# SOME RESULTS ON COMBINATORS IN THE SYSTEM TRC 

Thomas Jech<br>The Pennsylvania State University


#### Abstract

We investigate the system TRC of untyped illative combinatory logic that is equiconsistent with New Foundations. We prove that various unstratified combinators do not exist in TRC.


Introduction. We prove some results in the axiomatic system TRC introduced in [3]. The system TRC (for 'type-respecting combinators') is an untyped system of combinatory logic, in the sense of [1], [2]. TRC is a first order theory of functions (combinators) with equality and is illative, i.e. capable of expressing notions of propositional logic. Moreover, it is combinatorially complete for stratified combinators. The main interest of TRC is that it is equiconsistent with the theory NF [6], Quine's 'New Foundations'. As the consistency of NF remains an open problem, so does the consistency of TRC.

The objects of study of a combinatory logic are combinators. We denote $x y$ the application of the combinator $x$ to the combinator $y$, and adopt the convention that $x y z=(x y) z$.

The language of TRC has (in addition to equality and the binary function $x y$ ) constants Abst, $E q, p_{1}$ and $p_{2}$, and functions $k(x)$ and $\langle x, y\rangle$. The axioms of TRC are the following:
I. $k(x) y=x$.
II. $p_{i}\left\langle x_{1}, x_{2}\right\rangle=x_{i}$ for $i=1,2$.
III. $\left\langle p_{1} x, p_{2} x\right\rangle=x$.
IV. $\langle x, y\rangle z=\langle x y, x z\rangle$
V. Abst $x y z=x k(y)(y z)$.

[^0]VI. $E q\langle x, y\rangle=p_{1}$ if $x=y ; E q\langle x, y\rangle=p_{2}$ if $x \neq y$.
VII. If for all $z, x z=y z$, then $x=y$.
VIII. $p_{1} \neq p_{2}$.

Axiom I postulates the existence of constant functions. Axioms II-IV describe the pairing function $\langle x, y\rangle$ and the projections $p_{1}$ and $p_{2}$. Abst is the abstraction combinator and $E q$ is the characteristic function of equality. Axiom VII is the axiom of extensionality.

Let $I=\left\langle p_{1}, p_{2}\right\rangle$; from III and IV it follows that $I$ is the identity function $I x=x$.
Classical combinatory logic [2,3] employs combinators $I, K$ and $S$, where

$$
I x=x \quad K x y=x \quad S x y z=x z(y z)
$$

It has a powerful abstraction property: for every term $t$ and a variable $x$, there is a term $\lambda x t$ in which $x$ does not occur, such that for every term $s$,

$$
(\lambda x t) s=t[s / x]
$$

This guarantees, among others, the existence of a fixed point for every combinator, and implies that simple notions of propositional logic cannot be represented by combinators. Suppose that $N e g$ is the negation combinator, and consider $u=\lambda x(N e g(x x))$. Then $u u=N e g(u u)$.

The theory TRC is an illative theory, in the sense that it can encode notions of propositional logic. It also has an abstraction property (Theorem 1 of [3]). The term $\lambda x t$ can be constructed for every $t$ in which $x$ occurs with no type other than 0 . (For details about typing see [3].) It follows that TRC proves the existence of all stratified combinators. Examples of stratified combinators are $x(y x), x y(y z), y(x y z)$ : in $y(x y z), z$ has type $0, y$ has type 1 and $x$ has type 2. (In fact, Abst Abst Ixy $=x(y x)$, Abst (AbstAbst) $x y z=x y(y z)$, and AbstAbst $x y z=y(x y z))$.

We will show in Section 3 that (with the exception of $I$ ) the standard combinators used in classical combinatory logic do not exist in TRC. We shall give many examples of unstratified combinators whose existence contradicts the axioms of TRC.

In searching for proofs of the various results in TRC, we used a computer extensively and used the automated theorem prover OTTER [5].

