A Buchholz rule for modal fixed point logics

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Abstract. Buchholz's $\Omega_{\mu+1}$-rules provide a major tool for the proof-theoretic analysis of arithmetical inductive definitions. The aim of this paper is to put this approach into the new context of modal fixed point logic. We introduce a deductive system based on an Ω-rule tailored for modal fixed point logic and develop the basic techniques for establishing soundness and completeness of the corresponding system. In the concluding section we prove a cut elimination and collapsing result similar to that of Buchholz [3].

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1. Introduction

Buchholz's $\Omega_{\mu+1}$-rules play a prominent role in the proof-theoretic analysis of (iterated) arithmetical inductive definitions. However, unlike the ω-rule of arithmetic, which branches over the natural numbers, the $\Omega_{\mu+1}$-rules branch over certain classes of derivations. Let us cite Buchholz [3] to introduce the Ω_1-rule and give a motivation for it: 'According to the intuitionistic interpretation of implication a proof of $Pn \to C$ consists of a construction Π which transforms any proof X of Pn into a proof Π_X of C. This may serve as a motivation for the following inference rule: If for each direct proof X of Pn t_X is a deduction of C, then $(t_X)_{X \in \mathcal{P}n}$ is a deduction of $Pn \to C$.' In this statement Pn means that n belongs to the least fixed point P, and $\mathcal{P}n$ is the collection of all direct proofs of Pn.

Buchholz introduced the $\Omega_{\mu+1}$-rules for the proof-theoretic analysis of (iterated) inductive definitions, see [3, 5]. They soon turned out to be of fundamental interest in proof theory and are, among other applications, a basis for 'ordinal free' consistency proofs. For important work about Buchholz's $\Omega_{\mu+1}$-rules see, for example, Aehlig [1], Gordeev [6], and Towsner [11].

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In the present paper we are not concerned with the analysis of fixed points in fragments of second order arithmetic but show that a related method can also be applied in the area of modal fixed point logics. Such systems occur in many different forms and in many different contexts. To give some examples, let us mention temporal logics like **LTL** and **CTL**, epistemic logics like the logic of common knowledge, and program logics like **PDL**. All these logics are subsumed by the propositional modal μ-calculus.

The article Jäger, Kretz, and Studer [7] presents and studies an infinitary version of the full propositional modal μ-calculus which treats greatest fixed points by an infinitary rule reminiscent of the ω-rule in arithmetic. Here we develop the basic machinery for employing Buchholz's Ω_1-rule in a modal logic context and prove soundness and completeness of the deductive system with our Ω-rule. Therefore and in order to focus on the basic ideas, we confine ourselves to the theory $\mathbf{M_1}$ of non-iterated least fixed points of positive modal formulae. Extensions to systems permitting iterated and nested modal fixed points are planned for subsequent publications.

In the following section we introduce the syntax and semantics of $\mathbf{M_1}$. Then we present the corresponding deductive system $\mathbf{M_1^\infty}$ which is based on the Ω-rule. To show this rule at work, we derive in Section 4 the usual induction rule within $\mathbf{M_1^\infty}$. The central results of our paper are the completeness and soundness proofs for $\mathbf{M_1^\infty}$. We first establish completeness by a canonical counter-model construction and then make use of the finite model property of $\mathbf{M_1}$ to prove soundness of $\mathbf{M_1^\infty}$. In the concluding section we prove a cut elimination and collapsing result similar to that of Buchholz [3].

2. Syntax and semantics of $\mathbf{M_1}$

We begin this section with introducing the basic language $\mathcal{L}_0$ and then turn to its extension $\mathcal{L}_1$, which is the language of the theory $\mathbf{M_1}$. Let

$$\textsc{Prop} := \{\mathsf{X}, {\sim}\mathsf{X}, \mathsf{p}, {\sim}\mathsf{p}, \mathsf{q}, {\sim}\mathsf{q}, \mathsf{r}, {\sim}\mathsf{r}, \ldots\}$$

be a countable set of atomic propositions with X playing a special rôle later. Further, let $\mathsf{M} := \{1, \ldots, h\}$ be a finite set of indices.

