Coreflections in Algebraic Quantum Logic*

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Abstract

Various generalizations of Boolean algebras are being studied in algebraic quantum logic, including orthomodular lattices, orthomodular posets, orthoalgebras and effect algebras. This paper contains a systematic study of the structure in and between categories of such algebras. It does so via a combination of totalization (of partially defined operations) and transfer of structure via coreflections.

1 Introduction

The algebraic study of quantum logics focuses on structures like orthomodular lattices, orthomodular posets, orthoalgebras and effect algebras, see for instance [3, 4, 8]. This paper takes a systematic categorical look at these algebraic structures, concentrating on (1) relations between these algebras in terms of adjunctions, and (2) categorical structure of the categories of these algebras. Typical of these algebraic structures is that they involve a partially defined sum operation $\ovee$ that can be interpreted either as join of truth values (in orthomodular lattices/posets) or as sum of probabilities (in effect algebras).

The leading example of such a partially defined sum $\ovee$ is addition on the (real) unit interval $[0, 1]$ of probabilities: for $x, y \in [0, 1]$ the sum $x \ovee y = x + y$ is defined only if $x + y \leq 1$. Because this operation $\ovee$ is so fundamental, the paper takes the notion of partial commutative monoid (PCM) as starting point. An effect algebra, for instance, can then be understood as an orthosupplemented PCM, in which for each element x there is a unique element $x^\perp$ with $x \ovee x^\perp = 1$.

The paper studies algebraic quantum logics via a combination of:

- totalization of the partially defined operation $\ovee$ into a richer algebraic structure, forming a coreflection with the original (partial) structures. Such a coreflection is an adjunction where the left adjoint is a full and faithful functor;
- transfer of structure along these coreflections. It is well-known (see [1, I, prop. 3.5.3]) that limits and colimts can be transferred from one category to another if there is a coreflection between them. Here we extend this result to include also transfer of adjunctions and of monoidal structure.

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The paper concentrates on the following sequence of categories and inclusion functors, starting with Boolean algebras and ending with partial commutative monoids.

$$\mathbf{BA} \hookrightarrow \mathbf{OML} \hookrightarrow \mathbf{OMP} \hookrightarrow \mathbf{OA} \hookrightarrow \mathbf{EA} \hookrightarrow \mathbf{PCM} \tag{1}$$

The main results of this paper include the existence of:

- left adjoints to all these inclusion functors. Due to the page limit the constructions are ommited from this version of the paper.
- all limits and colimits in these categories
- symmetric monoidal structure in the categories $\mathbf{OA}$, $\mathbf{EA}$ and $\mathbf{PCM}$

Not all these results are new. Tensors for effect algebras are constructed explicitly in [2, 5]. Here they simply arise from a transfer result based on coreflections. We should add that the category $\mathbf{OML}$ of orthomodular lattices does not have a proper tensor [8].

2 Partial commutative monoids and effect algebras

Before introducing the main objects of study in this paper we first recall some basic notions about commutative monoids and fix notation.

The free commutative monoid on a set A is written as $\mathcal{M}(A)$. It consists of finite multisets $n_1a_1+\cdots+n_ka_k$ of elements $a_i \in A$, with multiplicity $n_i \in \mathbb{N}$. Such multisets may be seen as function $\varphi : A \to \mathbb{N}$ with finite support, i.e. the set $\mathsf{sup}(\varphi) = \{a \in A \mid \varphi(a) \neq 0\}$ is finite. The commutative monoid structure on $\mathcal{M}(A)$ is then given pointwise by the structure in $\mathbb{N}$, with addition $(\varphi + \psi)(a) = \varphi(a) + \psi(a)$ and zero element $0(a) = 0$. These operations can be understood as join of multisets, with 0 as empty multiset.

