelessness may perhaps explain why Weierstrass nowhere cites Eisenstein's work. Furthermore, already Cauchy[24], ever since 1843, gave some useful formulas involving infinite products and infinite series with related convergence properties which maybe could have played a certain role in the 1859 Riemann paper in deducing some properties of that functional equation related to his ξ function (whose an earlier form was also known to Euler over a hundred years before Riemann, and to which Euler had arrived in the real domain by use of divergent series methods; see(Kolmogorov & Yushkevich 1996, Volume II, Chapter 2) and (Bateman & Diamond 2004, Chapter 8, Section 8.11)), also because of the simple fact that Riemann himself known very well Cauchy's work. Therefore, Weierstrass himself acknowledges, in different places, his debit both to Gauss and Cauchy, in achieving his celebrated results on entire function factorization theorem. Furthermore, following (Fouët 1904-07, Tome II, Chapter IX, Section 272), many mathematicians have acknowledged in Abel one of the most influential scholars who have contributed to the intellectual development of Weierstrass. On the other hand, following (Pólya 1930), also J-B. Fourier, in (Fourier 1830, Exposé synoptique, Nos. (15) and (16) III^e and IV^e, pp. 65-66), as early as the late 1820s, considered infinite products in algebraic questions inherent transcendental equations of the type $\phi(x) = 0$, with applications to the case $\sin x = 0$, whose outcomes could be therefore known to Riemann. To be precise, in the words of Pólya, he states a more general theorem which it is worth quoting verbally as follows

«III^e. Une fonction transcendante ou algébrique ϕx étant proposée, si l'on fait l'énumération de toutes les valeurs réelles ou imaginaires de x,

[24]In this regard, see (Bellacchi 1894, Chapter X), where an interesting discussion of the Jacobi series is made, amongst other things highlighting that already Abel, on the wake of what was done by Johann Bernoulli, had introduced infinite product expansions of certain elliptic functions (Jacobi's triple product) that later Jacobi, in turn, converted into infinite series by means of trigonometric arguments, so giving rise to new elliptic functions (see (Greenhill 1892, Chapter IX, Section 258) and (Remmert 1998, Chapter 1, Section 5)).

savoir $\alpha, \beta, \gamma, \delta$, *etc. qui rendent nulle la fonction* $f(x)$, *et si l'on désigne par* $f(x)$ *le produit* $\left(1-\dfrac{x}{\alpha}\right), \left(1-\dfrac{x}{\beta}\right), \left(1-\dfrac{x}{\gamma}\right), \left(1-\dfrac{x}{\delta}\right), \ldots$ *de tous les facteurs simples qui correspondent aux racines de l'equation* $\phi x = 0$, *ce produit pourra différer de la fonction* ϕx, *en sorte que cette fonction, au lieu d'étre équivalente* fx, *sera le produit d'un premier facteur* $f(x)$ *par un second* $F(x)$. *Cela pourra arriver si le second facteur* $F(x)$ *ne cesse point d'étre une grandeur finie, quelque valeur réelle ou imaginaire que l'on donne à* x , *ou si ce second facteur* $F(x)$ *ne devient nul que par la substitution de valeurs de* x *qui rendent infini le premier facteur* $f(x)$. *Et réciproquement si l'équation* $F(x)$ *des racines, et si elles ne rendent point infini le facteur* $f(x)$, *on est assuré que le produit de tous les facteurs du premier degré correspondant aux racines de* $\phi x = 0$ *équivaut à cette fonction* ϕx. *En effet:* 1^0 *s'il existait un facteur* fx *qui ne pût devenir nul pour aucune valeur réelle ou imaginaire de* x, *par exemple si* Fx *était une constante* A *et si* fx *était* $\sin x$, *toutes les racines de* $A \sin x = 0$ *seraient celles de* $\sin x = 0$, *et le produit de tous les facteurs simples correspondant aux racines de* $A \sin x = 0$ *serait seulement* $\sin x$, *et non* $A \sin x$. *Il en serait de méme si le facteur* Fx *n'était pas une constante* A. *Mais s'il pouvait exister un facteur* Fx *qui ne cesserait point d'avoir une valeur finie, quelque valeur réelle ou imaginaire que l'on attribuât à* x, *toutes les racines de l'équation* $\sin x Fx = 0$ *seraient celles de* $\sin x = 0$, *puisqu'on ne pourrait rendre nul le produit* $\sin x$. Fx *qu'en rendant* $\sin x$ *nul. Donc le produit de tous les facteurs correspondants aux racines de* $\phi x = 0$ *serait* $\sin x$, *et non* $\sin x Fx$. *On voit donc que dans ce second cas il serait possible que le produit de tous les facteurs simples ne donnât pas* ϕx. 2^0 *Si l'équation* $Fx = 0$ *a des racines, ou réelles ou imaginaires, ce qui exclut le cas où* Fx*: serait une constante* A, *ou serait un facteur dont la valeur est toujours finie, et si les racines de* $Fx = 0$ *rendeut* fx *infini, le produit* $fxFx$ *devient* $0/0$, *et peut avoir une valeur très-différente de* fx. *Mais si les racines de* $Fx = 0$ *donnent pour* fx *une valeur finie, le produit* $fxFx$ *deviendrait nul lorsque* $Fx = 0$*: donc l'enumération complète des racines de l'équation* $\phi x = 0$, *ou* $fxFx = 0$, *comprendrait les racines de* $Fx = 0$. *Or nous avons représenté par* fx *le produit de tous les facteurs simples qui répondent aux racines de* $\phi x = 0$*:*

il serait donc contraire à l'hypothèse d'admettre qu'il y a un autre facteur Fx, tel que les racines de $Fx = 0$ sont aussi des facteurs de $\phi x = 0$. Cela supposerait que l'on n a pas fait une enumération complète des racines de l'équation $\phi x = 0$, puisqu'on a exprimé seulement par fx le produit des facteurs simples qui correspondent aux racines de cette équation.

IV[e]*. Étant proposée une équation algébrique ou transcendante $\phi x = 0$ formée d'un nombre fini ou infini de facteurs réels ou imaginaires $\left(1-\frac{x}{\alpha}\right)$, $\left(1-\frac{x}{\beta}\right)$, $\left(1-\frac{x}{\gamma}\right)$, $\left(1-\frac{x}{\delta}\right)$, etc. on trouve le nombre des racines imaginaires, les limites des racines réelles, les valeurs de ces racines, par la méthode de résolution qui a été exposée dans les premiers livres et qui sera la même soit que la différentiation répétée réduise ϕx une valeur constante, soit que la différentiation puisse être indéfiniment continuée. L'équation $\phi x = 0$ a précisément autant de racines imaginaires qu'il y a de valeurs réelles de x qui , substituées dans une fonction dérivée intermédiaire d'un ordre quelconque, rendent cette fonction nulle, et donnent deux résultats de même signe pour la fonction dérivée qui la précède et pour celle qui la suit. Par conséquent si l'on parvient à prouver qu'il n'y a aucune valeur réelle de x qui, en faisant évanouir une fonction dérivée intermédiaire, donne le même signe à celle qui la précède, et à celle qui la suit, on est assuré que la proposée ne peut avoir aucune racine imaginaire. Par exemple en examinant l'origine de l'équation transcendante (1) $0 = 1 - \frac{x}{1} + \frac{x^2}{(1 \cdot 2)^2} - \frac{x^3}{(1 \cdot 2 \cdot 3)^2} + \frac{x^4}{(1 \cdot 2 \cdot 3 \cdot 4)^2} -$ etc. nous avons prouvé qu'elle est formé du produit d'un nombre infini de facteurs; et en considérant une certaine relation récurrente qui subsiste entre les coefficients des fonctions dérivées des divers ordres, on reconnaît qu'il est impossible qu'une valeur réelle de x substituée, dans trois fonctions dérivées consécutives, réduise la fonction intermédiaire à zéro, et donne deux résultats de même signe pour la fonction précédente et pour la fonction suivante. On en conclut avec certitude que l'équation (1) ne peut point avoir de racines imaginaires».*

Pólya says that no proof of this theorem, by Fourier or another mathematician, is known. In 1841, M.A. Stern gave an invalid proof, and repeated in greater detail some affirmations of Fourier. Since then, the question seems to have been neglected. However, to further emphasize the Weierstrass work on entire function factorization, we report textual words of Picard (see (Picard 1897) and (Dugac 1973, Section 5.1)), who is one of the founders of the theory of entire functions, as we will see later

«L'illustre analyste a publié en 1876 un mémoire sur la Théorie des Fonctions uniformes; ce mémoire, en faisant connaître à un public plus étendu les résultats développés depuis longtemps déjà dans l'enseignement du maître, a été le point de départ d'un très grand nombre de travaux sur la Théorie des Fonctions. Cauchy et ses disciples français, en étudiant les fonctions analytiques uniformes, n'avaient pas pénétré bien profondément dans l'étude de ces points singuliers appelés "points singuliers essentiels", dont le point $z = 0$ *pour la fonction* $\exp(1/z)$ *donne l'exemple le plus simple. Weierstrass, en approfondissant cette étude, a été conduit à un résultat qui est un des plus admirables théorèmes de l'Analyse moderne, je veux parler de la décomposition des fonctions entières en facteurs primaires. D'après le théorème fondamental de l'Algèbre, un polynôme peut être décomposé en un produit de facteurs linéaires; pour une fonction entière, c'est-à-dire pour une fonction uniforme continue dans tout le plan telle que sins),ne peut-on chercher à obtenir aussi une décomposition en facteurs? Cauchy avait obtenu sur ce sujet des résultats importants, mais sans le traiter dans toute sa généralité. Il était réservé à Weierstrass de montrer qu'une fonction entière peut être décomposée en un produit d'un nombre généralement infini de facteurs primaires, chacun de ceux-ci étant le produit d'un facteur linéaire par une exponentielle de la forme* $\exp(P(z))$, *où* $P(z)$ *est un polynôme. C'est sans doute en étudiant l'intégrale Eulérienne de seconde espèce que Weierstrass a été mis sur la voie de ce beau théorème, et nous rappellerons à ce sujet cet important résultat que l'inverse de cette intégrale est une fonction entière».*

Rolf Nevanlinna points out the main role played by Weierstrass in realizing a class of elementary analytic functions, amongst which Abelian and elliptic functions, whose construction has led Weierstrass to the creation of a general theory of entire and meromorphic functions of which one of the founding pillars is just the theorem on the decomposition into primary factors (see (Dugac 1973, Section 5.1)). On the other hand, Weierstrass himself was fully aware of the importance played by this result within the general context of complex function theory. Furthermore, as (Kudryavtseva et al. 2005, Section 7) point out, the two decades following the publication of the celebrated 1859 Riemann's paper, were largely uneventful. Weierstrass, who was eleven years older than Riemann, but whose rise to fame - from an obscure schoolteacher to a professor at Berlin - happened in a way very different from Riemann's one, began working and lecturing on complex numbers and the general theory of entire functions already during the 1860s, but it wasn't until 1876, when Weierstrass finally published his famous memoir, that mathematicians became aware of some of his revolutionary ideas and results. Weierstrass' factorization theorem, together with Riemann's memoir, set the stage for the great work of Hadamard and de la Valée-Poussin in the 1890s. In particular, Hadamard made more explicit and applicable Weierstrass' factorization theorem.

4.2 On the Riemann's contribution. In 1858, Riemann wrote his unique paper on number theory, which marked a revolution in mathematics. According to Laugwitz (1999, Introduction, Sections 4.1 and 4.2), real and complex analysis has always influenced Riemann work: algebraic geometry appears, in his works, as a part of complex analysis; he treats number theory with methods of complex function theory; he subsumes physical applications into partial differential equations; he replaces the usual axiomatic conception of geometry by his novel (Riemannian) geometry, which is a part or real analysis of several variables; and he develops the topology of manifolds as a new discipline derived from analysis. Riemann knew the elements of algebraic analysis according to J-L. Lagrange and L. Euler, through the lessons of his teacher, M.A. Stern (1807-1894), who was one of the last schoolmasters of the subject. Riemann handled the gamma function in a secure and self-confident way and has dealt with differential equations and recursions in the Euler's manner. The Stern lessons were of very fundamental importance to achieve many Riemann's results, even if the celebrated 1748 Euler's *Introductio in Analysin Infinitorum* was one of the most influential textbooks of the time. Nevertheless, Klein (1979, Chapter VI) states that Riemann began already to study elliptic and Abelian functions since the late 1840s, because this subject, in the meantime, has become of a certain vogue in Germany. In the 1855-56 winter term, following the Dirichlet's research lines, Riemann lectured on functions of a complex quantity, in particular elliptic and Abelian functions, while in the 1856-57 winter term he lectured on the same subject, but now with special regard to hypergeometric series and related transcendentals. These lectures, from which he drew publications on Abelian and hypergeometric functions, were partially repeated in the following semesters. Klein (1979, Chapter VI) points out that the years 1857-62 marked the high-point of Riemann's creativity. Moreover, Klein states that before to characterize the specific Riemannian function theory work, he wishes to put forward a remark that may cause some surprise: Riemann did much important work in the theory of functions that does not fit into the framework of his typical theory. Klein refers to the notable 1859 paper on the number of primes lower than a given magnitude, where it is introduced *«the Riemann zeta-function $\zeta(\sigma + it)$ given by an analytic expression, namely an infinite product. This product is converted into a definite integral, which can then be evaluated by shifting the path of integration. The whole procedure is function theory à la Cauchy»*. Therefore, according to what Felix Klein states, the mathematical background that was at the basis of the Riemannian analytic treatment of his ζ function, essentially lies on the Cauchy's theory of complex functions. This is also confirmed by (Bottazzini & Gray 2013, Chapter 5, Section 5.1), coherently with what has just been said above in regards to the importance played by Cauchy's work on the Riemann's one.

According to (Laugwitz 1999, Chapter 1), notwithstanding the era of ferment that concerned the 19th century mathematics, an autonomous and systematic account of the foundations of complex analysis is findable, for the first time, in the Riemann's works and lecture notes through the winter term 1855-56 to the winter term 1861-62, the latter having been published by the physicist Carl Ernst Abbe (1840-1905) in the summer term 1861 (see (Ullrich 2003) and references therein) when he was a student of W. Weber and Riemann in Göttingen. The only systematic and congruous historical attempt to organically recognize the various Riemann's lessons has been pursued by E. Neuenschwander in (Neuenschwander 1996). In any case, Riemann was fully imbedded into the real and complex analysis scenery of the first middle of the 19th century, which seen involved the outstanding figures of Cauchy, Weierstrass and Riemann himself, whose researches were intertwined amongst them more times. According to (Laugwitz 1999, Chapter 1, Section 1.1.5), just in connection with the drawing up of his paper on the same subject, Riemann was aware of the Weierstrass' papers on Abelian functions wrote between 1853 and 1856-57, for which it is evident that a certain influence of the latter on the Riemann's one - at least, as concerns such a period - there was, even if Weierstrass will publish these his works only later. Again following (Neuenschwander 1996), one of the key themes of the last *Sommersemestern* 1861 Riemann lectures on analytic functions (see sections 11-13), was the determination of a complex function from its singularities (section 13) mainly following Cauchy's treatment[25]. Then, he clarifies that this problem regards only single-valued functions defined on $\mathbb{C} \cup \{0\}$ whose unique singularities are poles (the names *pole* and *essential singularity* are respectively due to C.A. Briot and J.C. Bouquet and to Weierstrass). In turn, the resolution of this problem requires the previous knowledge of the zeros of the function which has to be determined. At first, Riemann considered the case of a function having a finite number of zeros and poles, then he went over the the next question, namely to determine a function with infinitely many zeros whose unique point of accumulation if ∞ (which, inter alia, concerns too the Riemann zeta function theory). But, again following (Laugwitz 1999, Chapter 1, Section 1.1.6), in doing so Riemann went over very close to the next Weierstrass' work on the infinite product representation of an entire function, using special cases to explain the general procedure.

[25]Amongst the first lecture notes on complex functions very close to the Riemann ideas and approach, there are those exposed in (Durège, 1896).

Detlef Laugwitz points out that Riemann has pursued this latter task in such a way that, by following his directions, one could immediately give a proof of the known Weierstrass' product theorem, even if Riemann ultimately failed in reaching the general case; nevertheless, for what follows, this last claim has a certain importance from our historical standpoint.

Indeed, in his renowned paper on the distribution of prime numbers[26], Riemann stated the following function

$$(1) \qquad \xi(t) \doteq \Big(\frac{1}{2}\Gamma\Big(\frac{s}{2}\Big)s(s-1)\pi^{-s/2}\zeta(s)\Big)_{s=1/2+it}$$

later called *Riemann ξ-function.* It is an entire function (see (Titchmarsh 1986, Chapter II, Section 2.12)). Riemann conjectured that $(\xi(t)=0) \Rightarrow (\Im(t)=0)$, that is to say, the famous *Riemann hypothesis* (RH), as it

[26]This paper was presented by Riemann, after his nomination as full professor in July 1859, to the Berlin Academy for his consequent election as a corresponding member of this latter. To be precise, following (Bottazzini 2003), just due to this election, Riemann and Dedekind visited Berlin, where they met E.E. Kummer, L. Kronecker and K. Weierstrass. According to (Dedekind 1876), very likely, it was just from this meeting that sprung out of the celebrated 1859 Riemann number theory paper that was, then, sent to Weierstrass himself, to be published in the November issue of the *Monatsberichte der Berliner Akademie.* Following (Neuenschwander 1981), during time Riemann spent in Berlin, in the years 1847-1849, he was particularly influenced by Dirichlet, Eisenstein and Jacobi. Riemann attended Dirichlet's lectures on, among other things, the theory of definite integrals and partial differential equations, as well as those of Eisenstein and Jacobi on elliptic integrals. Dirichlet recognized Riemann's extraordinary talent early on and contributed considerably to the advancement of his career from that moment. In the year 1852, for example, he discussed Riemann's dissertation with him and helped him to get the literature for his Habilitationsschrift. But quite early Riemann was also making himself familiar with the writings of the most important French mathematicians. In his first year of study 1846-47 in Göttingen, he borrowed from the university library, among other books, the *Cours d'analyse,* the *Calcul différentiel* and the *Exercices de mathématiques,* all by Cauchy, along with the *Traité des fonctions elliptiques* of Legendre. From the above it becomes clear that Riemann was suitable, as no other German mathematician then was, to effect the first synthesis of the "French" and the "German" approaches in function theory. In his introductory lectures on general complex function theory of 1861, Riemann dealt with the Cauchy Integral Formulae, the operations on infinite series, the power series expansion, the Laurent series, the analytic continuation by power series, the argument principle, the product representation of an entire function with arbitrarily prescribed zeros, the evaluation of definite integrals by residues, etc., besides the subjects known from his published papers. Furthermore, and this is an important witness for our ends, according to the mathematician Paul B. Garrett (see his Number Theory lessons at http://www.math.umn.edu/ garrett/), general factorizations of entire functions in terms of their zeros are due to K. Weierstrass, but sharper conclusions from growth estimates are due to J. Hadamard. In relation to his 1859 memoir, Riemann's presumed existence of a factorization for ξ function to see the connection between prime numbers and complex zeros of zeta function, was a significant impetus to Weierstrass' and Hadamard's study of products in succeeding decades.

will be called later. Whereupon, he stated that

«This function $\xi(t)$ is finite for all finite values of t, and allows itself to be developed in powers of tt as a very rapidly converging series. Since, for a value of s whose real part is greater than 1, $\log\zeta(s) = -\sum\log(1-p^{-s})$ remains finite, and since the same holds for the logarithms of the other factors of $\xi(t)$, it follows that the function $\xi(t)$ can only vanish if the imaginary part of t lies between $i/2$ and $-i/2$. The number of roots of $\xi(t) = 0$, whose real parts lie between 0 and T is approximately $= (T/2)\log(T/2)\pi - T/2\pi$; because the integral $\int d\log\xi(t)$, taken in a positive sense around the region consisting of the values of t whose imaginary parts lie between $i/2$ and $-i/2$ and whose real parts lie between 0 and T, is (up to a fraction of the order of magnitude of the quantity $1/T$) equal to $(T\log(T/2)\pi - T/2\pi)i$; this integral however is equal to the number of roots of $\xi(t) = 0$ lying within this region, multiplied by $2\pi i$. One now finds indeed approximately this number of real roots within these limits, and it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation. If one denotes by α all the roots of the equation $\xi(t) = 0$, one can express $\log\xi(t)$ as

$$\sum\log\left(1-\frac{tt}{\alpha\alpha}\right)+\log\xi(0) \tag{2}$$

for, since the density of the roots of the quantity t grows with t only as $\log t/2\pi$, it follows that this expression converges and becomes for an infinite t only infinite as $t\log\xi(t)$; thus it differs from $\log\xi(t)$ by a function of tt, that for a finite t remains continuous and finite and, when divided by tt, becomes infinitely small for infinite t. This difference is consequently a constant, whose value can be determined through setting $t = 0$. With the assistance of these methods, the number of prime numbers that are smaller than x can now be determined».

So, in his celebrated 1859 paper, Riemann himself had already reached an infinite product factorization of $\xi(t)$, namely the (2), which can be equivalently written as follows

$$(3) \quad \log \xi(t) = \sum_{\alpha} \log\Big(1 - \frac{tt}{\alpha\alpha}\Big) + \log \xi(0) = \log \xi(0) \prod_{\alpha} \Big(1 - \frac{t^2}{\alpha^2}\Big)$$

from which it follows that $\xi(t) = \xi(0) \prod_{\alpha} \left(1 - t^2/\alpha^2\right)$. Thus, questions related to entire function factorizations had already been foreshadowed in this Riemann work. Therefore, we now wish to outline the main points concerning the very early history of entire function factorization theorems, having taken the 1859 Riemann paper as an occasional starting point of this historical question, in which, inter alia, a particular entire function factorization - i.e. the (3) - had been used. In short, this 1859 Riemann paper has been a valuable *καιρός* to begin to undertake one of the many study's branch which may depart from this milestone of the history of mathematics, to be precise that branch concerning the entire function theory which runs parallel to certain aspects of the theory of Riemann zeta function, with interesting meet points with physics. One of the very few references which allude to these Riemann paper aspects is the article by W.F. Osgood in (Burkhardt et al. 1899-1927, Zweiter Teil, B.1.III, pp. 79-80), where, discussing of the genus of an entire function, an infinite product expansion of the function $\sin \pi s/\pi s$ is considered; to be precise, since Johann Bernoulli to Euler, the following form[27] had

[27] As $n \to \infty$ and $t \neq 0$, Weierstrass proved to be $\sin \pi t/\pi t = \prod_{n=-\infty}^{\infty}(1-t/n)e^{t/n}$ - see (Bellacchi 1894, Chapter XI). But, according to (Bellacchi 1894, Chapter XI) and (Hancock 1910, Chapter I, Arts. 13, 14), Cauchy was the first to have treated (in the *Exercises de Mathématiques, IV*) the subject of decomposition into prime factors of circular functions and related convergence questions, from a more general standpoint. Although Cauchy did not complete the theory, he however recognized that, if a is a root of an integral (or entire) transcendental function $f(s)$, then it is necessary, in many cases, to join to the product of the infinite number of factors such as $(1 - s/a)$, a certain exponential factor $e^{P(s)}$, where $P(s)$ is a power series in positive powers of s. Weierstrass gave then a complete treatment of this subject. On the other hand, besides what has already been said above, also in (Greenhill 1892, Chapter IX, Section 258)) it is pointed out that, since Abel's work, the infinite product expansions of trigonometric functions have been formal models from which to draw inspiration, by analogy, for further generalizations or extensions. Analogously, following (Fouët 1904-07, Tome II, Chapter IX, Section II, Number 279), *«Cauchy* [in the *Anciens Exercices de Mathématiques, 1829-1830*] *avait vu que, pour obtenir certaines transcendantes, il fallait multiplier le produit des binomes du premier degré du type* $X - a_n$ *par une exponentielle* $e^{g(x)}$, $g(x)$ *désignant une fonction entière. Même l'introduction de cette exponentielle ne suffit à donner l'expression générale des fonctions admettant les zéros* $a_1, ..., a_n, ...$ *que dans le cas où la*

already been deduced (see (Bellacchi 1894, Chapter XI))

$$\frac{\sin \pi s}{\pi s} = \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right)$$

which has genus 0. This is often said to be the *Euler's product formula* (see (Borel 1900, Chapter IV), (Tricomi 1968, Chapter IV, Section 8), (Remmert 1998, Chapter 1, Section 3) and (Maz'ya & Shaposhnikova 1998, Chapter 1, Section 1.10)), and might be considered as one of the first meaningful instances of infinite product expansion of an entire function, given by Euler in his 1748 *Introductio in Analysin Infinitorum,* through elementary analytic methods (see (Sansone 1972, Chapter IV, Section 1)). Furthermore, following (Fouët 1904-07, Chapter IX, Section III, Number 286), there have always been a close analogical comparison between the trigonometric functions and the Eulerian integrals (amongst which the one involved in the Gamma function) together their properties, a way followed, for instance, by E. Heine. Also looking at the Riemann's lectures on function theory through the 1855-56, 1856-57, 1857-58 *Wintersemestern* to the 1858-59 and 1861 *Sommersemestern* lessons - see (Neuenschwander 1996, Section 13) as regards the last ones - it would be possible to descry as well some Riemann's attempts to consider factorization product expansions whose forms seem to suggest, by analogy, a formula similar to (3). Moreover, also in the 1847 Eisenstein paper, surely known to Riemann, there is also a certain lot of work devoted to the study of the Euler's sine product formulas (see (Ebbinghaus et al. 1991, Chapter 5)) which perhaps have could contribute to stimulate the Riemann insight in finding some formulas used in his 1859 celebrated paper, first of all the (3). To be precise, be-

série $\sum_n |a_n|^{-1}$ *converge. converge. L'étude du développement des (Weierstrass) fonctions* $\mathcal{P}$ *et* σ *en produits infinis amena Weierstrass à s'occuper de cette question et fut ainsi l'occasion d'une de ses plus belles découvertes* [see, for example, (Lang 1987, Chapter 1, Section 2)]. *Ces recherches, exposées en 1874 par Weierstrass dans son cours à l'Université de Berlin, ont été publiées dans un Mémoire fondamental "Zur Théorie der eindeutigen analytischen Functionen" de 1876. Quinze ans auparavant, Betti, dans ses* Leçons *à l'Université de Pise (1859-1860), avait traité un problème analogue à celui résolu par Weierstrass, mais sans apercevoir toutes les conséquences de sa découverte; il en fit l'application au développement des fonctions eulériennes, trigonométriqucs et elliptiques, puis, laissant son Mémoire dans 1860 inachevé, il n'y pensa plus».*

cause of the close friendship between Eisenstein and Riemann, seen too what is said in (Weil 1989) (see also (Weil 1976)) about the sure influence of Eisenstein work on Riemann one, reaching to suppose that 1859 Riemann paper was just originated by Eisenstein influence, we would be inclined to put forwards the historical hypothesis that the deep and complete analysis and critical discussion of infinite products pursued in the long and rich paper[28] (Eisenstein 1847), surely played a decisive role in the dawning of ξ product expansion of Riemann paper, also thanks to the great mathematical insight of Riemann in extending and generalizing previous mathematical contexts in others. Furthermore, following (Genocchi 1860, N. 2), an infinite product factorization of the ξ function could be easily deduced from what is said in (Briot & Bouquet 1859, Book IV, Chapter II) about infinite product factorizations[29]. Following (Stopple 2003, Chapter 6, Section 6.1), Euler's idea is to write the function $\sin \pi x/\pi x$ as a product over its zeros, analogous to factoring a polynomial in terms of its roots. For example, if a quadratic polynomial $f(x) = ax^2 + bx + c$ has roots α, β different from 0, then we can write $f(x) = c(1 - x/\alpha)(1 - x/\beta)$. On the other hand, $\sin \pi x = 0$ when $x = 0, \pm 1, \pm 2, ...$ and since $\sin \pi x/x = 1 - \pi^2 x^2/6 + O(x^4)$, $\sin \pi x/\pi x \underset{x\to 0}{\to} 1$ and $\sin \pi x/\pi x = 0$ when $x = \pm 1, \pm 2, ...$, Euler guessed that $\sin \pi x/\pi x$ could have a factorization as an infinite product of the type (see also (Ebbinghaus et al. 1991, Chapter 5))

$$\frac{\sin \pi x}{\pi x} = \Big(1 - \frac{x}{1}\Big)\Big(1 + \frac{x}{1}\Big)\Big(1 - \frac{x}{2}\Big)\Big(1 + \frac{x}{2}\Big)... =$$

[28] Where, amongst other things, already a wide use of exponential factors was made for convergence reasons related to infinite products.

[29] It is noteworthy to highlight some historical aspects of Charles Briot (1817-1882) mathematical works which started in the mathematical physics context in studying the mathematical properties of light propagation in a crystallin medium, like the Ether (as it was supposed to be at that time, until the advent of Einstein's relativity), undertaking those symmetry conditions (chirality) soon discovered by Louis Pasteur about certain chemical crystalline substances. In this regard, C.A. Briot published, with J-C. Bouquet, a series of three research memoirs on the theory of complex functions first published in the Journal de l'École Polytechniquein then collected into a unique monograph published by L.J-B. Bachelier in Paris in 1856 (see also (Bottazzini & Gray 2013) as well as the e-archive http://gallica.bnf.fr for a complete view of all the related bibliographical items), to which more enlarged and complete treatises will follow later either in pure and applied mathematics (see also (Briot & Bouquet 1859)) as well as in physics.

$$= \left(1-\frac{x^2}{1}\right)\left(1-\frac{x^2}{4}\right)\left(1-\frac{x^2}{9}\right)\left(1-\frac{x^2}{16}\right)...$$

which will lead later to a valid proof of this factorization. Then, even in the context of the history of entire function factorization theorems, W.F. Osgood points out that already Riemann, just in his famous 1859 paper, had considered an entire function, the $\xi(s)$, as a function of s^2 with genus 0, taking into account the above mentioned Euler product formula for the sine function but without giving any rigorous prove of this fact, thing that will be done later by J. Hadamard in 1893 as a by-product of his previous 1892 outcomes on entire function theory. In the next sections, when we will deepen the works of Hadamard and Pólya on the entire function theory related to Riemann zeta function, we also will try to clarify, as far as possible, these latter aspects of the 1859 Riemann paper which mainly constitute one of the central cores of the present work. Following (Cartier 1993, I.1.d),

«Concernant les zéros de la fonction ζ, on doit à Riemann deux résultats fondamentaux dans son mémoire de 1859. Tout d'abord, Riemann ajoute un facteur $s(s-1)$ dans la fonction $\zeta(s)$; cela ne détruit pas l'équation fonctionnelle, mais lui permet d'obtenir une fonction entière

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)s(s-1)$$

car les deux pôles sont compensés (Aujourd'hui, on préfère garder la fonction méromorphe). Grâce à l'équation fonctionnelle, on montre facilement que les zéros de la fonction ζ sont situés dans la bande critique $0 < \Re s < 1$. Il est de tradition, depuis Riemann, d'appeler ρ ces zéros. La fonction ξ est désormais une fonction entière; si l'on connaît l'ensemble de ses zéros, on doit pouvoir la reconstituer. Riemann affirme alors que $\xi(s)$ s'écrit sous forme du produit d'une constante par un produit infini qui parcourt tous les zéros de la fonction ξ, chaque facteur s'annulant pour le zéro $s = \rho$ correspondant de $\zeta(s)$. Bien entendu, ce produit infini diverge mais - et c'est un point important - il converge si on le rend symétrique, i.e. si l'on regroupe judicieusement les facteurs. L'équation fonctionnelle montre en effet qu'on peut associer tout nombre ρ le nombre $1-\rho$ qui en est le symétrique, géométriquement, par *rapport* $1/2$. *De ce fait, si l'on regroupe dans ce produit infini le facteur correspondant à* ρ *et le facteur correspondant à* $1-\rho$, *on obtient un produit infini absolument convergent (le prime signifie que l'on ne prend qu'une fois chaque paire* $\rho, 1-\rho$*)*

$$\xi(s) \stackrel{30}{=} c\prod_{\rho}\Big(1-\frac{s}{\rho}\Big)\Big(1-\frac{s}{1-\rho}\Big).$$

Le premier problème majeur, dans le mémoire de Riemann de 1859, était de démontrer cette formule; il l'énonce, mais les justifications qu'il en donne sont très insuffisantes. L'objectif de Riemann est d'utiliser cette formule du produit pour en déduire des estimations très précises sur la répartition des nombres premiers. Si l'on note, suivant la tradition, $\pi(x)$ *le nombre (Anzahl) de nombres (Zahlen) premiers* $p < x$, *Legendre (1788) et Gauss (en 1792, mais jamais publié) avaient conjecturé qu'on avait* $\pi(x) \sim (x/\ln x)$. *Riemann donne des formules encore plus précises au moyen de sommations portant sur les zéros et les* $\pi(x)$. *En fait, il a fallu près de quarante ans pour que Hadamard (1896) et, indépendamment, de la Vallée-Poussin (1896) démontrent rigoureusement cette formule de développement en produit infini au moyen d'une théorie générale de la factorisation des fonctions entières - par des arguments qui étaient essentiellement connus d'Euler et de Riemann, en tout cas certainement de Riemann - et justifient ainsi rigoureusement la loi de répartition des nombres premiers. Hadamard donne la forme* $\lim_{x\to\infty}(x/\ln x)$, *et de la Vallée-Poussin donne la forme plus forte* $\pi(x) = \displaystyle\int_2^x \frac{dt}{\ln t} + O\big(xe^{-c\sqrt{\ln x}}\big)$ *pour une constante* $c > 0$*»*.

Following (Stopple 2003, Chapter 10, Section 10.1), it was Riemann to realize that a product formula for $\xi(s)$ would have had a great significance for the study of prime numbers. The first rigorous proof of this product formula was due to Hadamard but, as himself remember, it took almost three decades before he reached to a satisfactory proof of it. Likewise, also H.M. Edwards (1974, Chapter 1, Sections 1.8-1.19) affirms that the parts concerned with (2) are the most difficult portion of the 1859

Riemann's paper (see also (Bottazzini & Gray 2013, Chapter 5, Section 5.10)). Their goal is to prove essentially that $\xi(s)$ can be expressed as an infinite product, stating that

«[...] *any polynomial* $p(t)$ *can be expanded as a finite product* $p(t) = p(0)\prod_{\rho}(1 - t/\rho)$ *where* ρ *ranges over the roots of the equation* $p(t) = 0$ [*except that the product formula for* $p(t)$ *is slightly different if* $p(0) = 0$]; *hence the product formula (2) states that* $\xi(t)$ *is like a polynomial of infinite degree. Similarly, Euler thought of* $\sin x$ *as a "polynomial of infinite degree*[31]*" when he conjectured, and finally proved, the formula* $\sin x = \pi x \prod_{n\in\mathbb{N}}\left(1 - (x/n)^2\right)$. *On other hand,* [...] $\xi(t)$ *is like a polynomial of infinite degree, of which a finite number of its terms gives a very good approximation in any finite part of the plane.* [...] *Hadamard (in 1893) proved necessary and sufficient conditions for the validity of the product formula* $\xi(t) = \xi(0)\prod_{\rho}(1 - t/\rho)$ *but the steps of the argument by which Riemann went from the one to the other are obscure, to say the very least*».