## 2. Some Basic Facts on TRC.

In this sectioin we derive some simple equalities from the axioms of TRC, and use a selfreference argument to obtain some simple negative results. First we state some properties of the abstraction combinator (see also [4]):

Theorem 2.1. (a) $\operatorname{Abst}(\operatorname{Abst}(\operatorname{Abst} x))=\operatorname{Abst} x$.
(b) $\operatorname{Abst}(\operatorname{Abst} k(x))=k(x)$.
(c) Abst $k(k(x))=k(k(x))$.
(d) Abst $k(x) y z=x(y z)$.
(e) Abst $k(x) k(y)=k(x y)$.

Proof. The equalities are obtained by an application of the axioms defining Abst and $k(x)$ and the axiom of extensionality; e.g. to prove (a), we evaluate the term $\operatorname{Abst}(\operatorname{Abst}(\operatorname{Abst} x)) y z$ and compare it with Abst xyz.

The next theorem gives some properties of the pairing function and the projections:
Theorem 2.2. (a) $\langle k(x), y\rangle z=\langle x, y z\rangle,\langle x, k(y)\rangle z=\langle x z, y\rangle$.
(b) $k(\langle x, y\rangle)=\langle k(x), k(y)\rangle, k\left(p_{i} x\right)=p_{i} k(x)$ for $i=1,2$.
(c) $p_{i}(x y)=p_{i} x y$ for $i=1,2$.

Proof. (a) From Axiom IV.
(b) Calculate $\langle k(x), k(y)\rangle z$ and $p_{i} k(x) z$ and use extensionality.
(c) Let $x=\langle u, v\rangle$ and use Axiom IV.

Next we state some more properties of the combinator Abst:
Theorem 2.3. (a) Abst $\langle x, y\rangle=\langle$ Abst $x$, Abst $y\rangle$.
(b) Abst $p_{i}=k\left(p_{i}\right)$ and Abst $k\left(p_{i}\right)=p_{i}$, for $i=1,2$.
(c) Abst $I=k(I)$ and Abst $k(I)=I$.

Proof. (a) Using Axiom IV, show that Abst $\langle x, y\rangle u v=\langle$ Abst $x$, Abst $y\rangle u v$.
(b) Abst $p_{i} x y=p_{i} k(y)(x y)=k\left(p_{i} y\right)(x y)=p_{i} y$ by Theorem 2.2 b , and Abst $k\left(p_{i}\right) x y=$ $p_{i}(x y)=p_{i} x y$ by Theorems 2.1d and 2.2c.
(c) Abst Ixy $=y=k(I) x y$, by Axioms V and I, and Abst $k(I) x y=I(x y)=I x y$ by Theorem 2.1d.

We shall now turn to negative results. In Section 3 we shall present a number of combinators that do not exist in TRC. Each proof will use one of the following basic negative results that use self-reference:

Theorem 2.4. For every $x$,
(a) $E q\left\langle x, p_{2}\right\rangle \neq x$.
(b) $E q\left\langle k(x), k\left(p_{2}\right)\right\rangle \neq x$.
(c) $\left\langle E q x, p_{2}\right\rangle \neq x$.

Proof. (a) $E q\langle x, y\rangle$ is either $p_{1}$ or $p_{2}$, and $E q\left\langle p_{1}, p_{2}\right\rangle=p_{2}$ while $E q\left\langle p_{2}, p_{2}\right\rangle=p_{1}$.
The proof of (b) and (c) is similar.
It follows from the discussion on classical combinatory logic in Section 1 that not every combinator in TRC has a fixed point. Theorem 2.4 gives an explicit example, $\left\langle E q, k\left(p_{2}\right)\right\rangle x \neq x$ :

Corollary 2.5. The combinator $\left\langle E q, k\left(p_{2}\right)\right\rangle$ does not have a fixed point.
A standard fact of combinatory logic (cf. [1], [2]) states that if $M$ is the combinator $M x=x x$ then for every $u$, the composition of $u$ and $M$ is a fixed point of $u$. As the abstraction theorem for TRC in [3] provides for composition of combinators, it follows that $M$ does not exist in TRC. Here we give a direct proof:

Theorem 2.6. There is no $M$ such that $M x=x x$.
Proof. Let $t=A b s t k(E q)\left\langle M, k\left(p_{2}\right)\right\rangle$ and let $s=t t$. Then (using Theorems 2.1.d and 2.2a)

$$
\begin{aligned}
s=t t & =A b s t k(E q)\left\langle M, k\left(p_{2}\right)\right\rangle t \\
& =E q\left(\left\langle M, k\left(p_{2}\right)\right\rangle t\right) \\
& =E q\left\langle M t, p_{2}\right\rangle \\
& =E q\left\langle t t, p_{2}\right\rangle \\
& =E q\left\langle s, p_{2}\right\rangle
\end{aligned}
$$

contradicting Theorem 2.4a.
A similar argument, using Theorem 2.4 b , yields the following:
Theorem 2.7. There is no $K_{1}$ such that $K_{1} x=k(x x)$.
Proof. Let $t=\operatorname{Abst} k(E q)\left\langle K_{1}, k\left(k\left(p_{2}\right)\right)\right\rangle$, and $s=t t$. Then (by Theorems 2.1d and 2.2a)

$$
\begin{aligned}
s=t t & =\operatorname{Abst} k(E q)\left\langle K_{1}, k\left(k\left(p_{2}\right)\right)\right\rangle t \\
& =E q\left(\left\langle K_{1}, k\left(k\left(p_{2}\right)\right)\right\rangle t\right) \\
& =E q\left\langle K_{1} t, k\left(p_{2}\right)\right\rangle \\
& =E q\left\langle k(s), k\left(p_{2}\right)\right\rangle,
\end{aligned}
$$

contradicting Theorem 2.4b.
An immediate consequence of Theorem 2.7 is that neither $k(x)$ nor $\langle x, y\rangle$ can be replaced in TRC by a combinator (see also [4]).

Theorem 2.8. (a) There is no $K$ such that $K x=k(x)$.
(b) There is no $p$ such that $p x y=\langle x, y\rangle$.

Proof. (a) Given such $K$, let $K_{1}=$ AbstAbst $K$. Then (by Theorem 2.1d)

$$
\begin{aligned}
K_{1} x y & =\text { AbstAbst Kxy } \\
& =\text { Abst } K(x)(K x) y \\
& =x(K x y) \\
& =x x
\end{aligned}
$$

and so $K_{1} x=k(x x)$, contradicting Theorem 2.7.
(b) Given $p$, let $K=p_{1} p$, and then (by Theorem 2.2c)

$$
\begin{aligned}
K x y & =p_{1} p x y \\
& =p_{1}(p x) y \\
& =p_{1}(p x y) \\
& =p_{1}\langle x, y\rangle \\
& =x,
\end{aligned}
$$

contradicting (a).

We conclude this section with the following result that we use in Section 3.
Theorem 2.9. (a) There is no $u$ such that $u x=x k(x)$.
(b) There is no $u$ such that $u k(x)=x k(x)$.

Proof. (a) Given $u$, let $M=\operatorname{Abst}($ Abst $u) I$, and then

$$
\begin{aligned}
M x & =A b s t(\text { Abst } u) I x \\
& =A b s t u k(x) x \\
& =u k(x) x \\
& =k(x) k(k(x)) x \\
& =x x,
\end{aligned}
$$

contradicting Theorem 2.6.
(b) Given $u$, let $t=\operatorname{Abst} k(E q)\left\langle u, k\left(p_{2}\right)\right\rangle$ and $s=u k(t)$. Then we have (by Theorems 2.1.d and 2.2a)

$$
\begin{aligned}
s=u k(t) & =A b s t k(E q)\left\langle u, k\left(p_{2}\right)\right\rangle k(t) \\
& =E q\left(\left\langle u, k\left(p_{2}\right)\right\rangle k(t)\right) \\
& =E q\left\langle u k(t), p_{2}\right\rangle \\
& =E q\left\langle s, p_{2}\right\rangle
\end{aligned}
$$

contradicting Theorem 2.4a.