The mapping $A \mapsto \mathcal{M}(A)$ yields a left adjoint to the forgetful functor $\mathbf{CMon} \to \mathbf{Sets}$ from commutative monoids to sets. For a function $f : A \to B$ we have a homomorphism of monoids $\mathcal{M}(f) : \mathcal{M}(A) \to \mathcal{M}(B)$ given by $(\sum_i n_ia_i) \mapsto (\sum_i n_if(a_i))$, or more formally, by $\mathcal{M}(f)(\varphi)(b) = \sum_{a\in f^{-1}(b)} \varphi(a)$. The unit $\iota : A \to \mathcal{M}(A)$ of the adjunction may be written as $\iota(a) = 1a$. If $M = (M, +, 0)$ is a commutative monoid we can interpret a multiset $\varphi \in \mathcal{M}(M)$ over M as an element $[\![\, \varphi \,]\!] = \sum_{x\in\mathsf{sup}(\varphi)} \varphi(x) \cdot x$, where $n \cdot x$ is $x + \cdots + x$, n times. In fact, this map $[\![-]\!]$ is the counit of the adjunction mentioned before. Each monoid M carries a preorder $\preceq$ given by: $x \preceq y$ iff $y = x + z$ for some $z \in M$. In free commutative monoids $\mathcal{M}(A)$ we get a poset order $\varphi \preceq \psi$ iff $\varphi(a) \leq \psi(a)$ for all $a \in A$. Homomorphisms of monoids are monotone functions wrt. this order $\preceq$. This applies in particular to interpretations $[\![-]\!] : \mathcal{M}(M) \to M$.

Definition 2.1. A *partial commutative monoid*, or PCM, is a triple $(M, \ovee, 0)$ consisting of a set M, an element $0 \in M$ and a partially defined binary operation $\ovee$ such that the three axioms below are satisfied. We let the expression '$x \perp y$' mean '$x \ovee y$ is defined', and call such elements x, y orthogonal.

1. $x \perp y$ implies $y \perp x$ and $x \ovee y = y \ovee x$.
2. $y \perp z$ and $x \perp (y \ovee z)$ implies $x \perp y$ and $(x \ovee y) \perp z$ and $x \ovee (y \ovee z) = (x \ovee y) \ovee z$.

3. $0 \perp x$ and $0 \oslash x = x$.

An *effect algebra* is a PCM with a special element 1 and an additional unary operator $(-)^{\perp}$ called the orthosupplement such that

1. $x^{\perp}$ is the unique element such that $x \oslash x^{\perp} = 1$.
2. $x \perp 1$ implies $x = 0$.

An *orthoalgebra* is further required to satisfy the relation $x \perp x$ implies $x = 0$.

In some texts (*e.g.* [3, 5]) it is required that $0 \neq 1$ however we allow $\{0\}$ as an effect algebra since it will be the final object in the category $\mathbf{EA}$.

An obvious example of a PCM is the unit interval $[0,1]$ of real numbers, with $x \oslash y$ defined, and equal to $x+y$, iff this sum $x+y$ fits in $[0,1]$. It even forms an effect algebra with $x^{\perp} = 1-x$ but it's not an orthoalgebra since $\frac{1}{2} \perp \frac{1}{2}$.

Definition 2.2. We organize partial commutative monoids into a category $\mathbf{PCM}$ as follows. The objects are PCMs and homomorphisms $f : (M, \oslash, 0) \to (N, \oslash, 0)$ are (total) functions from M to N such that $f(0) = 0$, and for $x, y \in M$, $x \perp y$ implies $f(x) \perp f(y)$ and $f(x \oslash y) = f(x) \oslash f(y)$.

We also form the category $\mathbf{EA}$ of effect algebras. An effect algebra homomorphism is a PCM homomorphism such that $f(1) = 1$. This condition implies that effect algebra homomorphsims preserve the orthosuplement. The orthoalgebras form a full subcategory $\mathbf{OA}$ of the category of effect algebras.

Remark 2.3. Recall the interpretation $[\![\sum_i n_i x_i]\!] = \sum_i n_i \cdot x_i$ of a multiset $(\sum_i n_i x_i) \in \mathcal{M}(M)$ in a monoid M. In case M is a PCM, such an interpretation need not always exist. Over a PCM M we call a multiset φ an *orthogonal* multiset in M if the interpretation $[\![\varphi]\!] = \oslash_{x \in \mathsf{sup}(\varphi)} \varphi(x) \cdot x$ exists in M. Here we write $n \cdot x$ for the n-fold sum $x \oslash \cdots \oslash x$, assuming it exists.

Such orthogonal multisets may be seen as (multiset version of) 'tests' from [7]. A non-multiset version of a 'test' for M is a multiset test φ with $\varphi(x) \leq 1$ for each $x \in M$. Such a test can thus be identified with a finite subset $\{x_1, \ldots, x_n\} \subseteq M$ for which $x_1 \oslash \cdots \oslash x_n$ exists.

We shall write $\mathcal{O}r(M) \hookrightarrow \mathcal{M}(M)$ for the subsets of orthogonal multisets in M. This subset is downclosed ($\varphi \preceq \psi \in \mathcal{O}r(M)$ implies $\varphi \in \mathcal{O}r(M)$), and forms a PCM itself, with the interpretations $[\![-]\!]$ forming homomorphisms of PCMs $[\![-]\!] \colon \mathcal{O}r(M) \to M$.