The last sentence of this Edwards' quotation is historically quite interesting and would deserve further attention and investigation. Furthermore, H.M. Edwards states too that

«[...] *a recurrent theme in Riemann's work is the global characterization of analytic functions by their singularities. See, for example, the Inauguraldissertation, especially Article 20 of Riemann's Werke (pp. 37-39) or Part 3 of the introduction to the Riemann article "Theorie der Abel'schen Functionen", which is entitled "Determination of a function of a complex variable by boundary values and singularities". See also Riemann's introduction to Paper XI of the his collected works, where he writes about "* [...] *our method, which is based on the determination of functions by means of their singularities (Umtetigkeiten und Unendlich-*

[31] Following (Bottazzini & Gray 2013, Chapter 8, Section 8.5.1), amongst the functions that behave very like a polynomial, there is the Riemann ξ function. In this regard, see also what will be said in the next sections about Lee-Yang theorems and, in general, the theory concerning the location of the zeros of polynomials.

werden) [...]". Finally, see the textbook (Ahlfors 1979), namely the section 4.5 of Chapter 8, entitled "Riemann's Point of View"»,

according to which Riemann was therefore a strong proponent of the idea that an analytic function can be defined by its singularities and general properties, just as well as or perhaps better than through an explicit expression, in this regard showing, with Riemann, that the solutions of a hypergeometric differential equation can be characterized by properties of this type. In short, all this strongly suggests us the need for a deeper re-analysis of Riemann *œuvre* concerning these latter arguments, as well as a historical seek for the mathematical background which was at the origins of his celebrated 1859 number theory paper. From what has just been said, it turns out clear that a look at the history of entire function theory, within the general and wider complex function theory framework, is needed to clarify some of the historical aspects of this influential seminal paper which, as Riemann himself recognized, presented some obscure points. In this regard, also Gabriele Torelli (see (Torelli 1901, Chapter VIII, Sections 60-64)) claimed this last aspect, pointing out, in particular, the Riemann's ansätz according to which the entire function $\xi(t)$ is equal, via (3), to the Weierstrass' infinite product of primary factors without any exponential factor. As is well-known, this basic question will be brilliantly solved by J. Hadamard in his famous 1893 paper that, inter alia, will mark a crucial moment in the history of entire function theory (see (Maz'ya & Shaposhnikova 1998, Chapter 9, Section 9.2) and next sections).

4.3 An historical account of entire factorization theorems from Weierstrass onward. To begin, we wish to preliminarily follow the basic textbook on complex analysis of Giulio Vivanti (1859-1949), an Italian mathematician whose main research field was into complex analysis, becoming an expert of the entire function theory. He wrote some notable treatises on entire, modular and polyhedral analytic functions: a first edition of a prominent treatise on analytic functions appeared in 1901, under the title *Teoria delle funzioni analitiche,* published by Ulrico Hoepli in Milan, where the first elements of the theory of analytic functions, worked out in the late 19th-century quarter, are masterfully exposed into three main parts, giving a certain load to the Weierstrass' approach respect to the Cauchy's and Riemann's ones. The importance of this work immediately arose, so that a German edition was carried out, in collaboration with A. Gutzmer, and published in 1906 by B.G. Teubner in Leipzig, under the title *Theorie der eindeutigen analytischen funktionen. Umarbeitung unter mitwirkung des verfassers deutsch herausgegeben von A. Gutzmer* (see (Vivanti 1906)), which had to be considered as a kind of second enlarged and revised edition of the 1901 first Italian edition according to what Vivanti himself said in the preface to the 1928 second Italian edition, entitled *Elementi della teoria delle funzioni analitiche e delle trascendenti intere,* again published by Ulrico Hoepli in Milan, and wrote following the above German edition in which many new and further arguments and results were added, also as regards entire functions. Almost all the Vivanti's treatises are characterized by the presence of a detailed and complete bibliographical account of the related literature, this showing the great historical attention that he always put in drawing up his works. Therefore, he also was a valid historian of mathematics besides to be an able researcher (see (Janovitz & Mercanti 2008, Chapter 1) and references therein), so that his works are precious sources for historical studies, in our case as concerns entire functions. The above mentioned Vivanti's textbook on complex analysis has been one of the most influential Italian treatises on the subject. It has also had wide international fame thanks to its German edition.

Roughly speaking, the transcendental entire functions may be formally considered as a generalization, in the complex field, of polynomial functions (see (Montel 1932, Introduction) and (Levin 1980, Chapter I, Section 3)). Following (Vivanti 1928, Sections 134-135), (Markuševič 1988, Chapter VII) and (Pierpont 1914, Chapter VIII, Sections 127 and 140), the great analogy subsisting between these two last function classes suggested the search for an equal formal analogy between the corresponding chief properties. To be precise, the main properties of polynomials concerned either with the existence of zeros (Gauss' theorem) and the linear factor decomposition of a polynomial, so that it was quite obvious trying to see whether these could be, in a certain way, extended to entire functions. As regards the Gauss' theorem, it was immediately realized that it couldn't subsist because of the simple counterexample given by the fundamental elementary transcendental function e^x which does not have any zero in the whole of complex plane. On the other hand, just this last function will provide the basis for building up the most general entire function which is never zero, which has the general form $e^{G(x)}$, where $G(x)$ is an arbitrary entire function, and is said to be an exponential factor. Then, the next problem consisted in finding those entire functions having zeros and hence how it is possible to build up them from their zero set. In this regard, it is well-known that, if $P(z)$ is an arbitrary non-zero polynomial with zeros $z_1, ..., z_n \in \mathbb{C} \setminus 0$, having $z = 0$ as a zero with multiplicity λ (supposing $\lambda = 0$ if $P(0) = 0$), then we have the following well-known finite product factorization[32]

$$P(z) = Cz^{\lambda} \prod_{j=1}^{n} \left(1 - \frac{z}{z_j}\right) \tag{4}$$

where $C \in \mathbb{C} \setminus 0$ is a constant, so that a polynomial, except a constant factor, may be determined by its zeros. For transcendental entire functions, this last property is much more articulated respect to the polynomial case: indeed, whilst the indeterminacy for polynomials is given by a constant C, for transcendental entire functions it is larger and related to the presence of an exponential factor which is need to be added to warrant

[32] It is noteworthy the historical fact pointed out by Giuseppe Bagnera (1927, Chapter III, Section 12, Number 73), in agreement to what has been likewise said above, according to which already Cauchy himself had considered first forms of infinite product developments, after the Euler's work. Also Bagnera then, in this his work, quotes Betti's work on elliptic functions and related factorization theorems. Instead, it is quite strange that the Italian mathematician Giacomo Bellacchi (1838-1924) does not cite Betti, in his notable historical work on elliptic functions (Bellacchi 1894) in regards to entire function factorization theorems which are treated in the last chapter of this his work; this is also even more strange because Chapter XI of his book is centered around the 1851 Riemann dissertation on complex function theory, without quoting the already existed Italian translation just due to Betti. Furthermore, Bellacchi studied at the *Scuola Normale Superiore* of Pisa in the 1860s, for which it is impossible that he had not known Betti (see (Maroni 1924)). On the other hand, also (Loria 1950, Chapter XLIV, Section 741) refers that Weierstrass found inspiration for his factorization theorem, a result of uncommon importance according to Gino Loria, generalizing a previous Cauchy's formula: indeed, both Cauchy and Gauss are quoted at p. 120 of the 1879 French translation of the original 1876 Weierstrass paper. This, to further confirmation of what has been said above.

the convergence of infinite product development. A great part of history of the approach and resolution of this last problem is the history of entire function factorization. Nevertheless, we also wish to report what says Giacomo Bellacchi (1894, Chapter XI, Section 98) about this last problem. To be precise, he states that

«Se $a_1, a_2, a_3, ..., a_n, ...$ *simboleggino le radici semplici di una funzione olomorfa* $f(z)$*, ed il quoziente* $f(z) : \prod(z - a_n)$ *non si annulli per alcuna di esse, la sua derivata logaritmica* $\psi'(z) = f'(z)/f(z) - \sum(1/(z - a_n))$ *è olomorfa in tutto il piano; moltiplicando i due membri per dz ed integrando, Cauchy giunse alla formula* $f(z) = Ce^{\psi(z)} \prod(1 - z/a_n)$*, dove* C *è una costante»*

[*«If* $a_1, a_2, a_3, ..., a_n, ...$ *represent the simple roots of a holomorphic function* $f(z)$*, and the ratio* $f(z) : \prod(z - a_n)$ *is not zero for each root, then its logarithmic derivative* $\psi'(z) = f'(z)/f(z) - \sum(1/(z - a_n))$ *is holomorphic in the whole of plane; multiplying both sides by dz and integrating, Cauchy reached the formula* $f(z) = Ce^{\psi(z)} \prod(1 - z/a_n)$*, where* C *is a constant»*],

so that it seems, according to Bellacchi, that already Cauchy had descried the utility of exponentials as convergence-producing factors, in a series of his papers published in the Tome XVII of the *Comptes Rendus de l'Académie des Sciences (France)*; this supposition is also confirmed by Hancock (1910, Chapter I, Art. 14). Nevertheless, following (Vivanti 1928, Sections 135-141), the rise of the first explicit formulation of the entire function factorization theorem was given by Weierstrass in 1876 (see (Weierstrass 1876)) and was mainly motivated by the purpose to give a solution to the latter formal problem, concerning the convergence of the infinite product development of a transcendental entire function $f(z)$ having an infinite number of zeros, namely $z = 0$, with multiplicity λ, and $z_1, ..., z_n, ...$ such that $0 < |z_j| \leq |z_{j+1}|, z_j \neq z_{j+1} \quad j = 1, 2, ...$, trying to extend the case related to a finite number of zeros $z_1, ..., z_n$, in which such a factorization is given by

$$f(z)=e^{g(z)}z^{\lambda}\prod_{j=1}^{n}\Big(1-\frac{z}{z_j}\Big), \tag{5}$$

to the case of infinite zeros, reasoning, by analogy, as follows. The set of infinite zeros z_j is a countable set having only one accumulation point, that at infinite. Therefore, for every infinite increasing natural number sequence $\{\rho_i\}_{i\in\mathbb{N}}$, it will be always possible to arrange the zeros z_j according to their modulus in such a manner to have the following non-decreasing sequence $|z_1|\leq|z_2|\leq ...$ with $\lim_{n\to\infty}|z_n|=\infty$. In such a case, if one wants, by analogy, to extend (5) as follows

$$f(z)=e^{g(z)}z^{\lambda}\prod_{j=1}^{\infty}\Big(1-\frac{z}{z_j}\Big), \tag{6}$$

then it will not be possible to fully avoid divergence's problems inherent to the related infinite product. The first hint towards a possible overcoming of these difficulties, was suggested to Weierstrass (see (Weierstrass 1856a)) by looking at the form of the inverse of the Euler integral of the second kind[33] - that is to say, the *gamma function* - and given by

$$\frac{1}{\Gamma(z)}=z\prod_{j=1}^{\infty}\Big(1+\frac{z}{j}\Big)\Big(\frac{j}{1+j}\Big)^{z}=z\prod_{j=1}^{\infty}\Big(1+\frac{z}{j}\Big)e^{-z\log\frac{j+1}{j}}, \tag{7}$$

from which he descried the possible utility of the exponential factors there involved to, as the saying goes, force the convergence of the infinite product of the last equality; these his ideas concretized only in 1876 with the explicit formulation of his celebrated theorem on the entire function factorization.

As we have said above, Weierstrass (1856a) attributes, however, the infinite product expansion (7) to Gauss, but some next historical studies

[33]Following (Amerio 1982-2000, Volume 3, Part I), the first historical prototype of the Euler integral of the first kind was provided by the so-called *Beta function,* whilst the first historical prototype of the Euler integral of the second type was provided by Gamma function.

attribute to Euler this formula, that he gave in the famous 1748 *Introductio in Analysin Infinitorum.* Indeed, as has been said above, from the 1879 French translation of the original 1876 Weierstrass paper, it turns out that both Cauchy and Gauss are quoted (at page 120), before to introduce the primary factors. Nevertheless, P. Ullrich (1989, Section 3.5) says that the real motivation to these Weierstrass' results about entire function factorization were mainly due to attempts to characterize the factorization of quotients of meromorphic functions on the basis of their zero sets, rather than to solve the above problem related to the factorization of a polynomial in dependence on its zeros. Furthermore, Ullrich (1989, Section 3.5) observes too that other mathematicians dealt with questions concerning entire function factorization methods, amongst whom are just Enrico Betti and Bernard Riemann, the latter, in his important 1861 *sommersemestern* lectures on analytic functions, arguing, as has already been said, upon the construction of particular complex functions with simple zeros, even if, all things considered, he didn't give, according to Ullrich (1989, Section 3.5), nothing more what Euler done about gamma function through 1729 to his celebrated 1748 treatise on infinitesimal analysis[34]. Instead, as we have seen above, D. Laugwitz (1999, Chapter 1, Section 1.1.6) states that Riemann's work on meromorphic functions was ahead of the Weierstrass' one, having been carried out with originality and simplicity. To this point, for our purposes, it would be of a certain importance to deepen the possible relationships between Riemann and Weierstrass, besides to what has been said above: for instance, in this regard, Laugwitz (1999, Chapter 1, Section 1.1.5) says that Riemann was aware of the Weierstrass' works until 1856-57, in connection with the composition of his paper on Abelian functions, in agreement with what has been said in the previous sections. Again according to (Laugwitz 1999, Chapter 1, Section 1.1.6), one of the key themes of Riemann's work on complex function theory was the determination of a function from its singularities which, in turn, implies the approach of another problem, the one concerning the deter-

[34]Following (Lunts 1950), (Markušecič 1988, Chapter VII) and references therein, also Lobačevskij, since 1830s, made some notable studies on gamma function which preempted times.

mination of a function from its zeros. In this regard, Riemann limited himself to consider the question to determine a function with infinitely many zeros whose only point of accumulation is ∞. What he is after is the product representation later named after Weierstrass. Riemann uses a special case to explain the general procedure. He does it in such a way that by following his direction one could immediately give a proof of the Weierstrass product theorem. Therefore, it would be hoped a deeper study of these 1857-61 Riemann's lectures on complex function theory to historically clarify this last question which is inside the wider historical framework concerning the work of Riemann in complex function theory.

Furthermore, to this point, there seems not irrelevant to further highlight, although in a very sketchily manner, some of the main aspects of the history of gamma function. To this end, we follow the as many notable work of Reinhold Remmert (see (Remmert 1998)) which, besides to mainly be an important textbook on some advanced complex analysis topics, it is also a very valuable historical source on the subject, which seems to remember the style of the above mentioned Vivanti's textbook whose German edition, on the other hand, has always been a constant reference point in drawing up the Remmert's textbook itself[35]. Following (Davis 1959), (Remmert 1998, Chapter 2), (Edwards 1974, Chapter 1, Section 1.3), (Bourbaki 1963, Chapter XVIII), (Bourguet 1881), (Montgomery & Vaughan 2006, Appendix C, Section C.1), (Scriba 1981) and (Pradisi 2012, Chapter 3), amongst the many merging mathematical streams from which it arose, the early origins of gamma function should be above all searched into the attempts to extend the function $n!$ to real arguments starting from previous attempts made by John Wallis in his 1655 Arithmetica Infinitorum, to interpolate the values of a discrete sequence, say $\{u_n\}_{n\in\mathbb{N}}$, with an integral depending on a real parameter, say λ, such that it is equal to u_n for $\lambda = n$. In 1730, J. Stirling investigated the formula $\log(n!) = \log\Gamma(n+1)$ in his celebrated *Methodus*

[35]The usefulness of historical notes are recognized by Remmert making him what was said by Weierstrass, according to whom *«one can render young students no greater service than by suitably directing them to familiarize themselves with the advances of science through study of the sources»* (from a letter of Weierstrass to Casorati of the 21st of December 1868). Anyway, see (Davis 1959) for a complete history of gamma function.

differentialis, sive tractatus de summatione et interpolatione serierum infinitarum. In 1727, Euler was called by Daniel Bernoulli to join San Petersburg Academy of Science, becoming close co-workers. In the same period, also Christian Goldbach was professor in the same Academy, and it seems have been just him to suggest to Euler, on the wake of Wallis' work, to extend factorial function to non-integer values. So, from then onwards, Euler was the first to approach this last Wallis' problem since 1729, giving a first expression of this function, in a celebrated 13th of October 1729 letter to Goldbach (see also (Whittaker & Watson 1927, Chapter XII, Section 12.1) and (Sansone 1972, Chapter IV, Section 5)), providing a first infinite product expression of this new function, but only for real values. Gauss, who did not know Euler work[36], also taking into account Newton's work on interpolation (see (Schering 1881, Sections XI and XII)), around the early 1810s, considered as well complex values during his studies on the hypergeometric function (of which the Γ function is a particular case of it), denoting such a new function with Π, while it was Legendre, in 1814 (but (Jensen 1891) reports the date of 1809), to introduce a unified notation both for Euler and Gauss functions, denoting these latter with $\Gamma(z)$ and speaking, since then, of *gamma function.* Other studies on gamma function properties were pursued, amongst others, by Cauchy, Hermite, A.T. Vandermonde, A. Binet and C. Krampt around the late 1700s. Afterwards, in 1854, Weierstrass began to consider an Euler infinite product expansion of the function $1/\Gamma(z)$, that he denoted with $Fc(z)$ and is given by $1/\Gamma(z) = ze^{\gamma z}\prod_{j=1}^{\infty}(1+z/j)e^{-z/j}$, where γ is the well-known *Euler-Mascheroni constant*[37], from which he maybe recognized, for the first time, the importance of the use of exponential factors as infinite product convergence-producing elements. Following (Remmert 1998, Chapter 2) and references therein, Weierstrass considered the Euler product for $Fc(z)$ the starting point for the theory,

[36]This explaining why Weierstrass, as late as 1876, gave Gauss credit for the discovery of the Gamma function.

[37]Following (Sansone 1972, Chapter IV, Section 5), the γ constant was discovered by Euler in 1769, then computed by L. Mascheroni in 1790, hence by Gauss in 1813 and by J.C. Adams in 1878. See (Pepe 2012) for a contextual brief history of the Euler-Mascheroni constant, as well as (Sansone 1972, Chapter IV, Section 5).

being it, in contrast to $\Gamma(z)$, holomorphic everywhere in $\mathbb{C}$. Weierstrass said that to be pleased

«to propose the name "factorielle of u" and the notation $Fc(u)$ for it, since the application of this function in the theory of factorials is surely preferable to the use of the Γ-function because it suffers no break in continuity for any value of u and, overall [...], essentially has the character of a rational entire function». Moreover, Weierstrass almost apologized for his interest in the function $Fc(u)$, writing «that the theory of analytic factorials, in my opinion, does not by means have the importance that many mathematicians used attributed to it».

Weierstrass' factorielle Fc is now usually written in the form $ze^{\gamma z}\prod_{\nu\in\mathbb{N}}(1+z/\nu)e^{-z/}$, where γ is the Euler's constant. Furthermore, Weierstrass observed, in 1854, that the Γ-function is the only solution of the differential equation $F(z+1)=zF(z)$ with the normalization condition $F(1)=1$ that also satisfies the limit condition $\lim_{n\to\infty}(F(z+n)/n^zF(z))=1$.

However, according to (Whittaker & Watson 1927, Part II, Chapter XII, Section 12.1), the formula (7) had already been obtained either by F.W. Newman (see (Newman 1848)), starting from Euler's expression of gamma function given by (7). Moreover, following (Davis 1959), the factorization formula given by Newman for the reciprocal to gamma function was the starting point of the early Weierstrass' interest in studying gamma function, which will lead him then to approach the problem how functions, other than polynomials, may be factorized, starting from the few examples then available, among which sine function factorization and Newman formula, which however required a general theory of infinite products. But, following (Jensen 1891) and references therein, it turns out already Euler was reached the following expression for the Gamma function

$$\Gamma(s)=\frac{1}{s}\prod_{\nu=1}^{\infty}\frac{\left(1+\dfrac{1}{\nu}\right)^s}{\left(1+\dfrac{s}{\nu}\right)}, \tag{8}$$

who unfortunately replaced this excellent definition by definite integrals by which, in consequences, several of the formal properties of the Gamma function escaped his attention[38]. This 1729 Euler's formula is equivalent to the following one

$$\Gamma(s) = \lim_{n\to\infty} \frac{(n+1)!(n+a)^s}{s(s+1)...(s+n-1)} \tag{9}$$

which was provided by Gauss in 1813 who undoubtedly was not familiar with Euler's expression (8). Later on, the expression

$$\Gamma(s) = e^{Cs} s \prod_{\nu=1}^{\infty} \left(1+\frac{1}{\nu}\right)^s \tag{10}$$

[38] Following (Remmert 1998, Chapter 2, Section 3), Euler observed, as early as 1729, in his work on the Gamma function, that the sequence of factorials $1, 2, 6, 24, ...$ is given by the integral

$$n! = \int_0^1 (-\ln\tau)^n d\tau, \qquad n \in \mathbb{N}.$$

In general

$$\Gamma(z+1) = \int_0^1 (-\ln\tau)^z d\tau$$

whenever $\Re z > 1$. With z instead of $z+1$ and $t = -\ln\tau$, this yields the well-known equation

$$\Gamma(z) = \int_0^{\infty} t^{s-1} e^{-t} dt$$

for $z \in \mathbb{T} \doteq \{z; z \in \mathbb{C}, \Re z > 0\}$. This last improper integral was called *Euler's integral of the second kind* by Legendre in 1811, and it was a cornerstone of the rising theory of Gamma function, becoming the matter-subject of other scholars like R. Dedekind and, above all, H. Hankel, a Riemann's student who will give important contributions to the theory of Gamma function. In 1766, Euler systematically studied the integral

$$\int_0^1 x^{p-1}(1-x^n)^{\frac{q}{n}-1} dx = \int_0^1 \frac{x^{p-1}}{\sqrt[n]{(1-x^n)^{n-q}}} dx,$$

from which he derived the following improper integral

$$B(w,z) = \int_0^1 t^{w-1}(1-t)^{z-1} dt,$$

which is convergent in $\mathbb{T} \times \mathbb{T}$ and, after Legendre (still in 1811), called *Euler's integral of the first kind*. Later, in 1839, this integral will be called *beta function* by J.Ph. Binet who introduced too the notation $B(w,z)$ (see (Sansone 1972, Chapter IV, Section 5)). Euler knew as well, by 1771 at the latest, that the beta function could be reduced to the gamma function.

was due either to O.X. Schlömilch in 1843 (see (Schlömilch 1844; 1848) as well as to F.W. Newmann (see above), besides to have been rediscovered by Weierstrass in his famous 1856 memoir on analytical factorials (see (Weierstrass 1856a); see also (Burkhardt et al. 1899-1927, Band II, Erster Teil, Erste Hälfte, A.3, Nr. 12e) and (Remmert 1998, Chapter 2)).

Following (Vivanti 1928, Section 135-141), (Remmert 1998, Chapter 3) and, above all, (Bottazzini & Gray 2013, Section 6.7), Weierstrass extended the product (5) in such a manner to try to avoid divergence problems with the *ad hoc* introduction, into the product expansion, of certain forcing convergence factors. This attempt was successfully attended, since 1874, as a solution to a particular question - the one which may be roughly summarized as the attempt to build up an entire transcendental function with prescribed zeros - which arose within the general Weierstrass' intent to solve the wider problem to find a representation for a single-valued function as a quotient of two convergent power series. To be precise, he reached, amongst other things, the following main result

«Given a countable set of non-zero complex points $z_1, z_2, ...,$ such that $0 < |z_1| \leq |z_2| \leq ...$ with $\lim_{n\to\infty} |z_n| = \infty$, then it is possible to find, in infinite manners, a non-decreasing sequence of natural numbers $p_1, p_2, ...$ such that the series $\sum_{j=1}^{\infty} |z/z_j|^{p_j+1}$ be convergent for every finite value of z, in such a manner that the most general entire function which is zero, with their own multiplicity, in the points $z_1, z_2, ...,$ and has a zero of order λ in the origin, is given by[39]

$$f(z) = e^{g(z)} z^{\lambda} \prod_{j=1}^{\infty} \Big(1 - \frac{z}{z_j}\Big) E_j(z) \tag{11}$$

where $E_j(z) = (1-z)(\sum_{i=1}^{j} z^i/i)$ for $j \geq 1$ and $E_0(z) = 1 - z$, $g(z)$ being an arbitrary entire function, and the infinite product is absolutely convergent for each finite value of $|z|$».

[39] Historically, in relation to (8), the function $f(z)$ was usually denoted, d'après Weierstrass, by $G(z)$, whilst $z^{\lambda} \prod_{j=1}^{\infty}(1 - z/z_j)E_j(z)$ was named *canonical* (or *primitive*) *function* - see (Sansone 1972, Chapter IV, Section 3), where there are too many interesting historical notes.

The factors $E_j(z)$ will be later called *Weierstrass' factors,* whilst the numbers p_j will be called *convergence exponents*; finally, $e^{g(z)}$ is also called *Weierstrass' exponential factor.* The sequence $E_j(z)_{j\in\mathbb{N}_0}$ plays a very fundamental role in the Weierstrass' theorem: from the equation

$$1 - z = \exp(\log(1 - z)) = \exp(-\sum_{i\geq 1} z^i/i), \tag{12}$$

Weierstrass obtained the formula $E_j(z) = \exp(-\sum_{i>j} z^j/j)$ in proving convergence properties which, on the other hand, would have been easier obtained by means of the following estimates

$$|E_j(z) - 1| \leq |z|^{j+1}, \forall j \in \mathbb{N}_0, \forall z \in \mathbb{C}, |z| \leq 1 \tag{13}$$

that have been proved only later. Amongst the first ones to have made this, seems there having been L. Fejér (see (Hille 1959, Section 8.7)), but the argument appears as early as 1903 in a paper of Luciano Orlando[40] (1903) which starts from Weierstrass' theorem as treated by Borel's monograph on entire functions. As has already been said above, Weierstrass was led to develop his theory by the chief objective to establish the general expression for all analytic functions meromorphic in $\mathbb{C}$ except in finitely many points, reaching the scope after a series of previous futile attempts only in 1876, with notable results, spelt out in (Weierstrass 1876), concerning the class of transcendental entire functions. But what was new and sensational in the Weierstrass' construction was just the introduction and the application of the so-called *convergence-producing factors* (or *primary factors* or *Weierstrass' factors*) which strangely have no influence on the behavior and distribution of the zeros.

4.4 Towards the theory of entire functions, and other. In the necrology of Weierstrass, Poincaré (1899, Section 6) said that Weierstrass' major contribution to the development of function theory was

[40]Luciano Orlando (1887-1915) was an Italian mathematician prematurely died in the First World War - see the very brief obituary (Marcolongo 1918) as well as (Rouse Ball 1937, Appendix II, pp. 430-431). His supervisors were G. Bagnera and R. Marcolongo who led him to make researches in algebraic integral equation theory and mathematical physics.

just the discovery of primary factors. Also Hermite was, in a certain sense, astonished and intrigued from the introduction of this new Weierstrass' notion of prime factor, which he considered of capital importance in analysis and making later notable studies in this direction; he also suggested to Èmile Picard to do a French translation of the original 1876 Weierstrass' work, so opening a French research trend on this area. En passant, we also point out the fact that, from the notion of prime factor and from the convergence of the infinite product $\prod_{j\in\mathbb{N}} E_j(z/a_j)$, representing an entire transcendental function vanishing, in a prescribed way, in each a_j, Hilbert drew inspiration to formulate his valuable algebraic notion of prime ideal[41]. Following (Pincherle 1922, Chapter IX, Section 137), (Vivanti 1928, Section 136), (Burckel 1979, Chapter XI), (Remmert 1998, Chapters 3 and 6), (Ullrich 1989, Section 3.5) and (Bottazzini & Gray 2013, Sections 5.11.5 and 6.7), since the late 1850s, Enrico Betti[42] had already reached notable results, about convergence properties of infinite products of the type (6), very near to the Weierstrass' ones related to the resolution of a fundamental problem of entire function theory, the so-called *Weierstrass' problem*[43] (see (Pincherle 1922, Chapter IX, Section 137)). Betti exposed these outcomes in his celebrated 1859-60 Pisa lectures on advanced analysis entitled *La teorica delle funzioni ellittiche*

[41]Usually, the notion of prime ideal of the commutative algebra, with related operations, would want to be stemmed from the factorization of natural numbers.

[42]Following (Bottazzini 2003), the influence of Riemann's ideas on 19th-century Italian mathematical school had a great impulse thanks to the Betti's interest since 1850s. In 1858, as is well known, Betti, Brioschi and Casorati went in Göttingen to personally know Riemann and his ideas, translating many works of Riemann. Betti and Casorati were immediately aware of the innovative power of the new Riemann ideas in complex analysis, introducing in Italy, for the first time, such a theory with appreciated works and treatises.

[43]Following (Forsyth 1918, Chapter V, Section 50) and (Bottazzini & Gray 2013, Section 4.2.3.2), in relation to the infinite product expression of an entire transcendental function prior to 1876 Weierstrass' paper, attention should be also paid to a previous 1845 work of A. Cayley on doubly periodic functions. Furthermore, following (Tannery & Molk 1893, Section 85), into some previous 1847 works of G. Eisenstein on elliptic functions, some notable problems having to do with the construction of analytic functions with prescribed zeros as a quotient of entire functions with the involvement of certain transcendental entire functions of exponential type (similar to the Weierstrass problem as historically related to meromorphic functions), had already been considered. See also certain function's quotients stemmed from the developments of certain determinants given in (Gordan 1874). In any case, all these historical considerations confirm, once again, that the prolegomena of entire function factorization theorems should be searched in the general history of elliptic functions.

(see (Betti 1903-1913, Tomo I, XXII)), published in the Tomes III and IV of the *Annali di matematica pura ed applicata,* Series I, after having published, in the Tome II of these *Annali*, an Italian translation of the celebrated 1851 Riemann's inaugural dissertation on complex function theory, which can be considered as an introduction to his next lectures on elliptic functions. Indeed, in these latter, Betti, before all, places an Introduction on the general principles on complex functions, essentially based on these 1851 Riemann lectures. From the point 3. onward of this Introduction, Betti starts to deal with entire functions, their finite and infinite zeros (there called *roots*), as well as on possible quotients between them. In particular, taking into account what is said in (Briot & Bouquet 1859), he considered infinite products of the type $\prod_\rho(1-z/\rho)$, where ρ are the zeros of an entire function, with the introduction of a factor of the type e^w, where w is an arbitrary entire function, to make convergent this infinite product. Furthermore, Betti dealt with this type of infinite products starting to consider infinite product representations of the following particular function $es(z) = z\prod_{m=1}^\infty(m/(m+1))^z(1+z/m)$, which satisfies some functional equations and verifies the relation $\Gamma(z) = 1/es(z)$. Therefore, as Weierstrass too will do later, Betti started from the consideration of the infinite product expansion of the inverse of the gamma function for studying the factorization of entire functions. Therefore, Betti guessed the utility of the convergence factors having exponential form, looking at the infinite product expansion of Gamma function, similarly to what Weierstrass will do. Afterwards, Betti proved some theorems which can be considered particular cases of the next Weierstrass' results, concluding affirming that

«Da questi teoremi si deduce che le funzioni intere potranno decomporsi in un numero infinito di fattori di primo grado ed esponenziali, e qui comparisce una prima divisione delle funzioni intere. Quelle che hanno gl'indici delle radici in linea retta, e quelle che le hanno disposte comunque nel piano; le prime, che sono espresse da un prodotto semplicemente infinito, le chiameremo di prima classe, le seconde, che sono espresse da un prodotto doppiamente infinito, le diremo di seconda classe. Le funzioni di prima classe si dividono anch'esse in due specie,

la prima, che comprende quelle che hanno gl'indici delle radici disposti simmetricamente rispetto a un punto, e che possono esprimersi per un prodotto infinito di fattori di primo grado, le altre, che hanno gl'indici delle radici disposti comunque sopra la retta, le quali si decomporranno in fattori di primo grado ed esponenziali. Ogni funzione intera di prima classe della prima specie potrà decomporsi nel prodotto di più funzioni intere della stessa classe di seconda specie, e data una funzione della seconda specie se ne potrà sempre trovare un'altra che moltiplicata per la medesima dia per prodotto una funzione della prima specie. Le funzioni di seconda classe si dividono anch'esse in due specie; la prima comprenderà quelle che hanno gl'indici delle radici disposti egualmente nei quattro angoli di due assi ortogonali, in modo che facendo una rotazione intorno all'origine di un quarto di circolo, gl'indici di tutte le radici vengano a sovrapporsi, le quali funzioni possono esprimersi per un prodotto doppiamente infinito di fattori di primo grado; la seconda comprenderà quelle che hanno gl'indici disposti comunque, e si decompongono in un prodotto doppiamente infinito di fattori di primo grado e di fattori esponenziali. Data una funzione della seconda specie se ne potrà sempre trovare un'altra che moltiplicata per quella dia una funzione della prima specie».

[*«From these theorems, we deduce that entire functions might be decomposed into an infinite number of first degree factors and exponential factors, so that here there is a first classification of entire functions according to that their root's indexes lie along a line or are arbitrarily placed in the plane; the former are said to be of first class and are expressed by a simply infinite product, while the latter are said to be of second class and are expressed by a doubly infinite product. The functions of the first class are, in turn, classified into two kinds: the first one comprises those functions having the root's indexes symmetrically placed respect to a point and that can be expressed by an infinite product of first degree factors; the second one comprises those functions having root's indexes arbitrarily placed along a line and that can be expressed by an infinite product both of first degree factors and of exponential factors. Each entire function of first class and of first kind might be decomposed* *into the product of other entire functions of the same class and of the second kind; furthermore, given a function of the second kind, it is always possible to find another function that multiplied by the former, the product gives rise to another function of the first kind. Likewise, the functions of the second class are divided into two kinds: the first one comprises those functions having the root's indexes equally placed into the four angles of the two orthogonal cartesian axes in such a manner that all these are overlapped through a* $\pi/2$ *radian rotation around the origin, and are decomposable into a doubly infinite product of first degree factors; the second one includes those functions having the root's indexes arbitrarily placed and that are decomposable into a doubly infinite product of first degree factors and exponential factors. Furthermore, given a function of the second kind, it is always possible to find another function that multiplied by the former, the product gives rise to a function of first kind»*].