Definition 1 (Formulae of $\mathcal{L}_0$). The formulae of the language $\mathcal{L}_0$ are inductively defined as follows:

1. If P is an element of $\textsc{Prop}$, then P is a formula of $\mathcal{L}_0$.
2. If A and B are formulae of $\mathcal{L}_0$, then so are $(A \wedge B)$ and $(A \vee B)$.
3. If A is a formula of $\mathcal{L}_0$ and $i \in \mathsf{M}$, then $\Box_i A$ and $\Diamond_i A$ are also formulae of $\mathcal{L}_0$.

An *operator form* is a formula of $\mathcal{L}_0$ which does not contain the negated atomic proposition ${\sim}\mathsf{X}$. In the following we let $\mathcal{A}$ range over operator forms and associate a fresh atomic proposition $P_{\mathcal{A}}$ to any operator form $\mathcal{A}$.

Definition 2 (Formulae of $\mathcal{L}_1$). The formulae of the language $\mathcal{L}_1$ are inductively defined as follows:

1. If P is an element of Prop, then P is an (atomic) formula of $\mathcal{L}_1$.
2. For each operator form $\mathcal{A}$, $P_\mathcal{A}$ and $\sim P_\mathcal{A}$ are (atomic) formulae of $\mathcal{L}_1$.
3. If A and B are formulae of $\mathcal{L}_1$, then so are $(A \wedge B)$ and $(A \vee B)$.
4. If A is a formula of $\mathcal{L}_1$ and $i \in \mathsf{M}$, then $\Box_i A$ and $\Diamond_i A$ are also formulae of $\mathcal{L}_1$.

The *positive* formulae of $\mathcal{L}_1$ are those without occurrences of $\sim P_\mathcal{A}$ for any operator form $\mathcal{A}$.

Typically, we only speak of formulae if it is clear that we refer to formulae of $\mathcal{L}_1$; also, we often omit parentheses whenever there is no danger of confusion. Note that formulae are a priori in negation normal form. The negation $\neg A$ of a formula A is defined as usual by De Morgan's laws, the law of double negation, and the duality laws for modal operators. For any formulae A and B and an arbitrary but fixed element p of Prop different from X we set

$$A \rightarrow B \ := \ \neg A \vee B \qquad \text{and} \qquad \bot \ := \ \mathsf{p} \wedge {\sim}\mathsf{p}.$$

If P is an element of Prop, A a formula which does not contain occurrences of $\sim P$, and B an arbitrary formula, then we write $A[P := B]$ for the result of simultaneously substituting B for each occurrence of P in A. The finite iterations of an operator form $\mathcal{A}$ with respect to a given formula B are defined, for any natural number $i \geq 1$, as follows:

$$\mathcal{A}^1(B) := \mathcal{A}[\mathsf{X} := B] \text{ and } \mathcal{A}^{i+1}(B) := \mathcal{A}[\mathsf{X} := \mathcal{A}^i(B)].$$

To simplify the notation, we generally write $\mathcal{A}(B)$ instead of $\mathcal{A}^1(B)$.

Definition 3 (Kripke structure). A Kripke structure is a triple $\mathsf{K} = (S, R, \pi)$ consisting of a non–empty set S, a function R from M to $\mathcal{P}(S \times S)$, and a function π from Prop to $\mathcal{P}(S)$ such that $\pi(\neg P) = S \setminus \pi(P)$ for all $P \in$ Prop.

If K is the Kripke structure (S, R, π), we usually write $|\mathsf{K}|$ for the set of states S. The function R assigns a binary accessibility relation to each $i \in \mathsf{M}$. Furthermore, for a Kripke structure $\mathsf{K} = (S, R, \pi)$ and a set $T \subseteq S$, we define the Kripke structure $\mathsf{K}[\mathsf{X} := T]$ as the triple (S, R, π'), where $\pi'(\mathsf{X}) = T$, $\pi'(\sim\mathsf{X}) = S \setminus T$ and $\pi'(P) = \pi(P)$ for all other $P \in$ Prop.