Notice that $1x \in \mathcal{O}r(M)$, for $x \in M$, so that $\bigcup_{\varphi \in \mathcal{O}r(M)} \mathsf{sup}(\varphi) = M$. For a map $f \colon M \to N$ in PCM, if $\varphi = (\sum_i n_i x_i) \in \mathcal{O}r(M)$ then $\mathcal{M}(f)(\varphi) = (\sum_i n_i f(x_i)) \in \mathcal{O}r(N)$, by definition of 'morphism in PCM', and:

$$[\![\mathcal{M}(f)(\varphi)]\!] = [\![\textstyle\sum_i n_i f(x_i)]\!] = \oslash_i n_i \cdot f(x_i) = f(\oslash_i n_i \cdot x_i) = f([\![\varphi]\!]). \tag{2}$$

The following (standard) notion of 'partial equality' for partially defined operations is useful. We write:

$$\varphi \simeq_M \psi \quad \text{for} \quad \begin{cases} \text{if } [\![\varphi]\!] \text{ is defined, also } [\![\psi]\!] \text{ is defined and } [\![\varphi]\!] = [\![\psi]\!], \\ \qquad \text{and} \\ \text{if } [\![\psi]\!] \text{ is defined, so is } [\![\varphi]\!] \text{ and } [\![\psi]\!] = [\![\varphi]\!]. \end{cases} \tag{3}$$

The subscript 'M' in $\simeq_M$ is sometimes omitted when M is clear from the context.

Lemma 2.1. *Partial equality $\simeq_M \subseteq \mathcal{M}(M) \times \mathcal{M}(M)$, for a PCM M, is an equivalence and a congruence relation.*

3 Coreflections

We now turn to some technical results on coreflections which will be used for various constructions with PCMs and effect algebras.

Recall that a coreflection is an adjunction $F \dashv G$ where the left adjoint $F : \mathbf{A} \to \mathbf{B}$ is full and faithful, or equivalently, the unit $\eta : \mathrm{id} \to GF$ is an isomorphism. It is well-known (see [1]) that in this situation $\mathbf{A}$ is as complete and cocomplete as $\mathbf{B}$, see theorem 3.1. Below we shall see that $\mathbf{A}$ not only inherits limits and colimits from $\mathbf{B}$ but also adjunctions and a (symmetric) monoidal structure under some mild conditions.

Assume we have a coreflection $(F, G, \eta, \varepsilon) : \mathbf{A} \to \mathbf{B}$

Theorem 3.1. *Suppose $D : \mathbf{J} \to \mathbf{A}$ is a diagram in $\mathbf{A}$ and that $l_j : L \to FD_j$ (resp. $c_j : FD_j \to C$) is a limit (resp. colimit) of $F \circ D$ in $\mathbf{B}$ then $\eta_{D_j}^{-1} \circ G(l_j) : G(L) \to D_j$ ($G(c_j) \circ \eta_{D_j} : D_j \to G(C)$) is a limit (colimit) of D in $\mathbf{A}$.*

Next suppose that $\mathbf{B}$ comes equipped with a (possibly symmetric) monoidal structure $(\otimes, I)$. We wish to use the coreflection to define this same structure on $\mathbf{A}$. To do so we must assume that the functor $FG : \mathbf{B} \to \mathbf{B}$ is monoidal and that the counit ε is a monoidal natural transformation. We then define a bifunctor $\otimes^{\mathbf{A}} : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$ by

$$X \otimes^{\mathbf{A}} Y := G(FX \otimes^{\mathbf{B}} FY) \tag{4}$$

and also define $I^{\mathbf{A}} := G(I^{\mathbf{B}})$.

Theorem 3.2. *In the situation sketched above:*

1. *The tensor* (4) *yields (symmetric) monoidal structure on the category $\mathbf{A}$ in such a way that the functor $G : \mathbf{B} \to \mathbf{A}$ is automatically monoidal.*
2. *The functor $F : \mathbf{A} \to \mathbf{B}$ is strongly monoidal.*
3. *If $\mathbf{B}$ is monoidal closed, then so is $\mathbf{A}$ via $B \multimap^{\mathbf{A}} C = G(FB \multimap^{\mathbf{B}} FC)$.*

Now suppose we have a second coreflection $(F', G', \eta', \varepsilon') : A' \to B'$ as well as a map of adjunctions $(H : A \to A', K : B \to B')$ and set $J = G \circ K \circ F'$.