Then, Betti carries on treating entire functions in the first part of his lessons on elliptic functions, followed by a second part devoted to quotients of functions, mentioning either the paper (Weierstrass 1856a) and the paper (Weierstrass 1856b). Therefore, Betti's work on entire function factorization, made in the period 1860-63, was very forerunner of the Weierstrass' one: this is confirmed either by (Rouse Ball 1937, Appendix II, pp. 376-384)) and by (Federigo Enriques 1982, Book III, Chapter I, Section 6), in which it is pointed out that the fundamental Weierstrass' theorem on the factorization of entire transcendental functions from their zeros, had already been discovered by Betti, highlighting however as the Pisa's mathematician, with singular personal disinterestedness, wanted not claim it as due to him. Indeed, following Francesco Cecioni's comments about some works of Ulisse Dini (see (Dini 1953-59, Volume II)), it turns out that Betti's work could easily reach, only with very slight modifications, the same generality and abstraction of the Weierstrass' one, as Dini explicitly proved in (Dini 1881); furthermore, Dini proved too that Betti's work could be able to give a particular case, given in the years 1876-77, of the general Gösta Mittag-Leffler theorem - see (Mittag-Leffler 1884), (Vivanti 1928, Section 145), (Loria 1950, Chapter XLIV, Section 752) and (Bottazzini & Gray 2013, Section 6.7.6) - independently by what Weierstrass himself was doing in the same period, in regards to these latter arguments. Cecioni says that this Dini's work had already been worked out since 1880, whilst the Weierstrass' theorem was published in 1876 - see (Weierstrass 1876). Thus, much before, namely in 1860, Betti had proved, as we have already said, a particular but important case of this theorem, albeit he didn't go beyond, because the results achieved by him were enough to his pragmatic scopes concerning Abelian and elliptic functions[44], and, as also Pincherle (1922, Chapter IX, Section 135) has claimed, the Weierstrass' method was essentially the same of the Betti's one with slight modifications. In the years 1876-77, also G. Mittag-Leffler proved a particular case of a more general theorem that he will give later, to be precise in 1884, after a long series of previous works in which he gradually, through particular cases, reached the general form of this his theorem as nowadays we know it. In the meanwhile, Weierstrass reconsidered Mittag-Leffler's works, since the early 1880s, in relation to what himself have done on the same subject. Also F. Casorati (1880-82) had some interesting ideas similar to the Mittag-Leffler's ones, giving further contributions to the subject (see (Loria 1950, Chapter XLIV, Section 750)). Almost at the same time, amongst others, Ernst Schering (1881), Charles Hermite (1881), Émile Picard (1881), Felice Casorati (1882), Ulisse Dini (1881), Paolo Gazzaniga[45] (1882), Claude Guichard (1884) and Paul Painlèvé (1898a,b), achieved notable results

[44]In this regard, also Salvatore Pincherle (1899, Chapter IX, Section 175) reports that Betti solved the Weierstrass' problem in a quite general case.

[45]Some historical sources refer of Paolo Cazzaniga, whereas others refer of Paolo Gazzaniga, but, very likely, they are the same person, that is to say, Paolo Gazzaniga (1853-1930), an Italian mathematician graduated from Pavia University in 1878 under the supervision of Felice Casorati. In the years 1878-1883, he was interim assistant professor at Pavia, then he spent a period of study in Germany under the Weierstrass and Kronecker supervision. Afterwards, from 1888, he became professor at the high school Tito Livio in Padua, teaching too in the local University. He was also one of the most influential teachers of Tullio Levi-Civita during his high school studied. Gazzaniga's researches mainly concerned with applied algebra and number theory. Furthermore, Paolo Gazzaniga has to be distinguished from Tito Camillo Cazzaniga (1872-1900), a prematurely died Italian mathematician, graduated from Pavia University in 1896, whose researches concerned with matrix theory and analytic functions according to the research trend of Ernesto Pascal (1865-1940) during his teaching in Pavia. Both Tito Cazzaniga (see (Rouse Ball 1937, Appendix II, pp. 412-413)) and Paolo Gazzaniga are quoted in (Vivanti 1901) but not in (Vivanti 1928).

about the general problem to build up a complex function with prescribed singularities, although related to a generality degree less than that of the Mittag-Leffler results. Thus, the history of the Mittag-Leffler theorem makes too its awesome appearance within the general history of meromorphic functions, a part of which may be retraced in the same Mittag-Leffler 1884 paper in which, amongst other things, also the 1881 work of Ulisse Dini is quoted. However, both Schering (1881, Section XVI) and Casorati (1880-82, p. 269, footnote (***)), in discussing the above mentioned Mittag-Leffler results, quote Betti's work on Weierstrass' theorem; in particular, the former speaks of Betti's convergence factors and the latter states that

«Anche il sig. Dini, nella sua Nota sopra citata, dimostra questo teorema, riducendo lo studio del prodotto infinito a quello della serie dei logaritmi dei fattori; riduzione di cui s'era già valso felicemente, per il caso di distribuzione degli zeri a distanze non mai minori di una quantità fissa d, il sig. Betti nella Introduzione della sua Monografia delle funzioni ellittiche (Annali di Matematica, Tomo III, Roma, 1860), dove precede assai più oltre di Gauss nella via che mena al teorema del sig. Weierstrass».

[*«Also Mr. Dini, in his Note of above, proves this theorem, reducing the study of the infinite product to the study of the series of the logarithms of the factors; reduction, this, that had already been used by Mr. Betti in the Introduction to his monograph on elliptic functions (Annali di Matematica, Tome III, Rome, 1860) for the case of a distribution of zeros having reciprocal distances not less than a fixed quantity d; in doing so, he much foregoes Gauss in a fashion which leads to the theorem of Mr. Weierstrass».*]

Therefore, from the Mittag-Leffler's works onwards, together to all those works made by other mathematicians amongst whom are Dini, Schering, Casorati, Hermite, Picard, Cazzaniga, Guichard, Von Schaper[46],

[46]Hans Von Schaper, a doctoral student of Hilbert (see (Borel 1900, Chapitre II, p. 26)), whose

Painlèvé and Weierstrass himself, it starts the theory of entire transcendental functions whose early historical lines have been traced in the previous sections. In any case, with Mittag-Leffler, we have the most general theorems for the construction, by infinite products, of a meromorphic function with prescribed singularities (see (Bottazzini & Gray, Chapter 6, Section 6.7) for a deeper historical analysis of these representation theorems). On the other hand, following (Gonchar et al. 1997, Part I, Introduction) and (Vivanti 1901, Section 215), the above mentioned works by Weierstrass, Mittag-Leffler and Picard, dating back to the 1870s, marked the beginning of the systematic studies of the theory of entire and meromorphic functions. The Weierstrass and Mittag-Leffler theorems gave a general description of the structure of entire and meromorphic functions, while the representation of entire functions as an infinite product *à la* Weierstrass, served as basis for studying properties of entire and meromorphic functions. Following (Remmert 1998, Chapter 3, Section 1), Weierstrass developed his 1876 paper with the main objective to establish the general expression for all functions meromorphic in $\mathbb{C}$ except at finitely many points but, as said above, the really importance of Weierstrass' construction was the application of the convergence-producing factors which have no influence on the behavior of the zeros. The awareness that there exist entire functions with arbitrarily prescribed zeros revolutionized the thinking of function theorists. Suddenly, one could construct holomorphic functions that were not even hinted at in the classical framework. Nevertheless, this sort of freedom does not contradict the so-called solidarity of value behavior of holomorphic functions required by the identity theorem because, with the words of Remmert himself, the 'analytic cement' turns out to be pliable enough to globally bind locally prescribed data in an analytic way. Weierstrass left it to other the extension of his product theorem to regions in $\mathbb{C}$. So, as early as 1881, E. Picard considered, for the first time, Weierstrass'

doctoral dissertation thesis, entitled *Über die Theorie der Hadamardschen Funktionen und ihre Anwendung auf das Problem der Primzählen,* and defended at Göttingen in 1898, was just centered around the applications of 1893 Hadamard factorization theorem of entire function; in it, some further interesting properties on the order of an entire function, like the distinction between *real* and *apparent order,* were discussed as well.

products in regions different from $\mathbb{C}$, albeit he nothing said about convergence questions. In 1884, Mittag-Leffler proved existence theorems for more general regions but without quoting Picard's work, even if Edmund Landau (see (Landau 1918)) later will speak of the "well-known Picard/Mittag-Leffler product construction". Further generalization of Weierstrass' theorem was then given too by A. Pringsheim in 1915 (see (Burkhardt et al. 1899-1927, II.C.4, Nr. 26)). Following (Gol'dberg & Ostrovskiĭ 2008, Preface), the classical 1868 theorem of J. Sokhotski and F. Casorati, the above mentioned 1876 theorem of Weierstrass and the 1879 Picard theorem opened the theory of value distribution of meromorphic functions, while the works of J.L.W. Jensen and J. Petersen in the late 1890s, had great importance for the further developments of the theory of entire and meromorphic functions (see (Remmert 1998, Chapter 4, Section 3)) which started, in the same period, to gradually become a separate and autonomous mathematical discipline after the pioneering investigations mainly pursued by the French school of Laguerre, Hadamard, Poincaré, Lindelöff, Picard, Valiron and Borel, up until the Rolf Nevanlinna work of the early 1900s, which gave an almost definitive setting to the theory. All that will be in-depth studied in the next section, where we shall deal with the main lines of the history of entire and meromorphic functions whose theory basically starts just from the entire function factorization theorems. Following (Zhang 1993, Preface), in 1925, Nevanlinna established two main theorems that constituted the basis upon which build up the theory of value distribution of meromorphic functions, whilst, in 1929, by examining some examples, he recognized as well that there is an intrinsic relationship between the problem of exceptional values (deficient values are exceptional values under a certain kind of implication) and the asymptotic value theory. Moreover, Nevanlinna anticipated that the study of their relationship might help to clarify some of the profound problems of the theory of entire and meromorphic functions. From his product theorem, Weierstrass immediately deduced the theorem on quotient representation of meromorphic functions, attracting attention by this alone. From this work of the "celebrated geometer of Berlin", Poincaré worked out his 1883 famous theorem on the representability of every meromorphic function in $\mathbb{C}^2$ as the quotient $f(w, z)/g(w, z)$ of two entire functions in $\mathbb{C}^2$ and locally relatively prime everywhere, so giving rise to a new theory that, through the works of P. Cousin, T.H. Gronwall, H. Cartan, H. Behnke, K. Stein, K. Oka, J-P. Serre, H. Röhrl and H. Grauert, is still alive and rich today. With his product theorem, Weierstrass opened the door to a development that led to new insights in higher-dimensional function theory as well. In particular, the Weierstrass' product theorem was for the first time generalized to the case of several complex variables as early as 1894 by Pierre Cousin (1867-1933), a student of Poincaré, in (Cousin 1895) centered around his doctoral thesis whose main aim was that to generalize the above mentioned 1883 Poincaré theorem to higher dimensions and more general domains, so giving rise to the celebrated *I* and *II problem of Cousin,* solved by him for product domains of the type $X = B_1 \times ... \times B_n \subset \mathbb{C}^n$ (see (Maurin 1997, Part V, Chapter 6) and (Della Sala et al. 2006, Chapter 11, Section 6)). As Cousin himself says, the 1883 Poincaré theorem was the first successful attempt to extend Weierstrass results to analytic functions several complex variables: following (Dieudonné 1982, A VIII), that branch of mathematics known as "analytic geometry" is nothing but the modern form of the theory of analytic functions of several complex variable. Then, Cousin recalls too the attempts made by P. Appell and by S. Dautheville in the 1880s to extend, along the same line, the 1884 Mittag-Leffler work to the n complex variable case. En passant, then, we also note that the Weierstrass' entire function factorization theorem has had further remarkable applications in many other pure and applied mathematical contexts. In this place, we wish to point out another possible interesting historical connection. To be precise, following (Markuševič 1967, Volume II, Chapters 8 and 9), (Burckel 1979, Chapter VII) and (Remmert 1998, Chapter 4), a very similar problem to that considered by Weierstrass was the one considered in (Markuševič 1967, Volume II, Chapter 8, Theorem 8.5) where, roughly, a bounded analytic function with prescribed zeros is constructed by means of certain infinite products introduced by Wilhelm Blaschke (see (Blaschke 1915)), called *Blaschke products,* in relation to questions related to the well-known Giuseppe Vitali convergence theorem for sequences of holomorphic functions, and defined upon those complex numbers assigned as given zeros of that function that has to be determined. They form a special class of Weierstrass' products[47]. Edmund Landau (see (Landau 1918)) reviewed Blaschke's work in 1918 and simplified the proof by using a formula due to J.L.W. Jensen (see (Jensen 1898-99)). By means of the differentiation theorem of products of holomorphic functions, in 1929 R. Ritt was able to give a factorization of an holomorphic function at the origin, whose product is normally convergent into a disc about the origin (see (Remmert 1998, Chapter 1, Section 2)). In the proceedings collected in (Mashreghi & Fricain 2013), where remarkable applications of Blaschke's products in pure and applied mathematics questions (amongst which one concerning approximation of Riemann zeta function) are presented, we report what is said in the incipit of the Preface, according to which

«Infinite Blaschke products were introduced by Blaschke in 1915. However, finite Blaschke products, as a subclass of rational functions, has existed long before without being specifically addressed as finite Blaschke products. In 1929, R. Nevanlinna introduced the class of bounded analytic functions with almost everywhere unimodular boundary values. Then the term inner function was coined much later by A. Beurling in his seminal study of the invariant subspaces of the shift operator. The first extensive study of the properties of inner functions was made by W. Blaschke, W. Seidel and O. Frostman. The Riesz technique in extracting the zeros of a function in a Hardy space is considered as the first step of the full canonical factorization of such elements. The disposition of zeros of an inner function is intimately connected with the existence of radial limits of the inner function and its derivatives. For almost a century, Blaschke products have been studied and exploited by mathematicians. Their boundary behaviour, the asymptotic growth of various integral means of Blaschke products and their derivatives, their applications in several branches of mathematics in particular as solutions to extremal problems, their membership in different function spaces and

[47]See (Remmert 1998, Chapter 4), (Lang 1974, Chapter 15) and (Lang 1999, Chapter XIII) for technical details.

their dynamics are examples from a long list of active research domains in which they show their face».

Following (Borel 1900, Chapter I), the major difficult in applying the Weierstrass theorem is the determination of the exponential factors $\exp G(x)$, a hindrance that the next Hadamard work coped with success, whose pioneering work will turn out to be extremely useful also in physics: amongst all the possible applications to which such a work has given rise, we here only mention the use of entire function theory (following (Boas 1954)) made by Tullio Regge in achieving some notable properties of the *analytic S matrix* of potential scattering theory (see (Regge 1958)), which are closely connected with the distribution of the zeros of entire functions. In particular, Regge cleverly uses infinite product expansions of entire functions, amongst which the Hadamard expansion, in finding analytic properties of the analytic *Jost functions* as particular asymptotic solutions to non-relativistic Schrödinger equation (*S waves*). Finally, following (Maz'ya & Shaposhnikova 1998, Chapter 1, Section 1.10), we also notice that remarkable applications of some results of entire function theory, amongst which some results due to Hadamard, were also considered by Poincaré in his celebrated three volume work *Les Méthodes Nouvelles de la Mécanique Céleste* (see (Poincaré 1892-1899): to be precise, in (Poincaré 1892-1899, Tome II, Chapter XVII, Section 187), the author considers some entire function factorization theorems in solving certain linear differential equations also making reference to the well-known 1893 Hadamard results.

We wish to report some very interesting historical remarks made by Hermann Weyl in one of his last works, the monograph on meromorphic functions wrote in 1943 and reprinted in 1965 with the collaboration of his son, F. Joachim Weyl (see (Weyl & Weyl 1965)). Weyl says that the main motif underlying the drawing up of this his monograph was the work of Lars V. Ahlfors on meromorphic curves on complex plane, dating back to the late 1930s, and that Weyl wanted to reformulate extending it to a general Riemann surface. In (Weyl & Weyl 1965, Introduction), Weyl states an analogical parallel, that is to say, that meromorphic functions stand for entire functions as rational functions stand for polynomials, pointing out that degree is the most important characteristic of a polynomial, hence considering the usual decomposition into linear factors of a complex polynomial in dependence on its roots. A complex polynomial with n roots, say $a_1, ..., a_n$, counted with their multiplicity, may be written in the form

$$(\star_1) \qquad f(z) = kz^h \prod_{i=1}^{n-h} \Big(1 - \frac{z_i}{a_i}\Big)$$

if $h(\leq n)$ out of the n roots are equal to zero, and k is a non-zero constant. Then, Weyl considers the type of growth of a polynomial of degree n, given by an inequality of the form $|f(z)| \leq C|z|^n$, or, more precisely, by an asymptotic equation of the type $|f(z)|/|z|^n \to C$ as $|z| \to \infty$, where $C \neq 0$ is a constant. This means that $f(z)$ takes on the value ∞ with multiplicity n at $z = \infty$. Then, Weyl asks whether it is possible to make statements about entire functions on the basis of what is known about polynomials. Weyl points out that the perfect analogical extension is not possible simply because there exist entire functions which have no zeros, like e^z to mention the simplest one. Weyl hence goes on considering the problem of building up an entire function knowing its zeros ordered according to their nondecreasing modulus which are in a finite number in every finite region of complex plane, hence observing that this problem (named *Weierstrass' problem*) was first solved by Weierstrass in a paper of 1876 which is the starting point of many other investigations on entire and meromorphic functions. Then, Weyl observes that the next problem of determining the growth of an entire function through its canonical decomposition into primary factors according to Weierstrass, is not solvable by the simple knowledge of such a decomposition because it is related an arbitrary but finite region of complex plane. So, it was Poincaré, in 1883, to approach and solve, for the first time, such a problem in some special cases connected with the convergence of certain series related to the zeros of the given entire function. This last problem was then approached and solved, in more general cases, by E.N. Laguerre and E. Borel, introducing the notions of *genus* (or *genre*) and *order* of an entire function, hence Hadamard, in 1893, gave a converse to Poincaré theorem for entire functions of finite genus, whose form was later improved and extended by R. Nevanlinna. Just in regard to the novelties due to Hadamard, Weyl points out that the driving force for these Hadamard's investigations was the wish to obtain sufficient information about the zeros of the Riemann zeta function for establishing the asymptotic law for the distribution of prime numbers. This law states that the number $\pi(n)$ of primes less than n becomes infinite with $n \to \infty$ exactly as strongly as $n \ln n$, that is to say

$$(\star_2) \qquad \frac{\pi(n) \ln n}{n} \to 1 \qquad \text{as} \quad n \to \infty,$$

Riemann having shown how this prime number problem crucially depends on the zeros of his zeta function. In 1896, either Hadamard and de la Vallée-Poussin, independently of each other, were able to draw the conclusion $(\star_2)$ from 1893 Hadamard's results concerning entire functions. Afterwards, besides the problem to determine zeros of an entire function $f(z)$, Weyl considers too the problem to determine the distribution of the points $z \in \mathbb{C}$ satisfying the equation $f(z) = c$ for any preassigned complex value c, which Weyl calls *c-places,* the former problem being therefore the one determining the 0-places of $f(z)$. Weyl quotes E. Picard results in this direction, dating back 1880, hence the next results of G. Valiron, A. Wiman and Nevanlinna brothers of 1920s, until up L. Ahlfors results of 1930s. Once meromorphic function theory was established, by various research papers from 1920s onward and mainly due, among others, to E. Borel, A. Bloch, P. Montel, R. Nevanlinna, F. Nevanlinna, H. Cartan, T. Shimizu, O. Frostman, H. Weyl, J.F. Weyl, E. Ullrich, G. Hällström and J. Dufresnoy, started the so-called *theory of meromorphic curves,* which originated by the rough idea to consider homogeneous coordinates $x_0, x_1, ..., x_k$ of a k-dimensional projective space, as meromorphic functions $x_1/x_0, x_2/x_0, ..., x_k/x_0$ depending on a certain complex parameter z ranging over the whole complex plane except $z = \infty$ (see (Weyl & Weyl 1965, Chapter II, Section 2; Chapter III, Sections 2-4) and references therein), so that a meromorphic function $f = x_1/x_2$ may be considered as a meromorphic curve in a two-dimensional projective space.

5. Outlines of history of entire function theory

Following (Borel 1897), the Weierstrass' work on the decomposition of entire functions into primary factors, has greatly contributed to the study of the distribution of zeros of the entire functions. The notion of *genus* of an entire function, introduced by Laguerre, will turn out to be of fundamental importance to this end, as well as the analogous notion of *order* of an entire function, which nevertheless will turn out to be much more useful and precise than the former, above all thanks to the contributions of Poincaré, Hadamard and Picard (see (Borel 1900)). Above all, Hadamard's work will provide new avenues to the theory of entire functions and the related distribution laws of their zeros. Following (Bergweiler & Eremenko 2006), the theory of entire functions begins as a field of research in the works of Laguerre (see (Laguerre 1898-1905)), soon after the Weierstrass product representation became available. Laguerre then introduced the first important classification of entire functions, according to their genera. Following (Gil' 2010, Preface), one of the most important problems in the theory of entire functions is the problem of the distribution of the zeros of entire functions. Many other problems in fields close to the complex function theory, lead to this problem. The connection between the growth of an entire function and the distribution of its zeros was investigated in the classical works of Borel, Hadamard, Jensen, Lindelöf, Nevanlinna and others. On the other hand, following (Gonchar et al. 1997, Part I, Chapter 1), the infinite product representation theory of entire functions marked the beginning of the systematic study of their properties and structure, with the first works by Weierstrass and Hadamard. Following (Markuševič 1966, Preface), entire functions are the simplest and most commonly encountered functions: in high schools, we encounter entire functions (like polynomials, the exponential function, the sine and cosine, and so forth), meromorphic functions, that is, the ratios of two entire functions (like the rationale functions, the tangent and cotangent, and so on), and, finally, the inverse functions of entire and meromorphic functions (like fractional powers, logarithms, the inverse trigonometric functions, etc.). Following (Levin 1980, Chapter I), an entire function is a function of a complex variable holomorphic in the whole of the complex plane and consequently represented by an everywhere convergent power series of the type $f(z) = \sum_{i=0}^{\infty} a_i z^i$, these functions forming a natural generalization of the polynomials, and are therefore close to polynomials in their properties. The theorem of Weierstrass on the expansion of entire functions into infinite products provided the basic apparatus for the investigation of the properties of entire functions and it was the starting point for their classification. This theorem plays a fundamental role in the theory of entire functions (see (Saks & Zygmund 1952, Chapter VII, Section 2)), being it, roughly, the analogue of the theorem on the decomposition of polynomials into linear factors. Following (Tricomi 1968, Chapter IV, Section 8), this Weierstrass theorem plays a central role in the whole of the theory of entire functions whose even most recent developments are, more or less directly, reconnected to it. At approximately the same time as this celebrated work of Weierstrass, Laguerre studied the connection between entire functions and polynomials, and introduced the important concept of genus of an entire function. Since then, the theory of entire and meromorphic functions underwent to a notable development, becoming one of the many wide chapters of complex analysis, assuming an autonomous status. Amongst the many contributions to the theory, which will be briefly recalled below, the classical investigations of Borel, Hadamard and Lindelöff dealt with the connection between the growth of an entire function and the distribution of its zeros. The rate of growth of a polynomial as the independent variable goes to infinity is determined, of course, by its degree. Thus, the more roots a polynomial has, the greater its growth is. This connection between the set of zeros of the function and its growth can be generalized to arbitrary entire functions, the content of most of the classical theorems of the theory of entire functions consisting just in establishing relations between the distribution of the roots of an entire function and its asymptotic behavior as $z \to \infty$, to measure the growth of an entire function and the density of its zeros, a special growth scale having been introduced. Following (Evgrafov 1961, Chapter II, Section 1), the basic task of the theory of entire functions (at least, from the point of view of its applications to other domains of analysis) is to establish connections between the different characterizing elements of an entire function as, for example, between the coefficients, the behavior at infinity, and the zeros. It would hardly be mistaken to say that the most important task of all is to establish such connections for entire functions that are in some sense regular, that is, have regularly decreasing coefficients, or regularly distributed zeros, or a simple integral representation, or else a simple functional equation. However, the study of entire functions under such strong hypotheses is a very complicated task, and it is necessary to know those simpler and more general laws that are less exact but which hold under weaker hypothesis. Amongst all the elements of an entire function, it is customary to single out three as the most important ones: these are the Taylor coefficients, the zeros of the function and its behavior at infinity. The simplest characteristics of these elements are the rate of decrease of the coefficients, the number of zeros in the sphere $|z| < r$, and the rate of growth of the logarithm of the maximum modulus of the function in the ball $|z| \leq r$. It is customary to compare the logarithm of the maximum modulus of the function with certain very smooth functions, called *orders of growth.* In what follows, we shall try to treat these latter facts and notions from a deeper historical viewpoint.

Following (Burkhardt et al. 1899-1927, Dritter Teil, erste Hälfte, C.4, Nr. 26-36; Zweiter Teil, B.1.III) once again, the starting point of entire transcendental functions is just the 1876 Weierstrass paper (see (Weierstrass 1876)) in which, from well-known special cases treated by Cauchy[48] and Gauss regarding Γ function and trigonometric functions, an infinite product expansion of non-constant entire rational and transcendental functions was given. Therefore, the factorization theorems of entire functions have opened the way to a new chapter of complex function theory[49], that regarding the entire functions. As we have seen

[48]See (Cauchy 1829, pp. 174-213), namely the chapter entitled *Usage du calcul des résidus par l'évaluation ou la transformation des produits composés d'un nombre fini ou infini de facteurs,* as well as (Cauchy 1827, pp. 277-297), in which a method of decomposing a meromorphic function into simple fractions had already been given before Mittag-Leffler's work - see also (Saks & Zygmund 1952, Chapter VII, Section 4) and (Sansone 1972, Chapter IV, Section 8).

[49]Following (Della Sala et al. 2006), the term *complex analysis* is quite recent because it has

above, the historical pathways of Riemann zeta function theory and of entire function theory intertwined among them, for the first time, just with the introduction of the Riemann ξ function, and, thenceforth, there were other similar intersection points along the history of mathematics and its applications that we wish to consider in what follows. Therefore, it is needful to briefly outline the main historical points concerning entire function theory from Weierstrass onward. Almost all treatises on entire function theory start with a first chapter devoted to Weierstrass' factorization theorem: in this regard, for instance, the first monograph on the subject, that is to say (Borel 1900), just begins with a first chapter recalling the main points concerning Weierstrass' work on factorization product of entire functions, hence Borel goes on with a second chapter devoted to explain the Laguerre works upon what previously made by Weierstrass, and in which, among other things, the fundamental notions of *genus* and *order* of an entire function were introduced starting from the Weierstrass factorization theorem (see also (Sansone 1972, Chapter V, Section 8)). With respect to these appreciated Laguerre works and on the wake of those made, above all, by E. Cesaro, G. Vivanti, A. Bassi and D. Pizzarello[50] on those entire functions having arbitrary genus but devoid of exponential factors (see (Vivanti 1928) for a most complete bibliographical account of the contributions of these last authors), the third chapter of Borel's monograph deals with the fundamental 1883 Poincaré's work on entire functions, until up the celebrated Hadamard work outlined in the next chapter IV, to end with the Picard's contribution delineated in the final chapter V. As the author himself says, a natural continuation of Borel's monograph is (Blumenthal 1910), where a central chapter, the fourth one with a final Note II, deals with a general

been used, for the first time, in the International Mathematical Union Congress held at Vancouver in 1974, where a section specifically devoted to Complex Analysis was considered for exposing researches in the theory of holomorphic functions of one or more variables.

[50]Domenico Pizzarello was born in Scilla (Messina, IT) on August 3, 1873 from Gaetano and Teresa Bellantoni. He was graduated in Mathematics from the University of Rome on November 12, 1899. Then, he was assistant at the Infinitesimal Calculus chair of Professor Giulio Vivanti at the University of Messina. Afterwards, he taught in various Italian high schools until 1924, when he was appointed head of the Francesco Maurolico classical high school at Messina, where he passed away on July 23, 1943.

theory of canonical products as it turned out be until 1910s. Furthermore, O. Blumenthal himself contributed to the theory of entire functions (see (Valiron 1949, Chapter II, Section 3)).

In what follows, we mainly refer to (Borel 1900), (Vivanti 1928), (Sansone 1972, Chapter V), (Levin 1980, Chapter I) and references therein. Retaking into consideration the above mentioned Weierstrass' theorem, Laguerre (see (Laguerre 1882a,b,c; 1883; 1884)), from 1882 onwards, published some short but remarkable papers on certain concepts and properties of entire functions, amongst which the notion of genus. To be precise, Laguerre first defines j as the *genus* of the Weierstrass' factors $E_j(z)$, letting $\gamma(E_j(z)) = j$, then he calls *genus* (or *rank*) of the entire function $f(z)$ as given by (8), the number $p = \max\{\partial eg\ g(z), \sup\{\gamma(E_j(z)\}\}$, which may also be ∞ when $\sup\{\gamma(E_j(z)\} = \infty$ or, otherwise, when $g(z)$ is a transcendental entire function (so $\partial eg\ g(z) = \infty$). The importance of the natural numbers $\partial eg\ g(z)$ and $\sup\{j; E_j(z)\}$ with respect to the Weierstrass decomposition (8), had already been recognized by Weierstrass himself, but it was Laguerre the first who understood that their maximum value has instead more importance and usefulness from a formal viewpoint. Most of Laguerre's work was pursued on entire functions of genus zero and one as well as on the study of the distribution of the zeros of an entire function and its derivatives, taking constantly into account the comparison between polynomials and entire functions on the wake of what had already known about the determination of the zeros of the former. Following (Gonchar et al. 1997, Part I, Chapter 1, Section 1), entire functions are a direct generalization of polynomials but their asymptotic behavior has an incomparably greater diversity. The most important parameter characterizing properties of a polynomial is its degree. A transcendental entire function that can be expanded into an infinite power series can be viewed as a kind of polynomial of infinite degree, and the fact that the degree is infinite brings no additional information to the statement that an entire function is not a polynomial. That is why, to characterize the asymptotic behavior of an entire function, one must use other quantities and new notions, like those of order, genus, the maximum modulus $M_f(r)$, and so forth.

According to (Burkhardt et al. 1899-1927, Dritter Teil, erste Hälfte, C.4, Nr. 26-36), (Fouët 1904-07, Tome II, Chapter IX, Section II, Number 283), (Marden 1949; 1966) and (Pólya & Szegő 1998a, Part III; 1998b, Part V), just upon the possible analogical transfer of the known results about the theory of polynomials (above all results on their zeros, like Rolle's and Descartes' theorems - see (Marden 1949; 1966)) towards entire functions, the next work of Poincaré, Hadamard, Borel, as well as of E. Schou, E. Cesaro, E. Fabry, E. Laguerre[51], G.A.A. Plana, F. Chiò, A. Genocchi, C. Runge, C. Hermite, E. Maillet, E. Jaggi, C.A. Dell'Agnola, J. von Puzyna, M.L.M. de Sparre, C. Frenzel, M. Petrovitch, A. Winternitz and others, will be oriented (see (Vivanti 1906)) since the early 1900s till to the 1920s with pioneering works of E. Lindwart, R. Jentzsch, G. Grommer, N. Kritikos and, above all, G. Pólya. The first notable results in this direction were obtained both by E. Picard in the late 1870s, who dealt with the values of an arbitrary entire function, and by Poincaré in the early 1880s (see (Poincaré 1882; 1883) and (Sansone 1972, Chapter V, Section 14)), who established some first notable relations between the modulus of an entire function, its genus and the variations of the magnitude of its coefficients; Poincaré was too the first one to apply entire function theory methods to differential equations. Following (Markušević 1966, Preface), the so-called *Picard's little theorem* roughly asserts that the equation $f(z) = a$, where $f(z)$ is a transcendental entire function and a is a given complex number, has in general, an infinite set of roots. This theorem clearly can be regarded as the analog, to the infinite degree, of the Gauss' fundamental theory of algebra, according to which the number of roots of the equation $p(x) = a$, where $p(x)$ is a polynomial, is equal to the degree of the polynomial. Following (Vivanti 1928, Part III, Section 184), the Poincaré theorems were underestimated up to the 1892-93 Hadamard work (that will be discussed later), notwithstanding their importance for having opened the way to the study of the relations between the distribution of the zeros of an entire function and the sequence of its coefficients. The relevance of the zeros of an entire function is simply due to the fact that this last

[51] See (Laguerre 1898-1905, Tome I, p. 168) in which he retakes a notable result achieved by Hermite who, in turn, used previous methods found by G.A.A. Plana, A. Genocchi e F. Chiò on the zeros of algebraic equations.

is determined by factorization theorems of the Weierstrass' type. With Poincaré, the notion of order of an entire function is introduced as follows. First, Poincaré proved that, if $f(z)$ is an entire function of genus p (as defined above) and ρ is a positive integer greater than p such that $\sum_{n\in\mathbb{N}} r_n^{-\rho}$ is convergent, then, for every positive number α, there exists an integer $r_0(\alpha) > 0$ such that, for $|z| = r \geq r_0(\alpha)$, we have $|f(z)| < e^{\alpha r^{\alpha}}$ in $|z| = r$. Then, if $f(z)$ is an entire function, to characterize the growth of an entire function, we introduce a not-decreasing function as follows: let $M_f(r) = \max_{|z|=r} |f(z)|$ be the maximum value of $|f(z)|$ on the sphere having center into the origin and radius r. $M_f(r)$ is a continuous not-decreasing monotonic function of r, tending to $+\infty$ as $r \to \infty$. For a polynomial f of degree n, the following asymptotic relation holds $\ln M_f(r) \sim n \ln r$, so that $n = \lim_{r\to\infty} \ln M_f(r)/\ln r$, i.e., the degree of a polynomial is closely related to the asymptotics of $M_f(r)$. The ratio $\ln M_f(r)/\ln r$ tends to ∞ for all entire transcendental functions. That is why the growth of $\ln M_f(r)$ is characterized by comparing it, not with $\ln r$, but with faster growing functions, the most fruitful comparison being that with power functions. Thus, in order to estimate the growth of transcendental entire functions, one must choose comparison functions that grow more rapidly than powers of r. If one chooses functions of the form e^{r^k} $k \in \mathbb{N}$, as comparison functions, then an entire function $f(z)$ is said to be of *finite order* if there exist $k \in \mathbb{N}$ and $r_0(k) \in \mathbb{R}^+$ such that the inequality $M_f(r) < e^{r^k}$ is valid for sufficiently large values of $r > r_0(k)$, the greatest lower bound of such numbers k, say ρ, being said the *order* of the entire function $f(z)$; finally, further indices, introduced by E.L. Lindelöff, H. Von Schaper, A. Pringsheim and E. Borel in the early 1900s (with further contributions due to S. Minetti in 1927 - see (Vivanti 1906; 1928, Section 203)), and often called *Lindelöff indices,* have been introduced to estimate the rapidity of variation of the modulus of the zeros, of the coefficients and of the function $M_f(r)$ of a given entire function $f(z)$ (see (Borel 1900, Chapter III), (Vivanti 1928, Part III, Section 176) and (Levin 1980, Chapter I)).