Assume that we are given a Kripke structure K and a formula A. We are interested in the set of all states $\|A\|_\mathsf{K}$ which validate A. To determine this set, we first introduce the interpretations of all formulae of $\mathcal{L}_0$, then interpret the fixed point constants, and finally extend these denotations to all formulae of $\mathcal{L}_1$.

Definition 4 (Denotations). Let $\mathsf{K} = (S, R, \pi)$ be a Kripke structure.

1. For any formula A of $\mathcal{L}_0$, the set $\|A\|_{\mathsf{K}}$ is inductively defined as follows:

$$\begin{aligned}
\|P\|_{\mathsf{K}} &:= \pi(P) \text{ for all } P \in \textsc{Prop},\\
\|B \land C\|_{\mathsf{K}} &:= \|B\|_{\mathsf{K}} \cap \|C\|_{\mathsf{K}},\\
\|B \lor C\|_{\mathsf{K}} &:= \|B\|_{\mathsf{K}} \cup \|C\|_{\mathsf{K}},\\
\|\Box_i B\|_{\mathsf{K}} &:= \{w \in S : v \in \|B\|_{\mathsf{K}} \text{ for all } v \text{ such that } (w,v) \in R(i)\},\\
\|\Diamond_i B\|_{\mathsf{K}} &:= \{w \in S : v \in \|B\|_{\mathsf{K}} \text{ for some } v \text{ such that } (w,v) \in R(i)\}.
\end{aligned}$$

2. If $\mathcal{A}$ is an operator form, we first introduce the monotone operator

$$F^{\mathsf{K}}_{\mathcal{A}} : \mathcal{P}(S) \to \mathcal{P}(S) \quad \text{with} \quad F^{\mathsf{K}}_{\mathcal{A}}(T) := \|\mathcal{A}\|_{\mathsf{K}[\mathsf{X}:=T]}$$

   for all $T \subseteq S$. Based on this $F^{\mathsf{K}}_{\mathcal{A}}$ we now set

$$\|P_{\mathcal{A}}\|_{\mathsf{K}} := \bigcap\{T \subseteq S : F^{\mathsf{K}}_{\mathcal{A}}(T) \subseteq T\} \quad \text{and} \quad \|{\sim}P_{\mathcal{A}}\|_{\mathsf{K}} := S \setminus \|P_{\mathcal{A}}\|_{\mathsf{K}}.$$

3. For formulae A of $\mathcal{L}_1$ the denotations $\|A\|_{\mathsf{K}}$ are generated by lifting the denotations of the atomic formulae according to the clauses in the first part of this definition.

Clearly, by the famous Knaster-Tarski theorem, $\|P_{\mathcal{A}}\|_{\mathsf{K}}$ is the least fixed point of $F^{\mathsf{K}}_{\mathcal{A}}$.

A formula A is called *satisfiable* if there exists a Kripke structure K such that $\|A\|_{\mathsf{K}}$ is non-empty. A formula A is said to be *valid* if for every Kripke structure K we have $\|A\|_{\mathsf{K}} = |\mathsf{K}|$; this is denoted by $\models A$. Finally, we say that a finite set Γ of formulae is valid, or $\models \Gamma$, if $\models \bigvee \Gamma$ for the disjunction $\bigvee \Gamma$ of the elements of Γ.

Given any Kripke structure K and an operator form $\mathcal{A}$, we will later also need the approximations of the least fixed point of the monotone operator $F^{\mathsf{K}}_{\mathcal{A}}$. Thus we inductively define for all ordinals α

$$I^{<\alpha}_{\mathcal{A},\mathsf{K}} := \bigcup_{\beta<\alpha} I^{\beta}_{\mathcal{A},\mathsf{K}} \quad \text{and} \quad I^{\alpha}_{\mathcal{A},\mathsf{K}} := F^{\mathsf{K}}_{\mathcal{A}}(I^{<\alpha}_{\mathcal{A},\mathsf{K}}).$$

Then we set $I_{\mathcal{A},\mathsf{K}} := \bigcup_{\alpha \in On} I^{\alpha}_{\mathcal{A},\mathsf{K}}$ and recall the following classical results which are discussed, for example, in the textbooks Moschovakis [9] and Barwise [2] or follow directly from what is treated there.