$$\begin{tikzcd}
\mathbf{B} \arrow[r, bend right, "K"'] \arrow[d, bend left, "G"] \arrow[d, phantom, "\dashv"] & \mathbf{B}' \arrow[l, bend right, "M"'] \arrow[d, bend left, "G'"] \arrow[d, phantom, "\dashv"] \\
\mathbf{A} \arrow[u, bend left, "F"] \arrow[r, bend right, "H"'] & \mathbf{A}' \arrow[l, dashed, "J"'] \arrow[u, bend left, "F'"]
\end{tikzcd}$$

Theorem 3.3. *If M is a left (resp. right) adjoint to K then J is a left (right) adjoint to H.*

4 The categorical structure of $\mathbf{PCM}$ and $\mathbf{EA}$

4.1 Totalization

In this section we'll embed PCMs and effect algebras into algebraic structures with total operations which are easier to work with. Categorically this will take the form of a coreflection so that we can use the results from the previous section. The first construction presented below is similar to the unigroups of Foulis, Greechie and Bennet [6]. However we use monoids instead of groups. While unigroups essentialy only work for interval effect algebras our barred commutative monoids (BCMs) works for all effect algebras. Also the products BCMs is just the cartesian product, unlike for unigroups [6].

Definition 4.1. Define a category $\mathbf{DCM}$ of *downsets in commutative monoids* as follows. Its objects consist of pairs (M,U) where M is a commutative monoid and $U \subseteq M$ is a nonempty downclosed subset of M: $0 \in U$, and $a \in U$ and $b \preceq a$ implies $b \in U$. The morphisms $f : (M,U) \to (N,V)$ consist of monoid homomorphisms $f : M \to N$ such that $f(U) \subseteq V$.

Definition 4.2. Define a totalization functor $\mathcal{T}o : \mathbf{PCM} \to \mathbf{DCM}$ as follows. Let $M = (M, \otimes, 0)$ be a partial commutative monoid and put

$$\mathcal{T}o(M) \quad := \quad (\,\mathcal{M}(M)/\!\sim,\ \{[1x]_\sim \mid x \in M\}\,)$$

where $\sim$ is the smallest congruence such that $1x + 1y \sim 1(x \otimes y)$, for all $x, y \in M$ with $x \perp y$, and $0_{\mathcal{M}(M)} \sim 1(0_M)$. Thus, $\varphi \sim 1[\![\,\varphi\,]\!]$, for each $\varphi \in \mathcal{O}r(M)$. Note that $[1x]_\sim + [1(0_M)]_\sim = [1x] = [1(x \otimes 0_M)]_\sim$ in $\mathcal{T}o(M)$.

For $f : M \to N$ a homomorphism of partial commutative monoids define $\mathcal{T}o(f)$ by $\mathcal{T}o(f)([\sum_i n_i x_i]) = [\sum_i n_i f(x_i)]$.

We'll usually omit the square brackets when denoting elements of $\mathcal{T}o(M)$, so we simply write $1x$ instead of $[1x]_\sim$, and $[\![\,\varphi\,]\!]$ instead of $[\![\,[\varphi]_\sim\,]\!]$. To see that $\mathcal{T}o$ is indeed a well-defined functor note that $\sim \subseteq \simeq$ where $\simeq$ is as described in Remark 2.3 so that $\{1x \mid x \in M\}$ is indeed a downset in $\mathcal{T}o(M)$. To see that $\mathcal{T}o(f)$ is well-defined note:

$$\begin{aligned}\mathcal{T}o(f)(1x+1y) &= \mathcal{T}o(f)(1x) + \mathcal{T}o(f)(1y) = 1f(x) + 1f(y)\\ &= 1(f(x) \otimes f(y)) = 1(f(x \otimes y)) = \mathcal{T}o(f)(1(x \otimes y))\end{aligned}$$

whenever $x \perp y$.

Definition 4.3. Define a functor $\mathcal{P}a : \mathbf{DCM} \to \mathbf{PCM}$ by $\mathcal{P}a(M,U) = U$, for (M,U) a monoid with downset; clearly $0 \in U$, and for $x, y \in U$ we set $x \perp y$ iff $x+y \in U$, and then $x \otimes y = x+y$. For a homomorphism $f : (M,U) \to (N,V)$ define $\mathcal{P}a(f) = f|_U : U \to V$.

We use the fact that U is a downset in M to show that $\mathcal{P}a(M,U)$ is a PCM. Commutativity is obvious because M is commutative. Furthermore if $x \otimes (y \otimes z)$ is defined then $x+y+z \in U$ and because U is a downset $x+y \in U$ and so $(x \otimes y) \otimes z$ is also defined and equal to $x \otimes (y \otimes z)$.