With the pioneering works of Jacques Hadamard (see (Hadamard 1892; 1893)), deepening of the previous results, as well as new research directions, were pursued. If Poincaré was the first to apply the early results of entire function theory to the study of differential equations, so Hadamard was the first to explicitly consider applications of the theory of entire functions to the number theory, just working upon what previously made by Riemann on the same subject. Following (Borel 1900, Chapter III) and (Maz'ya & Shaposhnikova 1998, Chapter 1, Section 1.10; Chapter 9, Section 9.2), the 1893 work of Hadamard roughly consisted in finding relations between the behavior of the coefficients and the distribution of the zeros of an entire function as well as in providing more explicit formulas of the Weierstrass type for functions growing slower than $\exp(|z|^{\lambda})$, so becoming easier to prove the absence of exponential factors in the case of the Riemann ξ function. Following (Maz'ya & Shaposhnikova 1998, Chapter 9, Section 9.2), the 1893 Hadamard memoir is divided into three parts. The first one, after having improved some previous results achieved by Picard (and mentioned above), is mostly devoted to the relationships between the rate of growth of $M_f(r)$ and the decreasing law of the coefficients c_n of the Taylor expansion of the given entire function $f(z)$. At the beginning, Hadamard found a majorant for $M_f(r)$ described in terms of the sequence of the coefficients c_n, noting, for example, that if $|c_n| < (n!)^{-1/\alpha}$, $\alpha > 0$, then $M_f(r) < e^{Hr^{\alpha}}$ for some constant H. Then, he considered the inverse problem, already approached by Poincaré, to find the law of decreasing of the coefficients departing from the law of growth of the function, extending Poincaré method in order to include functions satisfying $M_f(r) < e^{V(r)}$, where $V(r)$ is an arbitrary positive increasing unbounded function. Then, as the central goal of the paper in the aim of the author, Hadamard deals with an improvement of the Picard theorem, but it will be Borel, in 1896, to give a general prove of it, valid for every entire function. Following (Gonchar et al. 1997, Chapter 5, Section 1), 1879 Picard famous theorem is concerned with the problem of the distribution of the values of entire functions and it may be considered as one of the starting points of the theory of the distribution of the values of meromorphic functions which then began to develop only in the 1920s with the pioneering works of R.

Nevanlinna, albeit its very early starting point was the following formula

$$\log \frac{r^n|f(0)|}{|z_1...z_n|} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta,$$

due either to J.L.W. Jensen (see (Jensen 1898-99)) and J. Petersen (see (Petersen 1899)) but already known to Hadamard since the early 1890s (see[52] (Maz'ya & Shaposhnikova 1998, Chapter 9, Section 9.2)), where $z_1, z_2, ..., z_n, ...$ are the zeros of $f(z)$, $|z_1| \leq |z_2| \leq ...$ and $|z_n| \leq r \leq |z_{n+1}|$. This formula was called *Poisson-Jensen formula* by R. Nevanlinna around 1920s who, later, will give an extended version of it, today known as *Jensen-Nevanlinna formula* (see (Zhang 1993, Chapter I)).

Following (Maz'ya & Shaposhnikova 1998, Chapter 9, Section 9.2), in the second part of the 1893 memoir, as we have already said above, Hadamard considers a question converse to the one treated by Poincaré, that is to say, what information on the distribution of zeros of an entire function can be derived from the law of decreasing of its coefficients? In particular, he shows that the genus of the entire function is equal to the integer part $[\lambda]$ of λ provided by $|c_n|(n!)^{-1/(\lambda+1)} \underset{n\to\infty}{\longrightarrow} 0$ with λ, in general, not integer[53]. From this statement, one concludes that a function $f(z)$ has genus zero if $M_f(r) < e^{Hr^\alpha}$ holds with $\alpha < 1$. Hadamard's theorem, nevertheless, is less precise for the case when λ is integer, because, in this case, the function may have genus either λ or $\lambda + 1$. Hadamard's result

[52]With this historical remark, we might answer to a query expressed in (Davenport 1980, Chapter 11, p. 77, footnote [1]) about the use of Jensen's formula in proving Hadamard factorization theorem, where textually the author says that *«strangely enough, Jensen's formula was not discovered until after the work of Hadamard»*. Also H.M. Edwards, in (Edwards 1974, Chapter 2, Section 2.1, footnote[1]), about the Hadamard proof of 1893 memoir, affirms that *«A major simplification is the use of Jensen's theorem, which was not known at the time Hadamard was writing»*. Nevertheless, there are historical proves which state the contrary, amongst which a witness by a pupil of Hadamard, Szolem Mandelbrojt (1899-1983), who, in (Mandelbrojt 1967, p. 33), states that Hadamard was already in possession of Jensen's formula before Jensen himself, but did not not publish it, since he could not find for it any important application (see also (Narkiewicz 2000, Chapter 5, Section 5.1, Number 1)). The first part of the Volume 13, Issue 1, of the year 1967 of the review *L'Enseignement Mathématique*, was devoted to main aspects of Hadamard mathematical work, with contributions of P. Lévy, S. Mandelbrojt, B. Malgrange and P. Malliavin.

[53]Poincaré proved that, if $f(z)$ is an entire function of genus p such that $f(z) = \sum_{n\in\mathbb{N}_0} c_n z^n$, then $(n!)^{1/(1+p)}c_n \underset{n\to\infty}{\longrightarrow} 0$ (see (Sansone 1972, Chapter V, Section 9)). Further studies on entire functions of non-integral order were also attained by L. Leau in 1906.

was improved by Borel in 1897 (see also (Borel 1900)), who used two important characteristic parameters of an entire function, namely the order ρ (that, d'après Borel, he called *apparent order*) and the exponent of convergence of zeros p (said to be the *real order*, in his terminology borrowed by Von Schaper). The order is the upper lower bound ρ of the numbers α such that $M_f(r) < e^{Hr^\alpha}$, its explicit expression being given by

$$\rho = \limsup_{r\to\infty} \frac{\ln\ln M_f(r)}{\ln r}$$

which might therefore be taken as the definition of the order of the function f; the quantity instead $\lambda_\rho \doteq \liminf_{r\to\infty} \ln\ln M_f(r)/\ln r$ is said to be the *lower order* of f. For a polynomial we have $\rho = 0$, while for the transcendental functions $\exp z, \sin z, \exp(\exp z)$ the order is respectively $1, 1$ and ∞. If we have $\rho < \infty$, then the quantity $\sigma \doteq \limsup_{r\to\infty} r^{-\rho} \ln M_f(r)$ is called the *type value* of the entire function f. The *exponent of convergence* of the zeros, say p, is defined as the upper lower bound of those $\lambda > 0$ for which the series $\sum_n |z_n|^{-(\lambda+1)}$ converges. One can also check that the exponent of convergence is also provided by $\mu = \limsup(\ln n/\ln|z_n|)$. Hadamard proved that $\rho \geq p$, often said to be the *first Hadamard theorem* (see (Sansone 1972, Chapter V, Section 3)). If an entire function has only a finite number of zeros, then we say that it has exponent of convergence zero. Thus, while the order characterizes the maximal possible growth of the function, the exponent of convergence p is an indicator of the density of the distribution of the zeros of $f(z)$. Therefore, the Hadamard's refinement of Weierstrass formula (11) by using the notion of order, states that, if f is an entire function of finite order ρ, then the entire function $g(z)$ in (11) is a polynomial of degree not higher than $[\rho]$. As we have been said above, Borel obtained a kind of converse to this result by showing how the order can be found from the factorization formula, his theorem stating that, if $\mu < \infty$ and $g(z)$ is the polynomial appearing in (11), then $f(z)$ is an entire function of order $p = \max\{\mu, q\}$. Finally, the lst third part his memoir, Hadamard applied his results on the genus of an entire function, achieved in the first part, to the celebrated Riemann zeta function. To be precise, Riemann reduced the study of the zeta function to that of the even entire function ξ defined by $\xi(z) = z(z-1)\Gamma(z/2)\zeta(z)/2\pi^{z/2}$, and writing ξ as the series $\xi(z) = b_0 + b_2 z^2 + b_4 z^4 + ...$, Hadamard proved the inequality $|b_m| < (m!)^{-1/2-\varepsilon}, \varepsilon > 0$, thus verifying, for $1/\alpha = 1/2 + \varepsilon$, the following estimate $|c_n| < (n!)^{-1/\alpha}, \alpha > 0$ deduced in the first part of his memoir and briefly mentioned above, so that it follows that the genus of ξ, as a function of z^2, is equal to zero and that

$$\xi(z) = \xi(0) \prod_{k=1}^{\infty} \Big(1 - \frac{z^2}{\alpha_k^2}\Big)$$

where the α_k are the zeros of ξ, this last property, as is well-known, having already been provided by Riemann, in his celebrated 1859 paper, but without a rigorous proof.

Following (Gonchar et al. 1997, Part I, Chapter 1, Section 1), the classical Weierstrass theorem is well-known on the representation of an entire function with a given set of zeros in the form of an infinite product of Weierstrass primary factors. In the works of Borel and Hadamard on entire function of finite order, the Weierstrass theorem was significantly improved, showing that the genus of the primary factors could be one and the same, in the representation of an entire function only a finite number of parameters being not defined by the set of zeros. As early as the turn of the 20th century, the theory of factorization of entire functions was regarded as fully completed, albeit in a series of works started in 1945, M. Dzhrbashyan and his school constructed a new factorization theory, as well as H. Behnke and K. Stein extended, in 1948, factorization theorem to arbitrary non-compact Riemann surfaces (see (Remmert 1998, Chapter 4, Section 2)). The remarkable work of Behnke and Stein (see (Behnke & Stein 1948)) revaluated the role of the so-called *Runge sets* in the theory of non-compact Riemann surfaces, demonstrating a Runge type theorem. Following (Maurin 1997, Part V, Chapters 3 and 6), Carl Runge (1856-1927) gave fundamental contributions, between 1885 and 1889, to the theory of complex functions, proving a basic result, in which he introduced particular sets later called *Runge's sets,* regarding the approximation of holomorphic functions by a sequence of polynomials, almost in the same years in which Weierstrass gave his as much notable theorem on the approximation of a function on interval by polynomials. From Runge outcomes, hence also from Behnke-Stein ones, it follows much of the representation theorems for meromorphic functions due to Weierstrass and Mittag-Leffler. To point out the central result of the 1893 Hadamard paper, we recall, following (Levin 1980, Chapter I), the Weierstrass theorem, namely that every entire function $f(z)$ may be represented in the form

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\omega} G\Big(\frac{z}{a_n}; p_n\Big) \qquad (\omega \leq \infty)$$

where $g(z)$ is an entire function, a_n are the non-zero roots of $f(z)$, m is the order of the zero of $f(z)$ at the origin, and $G(u;p) = (1-u)\exp(u + u^2/2 + ... + u^p/p)$ is the generic *primary factor.* The sequence of numbers p_n is not uniquely determined and, therefore, the function $g(z)$ is not uniquely determined either. After Laguerre work, the representation of the function $f(z)$ is considerably simpler if the numbers a_n satisfy the following supplementary condition, that is, the series $\sum_{n\in\mathbb{N}} |a_n|^{-(\lambda+1)}$ converges for some positive λ. In this case, let p denote the smallest integer $\lambda > 0$ for which the series $\sum_{n\in\mathbb{N}} |a_n|^{-(\lambda+1)}$ converges. Thus, also the infinite product $\prod_{n\in\mathbb{N}} G(z/a_n;p)$ converges uniformly: it is called a *canonical product,* and the number p is called, following B.J. Levin, the *genus* of the canonical product, or else, following Borel, the exponent of convergence of the zeros a_n. If $g(z)$ is a polynomial, $f(z)$ is said to be an entire function of *finite genus.* If q is the degree of the polynomial $g(z)$, the largest of the numbers p and q is called the *genus* of $f(z)$. If $g(z)$ is not a polynomial or if the series $\sum_{n\in\mathbb{N}} |a_n|^{-(\lambda+1)}$ diverges for all the values of $\lambda > 0$, then the genus is said to be *infinite.* The representation of an entire function as an infinite product makes it possible to establish a very important dependence between the growth of the function and the density of distribution of its zeros. As a measure of the density of the sequence of the points a_n, having no finite limit point, we introduce, d'après Borel (see (Borel 1900, Chapter II)), the convergent exponent of the sequence $a_1, a_2, ..., a_n, ...$, with $a_n \neq 0$ definitively and

$\lim_{n\to\infty} a_n = \infty$, which is defined by the greatest lower bound of the numbers $\lambda > 0$ for which the series $\sum_{n\in\mathbb{N}}(1/|a_n|^{\lambda+1})$ converges. Clearly, the more rapidly the sequence of numbers $|a_n|$ increases, the smaller will be the convergent exponent, which may be also zero. A more precise description of the density of the sequence $\{a_n\}_{n\in\mathbb{N}}$, than the convergence exponent is given by the growth of the function $n(r)$, said to be *zero-counting function,* equal to the number of points of the sequence in the circle $|z| < r$, so that by the *order* of this monotone function we mean the number $\rho_1 = \limsup_{r\to\infty}(\ln n(r)/\ln r)$, and by the *upper density* of the sequence $\{a_n\}$, we mean the number $\Delta = \limsup_{r\to\infty}(n(r)/r^{\rho_1})$; if the limit exists, then Δ is simply called the *density* of the sequence $\{a_n\}$. Classical results on the connection between the growth of an entire function and the distribution of its zeros mainly describe the connection between $\ln M_f(r)$ and the zero-counting function $n(r)$. If f is a polynomial, then $\lim_{r\to\infty} n(r) = n$ if and only if $\ln M_f(r) \sim n \ln r$, whereas no simple connection exists between the asymptotic behavior of $\ln M_f(r)$ and $n(r)$ for entire transcendental functions. It is possible to prove that the convergent exponent of the sequence $\{a_n\}$, with $\lim_{n\to\infty} |a_n| = \infty$, is equal to the order of the corresponding function $n(r)$. Borel moreover proved that the order ρ of the canonical product $\Pi(z) = \prod_{n\in\mathbb{N}} G(z/a_n; p)$, does not exceed the convergence exponent ρ_1 of the sequence $\{a_n\}$, even better $p = \rho_1$ (*Borel theorem*; see also (Sansone 1972, Chapter V, Section 6)). Hadamard's factorization theorem is a refinement concerning the representation of entire functions of finite order, and is one of the classical theorems of the theory of entire functions. This theorem states that an entire function $f(z)$ of finite order ρ and genus p, can be represented in the form

$$f(z) = z^m e^{P(z)} \prod_{n=1}^{\omega} G\Big(\frac{z}{a_n}; p\Big) \qquad (\omega \le \infty),$$

where a_n are the non-zero roots of $f(z)$, $p \le \rho$, $P(z)$ is a polynomial whose degree q does not exceed $[\rho]$, and m is the multiplicity of the zero at the origin. This theorem, hence, states that the genus of an entire function does not exceed its order. Sometimes, the factor $e^{P(z)}$ is also called *external exponential factor* (see (Vivanti 1928) and (Sansone 1972, Chapter V, Section 5)). Following (Maz'ya & Shaposhnikova 1998, Chapter 9, Section 9.2), Borel obtained as well a sort of converse to this result by showing how the order can be found from the factorization formula, stating as follows: if $p < \infty$ and $P(z)$ is a polynomial of degree q, then $f(z)$ is a function of order $\rho = \max\{p, q\} = \max\{\rho_1, q\}$ (via Borel theorem). Finally, we recall that in this 1893 Hadamard memoir, further estimates for the minimum of the modulus of an entire function were also established (forming the so-called *second Hadamard theorem*), upon which, then, Borel (see (Borel 1900)), P. Boutroux, E. Maillet, A. Kraft, B. Lindgren, G. Faber, A. Denjoy, F. Schottky, E. Lindelöf, J.L.W. Jensen, J.E. Littlewood, G. Hardy, W. Gross, R. Mattson, G. Rémoundos, O. Blumenthal, R. Mattson, E. Landau, C. Carathéodory, A. Wahlund, G. Pólya, A. Wiman, P. Fatou, P. Montel, T. Carleman, L. Bieberbach, F. Iversen, E. Phragmèn, A. Pringsheim, E.F. Collingwood, R.C. Young, J. Sire, G. Julia, A. Hurwitz, G. Valiron and others will work on, at first providing further improvements to the estimates both for the minimum and the maximum of the modulus of an entire function and its derivatives (see (Burkhardt et al. 1899-1927, Dritter Teil, erste Hälfte, C.4, Nr. 26-36) and (Sansone 1972, Chapter V, Sections 4, 13 and 16)), till to carry out a complete, rich and autonomous chapter of complex analysis. Later studies on entire functions having integral order were also accomplished, in the early 1900s, above all by A. Pringsheim as well as by E. Lindelöf and E. Phragmèn who defined what is known as *Phragmèn-Lindelöf indicator* of an entire function which will be the basic characteristic of growth of an entire function of finite order (see (Ostrovskiĭ & Sodin 1998, Section 3)). Anyway, the description of the state-of-the-art of the theory of entire functions until 1940s, may be found above all in the treatise (Valiron 1949), as well as in the last editions of the well-known treatise (Whittaker & Watson 1927). Furthermore, it is also useful to look at the notes by G. Valiron, to the 1921 second edition of Borel's treatise on entire functions (i.e., (Borel 1900)), that is to say (Borel 1921), where, at the beginning of the Note IV, Valiron says that

«La théorie des fonctions entières a fait l'object d'un três grand

nomhre de travaux depuis la publication des Mémoires fondamentaux de J. Hadamard et E. Borel. Plus de cent cinquante Mémoires ou Notes ont été publiés entre 1900, dale de la première édition des Leçons sur les fonctions entières, et 1920; beaucoup de ces travaux ont leur origine dans les suggestions de E. Borel. On peut répartir ccs recherches en quatre groupes: 1º. Étude de la relation entre la croissance du module maximum et la croissance de la suite des coefficients de la fonction et démonstrations élémentaires du théorème de Picard; 2º. Études directes de la relation entre la suite des zéros et la croissance du module maximum; 3º. Recherches sur les fonctions inverses et généralisations du théorème de Picard; 4º. Recherches de nature algébrique et étude des fonctions d'ordre fini considérées comme fonctions limites d'une suite de polynomes. Il eût été difficile de donner dans quelques pages un aperçu des travaux particuliers de chaque auteur, certaines questions ayanl été traitées simultanément ou d'une façon indipendante par plusieurs mathématiciens [...]*».*

Afterwards, Valiron briefly exposes the main results achieved by those mathematicians whose names have been just recalled above, a more detailed treatment being given in his treatise (Valiron 1949) which covers the European area until up mid-1900s. After such a period, a great impulse to the theory of entire functions was given by Russian school which grew up around Boris Yakovlevich Levin (1906-1993) whose scientific and human biography may be found in the preface to (Levin 1996). Herein, we give a very brief flashing out on the research work on entire function theory achieved by Russian school, referring to (Ostrovskii & Sodin 1998; 2003) for a deeper knowledge. The fundamental problem in the theory of entire functions is the problem of the connection between the growth of an entire function and the distribution of its zeros, a basic characteristic of growth of an entire function of finite order being the so-called *Phragmén-Lindelöf indicator* (see (Phragmén & Lindelöf 1908)) defined by

$$(14)\qquad h(\varphi, f) = \limsup_{r\to\infty} r^{-\rho(r)} \ln|f(re^{i\varphi})|, \qquad \varphi\in[0,2\pi].$$

The systematic study of the connection of the indicator with the distribution of zeros, started in the 1930s with the Russian school leaded by Levin and Mark G. Kreĭn. Following (Levin 1980, Chapter VIII), the representation of an entire function by a power series shows the simple fact that any entire function is the limit of a sequence of polynomials which converges uniformly in every bounded domain. If we impose on the polynomials which are approaching uniformly the given entire function the additional requirement that their zeros belong to a certain set, then the limit functions will form a special class, depending on the set. The first notable results in this direction were due to Laguerre (see (Laguerre 1898-1905, Tome I, pp. 161-366)), who gave a complete characterization of the entire functions that can be uniformly approximated by polynomials, distinguishing two chief cases: the first one (I) in which the zeros of these polynomials are all positive, and the second one (II) in which these zeros are all real. In this latter case, a proof of his theorem was later given by G. Pólya (see (Pólya 1913)), while a more complete investigation of the convergence of sequences of such polynomials was carried out by E. Lindwart and Pólya (see (Lindwart & Pólya 1914)), showing, in particular, that in the two above just mentioned cases I and II (as well as in more general cases), the uniform convergence of a sequence of polynomials, in some disk $|z| < R$, implies its uniform convergence on any bounded subset of the complex plane. Now, the main results achieved in the theory of representation of an entire function by a power series, namely that any entire function is the limit of a sequence of polynomials which converges uniformly in every bounded domain, in turn refer to the theory of approximation of entire functions by polynomials whose zeros lie in a given region, say G, of the open or closed upper complex half-plane. Besides important results achieved by E. Routh and A. Hurwitz in the 1890s, the basic algebraic fact in this domain is a theorem stated by C. Hermite in 1856 (see (Hermite 1856a,b)) and C. Biehler (see (Biehler 1879)), and nowadays known as *Hermite-Biehler theorem,* which provides a necessary and sufficient condition for a polynomial of the type $\omega(z) = P(z) + iQ(z)$, with P and Q real polynomials, not have any root in the closed lower half-plane $\Im z \leq 0$, imposing conditions just on P and Q. In 20th century, the

Russian school achieved further deep results along this direction and, in carrying over the Hermite-Biehler criterion to arbitrary entire functions, an essential role is played by particular classes of entire functions, introduced by M.G. Kreĭn in a 1938 work devoted to the extension of some previous Hurwitz criteria for zeros of entire functions (see (Ostrovskiĭ 1994)), and said to be *Hermite-Biehler classes (HB* and $\overline{HB}$ *classes).* An entire function $\omega(z)$ is said to be a function of class HB [respectively $\overline{HB}$] if it has no roots in the closed [open] lower half-plane $\Im z \leq 0$, and if $|\omega(z)/\bar{\omega}(z)| < 1$ [$|\omega(z)/\bar{\omega}(z)| \leq 1$] for[54] $\Im z > 0$. On the basis of results achieved by M.G. Kreĭn, N.N. Meĭman, Ju.I. Neĭmark, N.J. Akhiezer, L.S. Pontrjagin, B.Ja. Levin, N.G. Čebotarev and others, around 1940s and 1950s, simple criteria for an entire function to belong to the class HB, as well as representation theorems for elements of this class of entire functions using special infinite products, were provided (see (Levin 1980, Chapter VII)). A polynomial which has no zeros in open lower half-plane will be called an *H-polynomial.* Then, the so-called *Laguerre-Pólya class (LP class)* is given by a particular class of entire functions obtained as limit of a sequence of H-polynomials uniformly converging in an angular δ-neighborhood of the origin, hence in an arbitrary bounded domain (see (Levin 1980, Chapter VIII)) through a criterion called *Laguerre-Pólya theorem* due to previous outcomes obtained by Laguerre in the late 1890s. The classical Laguerre-Pólya theorem asserts that an entire function f belongs to this class if and only if

$$f(z) = e^{-\gamma z^2+\beta z+\alpha} z^m \prod_n \Big(1 - \frac{z}{z_n}\Big) e^{\frac{z}{z_n}} \tag{15}$$

where all z_n, α and β are real, $\gamma \leq 0$, $m \in \mathbb{N}_0$ and $\sum_n |z_n|^{-2} < \infty$. Following (Bergweiler et al., 2002), in passing we recall that Laguerre-Pólya class LP coincides with the closure of the set of all real polynomials with only real zeros, with respect to uniform convergence on compact subsets of the plane. This is just what was originally proved by Laguerre in (Laguerre 1882c) for the case of polynomials with positive zeros and

[54] Here we understand by $\bar{\omega}(z)$ the entire function obtained from $\omega(z)$ by replacing all the coefficients in its Taylor series by their conjugates.

by Pólya in (Pólya 1913) in the general case. It follows that LP class is closed under differentiation, so that all derivatives of a function $f \in LP$ have only real zeros. Pólya, in (Pólya 1913), also asked whether the converse is true, that is to say, if all derivatives of a real entire function f have only real zeros then $f \in LP$. This conjecture was later proved by S. Hellerstein and J. Williamson in 1977 (see (Bergweiler et al., 2002) and references therein). Other notable results for entire functions belonging to LP class were achieved by E. Malo in the late 1890s and by G. Pólya, J. Egerádry, E. Lindwart, A.I. Markuševič, I. Schur, J.L.V. Jensen, O. Szàsz, J. Korevaar, M. Fekete, E. Meissner, E. Bálint, D.R. Curtiss, J. Grommer, M. Fujiwara, E. Frank, S. Benjaminowitsch, K.T. Vahlen, A.J. Kempner, I. Schoenberg, S. Takahashi, N. Obrechkoff and others, between the 1910s and the 1950s (see (Levin 1980, Chapter VIII) and (Marden 1949)). For other interesting historical aspects of entire function theory see also (Korevaar 2013) and references therein, while as regard history of mathematics in Russian area, see (Demidov 2002).

Finally, a notable work on after the mid-1900s entire function theory developments has surely been the one achieved by Louis de Branges since 1950s with his theory of Hilbert spaces of entire functions, culminated in the treatise (de Branges 1968). In the intention of the author expressed in the Preface to the latter, anyone approaches Hilbert spaces of entire functions for the first time will see the theory as an application of the classical theory of entire functions. The main tools are drawn from classical analysis, and these are the Phragmén-Lindelöf principle[55] (see (Phragmén & Lindelöf 1908)), the Poisson representation of positive harmonic functions, the factorization theorem for functions of Pólya class, Nevanlinna's theory of functions of bounded type, and the Titchmarsh-Valiron theorem relating growth and zeros of entire functions of exponential type. The origins of Hilbert spaces of entire functions are found in a theorem of Paley and Wiener that characterizes finite Fourier transforms as entire functions of exponential type which are square integrable on the real

[55]Besides Hadamard's work to be greatly influenced by Riemann 1859 paper, also E. Phragmén and E. Lindelöf work, in the very late 19th century and early 20th, was influenced too by this Riemann paper, once it became ever-more-clear that the difficulties in proving the Riemann Hypothesis were substantial (by a private communication with Professor Paul Garrett).

axis. This result has a striking consequence which is meaningful without any knowledge of Fourier analysis. The identity

$$\int_{-\infty}^{+\infty} |F(t)|^2 dt = \frac{\pi}{a} \sum_{-\infty}^{+\infty} |F(n\pi/a)|^2$$

which holds for any entire function $F(z)$ of exponential type at most a which is square integrable on the real line. The formula is ordinarily derived from a Fourier series expansion of the Fourier transform of $F(z)$. In the fall of 1958, de Branges discovered an essentially different proof which requires nothing more than a knowledge of Cauchy's formula and basic properties of orthogonal sets. The identity is a special case of a general formula which relates mean squares of entire functions on the whole real axis to mean squares on a sequence of real points. Certain Hilbert spaces, whose elements are entire functions, enter into the proof of the general identity. Since such an identity has its origins in Fourier analysis, de Branges conjectured that a generalization of Fourier analysis was associated with these spaces, spending the years 1958-1961 to verify this conjecture. The outlines of this de Branges theory are best seen by using the invariant subspace concept. The theory of invariant subspaces sprung out of some early studies of the end of 19th Century on the zeros of polynomials and their generalization by C. Hermite and T.J. Stieltjes, just after the Riemann conjecture (see (de Branges 1968; 1986)). The next axiomatization of integration just due to Stieltjes in the last years of 19th century, greatly contributed to settling up these studies, above all thanks to the work of Hilbert. A fundamental problem is to determine the invariant subspaces of any bounded linear transformation in Hilbert space and to write the transformation as an integral in terms of invariant subspaces: this is one of the main problems of spectral analysis. A similar problem can be stated for an unbounded or partially defined transformation once the invariant subspace concept is clarified. To this purpose, it may help to say that there exist invariant subspaces appropriate for a certain kind of transformation, the theory of Hilbert spaces of entire functions being the best behaved of all invariant subspace theories. Moreover, nontrivial invariant subspaces always exist for nontrivial transformations; invariant subspaces are totally ordered by inclusion. The transformation admits an integral representation in terms of its invariant subspaces, this representation being stated as a generalization of the Paley-Wiener theorem and of the Fourier transformation. Hilbert spaces of entire functions also have other applications, an obvious area being the approximation by polynomials of entire functions of exponential type. On the other hand, it was just through such problems that de Branges discovered such spaces. Although it is easy to construct entire functions with given zeros, it is quite difficult to estimate the functions so obtained. To this end, de Branges used the extreme point method to construct nontrivial entire functions whose zeros lie in a given set and whose reciprocals admit absolutely convergent partial fraction decompositions. A classical problem is to estimate an entire function of exponential type in the complex plane from estimates on a given sequence of points, so de Branges constructed Hilbert spaces of entire functions of exponential type with norm determined by what happens on a given sequence of real points.

6. On some applications of the theory of entire functions

6.1. On the applications of entire function theory to Riemann zeta function: the works of J. Hadamard, H. Von Mangoldt, E. Landau, G. Pólya, and others. Following (Valiron 1949, Chapter I), the early origin of the general theory of entire functions, that is to say of functions which are regular throughout the finite portion of the plane of the complex variable, is to be found in the work of Weierstrass. He shown that the fundamental theorem concerning the factorization of a polynomial can be extended to cover the case of such functions, and that in the neighborhood of an isolated essential singularity the value of a uniform function is indeterminate. These two theorems have been the starting point of all subsequent research. Weierstrass himself did not complete his second theorem, this having been done in 1879 by Picard who proved that in the neighborhood of an isolated essential singularity a uniform function actually assumes every value with only one possible exception. Much important work, the earliest of which was due to Borel, has been done in connection with Picard's theorem; and the consequent introduction of new methods has resulted in much light being thrown on obscure points in the theory of analytic functions. The notion of the genus of a Weierstrassian product was introduced and its importance first recognized by Laguerre, but it was not until after the work of Poincaré and Hadamard had been done that any substantial advance was made in this direction. Here also Borel has enriched the theory with new ideas, and his work has done much to reveal the relationship between the two points of view and profoundly influenced the trend of subsequent research. The theory of entire functions, or more generally of the functions having an isolated singularity at infinity, may be developed in two directions. On the one hand, we may seek to deduce from facts about the zeros information concerning the formal factorization of an entire function; on the other hand, regarding the problem from the point of view of the theorems of Weierstrass and Picard, we may endeavor to acquire a deeper insight into the nature of the function by investigating the properties of the roots of an equation of the type $f(z) - a = 0$, where $f(z)$ is an entire function. The study of the zeros of these functions thus serves a double purpose, since it contributes to advance the theory along both these avenues. The first step consists in giving all the theorems due to Hadamard and Borel concerning the formal factorization of an entire function, and then proceed towards a direct investigation of the moduli of its zeros by the methods provided by Borel, the resulting outcomes bringing out very clearly the close relationships existing between these two points of view. Along this treatment, the Jensen work plays a fundamental role. Apart Weierstrass' work, the Hadamard one on factorization of entire functions started from previous work of Poincaré but was inspired by Riemann 1859 paper, to be precise by problematic raised by Riemann ξ function and its properties. The next work of Borel, then, based on Hadamard one. In this section, nevertheless, we outline only the main contributions respectively owned to Hadamard, Edmund Landau and George Pólya, the only ones who worked on that meeting land between the theory of Riemann zeta function and the theory of entire functions.

• *The contribution of J. Hadamard.* In section 5.1 of the previous chapter, we have briefly outlined the main content of the celebrated 1893 Hadamard memoir, where in the last and third part he deals with Riemann ξ function. Now, in this section, we wish to start with an historical deepening of this memoir, to carry on then with other remarkable works centered on the applications of entire function theory to Riemann zeta function issues, amongst which those achieved by H. Von Mangoldt, E. Landau, G. Pòlya, and others. Following (Narkiewicz 2000, Chapter 5, Section 5.1, Number 1), the last twenty years of the 19th century seen a rapid progress in the theory of complex functions, summed up in the monumental works of Émile Picard and Camille Jordan. The development of the theory of entire functions, started with pioneering 1876 Weierstrass work and rounded up by Hadamard in 1893, revived the interest in Riemann's memoir and forced attempts to use these new developments to solve questions left open by Riemann. This led to the first proofs of the *Prime Number Theorem* (PNT), early conjectured in the late 1700s by Gauss and Legendre independently of each other, in the form (d'après E. Landau) $\theta(x) = \sum_{p \leq x} \ln x = (1 + o(1))x$ obtained independently by Hadamard and de la Vallée-Poussin in 1896. They both started with establishing the non-vanishing of $\zeta(s)$ on the line $\Re s = 1$ but obtained this result in completely different ways, but with equivalent results (as pointed out by Von Schaper in his PhD dissertation of 1898). Also the deduction of the Prime Number Theorem from that result is differently done by them, even if, according to (Montgomery & Vaughan 2006, Chapter 6, Section 6.3) and (Bateman & Diamond 1996), the methods of Hadamard[56] and de la Vallée-Poussin depended on the analytic continuation of $\zeta(s)$, on bounds for the size of $\zeta(s)$ in the complex plane, and on Hadamard theory of entire functions. Anyway, also (Ayoub 1963, Chapter II, Section 6) claims that the original 1896 proof by de la Vallée-Poussin made use of the product formula provided by Hadamard work of 1893, in deducing an expression for $\zeta'(s)/\zeta(s)$. Likewise, in (Chen 2003, Chapter 6, Section 6.1), the author comments that Hadamard product representation played an important role in the first proof of prime number theorem. Hadamard). Finally, as also pointed out in (Itô 1993, Article 429, Section B), almost all the outcomes delineated above concerning entire functions, originated in the study of the zeros of the Riemann zeta function and constitute the beginning of the theory of entire functions. Therefore, due to its fundamental importance, we shall return back again in discussing upon this 1893 Hadamard work.