Theorem 5. *Let $\mathcal{A}$ be any operator form.*

1. *For every Kripke structure K we have $I_{\mathcal{A},\mathsf{K}} = \|P_{\mathcal{A}}\|_{\mathsf{K}}$ and, for any natural number n,*

$$\models \mathcal{A}^n(\bot) \to P_{\mathcal{A}}.$$

2. *If K is a Kripke structure and $|\mathsf{K}|$ a set of at most k elements, then we also have*

$$\|P_{\mathcal{A}}\|_{\mathsf{K}} = \|\mathcal{A}^k(\bot)\|_{\mathsf{K}}.$$

$\mathbf{M_1}$ is the Hilbert-style formalization of the non-iterated propositional modal μ-calculus. It is obtained by extending the multi-modal version of the logic **K** by closure axioms for the fixed point constants and corresponding induction rules.

Logical axioms of $\mathbf{M_1}$. For all propositional tautologies A, all formulae B and C, and all $i \in \mathsf{M}$:

$$A \tag{taut}$$

$$\Box_i B \wedge \Box_i (B \to C) \;\to\; \Box_i C \tag{K}$$

Logical rules of $\mathbf{M_1}$. For all formulae A and B and all $i \in \mathsf{M}$:

$$\frac{A \qquad A \to B}{B} \tag{mp}$$

$$\frac{A}{\Box_i A} \tag{nec}$$

Closure axioms of $\mathbf{M_1}$. For all operator forms $\mathcal{A}$:

$$\mathcal{A}(P_{\mathcal{A}}) \to P_{\mathcal{A}} \tag{cl}$$

Induction rules of $\mathbf{M_1}$. For all operator forms $\mathcal{A}$ and all formulae B:

$$\frac{\mathcal{A}(B) \to B}{P_{\mathcal{A}} \to B} \tag{ind}$$

Provability of a formula A in $\mathbf{M_1}$ is defined as usual and denoted by $\mathbf{M_1} \vdash A$. It is an easy exercise to check that $\mathbf{M_1}$ is sound with respect to the semantics introduced above.

Theorem 6 (Soundness of $\mathbf{M_1}$). *For any formula A we have*

$$\mathbf{M_1} \vdash A \quad \Longrightarrow \quad \models A.$$

The completeness of $\mathbf{M_1}$ is a more delicate matter. In general it is rather challenging to show the completeness of Hilbert style deductive systems for modal fixed point logics. Walukiewicz's completeness proof of the modal μ-calculus in [12] is very technical and employs automata-theoretic reductions. Santocanale and Venema [10], on the other hand, present an algebraic completeness proof for so-called flat modal fixed point logics. They show that a Hilbert style system with induction rules and closure axioms is complete if only fixed points over aconjunctive or disjunctive formulae are considered. However, to treat arbitrary operator forms they need more general axioms and rules.

3. The infinitary system $\mathbf{M_1^\infty}$

In this section we introduce the infinitary system $\mathbf{M_1^\infty}$. Basically, it is obtained from a Tait-style reformulation of the modal logic $\mathbf{K}$ by adding a closure rule for all fixed point constants and a variant of Buchholz's Ω_1-rule, tailored for our modal context. However, before turning to $\mathbf{M_1^\infty}$ we introduce a rank function to measure the complexities of our formulae.

Definition 7 (Rank of a formula).

1. $\mathsf{rk}(P_{\mathcal{A}}) := \mathsf{rk}(\sim P_{\mathcal{A}}) := 0$ for each operator form $\mathcal{A}$,

2. $\mathsf{rk}(P) := 1$ for each $P \in \textsc{Prop}$,
3. $\mathsf{rk}(A \wedge B) := \mathsf{rk}(A \vee B) := \max(\mathsf{rk}(A), \mathsf{rk}(B)) + 1$ for all formulae A and B,
4. $\mathsf{rk}(\Box_i A) := \mathsf{rk}(\Diamond_i A) := \mathsf{rk}(A) + 1$ for all $i \in \mathsf{M}$ and all formulae A.

