

DEGREES OF PARTIAL FUNCTIONS

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In this paper we consider a new notion of relative recursion on partial functions, which allows for an easy definition of the recursive infimum of two functions.

1 Let U be the set of partial mappings: $\omega \rightarrow \{0, 1\}$. We write $f(x) = *$ if $f(x)$ is undefined. U will be the universum for our recursion theory. Computations are introduced using Kleene brackets [1]. So we have a relation

$$\{e\}^\alpha(\vec{m}) \cong n$$

where $e, m_i, n \in \omega, \alpha \in U$. A computation $\{e\}^\alpha(\vec{m})$ is undefined if either it never stops or it uses $\alpha(n)$ for an n s.t. $\alpha(n) = *$.

1.1 Definition α is recursive in β ($\alpha \leq \beta$) if for some $e \in \omega$

$$\alpha \subseteq \lambda x \cdot \{e\}^\beta(x).$$

It is easy to see that \leq is a transitive relation on U . We write $\alpha \equiv \beta$ if $\alpha \leq \beta$ and $\beta \leq \alpha$. Of course \equiv is an equivalence relation. The equivalence classes are called degrees. The lowest degree, 0, is the degree of the partial recursive functions.

Motivation We see $\alpha \in U$ as an object containing information (concerning its arguments). If $\alpha \subseteq \beta$ then β contains at least as much information as α does. Hence we insist to have $\alpha \leq \beta$ in this case. A similar argument holds if $\alpha = \lambda x \cdot \{e\}^\beta(x)$ for some e . These two requirements generate \leq .

As the total functions are included in U , U/\equiv has cardinality 2^{\aleph_0} , cf. [2]. On the other hand some equivalence classes of \equiv do have cardinality 2^{\aleph_0} themselves. It is not difficult to find α which is not equivalent to any total function. Furthermore a straightforward spoiling construction shows that there are no minimal degrees in U . Some motivation for considering U lies in the following theorem.

1.2 Definition 1 - $sc(\alpha)$ is the set of *total* functions recursive in α .

1.3 Theorem Let $V \subseteq \omega^\omega$ be countable and closed under recursion. Then for some $\alpha \in U$ $V = 1 - sc(\alpha)$.

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Proof: Straightforward construction.

Example Let nF be a type $-n$ functional. Let $\alpha = \lambda p \cdot \{p\}({}^nF)$. Then $1 - sc(\alpha) = 1 - sc({}^nF)$.

(The following problem is open: $\forall \alpha \exists {}^4F [1 - sc(\alpha) = 1 - sc(F)]?$)

2 In U/\equiv we can define easily the supremum and infimum of two degrees.

2.1 Definition (i) for $\alpha, \beta \in U$, $\alpha \vee \beta$ is defined by

$$\alpha \vee \beta(n) = \begin{cases} \text{if } n = 2k \text{ then } \alpha(k) \\ \text{if } n = 2k + 1 \text{ then } \beta(k) \end{cases}$$

(ii) for $\alpha, \beta \in U$, $\alpha \wedge \beta$ is defined by

$$\alpha \wedge \beta(n) = \alpha \wedge \beta(\langle x, y, z \rangle) = \begin{cases} \text{if } \{x\}^\alpha(z) = p = \{y\}^\beta(z) \text{ then } p(p \in (0, 1)) \\ * \text{ otherwise.} \end{cases}$$

2.2 Theorem \vee and \wedge give sup and inf in U/\equiv .

Proof: \vee : Suppose $\alpha \leq \gamma$ and $\beta \leq \gamma$ then $\alpha \vee \beta \leq \gamma$ immediately.

\wedge : Suppose $\gamma \leq \alpha$ and $\gamma \leq \beta$. Choose x, y such that

$$\gamma \subseteq \lambda n \cdot \{x\}^\alpha(n) \text{ and } \gamma \subseteq \lambda n \cdot \{y\}^\beta(n).$$

Clearly $\gamma \subseteq \lambda n \cdot \{x\}^\alpha(n) \cap \lambda n \cdot \{y\}^\beta(n)$. Now $\gamma \subseteq \lambda n \cdot \alpha \wedge \beta(\langle x, y, n \rangle)$. Hence $\gamma \leq \alpha \wedge \beta$.

An interesting problem is whether or not there exist α and β of non-zero degree such that $\alpha \wedge \beta$ is of degree zero. We were unable to solve this question. However we can solve it for an operation Δ which assigns, in a natural way, a function $\alpha \Delta \beta$ to α and β which is below $\alpha \wedge \beta$.

2.3 Definition

$$\alpha \Delta \beta(\langle x, y \rangle) = \begin{cases} \text{if } \alpha(x) = \beta(y) = p \text{ then } p \\ \text{else } * \end{cases}$$

$\alpha \Delta \beta$ allows one to compare all values of α with all values of β .