The members of the evaluation's commission of the annual *grand prix des sciences mathématiques* raffled by the French Academy of Sciences, namely Jordan, Poincaré, Hermite, Darboux and Picard, decided to award the celebrated 1893 Hadamard paper for having put attention to certain apparently minor questions treated by Riemann in his famous paper on number theory, from which arose new and unexpected results of entire function theory, as already said in the previous sections. In their report (see (Jordan et al. 1892)), published at the pages 1120-1122 of the Tome CXV, Number 25 of the *Comptes Rendus de l'Academie des*

[56]Furthermore, in (Ayoub 1963, Chapter II, Notes to Chapter II), the author says that it is worthwhile noting that Hadamard based his proof on that given by E. Cahen in (Cahen), where it is assumed the truth of the Riemann hypothesis, ascribing the ideas of his proof to G.H. Halphen.

Sciences de Paris in the year 1892, the relevance of the new complex function $\zeta(s)$ for studying number theory issues, introduced by Riemann in his 1859 celebrated seminal paper, have been pointed out together some its chief properties, just by Riemann himself but without providing any rigorous proof of them. In 1885, George H. Halphen (1844-1889) (see (Narkiewicz 2000, Chapter 4, Section 4.3) and references therein), referring to the latter unsolved Riemann questions, wrote that

«Avant qu'on sache établir le théorème de Riemann (et il est vraisemblable que Riemann ne l'a pas su faire), il faudra de nouveaux progrès sur une notion encore bien nouvelle, le genre des transcendantes entières».

Thus Hadamard, within the framework of the new entire function theory and in agreement with the above Halphen's consideration[57], proved one of these, determining the genus of the auxiliary $\xi(s)$ function which is as an entire function of the variable s^2 having genus zero but, at the same time, from an apparently minor issue (drawn from number theory), opening the way to new and fruitful directions in entire function theory (see also (Jordan et al. 1892, p. 1122)). Therefore, as it has been many times said above, the theory of entire function has plainly played - and still plays - a crucial and deep role in Riemann's theory of prime numbers whose unique 1859 number theory paper has been therefore one of the chief input for the development, on the one hand, of the number theory as well as, on the other hand, of the entire function theory itself with the next study of the Riemann ξ function and its infinite product factorization, opened by Hadamard work. Hence, we go on with a more particularized historical analysis of the 1893 celebrated Hadamard paper starting from the 1859 Riemann original memoir. Before all, we briefly recall the main points along which the Hadamard memoir lays down, referring to the previous chapter for more information. To be precise, Hadamard starts with the consideration that the decomposition of an entire function $f(x)$ into primary factors, achieved via Weierstrass' method

[57]Although Hadamard never quoted Halphen in his 1893 paper.

as follows

$$(16) \qquad f(x) = e^{g(x)} \prod_{j=1}^{\infty} \Big(1 - \frac{x}{\xi_j}\Big) e^{Q_j(x)},$$

leads to the notion of genus of an entire function, taking into consideration a result as early as achieved by Poincaré in 1883 (and already mentioned in the previous chapter), namely that, given an entire function of genus p, then the coefficient of x^m, say c_m, multiplied by $\sqrt[p+1]{m!}$, tends to zero as $m \to \infty$, as well as outcomes achieved by Picard, Hadamard expresses the intention to complete this Poincaré result, trying to find the general possible relations between the properties of an entire function and the laws of decreasing of its coefficients. In particular, as we have already been said in the previous chapter, Hadamard proves that, if c_m is lower than $(m!)^{-\frac{1}{p}}$, then it has, in general, a genus lower than p. Hence, in the first and second parts of the memoir, Hadamard proceeds finding relations between the decreasing law of the coefficients of the Taylor expansion of the given entire function $f(x)$ and its order of magnitude for high values of the variable x. Then, in the third and last part of his memoir, Hadamard applies what has been proved in the previous parts, to Riemann ξ function. Precisely, Hadamard reconsiders the 1859 Riemann paper in which he first introduces the function $\zeta(s)$ to study properties of number theory, from which he then obtained a particular entire function $\xi(s)$ defined by

$$(17) \qquad \xi(x) = \frac{1}{2} - \big(x^2 + \frac{1}{4}\big) \int_1^{\infty} \Psi(t) t^{-\frac{3}{4}} \cos\big(\frac{x}{2} \ln t\big) dt$$

with $\Psi(t) = \sum_{n=1}^{\infty} e^{-n^2 \pi t}$. Therefore, next Riemann's analysis lies on the main fact that such a function ξ, considered as a function of x^2, he says to have genus zero, but without providing right prove of this statement. The proof will be correctly given by Hadamard in the final part of his memoir through which a new and fruitful road to treat entire functions, through infinite product expansion, was opened.

From a historical viewpoint, the first extended treatises on history of number theory appeared in the early 1900s, with the two-volume treatise of Edmund Landau (1877-1933) (see (Landau 1909)) and the three-volume treatise of Leonard Eugene Dickson (1874-1954) (see (Dickson 1919-23)). In the preface to the first volume of his treatise, Landau highlights the great impulse given to the analytic number theory with the first rigorous outcomes achieved by Hadamard since the late 1880s on the basis of what exposed by Riemann in his 1859 celebrated paper, emphasizing the importance of the use of entire function method in number theory. For instance, in (Landau 1909, Band I, Erstes Kapitel, § 5.III-IV; Zweites Kapitel, § 8; Fünfzehntes Kapitel), infinite product expansions à la Weierstrass are extensively used to factorize Riemann ξ function, until Hadamard work. But, the truly first notable extended report (as called by Landau who quotes it in the preface to the first volume of his treatise) on number theory was the memoir of Gabriele Torelli (1849-1931), the first complete survey on number theory that was drawn up with a deep and wide historical perspective not owned by the next treatise on the subject. Also G.H. Hardy and E.M. Wright, in their monograph (Hardy & Wright 1960, Notes on Chapter XXII), state that *«There is also an elaborate account of the early history of the theory in Torelli, Sulla totalità dei numeri primi, Atti della R. Acad. di Napoli, (2) 11 (1902) pp. 1-222»,* even if then little attention is paid to Riemann's paper (with which analytic number theory officially was born (see (Weil 1975)), to which a few sections of chapter XVII are devoted. Therefore, herein we will recall the attention on this work, focusing on those arguments which are of our historical interest. Following (Marcolongo 1931) and (Cipolla 1932), Torelli[58] started his academic career at the University of Palermo in 1891, as a teacher of Infinitesimal Calculus and Algebraic Analysis. Afterwards, he moved to the University of Naples in 1907, as a successor of Ernesto Cesàro, until up his retirement in 1924. He chiefly made notable researches in algebra and infinitesimal calculus, upon which he wrote valid treatises. But, the most notable achievement of Torelli was memoir, entitled *Sulla totalità dei numeri primi fino ad un limite assegnato,* which is a monograph on the subject drawn up by the author when

[58]Not to be confused with Ruggero Torelli (1884-1915), his son, who gave remarkable contributions to complex algebraic geometry amongst which, for example, the celebrated *Torelli theorem* on projective algebraic curves (see (Maurin 1997)).

he was professor at the University of Palermo, for a competition called by the *Reale Accademia delle Scienze fisiche e matematiche di Napoli.* To be precise, as himself recall in the second cover of this monograph (with a dedication to Francesco Brioschi), such a work was drawn up to ask to the following question:

«Esporre, discutere, e coordinare, in forma possibilmente compendiosa, tutte le ricerche concernenti la determinazione della totalità dei numeri primi, apportando qualche notevole contributo alla conoscenza delle leggi secondo le quali questi numeri si distribuiscono tra i numeri interi».

[*«Explain, discuss and coordinate, in a compendious manner, the state of the art of all the researches concerning the determination of the totality of prime numbers, also personally concurring with some notable contribution to the knowledge of the distribution laws of prime numbers».*]

Until the publication of Edmund Landau treatise, Torelli report was the only monograph available at that time which covered the subject from Legendre work onwards. This work, for his novelty and importance, won the competition announced by *Reale Accademia delle Scienze fisiche e matematiche di Napoli* , hence it was published in the related Academy Acts (see (Torelli 1901)). As we have already said above, this monograph puts much attention and care to the related historical aspects, so that it is a valuable historiographical source for the subject. For our ends, we are interested in the chapter VIII, IX and X, where the works of Riemann and Hadamard are treated with carefulness.

Since Euclid times, one of the central problems of mathematics was to determine the totality of prime numbers less than a given assigned limit, say x. In approaching this problem, three main methods were available: a first one consisting in the effective explicit enumeration of such numbers, a second one which tries to determine this totality from the knowledge of a part of prime numbers, and a third one consisting in building up a function of x, say[59] $\theta(x)$, without explicitly knowing prime numbers, but whose values provide an estimate of the totality of such numbers less than x. To the first method, which had an empirical nature, little by little was supplanted by the other two methods, which were more analytical in their nature. Around the early 1800s, the method for determining the number of the primes through the third method based on $\theta(x)$ function, was believed the most important one even if it presented great formal difficulties of treatment. After the remarkable profuse efforts spent by Fermat, Euler, Legendre, Gauss, Von Mangoldt and Lejeune-Dirichlet, it was P.L. Tchebycheff[60] the first one to provide a powerful formal method for the determination of the function $\theta(x)$, even if he gave only asymptotic expressions which were unable to be used for finite values, also at approximate level. The function $\theta(x)$ plays a very fundamental role in solving the problem of determining the distribution laws of prime numbers, so that an explicit albeit approximate expression of it, was expected. This problem, however, constituted a very difficult task because such a function was extremely irregular, with an infinite number of discontinuities, and with the typical characteristic that, such a formula for $\theta(x)$, couldn't explicitly provide those points in which the prime numbers were placed. It was Riemann, in his famous 1859 paper, to give, through Cauchy's complex analysis techniques, a formula thanks to which it was possible, in turn, to deduce a first approximate expression for the θ function, valid for finite values of the variable. In any case, leaving out the formal details, Riemann was induced to introduce a complex function, that is to say the ζ function, to treat these number theory issues, bringing back the main points of the question to the zeros of this function. To be precise, for the determination of the

[59] It will be later called *prime number counting function,* and seems to have been introduced by Tchebycheff (see (Fine & Rosenberger 2007, Chapter 4, Section 4.3)). To be precise, $\pi(x) = \sum_{p\leq x} 1$ is the counting function for the set of primes not exceeding x, while $\theta(x) = \sum_{p\leq x} \ln x$ (see (Nathanson 2000, Chapter 8, Section 8.1)) is a Tchebycheff function.

[60] Besides, under advice of the mathematician Giuseppe Battagliani (1826-1894), in 1891 an Italian translation of an important monograph of Tchebycheff, with Italian title *Teoria delle congruenze* (see (Tchebycheff 1895)), was undertaken by Iginia Massarini, the first Italian woman to be graduated in mathematics from the University of Naples in 1887. Such a monograph, many times is quoted by Torelli in his memoir.

non-trivial zeros of this function, Riemann considered another function obtained by ζ through the functional equation to which it satisfies, the so-called Riemann ξ function, namely

$$\xi(t) = \frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\zeta(s) \tag{18}$$

which is an even entire function of t if one puts $s = 1/2 + it$. About such a function, Riemann enunciates the following three propositions

1. the number of zeros of $\xi(t)$, whose real part is comprised into $[0, T]$ with $T \in \mathbb{R}^+$, is approximately given by $(T/2\pi)\log(T/2\pi) - (T/2\pi)$,

2. all the zeros α_i of $\xi(t)$ are real,

3. we have a decomposition of the type

$$\xi(t) = \xi(0)\prod_{\nu=1}^{\infty}\Big(1 - \frac{t^2}{\alpha_i^2}\Big),$$

but without giving a correct proof of them. The points 1. and 3. will be proved later by Von Mangoldt and Hadamard, while the point 2. is the celebrated *Riemann hypothesis* which still resist to every attempt of prove or disprove. But, notwithstanding that, Riemann, assuming as true such three propositions, goes on in finding an approximate formula for the prime number counting function $\theta(x)$. For the proof of many other results of number theory discussed in his memoir, many times Riemann makes reference to such a ξ function and its properties, but without giving any detailed and corrected development of their prove, so constituting a truly seminal paper upon which a whole generation of future mathematicians will work on, amongst whom is Hadamard.

Soon after the appearance of Riemann memoir, Angelo Genocchi (see (Genocchi 1860)) published a paper in which contributed to clarify some obscure points of 1859 Riemann paper as well as observed some mistakes in Riemann memoir about ξ function, giving a detailed development of its expression as an entire function as deduced from a (Jacobi) theta function[61] transformation of the ζ function, hence pointing out some remarks concerning its infinite product factorization and related properties, referring to the well-known Briot and Bouquet treatise (see (Briot & Bouquet 1859)) as regards the infinite product factorization of the ξ function from the knowledge of the sequence of its zeros α_i. Nevertheless, this Genocchi's remark is not enough to give a rigorous and complete proof of the proposition 3. of above, thing that will be accomplished later by Hadamard in 1893. The words of Torelli (see (Torelli 1901, Chapter IX, Section 74)), in this regard, are very meaningful. Indeed, he says that Riemann assumed to be valid only statements 1. and 3. of above, while the statements 2. was considered to be uninteresting to the ends that Riemann wished to pursue. Hadamard was the first one to cope the very difficult task to solve the Riemann statements 1. and 3., achieving this with success in 1892 with a memoir presented to the Academy of Sciences of Paris. Indeed, he was able to brilliantly prove statement 3., from which he derived also of statement 1. as a corollary stated in the last part of the next paper *Étude sur les propriétés des fonctions entières et en particulier d'une fonctions considérée par Riemann,* which was published in 1893. As the title itself shows, the Hadamard work is a very new chapter of complex analysis in which are treated general properties of entire functions, so opening new directions in the theory of entire functions. In 1898, Hans Von Schaper, in his dissertation entitled *Über die Theorie der Hadamardschen Funktionen und ihre Anwendung auf das Problem der Primzahlen,* under the supervision of Hilbert, as well as Borel, in his important work *Leçons sur les fonctions entières* (see (Borel 1900)), reconsider this notable Hadamard paper as a starting point for a further deepening of the theory of entire functions of finite order, in particular simplifying the original proof of the above Riemann statement 3., as given by Hadamard in his 1893 paper. And in his report, Torelli considers Von Schaper proof of this statement, adopting the following terminology. If $a_1, a_2, ..., a_\nu, ...$ are non-zero complex numbers

[61] That is to say, $\vartheta(u) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 u}$, while $\omega(u) = \sum_{n\in\mathbb{N}} e^{-\pi n^2 u}$. These two function were introduced by Tchebychev in the 1850s (see (Goldstein 1973)).

arranged according to a non-decreasing modulus sequence tending to ∞, such that there the integer number $k+1$ is the lowest one such that $\sum_{\nu=1}^{\infty}(1/|a_\nu|^{k+1})$ converges, then we may consider the following absolutely and uniformly convergent infinite product

$$G(z) = \prod_{\nu=1}^{\infty}\Big(1-\frac{z}{a_\nu}\Big)e^{\sum_{j=1}^{k}\frac{1}{j!}\frac{z^j}{a_\nu^j}}. \tag{19}$$

Every entire function $F(z)$ having zeros $a_1, a_2, ..., a_n, ...$, must have the form $e^{H(z)}G(z)$, where $G(z)$ is given by (17) and $H(z)$ is also an entire function which may be a polynomial as well. When $H(z)$ is a polynomial of degree q, then the integer number $p = \max\{q,k\} < \infty$ is called, by Borel, the *genus* of the entire function $F(z)$, while Von Schaper speaks of *height* of $F(z)$; the entire functions of finite order are called *Hadamard's functions* by Von Schaper. If it is not possible to reduce $H(z)$ to a polynomial, or if the sequence of zeros a_ν of the given entire function $F(z)$ is such that the above integer $k+1$ does not exist, then we will say that $F(z)$ has *infinite genus.*

We are interested in entire functions of finite order, of which other two parameters have to be defined as follows. The upper lower bound of the integer numbers[62] k (or λ) such that, for any arbitrarily fixed $\varepsilon \in \mathbb{R}^+$, $\sum_{\nu=1}^{\infty}(1/|a_\nu|^{k+\varepsilon})$ converges and $\sum_{\nu=1}^{\infty}(1/|a_\nu|^{k-\varepsilon})$ diverges, is said to be (after Von Schaper) the *exponent of convergence* of the sequence of the zeros a_ν, or (after Borel) the *real order* of the function $F(z)$. The upper lower bound of the integer numbers[63] ρ such that, for any arbitrarily fixed $\varepsilon \in \mathbb{R}^+$, we have $|F(z)| < \exp(|z|^{\rho+\varepsilon})$ from a certain value of $|z|$ onwards, and $|F(z)| < \exp(|z|^{\rho-\varepsilon})$ into an infinite number of points z arbitrarily far[64], is called (after Borel) the *apparent order* of the function $F(z)$, while Von Schaper says that $F(z)$ is of the *exponential type* $\exp(|z|^\rho)$. Afterwards, Von Schaper and Borel proved a series of notable properties concerning the possible relationships between the above de-

[62] Which will be denoted by the same letter.

[63] Which will be denoted by the same letter.

[64] Taking into account the modern notions of lower and upper limit of a function, these last notions are nothing but those exposed in section 5.1.

fined four parameters p, q, k, ρ, which are summarizable as follows. We have the following properties

- $p \leq \rho$;
- If ρ is not an integer number, then $k = \rho$ and $p = [\rho]$ (= integer part of ρ);
- If ρ is an integer number, the the genus p is equal to ρ or to $\rho - 1$. We have $p = \rho - 1$ is and only if $q \leq \rho - 1$ and $\sum_{\nu=1}^{\infty}(1/|a_\nu|^k) < \infty$;
- $\rho = \max\{k, q\}$.

From all that, Torelli retraces the original Hadamard treatment of Riemann ξ function taking into account the just quoted above results achieved by Von Schaper and Borel, reaching to prove that the following even entire function in the variable t

$$\xi(t) = \frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{20}$$

is an entire function that, with respect to t^2, has an apparent order which cannot exceed 1/2, so that it follows that it has genus zero with respect to the variable t^2, if one puts $s = 1/2 + it$. Therefore, the infinite product expansion of the above Riemann statement 3., is now proved. Furthermore, the absence of exponential factors in this expansion into primary factors of the function $\xi(t)$, implies the existence of, at least, one root for $\xi(t) = 0$, while the expansion into an infinite series of increasing powers of t^2, implies too that such roots are in an infinite number. We have already said that Hadamard himself, in his celebrated 1893 memoir, also proved the above Riemann statement 1., even if further improvements were achieved later by H. Von Mangoldt (see (Von Mangoldt 1896)), J. Franel (see (Franel 1896)) and Borel (see (Borel 1897)). In conclusion, the pioneering Hadamard work contained in his 1893 memoir, has finally proved two out of the three above Riemann statements, namely the 1. and 3., to which other authors have later further contributed with notable improvements and extensions.

Following (Torelli 1909, Chapter VIII, Sections 71 and 72; Chapter IX, Section 77), as regards, instead, Riemann statement 2., that is to say, what will be later known as the *Riemann hypothesis,* first attempts to approach the solution of the equation $\xi(t) = 0$, were made by T.J. Stieltjes (see (Stieltjes 1885)), J.P. Gram (see (Gram 1895)), F. Mertens (see (Mertens 1897) and J.L.W. Jensen (see (Jensen 1898-99)). We are interested in the Jensen's work for its historical role played in the development of entire function theory, of which we have already said something about this in section 5.1. where it has been pointed out what fundamental role played this Jensen's work in the early developments of the theory of value distribution of entire and meromorphic functions as opened by R. Nevanlinna work. To be precise, in this Jensen's work of 1898, the author considers a meromorphic function, say $f(z)$, defined into a region of complex plane, say D, containing the zero and where such a function is neither zero nor infinite. Let $a_1, ..., a_n$ be the zeros and $b_1, ..., b_m$ be the poles of the function $f(z)$, counted with their respective multiplicity and supposed to be all included into a circle, say $\mathcal{C}_r$, given by $|z| \leq r$ centered in 0 and with radius r such that $\mathcal{C}_r \subseteq D$. Then, Jensen easily proves the following formula

$$\text{(21)} \qquad \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})|d\theta = \ln|f(0)| + \ln\frac{r^{n-m}|b_1 \cdot ... \cdot b_m|}{|a_1 \cdot ... \cdot a_n|}.$$

Now, Jensen argues that, if $f(z)$ is an entire function, then r may be chosen arbitrarily large in such a manner $\mathcal{C}_r$ does not contain any zero, so that the second term in the right hand side of (19) reduces only to the first, constat term. In doing so, we have thus a simple criterion for deciding on the absence or not of zeros within a given circle of complex plane. Once Jensen stated that, he finishes the paper announcing to have proved, through his previous researches on Dirichlet's series, that the function $\xi(t)$ does not have any zero within an arbitrary circle centered into the finite imaginary axis and comprehending the zero, which implies that $\xi(t) = 0$ has only real zeros, as Riemann conjectured. On the other hand, as we have already pointed out in the previous section 5.1., it is just from this Jensen formula that started the theory of value distribution of entire and meromorphic functions which was built by the pioneering work of Rolf Nevanlinna of 1920s. Often, in many treatise on entire and meromorphic functions, Jensen formula is the first key element from which to begin. Indeed, following (Zhang 1993, Chapter I), the theory of entire and meromorphic functions starts with Nevanlinna theory which, in turn, is based either on a particular transformation of the formula (19), called *Poisson-Jensen formula* by Nevanlinna[65], and on the previous works made by Poincaré, Hadamard and Borel on entire functions. Therefore, another central starting point of the theory of entire and meromorphic functions, as the one just examined above and due to Jensen, relies on the prickly problematic raised by Riemann ξ function. Lastly, the appreciated Torelli's monograph comes out with the following textual words

«Come conclusione di questo capitolo e dell'intero lavoro, si può senza alcun dubio affermare che la memoria di Riemann, insieme alle esplicazioni e i complimenti arrecati da Hadamard, Von Mangoldt, e de la Vallèe-Poussin, resta tuttora come il faro, che guidar possa nella scoperta di quanto ancora v'è di ignoto nella Teoria dei Numeri primi».

[*« As a conclusion of this chapter as well as of the whole work, surely we may state that the Riemann memoir, together all the explications and complements due to Hadamard, Von Mangoldt, and de la Vallèe-Poussin, still remains as that lighthouse which can drive towards the discovery of what yet is unknown in the theory of prime numbers»*].

Following (Ingham 1964, Introduction, 6.), as we have already said above Riemann enunciated a number of important theorems concerning the zeta function - i.e., the above Riemann statements 1., 2., and 3. - together with a remarkable relatioship connecting the prime number counting function with its zeros, but he gave in most cases only insufficient indications of proofs. These problems raised by Riemann memoir inspired, in due course, the fundamental researches of Hadamard in the

[65]Because either the original Jensen formula (19) and another formula of potential theory (*Poisson formula*) due to D.S. Poisson, are special cases of this formula due to Nevanlinna.

theory of entire functions, the results of which at last removed some of the obstacles which for more than thirty years had barred the way to rigorous proofs of Riemann statements. The proofs sketched by Riemann were completed (in essentials), in part by Hadamard himself in 1893, and in part by Von Mangoldt in 1894. Following (Ingham 1964, Introduction, 6.), as we have already said above Riemann enunciated a number of important theorems concerning the zeta function - i.e., the above Riemann statements 1., 2., and 3. - together with a remarkable relatioship connecting the prime number counting function with its zeros, but he gave in most cases only insufficient indications of proofs. These problems raised by Riemann memoir inspired, in due course, the fundamental researches of Hadamard in the theory of entire functions, the results of which at last removed some of the obstacles which for more than thirty years had barred the way to rigorous proofs of Riemann statements. The proofs sketched by Riemann were, in essentials, completed in part by Hadamard himself in 1893, and in part by Von Mangoldt in 1894. These discoveries due to Hadamard prepared the way for a rapid advance in the theory of the distribution of primes. The so-called *prime number theorem,* according to which $\theta(x) \sim x/\ln x$, was first proved in 1896 by Hadamard himself and by de la Vallée-Poussin, independently and almost simultaneously, the proof of the former having used the results achieved in his previous 1893 memoir. Out of the two proofs, Hadamard one is the simpler, but de la Vallée-Poussin, in another paper published in 1899 (see (de la Vallée-Poussin 1899-1900)), studied in great detail the question of closeness of approximation as well as gave further improvements to prime number theorem. Finally, either de la Vallée-Poussin work (see (de la Vallée-Poussin 1896)) and Von Mangoldt one (see (Von Mangoldt 1896) and (Von Mangoldt 1905)), used the results on entire function factorization achieved by Hadamard in his 1893 memoir (see (Maz'ya & Shaposhnikova 1998, Chapter 10, Section 10.1)). Following (Kudryavtseva 2005, Section 7), Riemann's paper is written in an extremely terse and difficult style, with huge intuitive leaps and many proofs omitted. This led to (in retrospect quite unfair) criticism by E. Landau and G.H. Hardy in the early 1900s, who commented that Riemann had only made conjectures and had proved almost nothing. The situation was greatly clarified in 1932 when C.L. Siegel (see (Siegel 1932)) published his paper, representing about two years of scholarly work studying Riemann's left over mathematical notes at the University of Göttingen, the so-called *Riemann's Nachlass.* From this study, it became clear that Riemann had done an immense amount of work related to his 1859 memoir and that never appeared in it. One conclusion is that many formulas that lacked sufficient proof in 1859 paper were in fact proved in these notes. A second conclusion is that the notes contained further discoveries of Riemann that were never even written up in the original memoir. One such is what is now called the *Riemann-Siegel formula,* which Riemann had written down and that Siegel (with great difficulty) was able to prove (see (Karatsuba 1994) and (Edwards 1974)). This formula, in essentials, arises from an Hankel integral type expression for $\xi(s)$, and gives a refined method to calculate $\xi(1/2+it)$, in comparison to previous ones. In any way, after Hadamard work on Riemann ξ function, only a few authors have put considerable attention to it, amongst whom are E. Landau and G. Pólya. Notwithstanding that, in the following sections we wish briefly to retrace the historical path which gather those main works on Riemann ξ function which will lead to a particular unexpected result of mathematical physics, to be precise belonging to statistical mechanics which, in turn, has opened a new possible line of approach Riemann hypothesis.

In conclusion, we wish to textually report what retrospectively Hadamard himself says about his previous work on entire function, following (Hadamard 1901, Chapter I), that is to say

«Les formules démontrées dans ma thèse relativement aux singularités polaires[66] *ont trouvé une application immédiate dans un Mémoire*

[66] To be precise, we report an excerpt of the last part of the Hadamard discussion about his 1892 thesis on Taylor development of functions (see (Hadamard 1892)), from which he starts to discuss his next work on entire functions. He states that *«A partir de la publication de ma Thèse, l'attention des géomètres s'est portée sur ce sujet. Grâce aux travaux de MM. Borel, Fabry, Leau, Lindelöf, et à la découverte de M. Mittag-Leffler, cette théorie, qui n'existait pas en 1892, forme aujourd'hui un chapitre assez important de la Théorie des fonctions, celui de tous (avec la théorie des Fonctions entières dont il va être question plus loin) qui, dans ces dernières années, a acquis à la Science le plus grand nombre de résultats. Une très grande partie de ceux-ci ont d'ailleurs été obtenus par*

auquel l'Académie a décerné, en 1892, le grand prix des Sciences mathématiques. La question posée par l'Académie, et qui portait sur une fonction employée par Riemann, soulevait un problème général: celui du genre des fonctions entières. Ou sait que la notion de genre est liée au théorème de Weierstrass d'après lequel toute fonction entière $F(x)$ peut être mise sous forme d'un produit de facteurs (facteurs primaires)

$$F(x) = e^{G(x)} \prod_n \left[\left(1 - \frac{x}{a_n}\right) e^{P_n\left(\frac{x}{a_n}\right)} \right]$$

(où $G(x)$ est une nouvelle fonction entière et les $P(x)$ des polynômes): décomposition analogue à celle d'un polynôme en ses facteurs linéaires.

Si l'on peut s'arranger les polynômes $P(x)$ soient de degré E au plus, la fonction $G(x)$ se réduisant elle-même à un polynôme de degré égal ou inférieur à E, la fonction $F(x)$ est dite (Il est sous-entendu que E doit être le plus petit entier satisfaisant aux conditions indiquées) de genre E. Il est nécessaire pour cela (mais non suffisant) que les racines $a_1, a_2, ..., a_n$ de l'équation $F(x) = 0$ ne soient pas trop rapprochées les unes des autres: la série $\sum(1/|a_n|)^{E+1}$ doit être convergente. M. Poincaré a donné, en 1883 (Bull. de la Soc. math. de France), une condition nécessaire pour qu'une fonction $F(x)$ soit de genre E; cette condition est que les coefficients du développement de $F(x)$ décroissent au moins aussi vite que les valeurs successives de $1/(m!)^{1/(E+1)}$. Cette condition nécessaire était-elle la condition nécessaire et suffisante pour que la fonction fût au plus de genre E? Etant donnée la manière compliquée dont les racines d'une équation dépendent de ses coefficients, il semblait hautement improbable que la réponse fût aussi simple, ni surtout qu'elle fût aisée à obtenir. C'était elle qui avait manqué à Halphen pour continuer les recherches qu'il avait commencées en 1883,

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le développement des méthodes mêmes que j'avais indiquées. Je n'ai pas perdu de vue, dans la suite, cette catégorie de questions, et, en 1897, j'ai démontré, également par la considération d'une intégrale définie, un théorème qui fait connaître les singularités possibles de la série $\sum a_i b_i x^i$ quand on connaît celles de la série $\sum a_i x^i$ et de la série $\sum b_i x^i$. Celte proposition dérive évidemment du même principe que le théorème mentionné en dernier lieu; comme lui, elle offre cet avantage de s'appliquer à toute l'étendue du plan. Aussi, ce travail a-t-il attiré l'attention des géomètres sur le principe en question et provoqué une nouvelle série de recherches ayant pour objet d'en obtenir de nouvelles applications».

sur les travaux de Riemann. La Commission (M. Picard, rapporteur) chargée déjuger le concours de 1892 rappelait, dans son Rapport, l'exemple d'Halphen et faisait observer combien il semblait peu vraisemblable au premier abord que l'on pût donner une réciproque au théorème de M. Poincaré. De son côté, ce dernier, dans le Mémoire précédemment cité, après s'être posé une question étroitement liée à la précédente, celle de savoir si la dérivée d'une fonction de genre E, ou la somme de deux fonctions de genre E, est également du même genre, ajoutait: «Ces théorèmes, en admettant qu'ils soient vrais, seraient très difficiles à démontrer».

Le problème qui consiste à déterminer le genre d'une fonction entière donnée par son développement en série de puissances se rattache d'une manière évidente aux recherches dont j'ai parlé jusqu'ici, puisque celles-ci ont pour objet général l'étude d'une série de Maclaurin donnée a priori. J'ai pu effectuer celle détermination on toulc rigueur dans le Mémoire soumis au jugement de l'Académie. Désormais, la théorie des fonctions entières est, au point de vue des zéros, toute parallèle à celle des polynômes. Le genre (ou, plus généralement, l'ordre décroissance) joue le rôle du degré, la distribution des zéros de la fonction étant en général réglée par ce genre comme le nombre des zéros d'un polynome par son degré. Dans un article ultérieur [see (Hadamard 1896c)], *j'ai précisé et simplifié la loi qui donne la croissance du module de la fonction lorsqu'on donne la suite des coefficients et qui joue un rôle important dans ces recherches.* [...] *Quant aux questions posées par M. Poincaré et relatives à la conservation du genre dans la dérivation ou dans les combinaisons linéaires, elles ne sont pas, il est vrai, résolues d'une façon tout à fait complète par les théorèmes dont je viens de parler, et ne sauraient, d'ailleurs, l'être par des méthodes de cette nature. Mais on peut dire qu'elles sont résolues en pratique. D'une part, en effet, les cas qui échappent aux méthodes précédentes sont tout exceptionnels, d'autre part, l'hésitation ne peut jamais être que d'une unité sur le genre cherché.*

Again, Hadamard goes on, recalling what follows

«Du théorème relatif au rayon de convergence d'une série entière dé-

coule celte conséquence: la condition nécessaire et suffisante pour qu'une série de Maclaurin représente une fonction entière est que la racine m^{ieme} *du coefficient de* x^m *tende vers 0. Les propriétés les plus importantes de la fonction entière sont liées à la plus ou moins grande rapidité avec laquelle a lieu cette décroissance des coefficients. L'étude de ces propriétés consiste tout d'abord dans l'établissement de relations entre cette loi de décroissance et les deux éléments suivants : 1º L'ordre de grandeur du module maximum de la fonction pour les grandes valeurs du module de la variable; 2º La distribution des zéros et la valeur du genre, laquelle est étroitement liée à cette distribution. Une partie de ces relations avait été établie dans le Mémoire cité de M. Poincaré: une limite supérieure des coefficients successifs avait pu être trouvée, connaissant l'une ou l'autre des deux lois qui viennent d'être énumérées. Mais on n'avait pas pu, depuis ce moment, obtenir les réciproques, c'est-à-dire déduire d'une limite supérieure supposée connue pour chaque coefficient les conséquences qui en découlent, d'une part quant à l'ordre de grandeur de la fonction ellemême, d'autre part quant à la distribution de ses zéros. C'est à l'établissement de ces conséquences qu'est principalement consacré le Mémoire couronné par l'Académie en 1892 et publié en 1893 au Journal de Mathématiques. J'ai ensuite précisé les premières dans la Note* [see (Hadamard 1896c)] *insérée au Bulletin de la Société Mathématique de France et dont j'ai également parlé dans l'Introduction. Quant aux zéros, les résultats contenus dans ma Thèse fournissaient aisément à leur égard cette conclusion simple: La loi de croissance des racines de la fonction entière* $\sum a_m x^m$ *est au moins aussi rapide que celle des quantités* $1/\sqrt[m]{|a_m|}$. *Pour étudier le facteur exponentiel, de nouvelles déductions ont, au contraire, été nécessaires. Ces déductions m'ont, en particulier, permis de démontrer, avec une extrême simplicité, le théorème de M. Picard sur les fonctions entières, pour toutes les fonctions de genre fini. La démonstration ainsi donnée s'étend d'elle-même, moyennant une restriction analogue, au théorème plus général du même auteur sur le point essentiel, ainsi que je l'ai montré depuis* [see (Hadamard 1896b)]. *On sait que mon Mémoire de 1893 a été le point de départ des si importants travaux de M. Borel, consacrés à la démonstra-*

tion du premier théorème de M. Picard sans restriction, et aussi de ceux de MM. Schou et Jensen. Outre les applications à la fonction $\zeta(s)$ et aux fonctions analogues, dont il me reste à parler, la proposition fondamentale de ce Mémoire a été utilisée par M. Poincaré dans une question relative aux déterminants infinis qui s'introduisent en Astronomie (Les méthodes nouvelles de la Mécanique céleste, t. II)».