Sequents are finite sets of formulae, and we use the capital Greek letters Γ, Δ, Π, and Σ to denote sequents. Often we write (for example) Γ, Δ, A, B for the union $\Gamma \cup \Delta \cup \{A, B\}$. A sequent is called *positive* if all its elements are positive, cf. Definition 2. For a sequent $\Delta = \{A_1, \ldots, A_n\}$, we define $\Diamond_i\Delta := \{\Diamond_i A_1, \ldots, \Diamond_i A_n\}$.

The infinitary system $\mathbf{M_1^\infty}$ is formulated as a Tait-style system which derives sequents rather than individual formulae. It comprises some basic axioms, rules for the propositional connectives and modal operators, closure rules for fixed point constants and cuts. When defining derivability in $\mathbf{M_1^\infty}$, also the Ω-rule comes into play.

Axioms of $\mathbf{M_1^\infty}$. For all sequents Γ and all P in $\textsc{Prop}$:

$$\Gamma, P, \neg P \tag{ax}$$

Propositional rules of $\mathbf{M_1^\infty}$. For all sequents Γ and all formulae A, B:

$$\frac{\Gamma, A, B}{\Gamma, A \vee B} \tag{$\vee$}$$

$$\frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B} \tag{$\wedge$}$$

Modal rules of $\mathbf{M_1^\infty}$. For all sequents Γ, Σ, all formulae A, and all $i \in \mathsf{M}$:

$$\frac{\Gamma, A}{\Diamond_i\Gamma, \Box_i A, \Sigma} \tag{$\Box$}$$

Closure rules of $\mathbf{M_1^\infty}$. For all sequents Γ and operator forms $\mathcal{A}$:

$$\frac{\Gamma, \mathcal{A}(P_{\mathcal{A}})}{\Gamma, P_{\mathcal{A}}} \tag{clo}$$

Cut rules of $\mathbf{M_1^\infty}$. For all sequents Γ and all formulae A:

$$\frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma} \tag{cut}$$

Here the formulae A and $\neg A$ are called the *cut formulae* of the cut. The rank of a cut is the rank of its cut formulae.

Definition 8 (Derivability in $\mathbf{M_1^\infty}$). We define $\mathbf{M_1^\infty} \vdash_k^\alpha \Gamma$ for all sequents Γ, ordinals α, and natural numbers k by induction on α.

1. If Γ is an axiom of $\mathbf{M_1^\infty}$, then we have $\mathbf{M_1^\infty} \vdash_k^\alpha \Gamma$ for all ordinals α and natural numbers k.
2. If $\mathbf{M_1^\infty} \vdash_k^{\alpha_i} \Gamma_i$ and $\alpha_i < \alpha$ for all premises Γ_i of a propositional rule, a modal rule, a closure rule, or a cut of rank less than k, then we have $\mathbf{M_1^\infty} \vdash_k^\alpha \Gamma$ for the conclusion Γ of this rule.

3. (Ω-rules) Let $\mathcal{A}$ be an operator form. If for all positive sequents Δ and for all natural numbers n there exists a $\beta_n < \alpha$ such that

$$\mathbf{M}_1^\infty \vdash_0^n \Delta, P_\mathcal{A} \quad \Longrightarrow \quad \mathbf{M}_1^\infty \vdash_k^{\beta_n} \Delta, \Gamma,$$

then $\mathbf{M}_1^\infty \vdash_k^\alpha \Gamma, {\sim}P_\mathcal{A}$.

To simplify the notation, we generally drop $\mathbf{M}_1^\infty$ and simply write $\vdash_k^\alpha \Gamma$ instead of $\mathbf{M}_1^\infty \vdash_k^\alpha \Gamma$. Hence $\vdash_0^\alpha \Gamma$ means that there exists a cut-free proof of Γ in $\mathbf{M}_1^\infty$ whose depth is bounded by α. Furthermore, $\nvdash_0 \Gamma$ is used as a shorthand for saying that there is no α such that $\vdash_0^\alpha \Gamma$; it thus states that there is no cut-free proof of Γ in $\mathbf{M}_1^\infty$. Finally, $\vdash_k^{<\alpha} \Gamma$ abbreviates that there exists an ordinal $\beta < \alpha$ for which $\vdash_k^\beta \Gamma$.