2.4 Theorem *There are $\alpha, \beta \in U$ such that $\alpha \neq 0, \beta \neq 0$ and $\alpha \Delta \beta \equiv 0$.*

Proof: We find α and β such that

(i) $\alpha \Delta \beta(\langle x, y \rangle)$ is defined implies $\alpha \Delta \beta(\langle x, y \rangle) = \{x\}(y)$ (hence $\alpha \Delta \beta \equiv 0$).

(ii) for all $n, \alpha \not\subseteq \lambda x \cdot \{n\}(x)$ and $\beta \not\subseteq \lambda x \cdot \{n\}(k)$.

α and β are defined in an infinite construction. We, in fact, define strictly increasing infinite sequences l_n and $l'_n (n \in \omega)$ and initial segments α_n, β_n of α and β with lengths $l_n, l'_n \cdot l_0 = l'_0 = 0, \alpha_0, \beta_0$ are the empty sequence. At step $2n$ we extend α_{2n-1} to ensure $\alpha_{2n} \not\subseteq \lambda x \cdot \{n\}(x)$. And at step $2n + 1$ we extend β_{2n} to ensure that $\beta_{2n+1} \not\subseteq \lambda x \cdot \{n\}(x)$ (and hence $\beta \not\subseteq \lambda x \cdot \{n\}(x)$). During the construction we ensure that the following two conditions remain satisfied.

a. Whenever $\alpha_n(i) = q = \beta_n(j)$ then $q = \{i\}(j)$ ($q \in \{0, 1\}$)

b. If $\alpha_n(i) \neq *$ then for some $t \in \omega$ we have: $\forall y > t \{i\}(y) \cong \{y\}(i)$.

The motivation of condition a. is clear. Condition b. helps us to do the odd steps (and complicates the even ones). We will now describe the steps:

Step 2n. Suppose β_{2n-1} is defined on $y_1 \dots y_k$ with values $z_1 \dots z_k$. Find, using the recursion theorem, an index $x \in \omega$ such that:

$$(i) \quad x > l_{2n-1}$$

$$(ii) \quad \{x\}(y) \cong \begin{cases} \text{if } y = y_j \text{ for some } j \leq k \text{ then } z_j \\ \text{else if } y \leq l'_{2n-1} \text{ then } * \\ \text{else } \{y\}(x). \end{cases}$$

We take $l_{2n} = x + 1$, $l'_{2n} = l'_{2n-1} \cdot \alpha_{2n}$ is found as follows:

$$\alpha_{2n}(i) = \begin{cases} \text{if } i < l_{2n-1} \text{ then } \alpha_{2n-1}(i) \\ \text{else if } i = x \text{ then } p \text{ else } *. \end{cases}$$

Here $p \in \{0, 1\}$ is a value such that $\{n\}(x) \neq p$ (this ensures $\alpha \not\subseteq \lambda x \{n\}(x)$). We must verify conditions a. and b.

a. Suppose $\alpha_{2n}(i) = q = \beta_{2n}(j)$ ($i \leq l_{2n}$, $j \leq l'_{2n}$) then $q = \beta_{2n-1}(j)$. Clearly $j = y_l$ for some $l \leq k$. Further we may assume that $i = x$. Thus: $q = \beta_{2n-1}(y_l) = z_l = \{x\}(y_l) = \{i\}(j)$.

b. Immediate by the construction of x .

Step 2n + 1. Let α_{2n} be defined on $x_1 \dots x_l$ with values $z_1 \dots z_l$. Choose $t > l'_{2n}$ such that for $i \leq l$ and $t' > t$:

$$\{x_i\}(t') \cong \{t'\}(x_i).$$

Using the recursion theorem we find an $y > t$ such that: $\{y\}(x_i) = z_i$ for $i \leq l$. We extend β_{2n} to β_{2n+1} by giving it a value q on y which ensures that $\beta_{2n} \not\subseteq \lambda x \{n\}(x)$. Again we must check conditions a. and b. (We have $l_{2n+1} = l_{2n}$, $l'_{2n+1} = y + 1$).

b. Immediate (no change).

a. Suppose $\alpha_{2n+1}(i) = q = \beta_{2n+1}(j)$, ($q \in \{0, 1\}$). We may assume that $j = y$ and $i = x_m$ for some $m \leq l$. $q = \alpha_{2n+1}(x_m) = z_m = \{y\}(x_m) = \{x_m\}(y) = \{i\}(y)$. This completes the proof of 2.4.

REFERENCES

- [1] Kleene, S. C., "Recursive functionals and quantifiers of finite type," *Transactions of the American Mathematical Society*, vol. 91, pp. 1-52.
- [2] Sacks, G. E., *Degrees of Unsolvability*, Princeton University Press, 1963.

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