## 3. Nonexistence of Various Combinators.

We will show that many standard classical combinators do not exist in TCR. Let us consider the following combinators; none of them is stratified. We use the list presented in [7], with several additions.

$$
\begin{array}{ccc}
B x y z=x(y z) & L x y=x(y y) & Q_{3} x y z=z(x y) \\
C x y z=x z y & L_{1} x y=y(x x) & R x y z=y z x \\
D x y z w=x y(z w) & M x=x x & S x y z=x z(y z) \\
F x y z=z y x & M_{1} x=x x x & T x y=y x \\
G x y z w=x w(y z) & M_{2} x=x(x x) & U x y=y(x x y) \\
H x y z=x y z y & O x y=y(x y) & V x y z=z x y \\
H_{1} x y=x y x & O_{1} x y=x(y x) & W x y=x y y \\
J x y z w=x y(x w z) & O_{2} x y=y(y x) & W_{1} x y=y x x \\
K x y=x & Q x y z=y(x z) & W_{2} x y=y x y \\
K_{1} x y=x x & Q_{1} x y z=x(z y) & W_{3} x y=y y x
\end{array}
$$

Below we prove that none of these combinators exist in TRC.
(3.1). $K_{1}, K, M$ and $J$ :

Theorems 2.6, 2.7 and 2.8 show that $K_{1}, K$ and $M$ do not exist. As for $J$, it is well known in combinatory logic (cf. [1]) that $\{I, J\}$ is combinatorially complete, and so $J$ cannot exist in TRC.
(3.2). $L, O, U$ and $W$ :

$$
M=L I=O I=U I=W I
$$

(3.3). $O_{2}$ and $M_{2}$ :

$$
\begin{aligned}
M=\operatorname{Abst}\left(O_{2} I\right) I & =\operatorname{Abst} M_{2} I: \\
\operatorname{Abst}\left(O_{2} I\right) I x & =O_{2} I k(x) x=k(x)(k(x) I) x=x x \\
\text { Abst } M_{2} I x & =M_{2} k(x) x=k(x)(k(x) k(x)) x=x x
\end{aligned}
$$

(3.4). $S$ and $O_{1}$ :
$O=S I$ and $S=A b s t \circ O_{1}$ (where $a \circ b$ is the composition, defined in TRC by $a \circ b=$ Abst $k(a)($ Abst $k(b) I))$ :

$$
\operatorname{Sxyz}=\operatorname{Abst}\left(O_{1} x\right) y z=O_{1} x k(z)(y z)=x(k(z) x)(y z)=x z(y z)
$$

(3.5). $T, C, G, Q_{1}$ and $Q_{3}$ :

$$
\begin{gathered}
K=\operatorname{Abst} T k(I) \text { and } T=C I=G I I=Q_{1} I=Q_{3} I: \\
K x=\operatorname{Abst} T k(I) x=T k(x)(k(I) x)=T k(x) I=\operatorname{Ik}(x)=k(x)
\end{gathered}
$$

(3.6). $B$ and $D$ :

$$
\begin{aligned}
K & =A b s t B I, B=D I: \\
K x y & =A b s t B I x y=B k(x)(I x) y=B k(x) x y= \\
& =k(x)(x y)=x
\end{aligned}
$$

(3.7). $R$ :
$K=R k(I) p_{1}\langle R, u\rangle R$, where $u$ is arbitrary:

$$
\begin{aligned}
K x y & =R k(I) p_{1}\langle R, u\rangle R x y \\
& =p_{1}\langle R, u\rangle k(I) R x y \\
& =R k(I) R x y \\
& =R x k(I) y \\
& =k(I) y x \\
& =I x=x
\end{aligned}
$$