Theorem 4.1. *The totalization functor* $\mathcal{T}o : \mathbf{PCM} \to \mathbf{DCM}$ *is a (full and faithful) left adjoint to* $\mathcal{P}a$. *Hence we have a coreflection.*

We now turn to a similar construction for effect algebras and orthoalgebras.

Definition 4.4. A *barred commutative monoid (or BCM)* $(M, +, 0, u)$ is a commutative monoid $(M, +, 0)$ that is positive *i.e.* $a + b = 0$ implies $a = b = 0$ together with an element $u \in M$ called the unit such that $a + b = a + c = u$ implies $b = c$.

An *orthobarred commutative monoid (or OBCM)* is a barred commutative monoid such that $a + a \preceq u$ implies $a = 0$.

The name barred commutative monoid comes from the fact that the unit forms a bar, below which certain properties must hold. However beyond this bar those porperties need not hold, for example the cancelation law holds for elements below the bar but it generally need not hold for arbitrary elements in a barred commutative monoid.

Definition 4.5. We form the category $\mathbf{BCM}$ of barred commutative monoids as follows. Let the objects be the BCMs $(M, +, 0, u)$ and let the homomorphisms $f : (M, +, 0, u) \to (M', +, 0, u')$ be monoid homomorphisms $M \to M'$ such that $f(u) = u'$.

We also construct the full subcategory $\mathbf{OBCM}$ of orthobarred commutative monoids.

We view the category $\mathbf{BCM}$ as a (non full) subcategory of $\mathbf{DCM}$ by taking the unit interval $\{a \in M \mid a \preceq u\}$ as the required downset. Similarly we view $\mathbf{EA}$ as a subcategory of $\mathbf{PCM}$.

Proposition 4.2. *We can restrict the functors $\mathcal{T}o$ and $\mathcal{P}a$ to $\mathbf{EA}$ and $\mathbf{BCM}$ and then to $\mathbf{OA}$ and $\mathbf{OBCM}$ and keep having coreflections. Thus we have the following row of coreflections.*

$$
\begin{tikzcd}
\mathbf{DCM} \arrow[d, bend left, "\mathcal{P}a"] & \mathbf{BCM} \arrow[l] \arrow[d, bend left, "\mathcal{P}a"] & \mathbf{OBCM} \arrow[l] \arrow[d, bend left, "\mathcal{P}a"] \\
\mathbf{PCM} \arrow[u, bend left, "\mathcal{T}o"] \arrow[u, phantom, "\dashv"] & \mathbf{EA} \arrow[l] \arrow[u, bend left, "\mathcal{T}o"] \arrow[u, phantom, "\dashv"] & \mathbf{OA} \arrow[l] \arrow[u, bend left, "\mathcal{T}o"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
$$

4.2 Limits and colimits

The categories $\mathbf{PCM}$, $\mathbf{EA}$, $\mathbf{DCM}$ and $\mathbf{BCM}$ will all turn out to be both complete and cocomplete. Products, coproducts and equalizers can be described directly in all categories. But coequalizers in $\mathbf{PCM}$ and $\mathbf{EA}$ are a different story. However thanks to theorem 3.1 it suffices to describe them in $\mathbf{DCM}$ and $\mathbf{BCM}$. We start with limits and colimits in $\mathbf{DCM}$. They are basically obtained as for commutative monoids.

Proposition 4.3. *Let I be some set and let $\{(M_i, U_i) \mid i \in I\}$ be a family of monoids with downsets, indexed by I.*

(a) The product of this family in $\mathbf{DCM}$ is:

$$(\prod_{i \in I} M_i,\ \{\phi \in \prod_{i \in I} M_i \mid \forall_{i \in I}.\, \phi(i) \in U_i\}),$$

where $\prod_{i \in I} M_i$ is the product of monoids. It consists of functions $\phi : I \to \bigsqcup_{i \in I} M_i$ with $\phi(i) \in M_i$ for all i, with the operation defined pointwise. Here $\bigsqcup$ denotes the disjoint union of the underlying sets.