Whereupon, in a brief but complete manner, Hadamard discusses his work on Riemann ζ function, stating first that

«Le dernier anneau de la chaîne de déductions commencée dans ma Thèse et continuée dans mon Mémoire couronné aboutit à l'éclaircissement des propriétés les plus importantes de la fonction $\zeta(s)$ de Riemann. Par la considération de cette fonction, Riemann détermine la loi asymptotique de fréquence des nombres premiers. Mais son raisonnement suppose : 1^o que la fonction $\zeta(s)$ a des zéros eu nombre infini; 2^o que les modules successifs de ces zéros croissent à peu près comme $n \ln n$; 3^o que, dans l'expression de la fonction auxiliaire $\xi(t)$ en facteurs primaires, aucun facteur exponentiel ne s'introduit. Ces propositions étant restées sans démonstration, les résultats de Riemann restaient complètement hypothétiques, et il n'en pouvait être recherché d'autres dans cette voie. De fait, aucun essai n'avait été tenté dans cet ordre d'idées depuis le Mémoire de Riemann, à l'exception : 1^o de la Note précédemment citée d'Halphen, qui était, en somme, un projet de recherches pour le cas où les postulats de Riemann seraient établis; 2^o d'une Note de Stieltjes, où ce géomètre annonçait une démonstration de la réalité des racines de $\xi(t)$, démonstration qui n'a jamais été produite depuis. Or les propositions dont j'ai rappelé tout à l'heure l'énoncé ne sont qu'une application évidente des théorèmes généraux contenus dans mon Mémoire.

Une fois ces propositions établies, la théorie analytique des nombres premiers put, après un arrêt de trente ans, prendre un nouvel essor; elle n'a cessé, depuis ce moment, de faire de rapides progrès. C'est ainsi que la connaissance du genre de $\zeta(s)$ a permis, tout d'abord, à M. Von Mangoldt d'établir en toute rigueur le résultat final du Mémoire de Riemann. Auparavant, M. Cahen avait fait un premier pas vers la solution du prob-

lème posé par Halphen; mais il n'avait pu arriver complètement au but: il fallait, en effet, pour achever de construire d'une façon inattaquable le raisonnement d'Halphen, prouver encore que la fonction ζ, n'avait pas de zéro sur la droite $R(s) = 1$. J'ai pu vaincre cette dernière difficulté en 1896, pendant que M. de la Vallée-Poussin parvenait de son côté au même résultat. La démonstration que j'ai donnée est d'ailleurs de beaucoup la plus rapide et M. de la Vallée-Poussin l'a adoptée dans ses publications ultérieures. Elle n'utilise que les propriétés les plus simples de $\zeta(s)$. En même temps, j'étendais le raisonnement aux séries de Dirichlet et, par conséquent, déterminais la loi de distribution des nombres premiers clans une progression arithmétique quelconque, puis je montrais que ce raisonnement s'appliquait de lui-même aux formes quadratiques à déterminant négatif. Les mêmes théorèmes généraux sur les fonctions entières ont permis, depuis, à M. de la Vallée-Poussin d'achever ce cycle de démonstrations en traitant le cas des formes a $b^2 - ac$ positif».

Then, Hadamard recalls furthermore that

«La détermination du genre de la fonction $\zeta(s)$ - et c'était d'ailleurs l'objet môme de la question posée par l'Académie - était nécessaire pour l'éclaircissement des points principaux du Mémoire principal de Riemann "Sur le nombre des nombres premiers inférieurs à une grandeur donnée". Cette détermination, qui avait été jusque-là cherchée en vain, s'eirectue sans aucune difficulté à l'aide des principes précédemment établis sur les fonctions entières. Aussi M. Von Mangoldt put-il peu après établir avec une entière rigueur les résultats énoncés par Riemann. Un seul point restait à élucider: la question de savoir si, conformément à une assertion énuse, en passant, par ce grand géomètre, les racines imaginaires de l'équation $\zeta(s) = 0$ sont toutes de la forme $1/2 + it$, t étant réel. Cette question n'a pas encore reçu de réponse décisive (le Mémoire dans lequel M. Jensen annonce qu'il donnera ce résultat n'ayant pas encore paru); mais j'ai pu en 1896 [see (Hadamard 1896a) and references therein] *établir que la partie réelle des racines dont il s'agit, laquelle n'est évidemment pas supérieure à l'unité, ne peut non plus, pour aucune d'elles, être égale à 1. Or ce résultat suffit pour établir les prin-*

cipales lois asymptotiques de la théorie des nombres premiers, de même que le résultat complet de Riemann conduirait à montrer (voir un Mémoire récent de M. Helge von Koch) que ces lois sont vraies à une erreur près, laquelle n'est pas seulement d'ordre inférieur à celui de la quantité considérée x*, mais est tout au plus comparable à* $\sqrt{x}$.

De plus le mode de démonstration que j'emploie n'utilise (pie les propriétés les plus simples de la fonction $\zeta(s)$*. Il en résulte que ce mode de démonstration s'étend sans grande difficulté aux séries analogues qui ont été ulilisées dans la théorie des nombres. J'ai fait voir en particulier, dans le même travail, qu'il s'applique aux séries qui servent à étudier la distribution des nombres premiers représentables soit par une forme linéaire (séries de Dirichlet*[67]*), soit par une forme quadratique définie. M. de la Vallée-Poussin parvenait en même temps au même résultat, mais par une voie moins rapide. Depuis, ce savant (tout en simplifiant son Analyse par l'emploi du mode de raisonnement que j'avais indiqué) a étendu ses recherches au cas des formes quadratiques indéfinies et aussi à celui où l'on donne à la fois une forme linéaire et une forme quadratique; de sorte que les mêmes principes relatifs aux fonctions entières servent de base à la solution générale de toutes les questions qui s'étaient posées relativement à la distribution des nombres premiers. Ce ne sont d'ailleurs pas les seules questions de Théorie des nombres pour la solution desquelles les théorèmes qui viennent d'être rappelés se soient montrés d'une importance essentielle. Je me contente de signaler, à cet égard, les Mémoires récents de MM. von Mangoldt, Landau, etc».*

In any event, Hadamard work until early 1900s was always influenced by 1859 Riemann's memoir: indeed, as remembers (Mandelbrojt & Schwartz 1965), for instance

«Hadamard's theorem on composition of singularities was proved in 1898. When stated without much rigour, it reads as follows. $\sum a_n b_n z^n$

[67] *Pour démontrer le théorème relatif à la distribution des nombres premiers dans une progression arithmétique, j'ai utilisé, en la complétant sur un point, la proposition de M. Lipschitz qui établit, pour les séries de Dirichlet, une relation fonctionnelle analogue à celle de Riemann-Schlömilch. J'ai été conduit, depuis le travail* [see (Hadamard 1897)], *à simplifier la démonstration de ce théorème.*

has no other singularities than those which can be expressed as products of the form $\alpha\beta$, where α is a singularity of $\sum a_n z^n$ and β a singularity of $\sum b_n z^n$. The theorem is proved by the use of Parseval's integral, which Hadamard adapted to Dirichlet series (in works of 1898 and 1928), not for the research of the singularities of the composite series, but for the study of interesting relationships between the values of Riemann's ζ function at different points, or between different types of ζ functions».

Then, Mandelbrojt and Schwartz recall further that

«The year 1892 is one of the richest in the history of Function Theory, since then not only did Hadamard's thesis appear, but also his famous work on entire functions, which enabled him, a few years later (in 1896), to solve one of the oldest and most important problems in the Theory of Numbers. The general results obtained, establishing a relationship between the rate of decrease of the moduli of the coefficients of an entire function and its genus (the converse to Poincaré's theorem), applied to the entire function $\xi(z)$, related to $\zeta(s)$, shows that its genus, considered as a function of z^2, is (as stated, but not proved correctly, by Riemann) zero. This relationship (for general entire functions) between the moduli of the zeros of an entire function and the rate of decrease of its coefficients is obtained by using the results of the Thèse [see (Hadamard 1892)], *and concerning the determinants $D_{n,m}$ of a suitable meromorphic function (the reciprocal of the considered entire function). This paper on entire functions was written for the Grand Prix de l'Académie des Sciences in 1892 for studying the function $\pi(x)$. As a matter of fact, the mathematical world in Paris was sure that Stieltjes would get the prize, since Stieltjes thought that he had proved the famous "Riemannische Vermutung", and it is interesting, I believe, to quote a sentence from Hadamard's extremely famous paper of 1896 with the suggestive title, "Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques". Hadamard writes: "Stieltjes avait démontré, conformément aux prévisions de Riemann, que ces zéros sont tous de la forme $1/2 + it$ (le nombre t étant réel), mais sa démonstration n'a jamais été publiée, et il n'a même pas été établi que la fonction ζ n'ait pas de zéros*

sur la droite $R(s) = 1$. C'est cette dernière conclusion que je me propose de démontrer.

The "modesty" and the grandeur, of the purpose: to prove that $\zeta(s) \neq 0$ for $\sigma = l$ $(s = \sigma + it)$, after the assertion that Stieltjes had "proved" the Riemannische Vermutung, are remarkably moving. The more so that, due to this proof, Hadamard could prove, in the same paper of 1896, the most important proposition on the distribution of primes: $\pi(x)$ being the number of primes smaller than x $(x > 0)$, $\pi(x) \sim x/\ln x$ $(x \to \infty)$. The event had certainly a great historical bearing. The assumption was made, at the beginning of the last century, by Legendre (in the form $\pi(x) = x/(\ln x - A(x))$, with $A(x)$ tending to a finite limit). Tchebycheff had shown that $0.92129 \leq \pi(x) \ln x/x \leq 1.10555$ ..., but did not prove that the expression tends to a limit, and there was no hope that his method could yield any such proof. Many mathematicians, Sylvester among them, were able, in using the same methods as Tchebycheff, to improve these inequalities. But there was nothing fundamentally new in these improvements. Let us quote Sylvester (in 1881) on this matter (quotation given by Landau). "But to pronounce with certainty upon the existence of such possibility ($\lim \pi(x) \ln x/x = 1$) we should probably have to wait until someone is born into the world as far surpassing Tchebycheff in insight and penetration as Tchebycheff proved himself superior in these qualities to the ordinary run of mankind". And, as Landau says, when Sylvester wrote these words Hadamard was already born. It should be pointed out that independently, and at the same time, de La Vallée-Poussin also proved the non-vanishing of ζ on $\sigma = 1$ and, thus, the prime-number theorem; however, Hadamard's proof is much simpler. Hadamard's study of the behavior of the set of zeros of $\zeta(s)$ is based on his result quoted above (proved in his paper of 1892, written for the Grand Prix), on the genus of $\xi(z)$. It seems to me of importance to insist upon the "chain of events" in Hadamard's discoveries: relationship between the position of the poles of a meromorphic function and the coefficients of its Taylor series; this result yields later the genus of an entire function by the rate of decrease of its Taylor coefficients; and from there, four years later, the important properties of $\zeta(s)$, and finally, as a consequence, the prime-number theorem. Clearly, one of the most *beautiful theories on analytic continuation, so important by itself, and so rich by its own consequences, seems to have been directed in Hadamard's mind, consciously or not, towards one aim: the prime-number theorem. He proved also the analogous theorems on the distribution of primes belonging to a given arithmetical progression, since by his methods he was able to study Dirichlet series which, with respect to these primes, play the same role as the ζ function plays with respect to all primes».*

We have above reported only two of the numerous witnesses about the important Hadamard work, namely (Mandelbrojt & Schwartz 1965) and (Mandelbrojt 1967), just those two strangely enough not quoted in (Maz'ya & Shaposhnikova 1998) (to which we however refer for the bibliographical richness and completeness on the subject), however the most complete and updated human and scientific biography devoted to the "living legend" in mathematics as Hardy defined Hadamard when introduced him to the London Mathematical Society in 1944 (see (Kahane 1991)). Finally, in (Ayoub 1963, Chapter II, Section 4), the author says that, relatively to (18) (see also next (54)), such a formula was not proved rigorously until about 1892 when Hadamard constructed his general theory of entire functions, and that Landau showed that it is possible to avoid this theory. Moreover, in (Meier & Steuding 2009), the authors, in relation to this Hadamard work of 1893, simply note that his general theory for zeros of entire functions forms now an important part of complex analysis, while (Rademacher 1973, Chapter 7, Section 56) points out that the existence of infinitely many non-trivial zeros of $\zeta(s)$ is usually proved through the application of Hadamard's theory of entire functions to the function $s(s-1)\Phi(s)$ where $\Phi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, whereas usual arguments connect it with the existence of infinitely many primes. Of course Hadamard's theory yields much more than merely the existence of infinitely many non-trivial zeros. Riemann conjectured, and Von Mangoldt proved in 1905 (see later), that the number $N(T)$ of zeros $\rho = \beta + i\gamma$ with $0 < \gamma \leq T$ is

$$N(T) = \frac{T}{2\pi}\Big(\ln\frac{T}{2\pi} - 1\Big) + O(\ln T),$$

and Hardy showed first in 1914 (see (Hardy 1914)) that infinitely many of these zeros lie on the middle line $\sigma = 1/2$ of the critical strip, while a better estimate in this direction was later provided by A. Selberg's theorem according to which the number $N(T)$ of zeros on $\sigma = 1/2$ satisfies $N_0(T) > AT \ln T$ for a certain positive A. Selberg's result states too that among all non-trivial zeros, those on the middle-line have a positive density.

• *The contributions of H. Von Mangoldt, E. Landau, and others.* After the pioneering 1893 work of Hadamard on Riemann ξ function, and its next fruitful application to prove prime number theorem by Hadamard himself, de la Vallée-Poussin and Von Mangoldt in the late 1890s, a few works were carried out on Riemann ξ function, except some researches which date back to early 1900s. Amongst the latter are the works of Edmund Landau (1877-1938), namely (Landau 1909a) and (Landau 1927). These works were accomplished by Landau after the publication of his famous treatise on the theory of prime numbers, namely (Landau 1909b), where, therefore, these are not quoted but, nevertheless, a deep and complete treatment of product formulas for $\xi(s)$ function is given (see (Landau 1909b, Band I, §§ 67-81)), mainly reporting what was known until then, that is to say, Hadamard work, as well as its applications to number theory issues by Hadamard himself, de la Vallée-Poussin and Von Mangoldt. In particular, Landau showed that, to estimate $\zeta'(s)/\zeta(s)$, it was enough to study behavior of $\zeta(s)$ for $\Re s > 0$ considering, as done by Riemann, the integral

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \ln \zeta(s) \frac{x^s}{s} ds,$$

where $a > 1$ and x is not an integer, avoiding entire function theory (see (Ayoub 1963, Chapter II, Sections 4 and 6)), applied to ξ function, according to 1893 Hadamard route. In any case, almost all the subsequent works on Riemann ξ function were turned towards applications to number theory, above all to estimate the number of zeros of Riemann ζ function within a given finite region of complex plane. Following (Burkhardt et al. 1899-1927, II.C.8, pp. 759-779), such estimates were achieved considering the following product formula

$$(s-1)\zeta(s)=\frac{1}{2}e^{bs}\frac{1}{\Gamma\left(\frac{s}{2}+1\right)}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}} \tag{22}$$

from which it is possible to deduce the following one

$$\frac{\zeta'(s)}{\zeta(s)}=b-\frac{1}{s-1}-\frac{1}{2}\frac{\Gamma\left(1+\frac{s}{2}\right)}{\Gamma\left(1+\frac{s}{2}\right)}+\sum_{\rho}\left(\frac{1}{\rho}+\frac{1}{s-\rho}\right) \tag{23}$$

which will play a fundamental role in the subsequent analytic number theory researches. As an ansatz already given by Riemann in his 1859 memoir and correctly proved, for the first time, by Hadamard, in 1893, through (20), it was possible to estimate the number of zeros of $\zeta(s)$ as follows (see (Landau 1909a, Section 2))

$$N(T)=\frac{1}{2\pi}T\ln T-\frac{1+\ln 2\pi}{2\pi}T+O(\ln T) \tag{24}$$

where $N(T)$ is the number of zeros $s+it$ of $\zeta(s)$ with $0<\sigma<1$ and $0<t\leq T$. Subsequently, Von Mangoldt (see (Von Mangoldt 1905)) improved this estimate through properties of either the ζ functional equation and the Gamma function, proving that

$$N(T)=\frac{1}{2\pi}T\ln T-\frac{1+\ln 2\pi}{2\pi}T+\frac{1}{2\pi i}\int_{-1+iT}^{2+iT}\frac{\zeta'(s)}{\zeta(s)}ds+O(1), \tag{25}$$

an estimate which will be further improved later by E. Landau in the years 1908-09 (see (Landau 1908; 1909a)) as well as by C. Hermite and J. Stieltjes in the early 1905 (see (Landau 1909a) and references therein). In any event, either Von Mangoldt and Landau have based their works on the Hadamard one upon entire functions, whereas the next works of R. Backlund (see (Backlund 1914; 1918)) gave too further improvements of this estimate without appealing to Hadamard work but simply on the basis of approximation properties of $\zeta(s)$. Further contributions to this subject, were also given later by J.P. Gram, H. Cramer, H. Bohr, J.L.W. Jensen, J.E. Littlewood, F. Nevanlinna and R. Nevanlinna (see (Borel 1921, Chapter IV), (Nevanlinna & Nevanlinna 1924) and references therein). Von Mangoldt, in the long-paper (Von Mangoldt 1896), considered Hadamard's work of 1893 for estimating the number of zeros of zeta function into a given finite region of the complex plane, starting from the previous estimate already provided by Hadamard himself, which will be extended and improved by Von Mangoldt, then deepening and extending the various number theory issues considered by Riemann in his 1859 original memoir, together to what will be achieved in the next papers (Von Mangoldt 1898; 1905) in which the author takes into account the well-known Hadamard's and de la Vallée-Poussin's works of 1896 even to number theory issues related to estimates of the number of zeros of Riemann ζ function via ξ function. Afterwards, Edmund Landau began to consider the previous work of Hadamard, de la Vallée-Poussin and Von Mangoldt on analytic number theory, drawing up a first long-paper in 16 sections (see (Landau 1908)) in which are gathered a great number of methods and applications of the entire function theory to number-theoretic questions and where, in particular, a detailed treatment of the Riemann ξ function is achieved, taking into account the related properties of entire functions - as, for example, exposed in the 1906 German translation of the G. Vivanti treatise (see (Vivanti 1901)), which is quoted in (Landau 1908, p. 199, footnote [52]) - even in view of their applications to prime number distribution theory questions, but with a considerable attention to Hadamard's papers of 1893 and 1896. This 1908 Landau paper may be considered as a first little monograph on analytic number theory applied to the distribution of prime numbers, which will be shortly afterwards followed by the more consistent treatise (Landau 1909b); in such a paper, Landau, amongst other things, improves and rectifies previous Von Mangoldt's formula to estimate the number of Riemann zeta function zeros through the ratio ζ'/ζ, as well as provides a proof of the so-called *explicit formula* for the difference $\pi(x)-\mathrm{li}(x)$ between the prime-counting function $\pi(x)$ and the function $\mathrm{li}(x) = \int_0^x \frac{dt}{\ln t}$. In the next Landau's paper of 1909 (see

(Landau 1909a)), the author starts with a consideration of a product expansion of the type (22) till to reach expression (23), on the basis of what already made in his previous paper of 1908. Landau begins with the consideration of some estimates achieved by Hadamard and Von Mangoldt, extending this to the case of a general Dirichlet's series. In the subsequent paper of 1927 (see (Landau 1927)), Landau starts with the consideration of a work due to the Indian mathematician K. Ananda-Rau (1893-1966), namely (Ananda-Rau 1924), saying that he was the first to consider the following type of Hadamard's infinite product expansion of Riemann ξ function

$$(s-1)\zeta(s) = e^{H(s)} \prod_{\rho} \Big(1 - \frac{s}{\rho}\Big) e^{\frac{s}{\rho}} \tag{26}$$

where $H(s)$ is an arbitrary linear function, then applying this case study to Riemann's ξ function and for further estimations of its zeros. The paper (Aranda-Rau 1924) mainly is centered around the getting of the equation (22) by means of Jensen's formula and some previous results achieved by Landau himself (to which Ananda-Rau refers mentioning (Landau 1909b)), rather that Hadamard's theory of entire functions (see (Narkiewicz 2000, Chapter 5, Section 5.1, Number 2)), with formal procedure which will be further improved by Landau himself in (Landau 1927). Finally, following (Titchmarsh 1986) and references therein, further works on Riemann ξ function were pursued by G.H. Hardy, E.K. Haviland, N. Koshlyakov, O. Miyatake, S. Ramanujan, A. Wintner (see (Wintner 1935; 1936; 1947)) and N. Levinson, most of them even referring to the previous pioneering 1893 work of J. Hadamard on entire functions.

• *The contribution of G. Pólya.* After the contributions by Von Mangoldt and Landau to Riemann ξ function as sketchily delineated above, for our historical ends we should further deepen the next contribution to Riemann ξ function due to George Pólya (1887-1985) in 1926, with the paper (Pólya 1926), and that will be the truly joint point between the entire function theory and the so-called Lee-Yang theorems. This paper, entitled *Bemerkung über die Integraldarstellung der Riemannschen*

ξ-*Funktion,* has been considered a minor contribution of Pólya to the Riemann zeta function theory but, as we will see later, it instead has contributed to open a new avenue in mathematical physics with very interesting methods and outcomes which, successively, will turn out to be useful also for some attempts to solve Riemann conjecture itself. Also following (Hejhal 1990, Introduction), Pólya's paper starts with the consideration of a Fourier integral representation of $\xi(1/2+it)$ to develop a very tantalizing result in the general direction of the Riemann hypothesis. To be precise, Pólya starts with the following transformation of the Riemann ξ function (see also (Titchmarsh 1986, Chapter X, Section 10.1))

$$\xi(iz) = \frac{1}{2}\Big(z^2 - \frac{1}{4}\Big)\pi^{-\frac{z}{2}-\frac{1}{4}}\Gamma\Big(\frac{z}{2}+\frac{1}{4}\Big)\zeta\Big(\frac{1}{2}+z\Big), \tag{27}$$

hence he considers the following integral transformation

$$\xi(z) = 2\int_0^\infty \Phi(u)\cos zu du \tag{28}$$

where

$$\Phi(u) = 2\pi e^{\frac{5u}{2}}\sum_{n=1}^{\infty}\big(2\pi e^{2u}n^2 - 3\big)n^2 e^{-n^2\pi e^{2u}} \tag{29}$$

with

$$\Phi(u) \sim 4\pi^2 e^{\frac{9u}{2}-\pi e^{2u}} \qquad \text{as} \quad u \to +\infty, \tag{30}$$

so that, due to the parity condition of this last asymptotic approximation, we have as well

$$\Phi(u) \sim 4\pi^2\Big(e^{\frac{9u}{2}} + e^{-\frac{9u}{2}}\Big)e^{-\pi(e^{2u}+e^{-2u})} \qquad \text{as} \quad u \to \pm\infty. \tag{31}$$

Afterwards, Pólya deals with an approximation formula for ξ obtained from (29) considering a finite sum of N terms rather than an infinite series, and with most of exponentials replaced by hyperbolic cosines (see

(Haglund 2009, Section 1) and (Balazard 2010)). Through this ansatz, Pólya proves that the resulting integral (28) has only real zeros when $N = 1$ (this result will asymptotically extended to an arbitrary N finite by D.A. Hejhal in (Hejhal 1990)). Pólya also showed that if we replace $\Phi(u)$ by any function which is not an even function of u, then the resulting integral (28) has only finitely many real zeros. In regard to the so-called *Riemann Vermutung* (i.e., the Riemann hypothesis), taking into account the asymptotic conditions (30) and (31), Pólya, on the basis of a personal discussion with E. Landau had in 1913, argues on the possible existence or not of real zeros of such an approximation of the ξ function and, to this end, he considers (28) with a finite approximation for Φ given by $N = 1$, under the asymptotic conditions (29) and (30), so obtaining

$$\xi^*(z) = 8\pi^2 \int_0^\infty \left(e^{\frac{9u}{2}} + e^{-\frac{9u}{2}}\right) e^{-\pi(e^{2u}+e^{-2u})} \cos zu du, \tag{32}$$

whence questions inherent real zeros of $\xi(z)$ reduce to questions inherent real zeros of ξ^*. Then, Pólya argues that asymptotically we have $\xi(z) \sim \xi^*(z)$ within an infinite angular sector comprehending the real axis, with vertex in 0 and symmetrically opening with respect to the real axis. Under this hypothesis, if $N(r)$ [respectively $N^*(r)$] is the number of zeros of ξ [respectively of ξ^*] within this angular sector, the we have $N(r) \sim N^*(r)$ with $N(r) - N^*(r) = O(\ln r)$. Pólya, therefore, reduces the study of real zeros of $\xi(z)$ to the study of real zeros of $\xi^*(z)$ (or to the study of imaginary zeros of $\xi^*(iz)$), at least for the case $N = 1$. To be precise, he considers ξ^* and, to analyze this function, he introduces an auxiliary entire function, namely the following one

$$\mathfrak{G}(z) = \mathfrak{G}(z; a) \doteq \int_{-\infty}^{+\infty} e^{-a(e^u+e^{-u})+zu} du, \tag{33}$$

where a is an arbitrary parameter even having positive values. Since we have

$$\xi^*(z) = 2\pi^2 \left\{ \mathfrak{G}\left(\frac{iz}{2} - \frac{9}{4}; \pi\right) + \mathfrak{G}\left(\frac{iz}{2} + \frac{9}{4}; \pi\right) \right\}, \tag{34}$$

it is evident that the study of the ξ^* function may be reduced to the study of the auxiliary function $\mathfrak{G}$, to be precise, the above question about the imaginary zeros of ξ^* function is reduced to the study of real zeros of the function $\mathfrak{G}(iz)$. Hence, Pólya keeps on with a detailed analysis of the various formal properties and possible functional representations of this auxiliary function $\mathfrak{G}$ in view of their applications to Riemann ξ^* function and its zeros, starting to consider the case in which

$$\xi^*(z) = 2\int_0^\infty \Phi^*(u)\cos zu du \tag{35}$$

and

$$\Phi^*(u) = \sum_{n=1}^{N} e^{-2\pi n^2 \cosh(2u)}[8\pi^2 n^4 \cosh(9u/2) - 12\pi n^2 \cosh(5u/2)] \tag{36}$$

as an approximation to $\Phi(u)$ for N finite. In particular, the case $N = 1$ is quite interesting, and Pólya was able to prove that all zeros of ξ^* function are real just in this case. Pólya also succeeded to find a Riemann-like estimate of the number of zeros of $\mathfrak{G}$, in the form

$$\frac{y}{\pi}\ln\frac{y}{a} - \frac{y}{\pi} + O(1). \tag{37}$$

Afterwards, Pólya introduces and proves two general lemmas which are need just for proving the main aim of the paper, that is to say, to evaluate the nature of the zeros of an approximation of the Riemann ξ function: the first one concerns general analytic function theory, while the second one regards instead entire functions. Due to the importance of the latter, we stress what Pólya says in this regard, and, let us say immediately, he attains a point in which necessarily intervenes Hadamard's factorization theorem of entire functions applied to certain representations of $\mathfrak{G}$, just to find its zeros. These Pólya lemmata will be just those formal outcomes which will lead to the prove of the theorems of Lee and Yang. To be precise, his first lemma, called *Hilfssatz I,* states that

«Let $F(u)$ be an analytic function which has real values for each real value of u, and furthermore let

$$(a) \qquad \lim_{u\to\infty} u^2 F^{(n)}(u) = 0$$

for $n = 0, 1, 2, \ldots$. Then, when $F(u)$ is not an even function, then we have

$$(b) \qquad G(x) = \int_0^\infty F(u) \cos xu du$$

for values of x great enough».

Hence, Pólya considers the case in which the function $F(u)$ of Hilfssatz I is of the type (31), so that the corresponding function $G(x)$ of (b) is an entire function to which is now applicable the well-known Hadamard factorization theorem of entire functions to find its zeros. To this end, extending to entire functions a previous result achieved by C. Hermite and C. Biehler for polynomials[68] (and already quoted in the previous chapter about HB class - see (Biehler 1879) and (Hermite 1856a,b; 1873)), Pólya considers a second lemma, called *Hilfssatz II,* which states that

«Let A be a positive constant and let G be an entire function of order 0 or 1, which has real values for real values of z, has, at least, on real zero, and has real zeros only. Then the function $G(z - ia) + G(z + ia)$ has real zeros only».

Due to the importance of this Pólya Hilfssatz II, herein we report the brief and elegant proof given by Pólya, following (Cardon 2002, Section 1). Applying Hadamard's factorization theorem to the entire function $G(z)$, we have

$$G(z) = cz^q e^{\alpha z} \prod_n \Big(1 - \frac{z}{\alpha_n}\Big) e^{\frac{z}{\alpha_n}}$$

[68] Following (Pólya & Szegő 1998a, Part III, Exercise 25), this result, due to Hermite and Biehler, is as follows. We assume that all the zeros of the polynomial $P(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n$ are in the upper half-plane $\Im z > 0$. Let $a_\nu = \alpha_\nu + i\beta_\nu$, with α_ν, β_ν real, $\nu = 0, 1, 2, \ldots, n$, and $U(z) = \alpha_0 z^n + \alpha_1 z^{n-1} + \alpha_{\nu-1} z + \alpha_0$, $V(z) = \beta_0 z^n + \beta_1 z^{n-1} + \beta_{\nu-1} z + \beta_0$. Then the polynomials $U(z)$ and $V(z)$ have only real zeros.

where $c, \alpha_1, \alpha_2, ...$ are real constant, $\alpha_n \neq 0$ for each n, $q \in \mathbb{N}_0$ and $\sum_n |\alpha_n|^2$ is convergent. When $z = x + iy$ is a zero of the function $G(z - ia) + G(z + ia)$, then we have $|G(z - ia)| = |G(z + ia)|$, so that, by means of the above Hadamard factorization

$$(\circ) \quad 1 = \Big|\frac{G(z - ia)}{G(z + ia)}\Big|^2 = \Big(\frac{x^2 + (y - a)^2}{x^2 + (y + a)^2}\Big)^q \prod_n \frac{(x - \alpha_n)^2 + (y - a)^2}{(x - \alpha_n)^2 + (y + a)^2}.$$

Now, if it were $y > 0$, then the right hand side of the last expression would be lesser than 1, whereas if it were $y < 0$, then the right hand side of the last expression would be greater than 1, and both of these cases are impossible, so $y = 0$ whence $G(z + ia) + G(z - ia)$ has only real zeros. Accordingly, it follows that $\mathfrak{G}(iz/2)$ has not real zeros when we apply Hilfssatz II to the function $G(z) = \mathfrak{G}(iz/2; \pi)$. From all this, related considerations follow for $\xi^*(z)$. In conclusion, we may say that Hadamard factorization theorem has played a crucial role in proving Hilfssatz II which, in turn, has helped in achieving the main aim of this Pólya's paper. Furthermore, a good part of the next literature on the argument, like that related to the work achieved by D.A. Cardon and co-workers (see, for example (Adams & Cardon 2007)), makes wide and frequent use of factorization theorems of the type Weierstrass-Hadamard. Later on, Pólya improved this result for finite values of $N > 1$ in (Pólya 1927a), this line of research results having been vastly generalized later by D.A. Hejal in (Hejal 1990), whilst Hilfssatz II was generalized by D.A. Cardon in (Cardon 2002). Following (Newman 1976), the problem of determining whether a Fourier transform has only real zeros arises in two rather disparate areas of mathematics: number theory and mathematical physics. In number theory, the problem is intimately associated with the Riemann hypothesis (see (Titchmarsh 1980, Chapter X)), while in mathematical physics it is closely connected with Lee-Yang theorems of statistical mechanics and quantum field theory. Finally, this 1926 work of Pólya was then commented by Mark Kac (1914-1984), when it was inserted into the *Collected Papers* of Pólya, of which we shall talk about in the next section.