(3.8). $V$ :

$$
K=A b s t(A b s t V A b s t) k(k(A b s t)):
$$

using Theorem 2.1.b, we have

$$
\begin{aligned}
K x & =A b s t(\text { Abst } V \text { Abst }) k(k(\text { Abst })) x \\
& =A b s t V \text { Abst } k(x) k(\text { Abst }) \\
& =V k(k(x))(\text { Abst } k(x)) k(\text { Abst }) \\
& =k(\text { Abst }) k(k(x))(\text { Abst } k(x)) \\
& =A b s t(\text { Abst } k(x)) \\
& =k(x)
\end{aligned}
$$

(3.9). $Q$ :

$$
\begin{aligned}
K_{1} & =\text { Abst } Q I: \\
K_{1} x y & =\text { Abst } Q I x y=Q \quad k(x) x y=x(k(x) y)=x x
\end{aligned}
$$

(3.10). $H_{1}, H, M_{1}$ and $W_{2}$ :

$$
M_{1} k(x)=H_{1} H_{1} k(x)=W_{2} W_{2} k(x)=x k(x)
$$

contradicting Theorem 2.9, and $H_{1}=H I$.
(3.11). $F$ and $W_{1}$ :

Let $u=\operatorname{Abst}(F z)$ Abst $k(x)$ (where $z$ is arbitrary) and $v=A b s t W_{1}$ Abst. Then $u k(x)=v k(x)=x k(x)$, contradicting Theorem 2.9: using Theorem 2.1d, we have

$$
\begin{aligned}
u k(x) & =\operatorname{Abst}(F z) \text { Abst } k(x) \\
& =F z k(k(x))(\text { Abst } k(x)) \\
& =\operatorname{Abst} k(x) k(k(x)) z \\
& =x(k(k(x)) z) \\
& =x k(x)
\end{aligned}
$$

and

$$
\begin{aligned}
v k(x) & =\text { Abst } W_{1} \text { Abst } k(x) \\
& =W_{1} k(k(x))(\text { Abst } k(x)) \\
& =A b s t k(x) k(k(x)) k(k(x)) \\
& =x(k(k(x)) k(k(x))) \\
& =x k(x)
\end{aligned}
$$

(3.12). $L_{1}$ :

Let $a=k\left(\left\langle E q, k\left(p_{2}\right)\right\rangle\right)$. Then for all $x$,

$$
\begin{aligned}
L_{1} a\left(L_{1} x\right) & =L_{1} x(a a) \\
& =a a(x x) \\
& =k\left(\left\langle E q, k\left(p_{2}\right)\right\rangle\right) a(x x) \\
& =\left\langle E q, k\left(p_{2}\right)\right\rangle(x x) \\
& =\left\langle E q(x x), p_{2}\right\rangle,
\end{aligned}
$$

which, by Theorem 2.4 c , is not equal to $x x$.
Now let $b=A b s t k\left(L_{1} a\right) L_{1}$. By Theorem 2.1d we have

$$
b b=A b s t k\left(L_{1} a\right) L_{1} b=L_{1} a\left(L_{1} b\right),
$$

a contradiction.
(3.13). $W_{3}$ :

Let $a=k\left(\left\langle E q, k\left(p_{2}\right)\right\rangle\right)$. Then for all $x$,

$$
\begin{aligned}
W_{3} x a & =a a x \\
& =\left\langle E q, k\left(p_{2}\right)\right\rangle x \\
& =\left\langle E q x, p_{2}\right\rangle,
\end{aligned}
$$

which, by Theorem 2.4 c , is not equal to $x$.
Now let $b=$ Abst $k\left(W_{3}\right)\left(W_{3} a\right)$. By Theorem 2.1d we have

$$
W_{3} a b=b b a=A b s t k\left(W_{3}\right)\left(W_{3} a\right) b a=W_{3}\left(W_{3} a b\right) a .
$$

Thus if above we let $x=W_{3} a b$, we get

$$
W_{3}\left(W_{3} a b\right) a \neq W_{3} a b
$$

a contradiction.

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Department of Mathematics, The Pennsylvania State University, 218 McAllister Bldg., University Park, PA 16802, U.S.A.

E-mail address: jech@math.psu.edu


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