More compactly, the Ω-rule can now be stated as: For all ordinals α and natural numbers k, if for all positive sequents Δ

$$\vdash_0^{<\omega} \Delta, P_\mathcal{A} \quad \Longrightarrow \quad \vdash_k^{<\alpha} \Delta, \Gamma,$$

then $\vdash_k^\alpha \Gamma, {\sim}P_\mathcal{A}$.

We conclude this section with the observation that any formula which has a finite proof in $\mathbf{M}_1^\infty$ which does not employ the Ω-rule can be proved in $\mathbf{M}_1$. This property plays a rôle later.

4. Embedding of M_1 into M_1^∞

The main purpose of this section is to show how the formal finite theory $\mathbf{M}_1$ can be embedded into the infinitary system $\mathbf{M}_1^\infty$. But in doing this, the reader will also acquire some skill in working with the Ω-rule and gain a better understanding of its proof-theoretic power.

We begin with stating standard weakening and inversion properties of $\mathbf{M}_1^\infty$. Both assertions of the following lemma are proved by induction on the depths of the respective derivations; details are left to the reader.

Lemma 9. *For all sequents Γ, Δ, all formulae A, B, all ordinals α, β, and all natural numbers m, n we have:*

1. $\alpha \leq \beta$, $m \leq n$, *and* $\vdash_m^\alpha \Gamma \quad \Longrightarrow \quad \vdash_n^\beta \Gamma, \Delta$.
2. $\vdash_m^\alpha \Gamma, A \vee B \quad \Longrightarrow \quad \vdash_m^\alpha \Gamma, A, B$.

Weakening and inversion are basic properties of our calculus $\mathbf{M}_1^\infty$, and typically we refrain from mentioning Lemma 9 when we use it.

Recall from the axioms of $\mathbf{M}_1^\infty$ that we have the tertium-non-datur for all atomic propositions from the set Prop. However, for the fixed point constants $P_\mathcal{A}$ it is not formulated as an axiom but requires a proof.

Lemma 10. *For all operator forms $\mathcal{A}$ we have*

$$\vdash_0^\omega \Gamma, {\sim}P_\mathcal{A}, P_\mathcal{A}.$$

Proof. For any positive sequent Δ – of course the following implication holds for arbitrary sequents as well – we trivially have

$$\vdash_0^{<\omega} \Delta, P_{\mathcal{A}} \quad \Longrightarrow \quad \vdash_0^{<\omega} \Delta, \Gamma, P_{\mathcal{A}}.$$

Hence a direct application of the Ω-rule yields $\vdash_0^{\omega} \Gamma, \sim P_{\mathcal{A}}, P_{\mathcal{A}}$. □

A finite proof of the tertium-non-datur for fixed point constants, i.e. a proof not making use of the Ω-rule, is not possible. With the previous lemma at hand, the tertium-non-datur for arbitrary formulae A is easily established by induction on $\mathsf{rk}(A)$.

Lemma 11. *For any sequent Γ and any formula A of rank n we have*

$$\vdash_0^{\omega+2n} \Gamma, \neg A, A.$$

Corollary 12. *For any sequent Γ and any operator form $\mathcal{A}$ of rank n we have*

$$\vdash_0^{\omega+2n+2} \Gamma, \mathcal{A}(P_{\mathcal{A}}) \to P_{\mathcal{A}}.$$

Proof. If $\mathcal{A}$ is an operator form of rank n, then it can be easily seen that $\mathsf{rk}(\mathcal{A}(P_{\mathcal{A}})) \leq n$. Hence the previous lemma implies

$$\vdash_0^{\omega+2n} \Gamma, \neg\mathcal{A}(P_{\mathcal{A}}), \mathcal{A}(P_{\mathcal{A}}),$$

and an application of (clo) yields

$$\vdash_0^{\omega+2n+1} \Gamma, \neg\mathcal{A}(P_{\mathcal{A}}), P_{\mathcal{A}}.$$

It only remains to apply $(\vee)$ in order to obtain our assertion. □

If Δ is a positive sequent and B an arbitrary formula, then $\Delta[P_{\mathcal{A}} := B]$ denotes the sequent which is obtained from the formulae in Δ by simultaneously replacing all occurrences of $P_{\mathcal{A}}$ by B.