(b) *The coproduct is given by*

$$\left(\coprod_{i\in I} M_i, \{\phi \in \coprod_{i\in I} M_i \mid \exists_{i\in I}.\, \phi(i) \in U_i \text{ and } \forall_j.\, i \neq j \Rightarrow \phi(j) = 0\}\right)$$

where $\coprod_{i\in I} M_i$ *is the monoid coproduct. It consists of functions* $\phi : I \to \bigsqcup_{i\in I} M_i$ *with* $\phi(i) \in M_i$ *and* $\{i \in I \mid \phi(i) \neq 0\}$ *is finite.* □

Next, let $f, g : (M, U) \to (N.V)$ *be two arrows in* $\mathbf{DCM}$*.*

(c) *The equalizer of* f *and* g *is* (E, W) *where* $E = \{m \in M \mid f(m) = g(m)\}$ *and* $W = E \cap U$*.*

(d) *The coequalizer of* f *and* g *is* $(N/\sim, \{[v] \mid v \in V\})$ *where* $\sim$ *is the smallest monoid congruence such that* $f(m) \sim g(m)$*.* □

Since $\mathbf{DCM}$ has all products and coproducts as well as equalizers and coequalizers we see that $\mathbf{DCM}$ is both complete and cocomplete.

Proposition 4.4. *Let* I *be a set and let* $\{M_i \mid i \in I\}$ *be a family of PCMs.*

(a) *The product of this family is given by the Cartesian product* $\prod_{i\in I} M_i$*.*

(b) *The coproduct is given by the disjoint union* $(\bigsqcup_{i\in I} M_i \setminus \{0\}) \cup \{0\}$ *with all the* 0 *elements identified.*

Now let $f, g : M \to N$ *be two PCM homomorphisms.*

(c) *The equalizer of* f *and* g *is* $E = \{m \in M \mid f(m) = g(m)\}$*.*

(d) *The coequalizer of* f *and* g *is* $\mathcal{P}_a(h) \circ \eta_N$ *where* $h : \mathcal{T}_o(N) \to \mathcal{T}_o(N)/\sim$ *is the coequalizer of* $\mathcal{T}_o(f)$ *and* $\mathcal{T}_o(g)$ *(cf. theorem 3.1).*

We now turn to $\mathbf{EA}$ and $\mathbf{BCM}$. Products and equalizers in $\mathbf{EA}$ and $\mathbf{BCM}$ are constructed in the same way as in $\mathbf{PCM}$ and $\mathbf{DCM}$. The products are just the Cartesian products with pointwise operations and the equalizers are just the set-theoretic ones.

Before tackling colimits we first study congruences on BCMs and see what consequences this has for effect algebras.

Definition 4.6. A *congruence* $\sim$ on a BCM E is an equivalence relation such that the following conditions hold:

(i) $a_1 \sim a_2$ and $b_1 \sim b_2$ implies $a_1 + b_1 \sim a_2 + b_2$;

(ii) $a + b \sim 0$ implies $a \sim 0$ and $b \sim 0$;

(iii) $a + b \sim u$ and $a + c \sim u$ implies $b \sim c$.

We'll denote the set of all congruences on E with $\mathrm{Cong}(E)$.

When the following condition also holds we call $\sim$ an orthocongruence:

(iv) $a + a + b \sim u$ implies $a \sim 0$.

Definition 4.7. If M is some monoid and $u \in M$ is an element and $\sim$ is a monoid congruence on M then we call $\sim$ a *bar congruence with respect to* u if it satisfies conditions (ii) and (iii) from definition 4.6. If it also satisfies condition (iv) then we call it an *orthocongruence with respect to* u.

Proposition 4.5. *(a) If $(E, +, 0, u)$ is a BCM and $\sim$ is a congruence on E then $E/\sim$ is a BCM with unit $[u]_\sim$. There exists a canonical surjective homomorphism $\pi : E \to E/\sim$ that sends x to $[x]_\sim$.*

(c) When $\sim$ is an orthocongruence then $E/\sim$ is a OBCM.

(d) When M is a monoid and $\sim$ is a bar congruence with respect to u then $M/\sim$ is an BCM with unit $[u]_\sim$.

The main advantage of $\mathbf{BCM}$ over $\mathbf{EA}$ (and of $\mathbf{DCM}$ over $\mathbf{PCM}$) is that the intersection of congruences is again a congruence. So it makes sense to talk about the smallest congruence containing a given relation.

We now move on to the construction of coproducts and coequalizers in $\mathbf{BCM}$ and $\mathbf{EA}$.

Proposition 4.6. *Let I be a nonempty set and let $(E_i)_{i \in I}$ be a collection of barred commutative monoids. Let $C := \coprod_{i \in I} E_i$ be the monoid coproduct of the E_i and let $\kappa_i : E_i \to C$ be the inclusions. Choose some $i \in I$ and let $\sim$ be the smallest bar congruence with respect to $\kappa_i(u)$ such that for all $j \in I$ we have $\kappa_j(u) \sim \kappa_i(u)$. The coproduct of the E_i in $\mathbf{BCM}$ is given by $C/\sim$. The choice of i doesn't make a difference.*

The empty coproduct also exists and is the initial object $(\mathbb{N}, +, 0, 1)$.