6.2. On the history of the theorems of T.L. Lee and C.N. Yang. In regard to the just above mentioned Pólya's paper, Kac says that, although this beautiful paper takes one to within an hair's breadth of Riemann's hypothesis, it didn't seem to have inspired much further work, and references to it, in the subsequent mathematical literature, were rather poor. Nevertheless, Kac says that, because of this, it may be of interest to refer that Pólya's paper did play a small, but perhaps not wholly negligible, part in the development of an interesting and important chapter in Statistical Mechanics, as we will see later. Instead, according to us, Pólya's paper played a notable, and not simply a small, role (though quite implicit) not only as regard the pioneering work of T.D. Lee and C.N. Yang in statistical mechanics of the early 1950s, but also for some next developments of Riemann zeta function theory, as witnessed by the latest researches on the subject as, for instance, those made by D.A. Cardon and co-workers (see (Adams & Cardon 2007) and references therein), in which Polya work is put into an interesting and fruitful relationship with Lee-Yang theorems in view of its applications just to Riemann zeta function. However, to begin in delineating the history of the Lee-Yang theorem, we first report the related comment and witness due to Yang himself and drawn from (Yang 2005, pp. 14-16)

«In the fall of 1951, T.D. Lee came to the Institute for Advanced Study, and we resumed our collaboration. The first problem we tackled was the susceptibility of the two-dimensional Ising model. As stated in a previous Commentary, the Onsager-Kaufman method yielded information about all eigenvectors of the transfer matrix. I had used some of that information to evaluate the magnetization, and I thought we might be able to use more of that information to evaluate the susceptibility by a second-order perturbation method, one order beyond that used to obtain equation (14) of the previous paper The Spontaneous Magnetization of a Two-Dimensional Ising Model, The Physical Review, 85, 808 (1952). *This led to a formula that was, so to speak, an order of magnitude more difficult to evaluate than the magnetization. After a few weeks of labor we gave up and turned our attention to the lattice gas, then to J. Mayer's theory of gas-liquid transitions, and finally to the unit-circle theorem.*

These considerations led to papers (Yang & Lee 1952) and (Lee & Yang 1952). The idea of the lattice gas was more or less in the minds of many authors (see reference 2 of (Lee & Yang 1952)). We firmed it up and elaborated on it because with the result of (Yang & Lee 1952), we were able to construct the exact two-phase region of a simple two-dimensional lattice gas. (We were especially pleased by the "law of constant diameter", which resembled the experimental "law of rectilinear diameter"). The two-phase region consists of flat portions of the $P - V$ *diagram, bounded by the liquid and gas phases. We were thus led very naturally to the question of why Mayer's theory of condensation gave isotherms that stayed flat into the liquid phase, instead of becoming curves in the liquid phase. Besides, Mayer theory of condensation was a milestone in equilibrium statistical mechanics, for it broke away from the mean field type of approach to phase transitions. It caused quite a stir at the Van der Waals Centenary Congress on November 26, 1937. Mayer's theory led to a number of papers by Mayer himself, by B. Kahn and G.E. Uhlenbeck, and by others in succeeding years. In the early 1940s I had attended a series of lectures by J.S. Wang in Kunming on these developments and had been very much interested in the subject ever since. Using the lattice gas model, for which we had a lot of exact information, Lee and I examined Mayer's theory as applied to this case. This led to a study of the limiting process in the evaluation of the grand partition function for infinite volume. Paper (Yang & Lee 1952) resulted from this study. It clarified the limiting process and made transparent the relationship between the various portions of an isotherm and the limiting process. In late 1952, after (Yang & Lee 1952) had appeared in print, Einstein sent Bruria Kaufman, who was then his assistant, to ask me to see him. I went with her to his office, and he expressed great interest in the paper. That was not surprising, since thermodynamics and statistical mechanics were among his main interests. Unfortunately I did not get very much out of that conversation, the most extensive one I had with Einstein, since I had difficulty understanding him. He spoke very softly, and I found it difficult to concentrate on his words, being quite overwhelmed by the nearness of a great physicist whom I had admired for so long. Back in the fall of 1951, Lee and I, in familiarizing ourselves with* *lattice gases, computed the partition function for several small lattices with 2, 3, 4, 5, etc. lattice points. We discovered to our amazement that the roots of the partition functions, which are polynomials in the fugacity, are all on the unit circle for attractive interactions. We were fascinated by this phenomenon and soon conjectured that it was a general theorem for a lattice of any size with attractive interactions. The theorem, later called the unit circle theorem, became the main element that was exploited in (Lee & Yang 1952) to discuss the thermodynamics of a lattice gas. Our attempt at proving the conjecture was a struggle, which I described in a letter to M. Kac, dated September 30, 1969, when he was writing for the Collected Papers of George Pólya. I quote now from that letter*

> When Lee arrived at Princeton, in the fall 1951, I was just recovering from my computation of the magnetization of the Ising model. I realized that the Ising model is equivalent to the concept of a lattice gas. So, we worked on that and finally produced our paper (Yang & Lee 1952). In the process of doing that, we discovered, by working on a number of examples, the conjectured unit circle theorem. We then formulated a physicist's "proof" based on no double roots when the strength of the couplings were varied. Very soon we recognized this was incorrect; and for, I would guess, at least six weeks we were frustrated in trying to prove the conjecture. I remember our checking into Hardy's book *Inequalities,* our talking to Von Neumann and Selberg. We were, of course, in constant contact with you all along (and I remember with pleasure your later help in showing us Wintner's work, which we acknowledged in our paper). Sometime in early December, I believe, you showed us the proof of the special case when all the couplings are there and are of equal strength, the case that you are now writing about in connection with Pólya's collected works. The proof was fine, but we were still stuck on the general problem. Then one evening around December 20, working at home, I suddenly recognized

that by making $z_l, z_2, \ldots$ independent variables and studying their motions relative to the unit circle one could, through an induction procedure, bring to bear a reasoning similar to the one used in your argument and produce the complete proof. Once this idea was there, it took only a few minutes to tighten up all the details of the argument. The next morning I drove Lee to pick up some Christmas trees, and I told him the proof in the car. Later on, we went to the Institute; and I remember telling you about the proof at a blackboard. I remember these quite distinctly because I'm quite proud of both the conjecture and the proof. I t is not such a great contribution, but I fondly consider it a minor gem.

The unit circle theorem was later generalized and extended to very interesting additional types of interactions[69]. *With the unit circle theorem, it appeared to Lee and me in early 1952 that we could somehow figure out or guess at the root-distribution function $g(\theta)$ on the unit circle (see (Lee & Yang 1952, Section V)) for the two-dimensional Ising model. We thought that, with the exact expressions for the free energy and the magnetization already known, we had powerful handles on the structure of $g(\theta)$. Unfortunately these handles were not powerful enough, and the exact form of $g(\theta)$ remains unknown today (the exact form of $g(\theta)$ is of course transformable into the exact partition function of the Ising model in a magnetic field). But our efforts in this direction did lead to two useful results. In listening to a seminar around the end of February, 1952, I learned about the new, ingenious combinatorial method of M.*

[69]The theory of phase transitions and its rigorous results, was then improved, generalized, enlarged and extended in many respects, through the works of D. Ruelle (see, for instance, (Ruelle 1969; 1994; 2000; 2010)), B.M. McCoy, T.T. Wu, T. Asano, M. Suzuki, M.E. Fischer, C.M. Newman, J.L. Lebowitz, R.B, Griffiths, E.R. Speer, B. Simon (see (Simon 1974) and references therein), E.H. Lieb, O.J. Heilmann, A.D. Sokal, D.G. Wagner, R.L. Dobrushin, G. Gallavotti, S. Miracle-Sole, D.W. Robinson, J. Fröhlich, P-F. Rodriguez (see (Frohlich & Rodriguez 2012)), J. Borcea, P. Brändén (see (Borcea & Brändén 2008; 2009a,b), (Brändén 2011) and references therein), M. Biskup, C. Borgs, J.T. Chayes, R. Kotecky, L.J. Kleinwaks, L.K. Runnels, J.B. Hubbard, A. Hinkkanen, C. Gruber, A. Hintermann, D. Merlini, and others. See (Georgii 2011, Bibliographical Notes) as well as (Gruber et al. 1977), (Baracca 1980, Appendice), (Lebowitz et al. 2012) and references therein. Of the interesting work of these authors, we shall deal with another, next paper.

Kac and J. Ward for solving the Ising problem without a magnetic field. It occurred to me during the seminar that, by a slight modification of the Kac-Ward method, one could find the partition function for the king model with an imaginary magnetic field $H = i\pi/2$*. This requires the evaluation of an* 8×8 *matrix, which Lee and I carried out in the next couple of days, arriving at equation (48) of (Lee & Yang 1952) for the free energy with* $H = i\pi/2$*. Comparing this expression with the known L. Onsager result for the same quantity for the case* $H = 0$*, Lee and I observed that they are very similar except for some sign changes and related alterations. Thus it seemed that the change* $H = 0 \div H = i\pi/2$ *is altogether minor. We therefore tried similar minor changes on the magnetization for* $H = 0$ *and tested the results by checking whether they were in agreement with the first few terms of a series expansion of the magnetization for* $H = i\pi/2$*. This was a very good method, and we soon arrived at equation (49) of (Lee & Yang 1952), which we knew was correct, but did not succeed in proving. It was finally proved by B.M. McCoy and T.T. Wu*[70] *in 1967».*

Following (Huang 1987, Chapter 9), after pioneering work of Lee and Yang, phase transitions are manifested in experiments by the occurrence of singularities in thermodynamic functions, such as the pressure in a liquid-gas system, or the magnetization in a ferromagnetic system, with N particles. Huang asks: How is it possible that such singularities arise from the partition function, which seems to be an analytic function of its arguments? Huang says that the answer lies in the fact that a macroscopic system is close to the idealized thermodynamic limit - i.e., the limit of infinite volume with particle density held fixed. As we approach this limit, the partition function can develop singularities, simply because the limit function of a sequence of analytic functions need not be analytic. Yang and Lee just proposed a definite framework for the occurrence of singularities in the thermodynamic limit. Due to its formal character, it

[70]In (McCoy & Wu 1967a), using higher mathematical techniques of complex analysis, like Wiener-Hopf method, Szegő's theorems for $N \times N$ Toeplitz determinants applied to determine magnetization $M(i\pi/2)$ as $N \to \infty$ (for $\beta = 1$), etc. See also (McCoy & Wu 1966; 1967b), (Cheng & Wu 1967) and (McCoy & Wu 1973).

belongs to a chapter of statistical physics sometimes known as "rigorous statistical mechanics" (see the 1969 Ruelle's monograph). Following (Ma 1985, Chapter 9), if the number of particles N of a given thermodynamical system is finite, then the calculation of the various thermodynamic potentials does not pose any problem. Although N is not infinite, it is nevertheless a very large number like 10^{23} (Avogadro's number), hence the problem of the $N \to \infty$ limit becomes a very important problem for the application of the basic assumption in thermodynamics, i.e. the so-called problem of the *macroscopic limit.* The rigorous mathematical analysis of this limit is a branch of statistical mechanics. The pioneering work in this topic is just the Yang-Lee theorem of 1950s, which was originally proposed for phase transitions. Following them, many have applied rigorous mathematical analyzes to describe phase transitions, because the model problems of phase transitions are not easily solvable and less than rigorous analysis is not reliable. However the application of the Yang-Lee theorem is quite universal. Following (McCoy & Wu 1973), the analyticity properties of the grand canonical partition function for Ising models correspond to qualitative features that appear in the thermodynamic limit which are not possible in a system with a finite number of particles. These analytic properties are intimately related to the physical notion of phase transition. The major reason for studying the two-dimensional Ising model (as, for example, masterfully exposed in (McCoy & Wu 1973)) is to attempt to make this connection more precise. Following (Domb & Green 1972, Chapter 2, II. and IV.), a mathematically "sharp" phase transition can only occur in the thermodynamical limit. It is also true in general that, only in the thermodynamic limit, the different statistical ensembles (i.e., microcanonical, canonical, and grand canonical) yield equivalent thermodynamic functions. Hence this limit permits a mathematically precise discussion of the question of phase transitions. The problem of proving the existence of a thermodynamic limit for the thermodynamic properties of a system of interacting particles seems first to have been discussed by L. Van Hove in 1949 in the case of a continuum classical gas with hard cores in the canonical ensemble, although the proof is incomplete due to an error in the appendix of the paper. Later, Yang and Lee, in 1952, considered the same system in the grand ensemble and L. Witten, in 1954, extended their proof with a relaxation of the condition of hard cores. Then, D. Ruelle, in 1963, proved the existence of limits in both the canonical and grand canonical case under a "strong-tempering" condition on the potential, and extended the results to quantum gases. Hence, R.L. Dobrushin and M.E. Fisher, in 1964, showed how Ruelle arguments could be extended to a more general class of potentials, and Fisher considered in some detail the possible class of domains tending to infinity for which a limit exists. The thermodynamic limit for lattice systems was discussed by R.B. Griffiths in 1964 for both classical and quantum systems. Additional results have been obtained in 1967 by G. Gallavotti and S. Miracle-Sole for classical lattice systems and by D.W. Robinson for quantum lattice systems (see (Domb & Green 1972, Chapter 2) for detailed bibliographical information). Lee and Yang opened the way to the rigorous theory of phase transitions, then included into the wider chapter of algebraic methods of statistical mechanics (see (Lavis & Bell 1999, Volume 2, Chapter 4)); moreover, for most recent formal developments of advanced statistical mechanics see also (McCoy 2010).

From a direct analysis of the original papers (also following the modern treatment given by (Huang 1987, Chapter 9)), it turns out that already in (Yang & Lee 1952, Section III) the authors make use of simple polynomial factorizations of the Weierstrassian type. Indeed, in (Yang & Lee 1952, Section II), it is considered a system of N particles filling a region of finite volume V, undergoing a two-body interaction by means of a potential of the type U, whose partition function, defined on the grand canonical ensemble in the complex variable $y = A\exp(\mu/kT)$ (fugacity), is given by

$$\mathcal{Q}_y = \sum_{N=0}^{M} \frac{Q_N}{N!} y^N \tag{38}$$

where

$$Q_N = \int ... \int_V \exp(-\frac{U}{kT}) d\tau_1 ... d\tau_N$$

where $U = \Sigma_{ij} u(r_{ij}) = \sum_{ij} u(|r_i - r_j|)$ is the sum of the various interaction potentials between the i-th and j-th particles, which undergo particular restrictions, and $M = M(V)$ is the maximum number of particles which can be crammed into the finite volume V. Then Yang and Lee consider the following limits for infinite volume

$$\frac{p}{kT} = \lim_{V\to\infty} \frac{1}{V} \ln \mathcal{Q}_y, \qquad \rho = \lim_{V\to\infty} \frac{\partial}{\partial \ln y} \frac{1}{V} \ln \mathcal{Q}_y \tag{39}$$

reaching two main theorems thanks to which it is possible to study these limits for potentials of the type[71]

$$U(r) = \begin{cases} \infty & \text{for } \ r < a, \\ -\infty < U(r) < -\epsilon & \text{for } \ a < r < r_0, \\ 0 & \text{for } \ r > r_0, \end{cases}$$

which is a reasonable approximation of a potential of the Lennard-Jones type. In such a case, Q_N converges and $\mathcal{Q}_y$ is a polynomial in y whose degree depends on V and whose coefficients are analytic functions of $\beta = 1/kT$, defined to be positive for real values of β. Accordingly, the zeros of $\mathcal{Q}_y$, in the complex plane y, are in a finite number and lie out of the positive real axis. Only in the thermodynamic limit $V \to \infty$ (or infinite volume limit), the zeros of $\mathcal{Q}_y$ are infinite and may approach positive real axis[72], with the appearance of singularities in the thermodynamic potentials. To be precise, in both limits $N \to \infty$ and $V \to \infty$ in such a way that the specific volume $v = V/N$ is bounded, some of the zeros of $\mathcal{Q}_y$ may approach the positive real axis, so giving rise to possible phase

[71]It is said to be an *hard-core potential.* Such a potential is related to an impenetrable sphere of radius a surrounded by an attractive potential with action radius r_0 and maximal deep ϵ. The occurrence of such an impenetrable potential implies that, for each fixed value of the total volume V, only a finite number of particles may be considered, and if $N_{max}(V) = M$ is the maximum of such a number, then we have that, when $N > M$, at least two particles are in touch, so the potential U is infinite, and $\mathcal{Q}_y = 0$. Therefore, $\mathcal{Q}_y$ is a polynomial of degree M (see (41)). Nevertheless, the thermodynamics of physical systems is ruled by the logarithm of the partition function, so that its zeroes broke analyticity of thermodynamic functions, so giving rise to singularities which, on its turn, are related to the occurrence of phase transitions.

[72]In this case, we feel allowable to refer to many theorems on the distribution of the zeros of entire functions like, for example, expounded in (Levin 1980, Chapters VII, VIII) and mainly regarding LP and HB classes of entire functions, some of which just provide necessary and/or sufficient conditions for zeros of certain entire functions approach positive real axis.

transitions. From these considerations applied to particular physical systems (amongst which ferromagnetic spin systems), Lee and Yang have worked out a phase transition model whose one of the main characteristics is having pointed out the close relationships between the existence of phase transitions and general properties of the related potential on the one hand, and between the thermodynamic limit and the occurrence of singularities of thermodynamic potentials on the other hand. These latter relationships are, on its turn, related to the occurrence of zeros of the partition function. In particular, for Ising models of ferromagnetic spin systems, the distribution of the zeros of the partition function takes a well-determined geometrical shape by means of the deduction of certain general theorems proved by Lee and Yang in their two seminal papers of 1952. To be precise, we are interested in two of these theorems, namely the so-called Theorem 1, according to which, for all positive real values of y, the first limit approaches, as $V \to \infty$, a limit which is independent of the shape of V, this limit being moreover a continuous, monotonically increasing function of y, and the so-called Theorem 2, which states that, if in the complex y plane a region R containing a segment of the positive real axis is always free of roots, then in this region as $V \to \infty$, all the quantities

$$(40) \qquad \frac{1}{V}\ln \mathcal{Q}_y, \quad \frac{\partial}{\partial \ln y}\frac{1}{V}\ln \mathcal{Q}_y, \quad \Big(\frac{\partial}{\partial \ln y}\Big)^2\frac{1}{V}\ln \mathcal{Q}_y, \quad ...,$$

approach limits which are analytic with respect to y. To study the limit of $\dfrac{\partial}{\partial \ln y}\dfrac{1}{V}\ln \mathcal{Q}_y$ we notice that $\mathcal{Q}_y$ is a polynomial in y of finite degree M. This is a direct consequence of the assumed impenetrable core of the atoms (roughly formalized by an hard-core potential U as done above). It is therefore possible to factorize $\mathcal{Q}_y$ and write

$$(41) \qquad \mathcal{Q}_y = \prod_{i=1}^{M}\Big(1 - \frac{y}{y_i}\Big)$$

where $y_1, ..., y_M$ are the roots of the algebraic equation $\mathcal{Q}_y(y) = 0$. Evidently none of these roots can be real and positive, since all the coefficients in the polynomial $\mathcal{Q}_y$ are positive. Following (Yang & Lee 1952, Section IV), by Theorem 2 it follows that, as V increases, these roots move about in the complex y plane and their number M increases (essentially) linearly with V. Their distribution in the limit $V \to \infty$ gives the complete analytic behavior of the thermodynamic functions in the y plane. On the other hand, the problem of phase transition is intrinsically related to the form of the regions R described in Theorem 2, and Lee and Yang discuss two main cases related to the geometry of this region R, and the related roots of $\mathcal{Q}_y(y) = 0$, around real y axis, reaching the conclusion that phase transitions of the system occur only at the points on the positive real y axis onto which the roots of $\mathcal{Q}_y(y) = 0$ close in as $V \to \infty$ (which entails $M \to \infty$ in (41)). For other values of the fugacity y, a single phase system is obtained. The study of the equations of state and phase transitions can thus be reduced to the investigation of the distribution of roots of the grand partition function. In many cases, as will be seen in (Lee & Yang 1952), such distributions will turn out to have some surprisingly simple regularities. The above mentioned theorems 1 and 2, will be proved respectively in Appendix I and II of (Yang & Lee 1952), considering arbitrary circles lying inside R. On the other hand, since the degree M of the polynomial $\mathcal{Q}_y$ is function of V, so $M \to \infty$ as $V \to \infty$, following (Ruelle 1969, Chapter 3, Sections 3.2, 3.4; Exercise 3.E), under the hypothesis that Hamiltonian operator of the physical system be bounded below (*stability condition*), we have that the partition function in the grand canonical ensemble[73] is given by the following entire function of the fugacity (or activity) z (corresponding to y of Lee and Yang notation) considered as a complex variable (in the notations of Ruelle)

$$\Xi(\Lambda, z, \beta) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \int ... \int_{\Lambda^n} dx_1 \, ... \, dx_n \exp[-\beta U(x_1, ..., x_n)] =$$

$$= \sum_{n=0}^{\infty} z^n Q(\Lambda, n, z), \qquad \text{with} \quad Q(\Lambda, 0, z) = 1$$

[73]Following (Huang 1987, Chapter 7, Section 7.3), such a function is often simply called the *grand partition function.*

which is of order at most 1, and of order 0 in the case of superstable potentials, that is, potentials U satisfying, into a cube Λ of volume V, the condition $U(x_1, ..., x_n) \geq n(-B+nC/V)$ for certain constants $B, C > 0$, in such last case Hadamard's factorization yielding

$$\Xi(z) = \prod_i \Big(1 - \frac{z}{z_i}\Big),$$

where z_i are the zeros of Ξ, and reducing to a polynomial when U is an hard-core potential (Lee-Yang case). Nevertheless, the construction of the partition function of a given physical system is one of the tricky task of statistical mechanics.

Thus[74], in (Yang & Lee 1952), the authors have seen that the problem of a statistical theory of phase transitions and equations of state is closely connected with the distribution of roots of the grand partition function. There, it was shown that the distribution of roots determines completely the equation of state, and in particular its behavior near the positive real axis prescribes the properties of the system in relation to phase transitions. It was also shown there that the equation of state of the condensed phases as well as the gas phase can be correctly obtained from a knowledge of the distribution of roots. While this general and abstract theory clarifies the problems underlying the statistical theory of phase transitions and condensed phases, it is natural to ask whether it also provides us with a means of obtaining practical approximation methods for calculating properties pertaining to phase transitions and condensed phases. The problem is clearly that of seeking for the properties of the distribution of roots of the grand partition function. At first sight, this appears to be a formidable problem, as the roots are in general complex and would naturally be expected to spread themselves for an infinite sample in the entire complex plane, or at least regions of the complex plane, and make it very difficult to calculate their distribution. We were quite surprised, therefore, to find that for a large class of problems of practical interest, the roots behave remarkably well in that they distribute themselves not all over the complex plane, but only on a fixed

[74]We here follow, almost verbatim, (Lee & Yang 1952).

circle. This fact will be stated as a theorem in Section IV of (Lee & Yang 1952) and proved in the Appendix, while implications of the theorem are discussed in Section V. Lee and Yang return to the general problem of the condensation of gases, and shall in the following apply the results of the previous paper (Yang & Lee 1952) to the problem of a lattice gas. There was then no loss of generality in confining their attention to a lattice gas, as a real continuum gas can be considered as the limit of a lattice gas as the lattice constant becomes infinitesimally small. The equivalence proved in Section II of (Lee & Yang 1952) states that the problem of a lattice gas is identical with that of an Ising model in a magnetic field, and that the grand partition function in the former problem is proportional to the partition function in the latter problem. It is convenient to introduce in the Ising model problem the variable $z = \exp(-2H/kT)$ which is proportional to the fugacity y of the lattice gas $y = \sigma z$, where σ is a constant. In terms of z the partition function $\exp(-NF/kT)$ of the Ising lattice is equal to $\exp(NH/kT)$ times a polynomial $\mathcal{P}$ in z of degree N, that is, $\exp(-NF/kT) = \mathcal{P}\exp(NH/kT)$ where $\mathcal{P} = \sum_{n=0}^{N} P_n z^n$. The coefficients P_n, are the contribution to the partition function of the Ising lattice in zero external field from configurations with the number of $\downarrow$ spin down equal to n. It should be noticed that $P_n = P_{n'}$ if $n+n' = N$, with each P_n real and positive. Furthermore, the roots of the polynomial $\mathcal{P}$ are never on the positive real z axis, and are in general complex. The results of (Yang & Lee 1952), show that if at a given temperature as N approaches infinity, the roots of the polynomial $\mathcal{P}$ do not close in onto the positive real axis in the complex z plane, the free energy F is an analytic function of the positive real variable z. Physically this means that the Ising model has a smooth isotherm in the I-H diagram (where I is the intensity of magnetization and H is the magnetic field) and that the corresponding lattice gas undergoes no phase transition at the given temperature. If, on the other hand, the roots of the polynomial $\mathcal{P}$ do close in onto the positive real z axis at the points $z = t_l, t_2, ...$, each of these points would correspond to a discontinuity of the isotherm in the I-H diagram of the Ising lattice and to a phase transition of the lattice gas. To study the problem of phase transitions of a lattice gas as well as of an

Ising model related to ferromagnetic spin systems, one therefore needs only to study the distribution in the complex z plane of the roots of the polynomial $\mathcal{P}$. The surprising thing is that under quite general conditions, this distribution shows a remarkably simple regularity, which may be stated in the form a theorem, say Theorem 3 (see (Lee & Yang 1952, Section IV)), stating that, if the interaction u between two gas atoms is such that $u = +\infty$ if the two atoms occupy the same lattice and $u \leq 0$ otherwise, then all the roots of the polynomial $\mathcal{P}$ lie on the unit circle in the complex z-plane. This theorem will be proved in Appendix II of (Lee & Yang 1952). Thus, for the interaction of the theorem 3, the roots of $\mathcal{P}$ lie on the unit circle, so its distribution as $N \to \infty$ may be described as a function $g(\theta)$ so that $Ng(\theta)d\theta$ is the number of roots with z between $e^{i\theta}$ and $e^{i(\theta+d\theta)}$, with $g(\theta) = g(-\theta)$ and $\displaystyle\int_0^\pi g(\theta)d\theta = 1/2$. The average density of a finite lattice gas is easily seen to be $\sum_k z/(z - \exp(i\theta_k))$ where $z = \exp(i\theta_k)$ are the zeros of the grand partition function. The results of (Yang & Lee 1952) show that this average density converges to an analytic function in z both inside and outside of the unit circle as the size of the lattice approaches infinity. It seems intuitively clear from this that the distribution of these roots should also approach a limiting distribution on the unit circle for an infinite lattice, this being indeed the case whose a rigorous mathematical proof exists in the literature (see[75] (Wintner 1934)), the authors acknowledging Professor Kac for have shown them the proof. After having considered a certain number of specific physical cases, the authors finish stating that the previous results have direct bearing on the distribution function $g(\theta)$ of the zeros of the partition function on the unit circle, showing too that the motion of the roots deploys toward the right along the unit circle as the temperature decreases. They also say that, since the relation between the distribution of roots of a polynomial and its coefficients is mathematically a very

[75] This work mainly deals with mathematical properties of asymptotic distributions σ of the values of certain fast periodic and quasi-periodic functions (above all following many E. Helly and H. Bohr works on this subject), by means of certain Cauchy transforms (see also (Müller-Hartmann 1977, Section II.B)). Yang and Lee pointed out what fundamental role has played the paper of Aurel Wintner of 1934, suggested them by M. Kac, in proving that roots of grand partition function are distributed along a unit circle.

complicated problem, it is therefore very surprising that the distribution should exhibit such simple regularities as proved in Theorem 3 which applies under very general conditions, so being tempted to generalize such outcomes. One cannot escape the feeling that there is a very simple basis underlying the theorem, with much wider application, which still has to be discovered. Finally, the authors express their gratitude to Professor M. Kac for many stimulating and very pleasant discussions from which they learned much in mathematics. The paper (Lee & Yang 1952) finishes with the Appendix II in which Theorem 3 is proved in a detailed manner. Usually, all the theorems contained in (Yang & Lee 1952) and (Lee & Yang 1952), are sometimes called *Lee-Yang theorems* (or *Yang-Lee theorems*), while some other times, Theorem 3 is the one to which is usually referred to the single expression *Lee-Yang theorem,* when it is declined in the singular. Often, the latter is also referred to as the *Lee-Yang circle theorem* (or *Lee-Yang unit circle theorem*). To summarize, these theorems therefore imply that the zeros of a finite physical system cannot lie in the positive real axis, with consequent absence of phase transitions. But a quite different situation arises when we deal with infinite systems, in such a case being possible that the zeros of the partition function may approach real axis and, in the thermodynamic limit, produce that catastrophic situation given by a phase transition. In the special case of a two-dimensional lattice Ising ferromagnetic system, Lee and Yang proved that such zeros laid all into a unitary circle of the complex plane of fugacity y, so that real axis is cut in $y = 1$ by this circle, corresponding therefore to a phase transition with zero magnetic field, while all the other positive real values of y are points of analyticity for the thermodynamical functions. This pioneering idea of Lee and Yang, albeit related to a particular case, will be the subject-matter of further researches meant to generalize or extend it.

But, as a further historical deepening of this case, we report the witness of M. Kac inserted into the comment to 1926 Pólya paper included into the the second volume of the *Collected Papers* of Pólya (see (Pólya 1926)). To be precise, Kac says that, in the fall of 1951 and in the spring of 1952, Yang and Lee were developing their theory of phase transitions which has since become justly celebrated. To illustrate the theory, they introduced the concept of a "lattice gas" and they were led to a remarkable conjecture which (not quite in its most general form) can be stated as follows. Let

$$G_N(z) = \sum_{\mu_k} \exp\big(\sum_{k,l=1}^{N} J_{kl}\mu_k\mu_l\big)\exp\big(iz\sum_{k=1}^{N}\mu_k\big) \tag{42}$$

where $J_{kl} \geq 0$ and the summation is over all 2^N sequences $(\mu_1, ..., \mu_N)$ with each μ_k assuming only values ± 1. Then, $G_n(z)$ has only real roots (*Lee-Yang theorem*). Textually, Kac tells that, when he first heard of this conjecture, he considered the simplest case $J_{k,l} = \nu/2$ for all k, l, and somehow Hilfssatz II of Pólya's paper came into his mind. Then, Kac shows how, by a slight modification of Polya proof, one can prove the Lee-Yang theorem in the above special case. First of all, for all N, $G_N(z)$ is an entire function of order 1 which assumes real values for real z. Note furthermore that

$$\Big(\frac{\nu}{2}\Big)\big(\sum_{k=1}^{N+1}\mu_k\big)^2 + iz\sum_{k=1}^{N+1}\mu_k =$$

$$= \Big(\frac{\nu}{2}\Big)\big(\sum_{k=1}^{N}\mu_k\big)^2 + (\nu\mu_{N+1} + iz)\sum_{k=1}^{N}\mu_k + iz\mu_{N+1} + \Big(\frac{\nu}{2}\Big)$$

and therefore

$$e^{-\nu/2}G_{N+1}(z) = e^{iz}G_N(z - i\nu) + e^{-iz}G_N(z + i\nu). \tag{43}$$

If z is a root of G_{N+1}, we have

$$|e^{2iz}|^2 = \Big|\frac{G_N(z + i\nu)}{G_N(z - i\nu)}\Big|^2, \tag{44}$$

and if we assume that G_N has only real roots, say $\alpha_1, \alpha_2, ...$, then, by Hadamard factorization theorem, we have

$$G_N(z) = ce^{\alpha z}\prod_{n=1}^{\infty}(1 - z/\alpha_n)e^{z/\alpha_n} \tag{45}$$

where c and α (as well as $\alpha_1, \alpha_2, ...$) are real. Equation (44) now becomes (upon setting $z = x + iy$)

$$e^{-4y} = \prod_{n=1}^{\infty} \frac{(\alpha_n - x)^2 + (y + \nu)^2}{(\alpha_n - x)^2 + (y - \nu)^2}. \tag{46}$$

Since $\nu > 0$, each term of the product (and hence the product itself) is greater than 1 if $y > 0$ and less than 1 if $y < 0$. On the other hand, $\exp(-4y)$ is less than 1 for $y > 0$ and greater than 1 for $y < 0$. Thus (46) can hold only if $y = 0$, i.e., all roots of G_{N+1} are also real. Since for $N = 2$ a direct check shows that all roots of G_2 are real, the theorem for all N follows by induction. Then, Kac refers that he showed this proof for the special case to Yang and Lee. A couple of weeks later, they produced their proof of the general theorem (in (Lee & Yang 1952, Appendix II)). Moreover, Kac also remembers Professor Yang telling him at the time that Hilfssatz II of Pólya, in the form discussed above, was one essential ingredient in their proof, as also recalled above. In any way, one immediately realizes that the key tool of the above Kac's argument, is just Hadamard factorization theorem.

Therefore, Pólya works (see (Pólya 1926a,b)) have opened new fruitful avenues in pure and applied mathematics. Indeed, according to (Dimitrov 2013), we consider the following question: suppose that K is a positive kernel which decays sufficiently fast at $\pm\infty$, supposing it belongs in the Schwartz class, and its Fourier transform $\mathcal{F}(z; K) \doteq \int_{-\infty}^{+\infty} e^{-izt} K(t) dt$ is an entire function. More generally, we consider positive Borel measures $d\mu$ with the property that $\mathcal{F}_\mu(z) \doteq \int_{-\infty}^{+\infty} e^{-izt} d\mu(t)$ defines an entire function. The problem to characterize the measures μ for which $\mathcal{F}_\mu$ has only real zeros has been of interest both in mathematics, because of the Riemann hypothesis, and in physics, because of the validity of the so-called general Lee-Yang theorem for such measures. It seems that Pólya was the first to formulate the problem explicitly in his works (Pólya 1926a,b), beginning, mutatis mutandis, with the following issue: What properties of the function $K(u)$ are sufficient to secure that the integral $2\int_{-\infty}^{+\infty} K(u)\cos zu du = \mathcal{F}(z)$ has only real zeros? The origin of this rather artificial question is the well-known hypothesis concerning the Riemann zeta function, as the author himself recognizes in (Pólya 1926a). If we put $K(u) = \Phi(u)$ as given by (29), then $\mathcal{F}(z)$ is nothing but that the Riemann ξ function. Since $\Phi(u)$ is an even kernel which decreases extremely fast, the above definition of Pólya for $\mathcal{F}$, in the case when K is even, is exactly the one for the Fourier transform. The Riemann's hypothesis, as formulated by Pólya himself, states that the zeros of ξ are all real. The efforts to approach Riemann hypothesis via ξ defined as a Fourier transform, have failed despite of the remarkable efforts due to Pólya, N.G. de Bruijn (see (de Bruijn 1950)) and many other mathematicians for two chief reasons. The first one is that the above question of Pólya still remains open, whilst the second one is that sufficient conditions for the kernels have turned out to be extremely difficult to be verified for Φ or simply do not hold for it. Finally, the notable work of C.M. Newman (see (Newman 1974) as well as (Kim 1999), (Ki & Kim 2003), (Ki et al. 2009), (Korevaar 2013) and references therein for a modern sight of the question and related arguments) based on an extension and generalization of the above pioneering work of T.D. Lee and C.N. Yang, has proved the latter to be equivalent to the above Pólya's question. Moreover, following (Korevaar 2013), de Bruijn and J. Korevaar were both inspired by work of Pólya on the zeros of entire functions. de Bruijn was fascinated by Pólya's results of 1926 on the zeros of functions given by trigonometric integrals, while Korevaar was attracted by other Pólya's papers on the approximation of entire functions by polynomials whose zeros satisfy certain conditions. All these articles by Pólya have been reproduced with commentary in the second volume of his *Collected papers.* Moreover, de Bruijn and Korevaar both published extensions of Pólya's work in Duke Mathematical Journal, referring to (Korevaar 2013) for a deeper historical analysis of all that and for other notable aspects of the history of entire function theory. Likewise, some works of D.A. Cardon and collaborators (see (Cardon & Nielsen 2003), (Cardon 2002; 2005), (Adams & Cardon 2007) and references therein) have fruitfully combined and fitted very well together, on the basis of certain extensions, formal comparative analogies and possible generalizations, the 1952 Lee-Yang formal approach to phase transitions with the original 1926 Pólya approach to Riemann ξ function revisited from the modern setting given by entire function theory as exposed in (Levin 1980), to be precise, reformulating the Pólya results within either the Hermite-Biehler and Laguerre-Pólya classes of entire functions with related distributions of zeros also using some tools drawn from stochastic and probabilistic analysis. Finally, very interesting attempts to apply Lee-Yang theorem for approaching Riemann conjecture have been pursued in (Knauf 1999) and (Julia 1994) (see also references therein quoted). Following (Borcea & Brändén 2009b, Introduction), the Lee-Yang theorem seems to have retained an aura of mystique. In his 1988 Gibbs lecture, Ruelle proclaimed: "I have called this beautiful result a failure because, while it has important applications in physics, it remains at this time isolated in mathematics". Ruelle's statement was apparently motivated by the fact that the Lee-Yang theorem also inspired speculations about possible statistical mechanics models underlying the zeros of Riemann or Selberg zeta functions and the Weil conjectures, but "the miracle has not happened". Nevertheless, only recently Lee-Yang theorem has received new attention from mathematician, as witnessed, for instance, by the recent works of J. Borcea and P. Brändén (see (Borcea & Brändén 2008; 2009a,b), (Brändén 2011) and references therein) whose research program makes often reference to Laguerre-Pólya, Hermite-Bielher and Pólya-Schur classes of complex functions. Indeed, recently Lee-Yang like problems and techniques have appeared in various mathematical contexts such as combinatorics, complex analysis, matrix theory and probability theory. The past decade has also been marked by important developments on other aspects of phase transitions, conformal invariance, percolation theory. However, as A. Hinkkanen has observed, the power in the ideas behind the Lee-Yang theorem has not yet been fully exploited: "It seems that the theory of polynomials, linear in each variable, that do not have zeros in a given multidisk or a more general set, has a long way to go, and has so far unnoticed connections to various other concepts in mathematics". Anyway, from a general overview of almost all these works concerning Riemann ξ function and related applications according to the line of thought opened by Polya's works of 1926 until up the new directions provided by Lee-Yang theorem on the wake of Pólya's work itself, it turns out that Weierstrass-Hadamard factorizations are the key formal tools employed in these treatments, besides to be the pivotal source from most of entire function theory sprung out, as well as to be crucial formal techniques widely employed in the modern treatment of the theory of polynomials and their zeros (see (Gil' 2010), (Fisk 2008) and (Rahman & Schmeisser 2002)).

In conclusion, we may state that two main results have been at the early origins of the Lee-Yang theorems, especially as regard the unit circle theorem, namely a 1934 paper of Aurel Wintner, from which Lee and Yang have drawn useful hints for determine the properties of the distribution function $g(\theta)$ and related geometrical settings of the zeros of grand partition function for the physical systems analyzed by them, and a trick used to prove a lemma due to Pólya, namely ($\circ$) of his Hilfssatz II (see previous section), thanks to which Lee and Yang proved the non-trivial basis of induction corresponding to the case $n = 2$ concerning the auxiliary polynomial $\mathfrak{B}(z_1, ..., z_n)$ - see (Lee & Yang 1952, Appendix II) - used for proving, by induction, a more general theorem than the Theorem 3 of section IV, that is to say, the unit circle theorem. Nevertheless, as Lee and Yang themselves point out at the beginning of Section V, Point A. of (Lee & Yang 1952), the distribution function $g(\theta)$ has been used only to estimate the number of zeros in the unit circle, once this last geometrical arrangement had already been determined by means of other routes (that is to say, via unit circle theorem), and not to properly determine this last settlement. In any event, Lee and Yang have been pioneers in opening a possible avenue in mathematical physics, even if their appreciated work dealt only with particular physical systems, still waiting general mathematical tools which would have generalized and extended this Lee and Yang model to a wider class of physical systems. This truly difficult task has been undertaken by other authors (recalled above), amongst whom are T. Asano[76], D. Ruelle, M.

[76]For instance, the interesting theory of *polynomial contractions* due to Taro Asano, might have fruitful applications in algebraic geometry, and vice versa, i.e., tools and methods of this last subject might turn out to be useful for statistical mechanics of phase transitions and its rigorous results,

Suzuki, C.M. Newman, E.H. Lieb, and A.D. Sokal, with very interesting results which, nevertheless, have not reached the expected goal. Nevertheless, just due to the great difficult to exactly determine the grand canonical partition function of an arbitrary thermodynamical system, often the Lee-Yang model runs well when is applied to the state equation and its possible singularities. Following (Ruelle 1988, Section 3), Lee-Yang theorem, conjectured on a physical basis related to ferromagnetic spin system, originally took some effort to prove. A later idea, due to T. Asano in extending Lee-Yang model to quantum case, now permits a different but short proof (see (Ruelle 1969, Chapter 5) as well as (Ruelle 1988, Appendix)) of this theorem. Notwithstanding its remarkable importance, Ruelle has nevertheless said of this beautiful result to be a failure because, while it has important applications in physics, it remains at this time isolated in mathematical physics and mathematics. In textual words of Ruelle, one might think of a connection with zeta functions (and the Weil conjectures), the idea of such a connection being not absurd but the miracle has not happened, so that one still does not know what to do with the circle theorem. Ruelle says too that this connection with Riemann zeta function and related conjecture is not fully meaningless because there exist interesting applications of certain ideas of statistical mechanics to differentiable dynamics, made possible by the introduction of Markov partitions which transform the problems of ergodic theory for hyperbolic diffeomorphisms or flows into problems of statistical mechanics on the "lattice" $\mathbb{Z}$. Among the many applications of the method, Ruelle mentions Ya.G. Sinai's beautiful proof that hyperbolic diffeomorphisms do not necessarily have a smooth invariant measure. Also, since the geodesic flow on manifolds of negative curvature is hyperbolic, one has the possibility of studying zeta functions of A. Selberg's type, and, using Markov partitions, these zeta functions are expressed as certain sorts of partition functions, which can be studied by statistical mechanics. Thus, one obtains for instance a kind of "prime number theorem" for the lengths of closed geodesies on a compact man-

always along the line outlined by Asano. In this regard, see the exposition given in (Ruelle 2007, Chapter 17). See also (Glimm & Jaffe 1987, Chapter 4) for other modern treatments and extension of Lee-Yang model, above all in relation to quantum field theory context.

ifold of negative curvature not necessarily constant (see (Mayer 1980, Chapter IV)).

6.3. On some other applications of the theory of entire functions, and all that. In reviewing the main moments of the history of Riemann zeta function and related still unsolved conjecture known as Riemann Hypothesis (RH), as for instance masterfully exposed in (Bombieri 2006) as well as in the various treatises, textbooks and survey papers on the subject (see, for instance, (Whittaker & Watson 1927), (Chandrasekharan 1958), (Ingham 1964), (Iviç 1985), (Titchmarsh 1986), (Patterson 1988), (Karatsuba & Voronin 1992), (Karatsuba 1994), (Edwards 2001), (Chen 2003), (Conrey 2003), (Gonek 2004) and (Borwein et al. 2008)), one realizes that a crucial point which would have deserved major historical attention is the one concerning Hadamard factorization theorem, which is the central point around which has revolved our attention and that has casted a precious bridge with entire function theory, opening a new avenue in complex analysis. This point has been sufficiently treated in the above sections which have seen involved the figures of Riemann, Weierstrass and Hadamard, so that we herein sum up, in passing, the main points of what has been before discussed in such a manner to be a kind of preamble of what will be said herein. As has been seen, Hadamard formulated his celebrated 1893 theorem as a continuation and completion of a previous 1883 theorem stated by Poincaré as regard the order of an entire function factorized according to the Weierstrass factorization theorem of 1876, applying the results so obtained to the Riemann ξ function which, in turn, had already been factorized by Riemann himself in his 1859 seminal paper. This celebrated Hadamard result was the pivotal point through which the entire function theory entered into the realm of Riemann zeta function. After Hadamard, it was then Pólya, in the 1920s, to achieve some further remarkable outcomes along this research's path emphasizing the entire function theory perspective of Riemann zeta function also making use of the 1893 Hadamard work, until to reach the recent outcomes of which we will briefly refer in this section. First of all, according to (Davenport 1980, Chapters 8, 11 and 12), in passing we recall that

«In his epoch-making memoir of 1860 (his only paper on the theory of numbers), Riemann showed that the key to the deeper investigation of the distribution of the primes lies in the study of $\zeta(s)$ as a function of the complex variable s. More than 30 years were to elapse, however, before any of Riemann's conjectures were proved, or any specific results about primes were established on the lines which he had indicated. Riemann proved two main results: (a) The function $\zeta(s)$ can be continued analytically over the whole plane and is then meromorphic, its only pole being a simple pole at $s = 1$ with residue 1. In other words, $\zeta(s) - (s-1)^{-1}$ is an integral function. (b) $\zeta(s)$ satisfies the functional equation

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{s}}\Gamma(\frac{1-s}{2})\zeta(1-s) \tag{47}$$

which can be expressed by saying that the function on the left is an even function of $s - 1$. The functional equation allows the properties of $\zeta(s)$ for $\sigma < 0$ to be inferred from its properties for $\sigma > 1$. In particular, the only zeros of $\zeta(s)$ for $\sigma < 0$ are at the poles of $\Gamma(s/2)$, that is, at the points $s = -2, -4, -6, \ldots$. These are called the trivial zeros. The remainder of the plane, where $0 < \sigma < 1$, is called the critical strip. [...] Riemann further made a number of remarkable conjectures, amongst which is the follows: the entire function $\xi(s)$ defined by (entire function because it has no pole for $\sigma \geq 1/2$ and is an even function of $s - 1/2$) has the product representation

$$\xi(s) = e^{A+Bs}\prod_{\rho}\Big(1-\frac{s}{\rho}\Big)e^{\frac{s}{\rho}} \tag{48}$$

where A and B are constants and ρ runs through the zeros of $\zeta(s)$ in the critical strip. This was proved by Hadamard in 1893. It played an important part in the proofs of the prime number theorem by Hadamard and de la Vallée-Poussin. [...] The next important progress in the theory of the $\zeta(s)$ function, after Riemann's pioneering paper, was made by Hadamard, who developed the theory of entire functions of finite order in the early 1890s and applied it to $\zeta(s)$ via $\xi(s)$. His results were used *in both the proofs of the prime number theorem, given by himself and by de la Vallée-Poussin, though later it was found that for the particular purpose of proving the prime number theorem, they could be dispensed».*

Following (Bombieri 2006), one of the main tools to study the mathematical properties of Riemann zeta function $\zeta(s)$ (hereafter RZF), defined by

$$\zeta(s) \doteq \sum_{n\in\mathbb{N}} \frac{1}{n^s}, \quad s \in \mathbb{C}, \quad \Re(s) > 1, \tag{49}$$

is the related *Riemann functional equation,* which was established in (Riemann 1858) and is defined as follows (see also (Titchmarsh 1986, Sections 2.4 and 2.6), (Katz & Sarnak 1999, Section 1))

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{s}}\Gamma(\frac{1-s}{2})\zeta(1-s). \tag{50}$$

According to (Motohashi 1997, Preface), ever since Riemann's mastery use of theta transformation formula in one of his proofs of the functional equation for the zeta-function, number-theorists have been fascinated by various interactions between zeta-function and automorphic forms (see also (Maurin 1997)). From a proper historical viewpoint, following (Cahen 1894, Introduction) and (Torelli 1901, Chapter VIII, Section 62), it seems have been O.X. Schlömilch, in (Schlömilch 1858), to provide a first form of functional equation satisfied by $\zeta(s)$. Furthermore, following (Davis 1959), around 1890s, it was discovered that first forms of the functional equation $\zeta(s) = \zeta(1-s)\Gamma(1-s)2^s\pi^{s-1}\sin(\pi s/2)$ seem to be already present in some Eulerian studies on gamma function ever since 1740s, where there is no proof of it but a verification of its validity only for integer values and for some rational number, like $1/2$ and $3/2$. Anyway, infinite products have been at the basis of the theory of Riemann zeta function since its institution: indeed, the primary relation upon which Riemann based his 1859 paper, is the celebrated *Euler's product* (given in the 1748 Euler's *Introductio in Analysin Infinitorum*)

$$\sum_{n\in\mathbb{N}} \frac{1}{n^s} = \prod_{p\in\mathcal{P}} \frac{1}{1-p^{-s}} \tag{51}$$

for each $s \in \mathbb{C}$, $\Re(s) > 1$, where $\mathcal{P} = \{p; p \in \mathbb{N}, p \text{ prime}, p > 1\}$. Following (Ingham 1964, Introduction, 6.), as has been said above, this latter Euler's identity was first used by Euler himself only for a fixed value, namely $s = 1$ (besides for some rational number), while Tchebycheff used it with s real. Subsequently, Riemann considered the left hand side of (13) as a complex function of s, called *zeta function,* denoted by $\zeta(s)$ and defined for $s \in \mathbb{C}$, $\Re(s) > 1$; afterwards, Riemann will give the analytical continuation of such a function to the whole complex plane through the above mentioned functional equation, obtaining a meromorphic function with only a simple pole at $s = 1$, and using it to study number theory questions through the right hand side of (4). It has been this putting into relationship number theory with complex analysis via (4), the first revolutionary and pioneering result[77] achieved by Riemann in his seminal paper.

As has been said above, from the symmetric form of (12) (see (Iviç 1986, Section 1.2), it is possible, in turn, to define the Riemann ξ function (Riemann 1858) as follows (see (Whittaker & Watson 1927, Section 13.4))

$$\xi(t) = \Big(1/2s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)\Big)_{s=\frac{1}{2}+it} \tag{52}$$

which is an even entire function of order one with simple poles in $s = 0, 1$, and whose zeros verify[78] $|\Im(t)| \leq 1/2$ (Riemann 1858). This last estimate was then improved in $|\Im(t)| < 1/2$ both by Hadamard (1896a)

[77]From an epistemological standpoint, this revolutionary idea's correlation, sets up by the two sides of equation (51), is quite similar to that provided, for example, by ***Einstein's field equations*** (1915) of General Relativity, $R_{\mu\nu} - (1/2)g_{\mu\nu}R = 8\pi G T_{\mu\nu}$ (in the natural units), which relates geometrical properties of space-time (on the left-hand side) with physical field properties (on the right-hand-side). Besides Riemann and Einstein, also H. Weyl was a pioneer in putting into relation conceptual areas before considered very far between them. This type of conceptual correlation of ideas is one of the main epistemological processes with which often scientific creativity expresses (see also what has been said by K. Maurin in the above Hors d'œuvre).

[78]Indeed, let $t = a + ib$ and $s = c + id$, so that, from $s = \frac{1}{2} + it$, it follows that $is = \frac{1}{2}i - t$,

and by de la Vallée-Poussin (1896), but independently of each other. The RH asserts that $\Im(t) = 0$, that is to say $t \in \mathbb{R}$. Following (Iviç 1989) and (Gonek 2004), it is plausible to conjecture that all the zeros of RZF, along the critical line, are simple, this assertion being supported by all the existing numerical evidences (see for example (van de Lune et al. 1986)). Subsequently, Hadamard (1893) gave a fundamental Weierstrass infinite product expansion of Riemann zeta function, of the following type (see, for example, (Karatsuba 1994, Chapter 1, Section 3.2) and (Bateman & Diamond 2004, Chapter 8, Section 8.3))

$$\xi(t) = ae^{bt} \prod_{\rho \in Z(\zeta)} \Big(1 - \frac{t}{\rho}\Big) e^{\frac{t}{\rho}} \tag{53}$$

where a, b are constants and $Z(\zeta)$ is the set of all the complex non-trivial zeros of the Riemann zeta function $\zeta(s)$, so that $Z(\zeta) \subseteq t; t \in \mathbb{C}, 0 < \Re(t) < 1$, with $cardZ(\zeta) = \infty$ (G.H. Hardy). This Hadamard paper was considered by H. Von Mangoldt (1854-1925) as *"the first real progress in the field in 34 years" since the only number theory Riemann 1859 paper* (see (Von Mangoldt 1896) and (Edwards 2001, Section 2.1)), having provided the first basic link between Riemann zeta function theory and entire function theory. Nevertheless, in relation to the Riemann zeta function, Hadamard work didn't have that right historical attention which it would have deserved, since a very few recalls to it have been paid in the related literature. From above Hadamard product formula, it follows an infinite product expansion of Riemann zeta function of the following type (see, for example, (Landau 1909, Band I, Erstes Kapitel, § 5.III-IV), (Ayoub 1963, Chapter II, Section 4), (Erdélyi 1981, Section 17.7), (Titchmarsh 1986, Section 2.12), (Narkiewicz 2000, Chapter 5, Section 5.1, Number

whence $t = \frac{1}{2}i - is = i(\frac{1}{2} - s)$, that is $t = a + ib = i(\frac{1}{2} - s) = i(\frac{1}{2} - (c + id)) = i(\frac{1}{2} - c - id) = i((\frac{1}{2} - c) - id) = d + i(\frac{1}{2} - c)$ whence $a = d$ and $b = \frac{1}{2} - c$, that is to say $\Re(t) = \Im(s), \Im(t) = \frac{1}{2} - \Re(s)$, whence $|\Im(t)| = |\frac{1}{2} - \Re(s)| \leq \frac{1}{2}$ since $c = \Re(s) \in [0, 1]$ in the critical strip. In fact, if $\frac{1}{2} \leq c \leq 1$, then $|\frac{1}{2} - c| = c - \frac{1}{2} \leq 1 - \frac{1}{2} = \frac{1}{2}$ because $c \leq 1$, whereas, if $0 \leq c \leq \frac{1}{2}$, then $|\frac{1}{2} - c| = \frac{1}{2} - c \leq \frac{1}{2}$ since $0 \leq c$, so that, anyway, we have $|\frac{1}{2} - c| \leq \frac{1}{2}$.

2) and (Voros 2010, Chapter 4, Section 4.3))

$$(54)\quad \zeta(s) = \frac{e^{(\ln 2\pi - 1 - \frac{\gamma}{2})s}}{2(s-1)\Gamma(1+\frac{s}{2})} \prod_{\rho \in Z(\zeta)} (1 - \frac{s}{\rho}) e^{\frac{s}{\rho}} = \Theta(s) \prod_{\rho \in Z(\zeta)} (1 - \frac{s}{\rho}) e^{\frac{s}{\rho}}$$

where γ is the Euler-Maclaurin constant. The function $\Theta(s)$ is non-zero into the critical strip $0 < \Re(s) < 1$, so that it is quickly realized that any question about zeros of $\zeta(s)$ might be addressed to the above infinite product factor, which is an entire function, and, likewise, as regard the above Hadamard product formula for ξ. Therefore, it seems quite obvious to account for the possible relationships existing with entire function theory, following this fruitful perspective opened by Hadamard. Out of the best treatises on entire function theory, there are those of Boris Ya. Levin (see (Levin 1980; 1996)). In particular, the treatise (Levin 1980) is hitherto the most complete one on the distributions of zeros of entire functions, which deserves a considerable attention. As regard, then, the above Hadamard product formula, in reviewing the main textbooks on Riemann zeta function, amongst which (Chandrasekharan 1958, Lectures 4, 5 and 6), (Titchmarsh 1986, Chapter II), (Iviç 1986, Section 1.3), (Patterson 1988, Chapter 3), (Karatsuba & Voronin 1992, Sections 5 and 6), (Edwards 2001, Chapter 2) and (Chen 2003, Chapter 6), it turns out that such a fundamental factorization, like the one provided by Hadamard, has been used to study some properties of this special function, for instance in relation to its Euler infinite product expansion or in relation to its growth order questions. But, in such treatises, it isn't exposed those results properly related to the possible links between Riemann zeta function theory and entire function theory, from Hadamard and Pólya works onward. Only recently, there have been various studies which have dealt with entire function theory aspects of Riemann ξ function d'aprés Pólya and Hadamrd work, and, in this regard, we begin mentioning some valuable considerations very kindly communicated to me by Professor Jeffrey C. Lagarias (private communication). He first says that, although there are strong circumstantial evidences for RH, no one knows how to prove it and no promising mechanism for a proof is currently known. In particular, there are many approaches to it, and it is not clear whether the complex variables approaches based on Laguerre-Pólya (LP) and Hermite-Biehler (HB) connections with Riemann zeta function theory via Riemann ξ function (see (Levin 1980, Chapters VII and VIII)) are going to get anywhere. He refers that, maybe, Pólya might have been the first to have established the LP connection on the basis of the previous work made by J.L.W. Jensen, and recalled in the previous sections. The truth of Riemann hypothesis requires that $\xi(s)$ falls into the HB class under suitable change of variable (see (Lagarias 2005)), even if Lagarias stresses the fact that this was already known for a long time, by which reason it requires further historical examination. Also Louis De Branges has made some interesting works in this direction, no matter by his attempts to prove Riemann hypothesis which yet deserve as well a certain attention because they follow a historical method, as kindly De Branges himself said to me (private communication). Nevertheless, Lagarias refers too that who has been the first to state this connection to HB class is historically yet not wholly clear. Further studies even along this direction have been then made, amongst others, by G. Csordas, R.S. Varga, M.L. Patrick, W. Smith, A.M. Odlyzko, J.C. Lagarias, D. Montague, D.A. Hejhal, D.A. Cardon, S.R. Adams and some other. Finally, Lagarias concludes stating that the big problem is to find a mechanism that would explain why the Riemann ξ function would fall into this HB class of functions.

Herein, we briefly remember the main lines of some of these works. For instance, the work (Cardon & de Gaston 2005) starts considering the Laguerre-Pólya class which, as is known, consists of the entire functions having only real zeros with Weierstrass products of the form

$$cz^m e^{\alpha z-\beta z^2}\prod_k\Big(1-\frac{z}{\alpha_k}\Big)e^{\frac{z}{\alpha_k}} \tag{55}$$

where c,α,β,α_k are real, $\beta\geq 0,\alpha_k\neq 0$, m is a non-negative integer, and $\sum_k(1/\alpha_k^2)<\infty$. An entire function belongs to LP if and only if it is the uniform limit on compact sets of a sequence of real polynomials having only real zeros (see (Levin 1980, Chapter VIII, Theorem 3)). One of the reasons for studying the Laguerre-Pólya class is its relationship to the Riemann zeta function. Let $\xi(s)=(1/2)s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$, where $\zeta(s)$ is the Riemann zeta function. Then $\xi(1/2+iz)$ is an even entire function of genus 1 that is real for real z. The Riemann hypothesis, which predicts that the zeros of $\xi(s)$ have real part $1/2$, can be stated as $\xi(1/2+iz) \in LP$. Furthermore, evidence suggests that most, if not all, of the zeros of $\xi(s)$ are simple. Hence, functions in LP with simple zeros are especially interesting also in issues concerning Riemann hypothesis. Then, following (Lagarias & Montague 2011, Section 1.1) and references therein, there have been many studies of properties of the Riemann ξ-function. This function motivated the study of functions in the LP class (see (Pólya 1927) and (Levin 1980, Chapter VIII)), to which the function $\xi(z)$ would belong if the Riemann hypothesis were true. It motivated the study of properties of entire functions represented by Fourier integrals that are real and bounded on the real axis (see (Pólya 1926a,b; 1927a) as well as (Titchmarsh 1980, Chapter X)) and related Fourier transforms (see (Wintner 1936)). It led to the study of the effect of various operations on entire functions, including differential operators and convolution integral operators, preserving the property of having zeros on a line, as well as various necessary conditions for the ξ-function to have real zeros, have been verified, amongst others by D. Craven, G. Csordas, W. Smith, P.P. Nielsen, D.A. Cardon, S.A. de Gaston, T.S.Norfolk, and R.S. Varga. In (Newman 1976), the author introduced a one-parameter family of Fourier cosine integrals, given for real λ by $\xi_\lambda(z) \doteq 2\int_0^\infty e^{\lambda u^2}\Phi(u)\cos zu du$ with $\Phi(u)$ given by (29). Here $\xi_0(z) = \xi(z)$ as given by (28), so this family of functions ξ_λ can be viewed as deformations of the ξ-function. It follows from a 1950 result of N.G. de Bruijn that the entire function $\xi_\lambda(z)$ has only real zeros for $\lambda \geq 1/8$. In (Newman 1976), the author proved that there exists a real number λ_0 such that $\xi_\lambda(t)$ has all real zeros for $\lambda \geq \lambda_0$, and has some non-real zeros for each $\lambda < \lambda_0$. The Riemann hypothesis holds if and only if $\lambda_0 \leq 0$, and C.M. Newman conjectured that the converse inequality $\lambda \geq 0$ holds. Newman also stated that his conjecture represents a quantitative version of the assertion that the Riemann hypothesis, if true, is just barely true. The rescaled value $\Lambda = 4\lambda_0$ was later named by Csordas, Norfolk and Varga, the *de Bruijn-Newman constant,* and they proved that $\Lambda \geq -50$. Successive authors obtained better bounds obtaining by finding two zeros of the Riemann zeta function that were unusually close together. Successive improvements of examples on close zeta zeros led to the lower bound $\Lambda > -2.7 \times 10^{-9}$, obtained by A.M. Odlyzko. Recently H. Ki, Y-O. Kim and J. Lee established that $\Lambda < 1/2$. The conjecture that $\Lambda = 0$ is now termed the *de Bruijn-Newman conjecture.* Odlyzko observed that the existence of very close spacings of zeta zeros, would imply the truth of the de Bruijn-Newman conjecture. In another direction, one may consider the effects of differentiation on the location and spacing of zeros of an entire function $F(z)$. In 1943 Pólya (see (Pólya 1943)) conjectured that an entire function F(z) of order less than 2 that has only a finite number of zeros off the real axis, has the property that there exists a finite $m_0 \geq 0$ such that all successive derivatives $F^{(m)}(z)$ for $m_0 \geq 0$ have only real zeros. This was proved by Craven, Csordas and Smith in 1987, with a new proof given by Ki and Kim in 2000. In 2005, D.W. Farmer and R.C. Rhoades have shown (under certain hypotheses) that differentiation of an entire function with only real zeros will yield a function having real zeros whose zero distribution on the real line is "smoothed". Their results apply to the Riemann ξ-function, and imply that if the Riemann hypothesis holds, then the same will be true for all derivatives $\xi^{(m)}(s) = d^m\xi(s)/ds^m, m \geq 1$. Various general results are given (Cardon & de Gaston 2004), while for more extensive informations about other researches on Riemann ξ function, we refer to (Lagarias & Montague 2011) and references therein. In any way, from what has just been said above, it turns out quite clear what fundamental role has played many part of Pólya work on Riemann ξ function in the development of entire function theory. We refer to (Korevaar 2013) and reference therein for more historical informations in this regard.

Finally, what follows is the content of a private communication with which Enrico Bombieri who has very kindly replied to my request to have some his comments and hints about some possible applications of entire function theory on Riemann zeta function theory. He kindly refers that, very likely, the 1893 Hadamard work was mainly motivated by the possible applications to Riemann zeta function, as we have above widely discussed. On the other hand, the general theory of complex and special functions had a great growth impulse just after the middle of 18th century above all thanks to the pioneering works of Weierstrass, H.A. Schwarz, Nevanlinna brothers and others. But Hadamard was the first to found a general theory which will receive its highest appreciation with the next works of Nevanlinna brothers. Afterwards, the attempts to isolate entire function classes comprising Riemann zeta function (properly modified to avoid its single poles in $s = 0, 1$) have been quite numerous (amongst which those by De Branges), with interesting results but unfruitful as regard the possible applications to Riemann hypothesis. Nevertheless, nowadays only a few mathematicians carry on along this path, amongst whom G. Csordas and co-workers with interesting works, besides those other scholars mentioned above. For instance, along the line of research opened by Pólya, in a recent conference, in which Bombieri was attended, Csordas proposed to consider the class of Mellin transformation $Mf(x)$ of fast decreasing functions f as $x \to \infty$ such that each $(xd/dx)^n f(x)$ has exactly n zeros for each $n \in \mathbb{N}_0$. Now, it would seem that the Riemann zeta function may be related with this class of functions, but, at this time, there is no exact proof of this idea to which Bombieri himself was pursuing through other ways. Many mathematicians have besides worked on Lee-Yang theorem area hoping to meet along their routes a possible insight for the Riemann zeta function and related conjecture, but after an initial enthusiasm, every further attempt didn't have any sequel. As regard, then, the general context of complex function theory, this reached its apex around 1960s, above all with the works achieved by the English school of W. Hayman on meromorphic functions and by the Russian school of B.Ya. Levin. However, it is noteworthy to mention the recent statistical mechanics approach which seems promising as regard zero distribution of Riemann zeta function whose behavior is however quite anomalous with respect other complex functions, and seems to follow a Gaussian Circular Unitary Ensemble (GCUE) law (see (Katz & Sarnak 1999) in relation to random matrix theory). Bombieri, then, finishes mentioning some very interesting results achieved, amongst others, by A. Beurling, B. Nyman and L. Báez-Duarte, hence concluding saying that, today, there still exists a little but serious group of researchers working on the relationships between

Riemann ξ function, its Fourier transform and entire function theory, d'aprés Pólya work. As regard what has just been said about GCUE law, following (Lagarias 2005), there is a great deal of evidence suggesting that the normalized spacings between the nontrivial zeros of the Riemann zeta function have a "random" character described by the eigenvalue statistics of a random Hermitian matrix whose size $N \to \infty$. The resulting statistics are the large N limit of normalized eigenvalue spacings for random Hermitian matrices drawn from the GUE distribution ("Gaussian unitary ensemble"). This limiting distribution is identical to the large N limit of normalized eigenvalue spacings for random unitary matrices drawn from the GUE distribution ("circular unitary ensemble"), i.e., eigenvalues of matrices drawn from $U(N)$ using Haar measure, and taking into account that the GUE and CUE spacing distributions are not the same for finite N. More precisely, one compares the normalized spacings of k consecutive zeros with the limiting joint probability distribution of the normalized spacings of k adjacent eigenvalues of random hermitian $N \times N$ matrices, as $N \to \infty$. The relation of zeta zeros with random matrix theory was first suggested by the work of H. Montgomery in 1973 which concerned the pair correlation of zeros of the zeta function. Montgomery's results showed (conditional on the Riemnan hypothesis) that there must be some randomness in the spacings of zeros, and were consistent with the prediction of the GUE distribution. Hence, A.M. Odlyzko, in 1987, made extensive numerical computations with zeta zeros, now up to height $T = 1022$, which show an extremely impressive fit of zeta zero spacings with predictions of the GUE distribution. The GUE distribution of zero spacings is now thought to hold for all automorphic L-functions, specifically for principal L-functions attached to $GL(n)$, (see (Katz & Sarnak 1999) and (Gonek 2004)). Further evidence for this was given by Z. Rudnick and P. Sarnak in 1996, conditionally on a suitable generalized Riemann hypothesis. They showed that the evaluation of consecutive zero gaps against certain test functions (of limited compact support) agrees with the GUE predictions. There is also supporting numerical evidence for certain principal L-functions attached to $GL(2)$. As regard, however, Riemann zeta function theory and random matrix theory, see (Borwein et al. 2008, Chapter 4, Section 4.3).

Finally, following (Conrey & Li 2000), the theory of Hilbert spaces of entire functions was developed by Louis de Branges in the late 1950s and early 1960s (see (de Branges 1968)). It is a generalization of the part of Fourier analysis involving Fourier transforms and the Plancherel formula. To be precise, the origins of Hilbert spaces of entire functions are found in a theorem of R.E. Paley and N. Wiener that characterizes finite Fourier transforms as entire functions of exponential type which are square integrable on the real axis. The known examples of Hilbert spaces of entire functions belong to the theory of special functions, a subject which is very old in relation to most of modern analysis. The foundations of the theory were laid by Euler in the century following the discovery of the calculus whose historical approach to the subject is already so well represented by the treatise (Whittaker & Watson 1927). In 1986 (see (de Branges 1986)), de Branges proposed an approach to the generalized Riemann hypothesis, that is, the hypothesis that not only the Riemann zeta function $\zeta(s)$ but also all the Dirichlet L-functions $L(s, \chi)$ with χ primitive, have their nontrivial zeros lying on the critical line $\Re s = 1/2$. In his 1986 paper (see (de Branges 1986)), de Branges said that his approach to the generalized Riemann hypothesis using Hilbert spaces of entire functions is related to the so-called Lax-Phillips theory of scattering, exposed in (Lax & Phillips 1976), where interesting applications to Riemann zeta function, following the so-called *Hilbert-Polya approach,* are exposed as well, but explaining too the difficulties of approaching the Riemann hypothesis by using the scattering theory (see also (Lax & Phillips 1989) and (Lapidus 2008)). However, also on the basis of what has been said above, de Branges' approach to Riemann hypothesis formulated for Hilbert spaces of entire functions has its early historical origins above all in a theorem on Fourier analysis due either to A. Beurling and P. Malliavin of the late 1950s (see (Beurling & Malliavin 1962)), and later improved by N. Levinson, as well as in other results achieved by M. Rosenblum, J. Von Neumann and N. Wiener. Finally, we refer to (Borwein et al. 2008) for an updated and complete survey of the most valuable approaches and attempts to solve RH, and all that, while we refer to (Lapidus 2008) and references therein, for a comprehensive, detailed and adjourned review of almost all attempts to approach RH through mathematical physics methods.

7. Conclusions

In a few words, here we sketchily outline a summary of the historical work made in this research study. To be precise, our initial subject-matter of historical investigation has been the so-called *Riemann ξ function,* a truly particular entire function formulated by Riemann in 1858 to study distribution of prime numbers, and then roughly expressed by himself as an infinite product. Just from this last factorization due to Riemann - who moreover didn't give any proof of this his statement - started our historical interest that has moved our intention toward a deeper historical investigation. Therefore, we have been forced to study the history of the techniques and tools regarding factorization of entire functions, until to reach first definitive results in this direction, that is to say, Weierstrass' and Hadamard's factorization theorems. Exactly, the Hadamard's pioneering work of 1893, rigorously proved for the first time Riemann's initial factorization of his ξ function, so opening the way to entire function theory. Therefore, the factorization of ξ function provided by Riemann in 1858, seems to have been the first starting point from which entire function theory sprung out as an autonomous chapter of complex function theory after the works by Weierstrass of 1876 and by Hadamard of 1893. On its turn, meanwhile entire function theory grew up, constant attention was being put toward possible applications to the various unsolved questions raised by the seminal 1858 paper of Riemann, amongst which the proof of prime number theorem and the estimates of prime numbers. In this regard, entire function theory aspects applied to Riemann ξ function, were considered, amongst others, by Hadamard, de la Vallée-Poussin, Landau, Von Mangoldt, and Pólya. In particular, the latter has published considerable works on entire function theory, some of which were concerned with integral representations of the Riemann ξ function, trying to solve RH in some approximation cases. In so doing, in 1926 Pólya obtained some results, amongst other just making also use of Hadamard factorization theorem, to prove some lemmata which will turn out to be useful in proving some rigorous results of statistical mechanics achieved by Lee and Yang in the 1950s which, in turn, opened new avenues to approach Riemann conjecture itself. This has been, in a few words, the historical line roughly followed above, which has seen mainly involved the figures of Riemann, Weierstrass, Hadamard, Pólya, Lee and Yang, together other notable mathematicians and physicists who inherently joined too the chief pathway routed by these latter scholars. So doing, we have touched both history of mathematics and history of physics, according to a modern perspective of history of science as, for example, recently demanded by J. Gray in (Gray 2011).

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