Proposition 4.7. *Let I be a set and let $(E_i)_{i \in I}$ be a collection of effect algebras. Their coproduct is given by their disjoint union with zeros and units identified.*

$$\coprod_{i \in I} E_i = (\bigsqcup_{i \in I} E_i \setminus \{0, 1\}) \cup \{0, 1\}$$

The operations are defined like for PCMs.

Proposition 4.8. *(a) If $f, g : E \to F$ are two BCM maps then their coequalizer is given by $\pi : F \to F/\sim$ where $\sim$ is the smallest congruence containing $\{(f(a), g(a)) \mid a \in E\}$.*

(b) For two effect algebra maps $f, g : E \to F$ their coequalizer is given by $\mathcal{T}_o(h) \circ \eta_E$ where $h : \mathcal{T}_o(F) \to \mathcal{T}_o(F)/\sim$ is the coequalizer of $\mathcal{T}_o(f)$ and $\mathcal{T}_o(g)$ in $\mathbf{BCM}$.

4.3 tensor products

Definition 4.8. Let M, N be two partial commutative monoids. A *bimorphism* (of PCMs) f is a function $f : M \times N \to L$ such that

$$\begin{aligned} f(m, n_1 \ovee n_2) &= f(m, n_1) \ovee f(m, n_2) && \text{whenever } n_1 \perp n_2 \\ f(m_1 \ovee m_2, n) &= f(m_1, n) \ovee f(m_2, n) && \text{whenever } m_1 \perp m_2 \\ f(m, 0) &= 0 = f(0, n) \end{aligned}$$

for all $m, m_1, m_2 \in M$ and $n, n_1, n_2 \in N$. An effect algebra bimorphism is a PCM bimorphism such that $f(1, 1) = 1$.

Let $(M, U), (N, V), (L, W)$ be commutative monoids with downsets. A *bimorphism* (of monoids with downsets) f is a function $f : M \times N \to L$ such that

$$\begin{aligned} f(m, n_1 + n_2) &= f(m, n_1) + f(m, n_2) \\ f(m_1 + m_2, n) &= f(m_1, n) + f(m_2, n) \\ f(m, 0) &= 0 = f(0, n) \\ f(u, v) &\in W \end{aligned}$$

for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$, $u \in U$ and $v \in V$. A bimorphism of barred commutative monoids is a bimorphism of monoids with downsets such that $f(u, u) = u$.

We recall the usual definition of tensor products in terms of bimorphisms.

Definition 4.9. Let M and N be PCMs (or effect algebras, monoids with downsets or BCMs). A tensor product of M and M is a pair (T, t) consisting of a PCM (effect algebra, $\ldots$) T and a universal bimorphism $t : M \times N \to T$ such that for every bimorphism $f : M \times N \to L$ there is a unique homomorphism $g : T \to L$ such that $f = g \circ t$.

Of course the tensor product is unique up to isomorphism should it exist. We will now construct the tensor product for all four categories $\mathbf{DCM}$, $\mathbf{PCM}$, $\mathbf{BCM}$, $\mathbf{EA}$.

We will write $\boxtimes$ for the tensor product in the category $\mathbf{CMon}$ of commutative monoids, with the universal bimorphism $M \times N \to M \boxtimes N$ given by $(m, n) \mapsto m \boxtimes n$.

Definition 4.10. Let $(M, U), (N, V) \in \mathbf{DCM}$, define:

$$(M, U) \otimes (N, V) = (M \boxtimes N, \downarrow \{u \boxtimes v \mid u \in U, v \in V\})$$

Theorem 4.9. *Let $(M, U), (N, V) \in \mathbf{DCM}$, $(M, U) \otimes (N, U)$ together with the map $\chi : (m, n) \mapsto m \boxtimes n$ forms the tensor product of (M, U) and (N, V).*

The category $\mathbf{DCM}$ is symmetric monoidal. The tensor unit is $(\mathbb{N}, \{0, 1\})$ and the coherence isomorphisms are inherited from $\mathbf{CMon}$. We want to apply proposition 3.2 to create a monoidal structure on $\mathbf{PCM}$. So we must show that $\mathcal{T}_o \circ \mathcal{P}_a$ is a monoidal functor.

Since $\varepsilon : \mathcal{T}_o\mathcal{P}_a(\mathbb{N}, \{0, 1\}) \to (\mathbb{N}, \{0, 1\})$ is an isomorphism we set $\zeta = \varepsilon^{-1}$. To construct $\xi : \mathcal{T}_o\mathcal{P}_a(M, U) \otimes \mathcal{T}_o\mathcal{P}_a(N, V) \to \mathcal{T}_o\mathcal{P}_a((M, U) \otimes (N, V))$ we use the bimorphism

$$\begin{aligned} \mathcal{T}_o\mathcal{P}_a(M, U) \times \mathcal{T}_o\mathcal{P}_a(N, V) &\to \mathcal{T}_o\mathcal{P}_a((M, U) \otimes (N, V)) \\ (\textstyle\sum n_i u_i, \sum m_j v_j) &\mapsto \textstyle\sum n_i m_j (u_i \otimes v_j) \end{aligned}$$

It's easy to check that this is natural and that ε is a monoidal natural transformation.

Thus we get a symmetric monoidal structure on $\mathbf{PCM}$ given by $M \otimes N = \mathcal{P}_a(\mathcal{T}_o(M) \otimes \mathcal{T}_o(N))$. This construction is in fact a tensor product in the sense of definition 4.9.

Theorem 4.10. *If M and N are PCMs then $\mathcal{P}_a(\mathcal{T}_o(M) \otimes \mathcal{T}_o(N))$ together with the map $(x, y) \mapsto 1x \boxtimes 1y$ is the tensor product of M and N.*

Definition 4.11. Let E, F be two barred commutative monoids. Define $E \otimes F := (E \boxtimes F)/{\sim}$, where $\boxtimes$ is the commutative monoid tensor product and $\sim$ is the smallest bar congruence with respect to $u \boxtimes u$. We'll denote the $\sim$ equivalence class of $e \boxtimes f$ by $e \otimes f$.

Theorem 4.11. *Let E, F be BCMs, $E \otimes F$ together with the map $\chi : (e, f) \mapsto e \otimes f$ forms the tensor product of E and F.*

Like $\mathbf{DCM}$ the category $\mathbf{BCM}$ is also symmetric monoidal and the functor $\mathcal{T}_o\mathcal{P}_a$ is monoidal. So $\mathbf{EA}$ is also symmetric monoidal and just like before this monoidal structure on $\mathbf{EA}$ is in fact a tensor.

Theorem 4.12. *If E and F are effect algebras then $\mathcal{P}_a(\mathcal{T}_o(E) \otimes \mathcal{T}_o(F))$ together with the map $(x, y) \mapsto 1x \otimes 1y$ is the tensor product of E and F.*

If we modify the construction in definition 4.11 slighty by using orthocongruences instead of bar congruences then we also find a tensor for the categories $\mathbf{OBCM}$ and $\mathbf{OA}$.

The categories $\mathbf{DCM}$ and $\mathbf{PCM}$ are in fact closed symmetric monoidal categories as we will see in a moment. $\mathbf{BCM}$ and $\mathbf{EA}$ are not. One can't give $\mathrm{Hom}(E, F)$ an effect algebra structure for arbitrary effect algebras E and F. If one tries to define $\ovee$ and $\perp$ pointwise, the following problem pops up $f^{\perp}(1) = f(1^{\perp}) = f(0) = 0$. This problem doesn't occur in $\mathbf{PCM}$ and $\mathbf{DCM}$.

Definition 4.12. Let $(M, U), (N, V) \in \mathbf{DCM}$ define an exponent:

$$(M, U) \multimap (N, V) := (\mathrm{Hom}_{\mathbf{CMon}}(M, N), \mathrm{Hom}_{\mathbf{DCM}}((M, U), (N, V)))$$

This exponent is again a commutative monoid with a downset in the obvious way.

For $M, N \in \mathbf{PCM}$ define $M \multimap N := \mathrm{Hom}_{\mathbf{PCM}}(M, N)$, where the PCM structure on $M \multimap N$ is as follows. For $f, g : M \to N$ we define $f \ovee g : M \to N$ by $(f \ovee g)(m) = f(m) \ovee g(m)$. Of course $f \ovee g$ is only defined when $f(m) \perp g(m)$ for all $m \in M$.

We view $\multimap$ as a bifunctor in the usual way.

Theorem 4.13. *For $M \in \mathbf{PCM}$ (or $\mathbf{DCM}$) the functor $M \multimap (-)$ is a right adjoint to the functor $(-) \otimes M$.*

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