Lemma 13. *Let Δ, Σ be positive sequents, let $\mathcal{A}$ be an operator form, and let B be an arbitrary formula. If*

$$\vdash_s^{\alpha} \mathcal{A}(B) \to B$$

and $k = \max(s, \mathsf{rk}(\mathcal{A}(B)) + 1)$, then we have, for all $m < \omega$, that

$$\vdash_0^{m} \Delta, \Sigma \quad \Longrightarrow \quad \vdash_k^{\alpha+m} \Delta, \Sigma[P_{\mathcal{A}} := B].$$

Proof. We show this assertion by induction on m and distinguish the following cases:

1. Δ, Σ is an axiom of $\mathbf{M}_1^{\infty}$. Then $\Delta, \Sigma[P_{\mathcal{A}} := B]$ is an axiom of $\mathbf{M}_1^{\infty}$, too.

2. Δ, Σ is the conclusion of a propositional or modal rule of $\mathbf{M}_1^{\infty}$. Then the claim follows immediately by the induction hypothesis.

3. Δ, Σ is the conclusion of a closure rule of $\mathbf{M}_1^{\infty}$ whose main formula is different from $P_{\mathcal{A}}$ or an element of Δ. Then the claim also follows immediately by the induction hypothesis.

4. Δ, Σ is the conclusion of a closure rule of $\mathbf{M_1^\infty}$ whose main formula is the formula $P_\mathcal{A}$ and an element of Σ. In this case we have

$$\vdash_0^{m_0} \Delta,\, \Sigma,\, \mathcal{A}(P_\mathcal{A})$$

for some $m_0 < m$. Thus the induction hypothesis yields

$$\vdash_k^{\alpha+m_0} \Delta,\, \Sigma[P_\mathcal{A} := B],\, \mathcal{A}(B).$$

On the other hand, in view of Lemma 9, our assumptions also imply

$$\vdash_s^{\alpha} \Delta,\, \Sigma[P_\mathcal{A} := B],\, \neg\mathcal{A}(B),\, B.$$

With a cut we therefore may conclude that

$$\vdash_k^{\alpha+m} \Delta,\, \Sigma[P_\mathcal{A} := B],\, B.$$

Because of $P_\mathcal{A} \in \Sigma$ we have $B \in \Sigma[P_\mathcal{A} := B]$, establishing our assertion also in this case.

Moreover, Δ, Σ cannot be the conclusion of an Ω-rule since it is a positive sequent. Hence we have treated all possible cases. □

Lemma 14. *Let $\mathcal{A}$ be an operator form and B an arbitrary formula. If*

$$\vdash_s^{\alpha} \mathcal{A}(B) \to B$$

and $k = \max(s, \mathsf{rk}(\mathcal{A}(B)) + 1)$*, then we also have*

$$\vdash_k^{\alpha+\omega+1} P_\mathcal{A} \to B.$$

Proof. In view of our assumptions and the previous lemma we know that for all positive sequents Δ

$$\vdash_0^{<\omega} \Delta,\, P_\mathcal{A} \quad \Longrightarrow \quad \vdash_k^{<\alpha+\omega} \Delta,\, B.$$

Hence an application of the Ω-rule yields $\vdash_k^{\alpha+\omega} {\sim}P_\mathcal{A}, B$. From that we deduce $\vdash_k^{\alpha+\omega+1} P_\mathcal{A} \to B$ by $(\vee)$. □

Corollary 12 tells us that the closure axioms for fixed point constants are provable in $\mathbf{M_1^\infty}$, the previous lemma shows that $\mathbf{M_1^\infty}$ is closed under the induction rules of $\mathbf{M_1}$. Hence it is routine to prove the following embedding theorem by induction on the derivations in $\mathbf{M_1}$.

Theorem 15. *Whenever a formula A is provable in $\mathbf{M_1}$, then there exist natural numbers m and s such that $\vdash_s^{\omega\cdot m} A$.*

5. Completeness of cut-free $\mathbf{M_1^\infty}$

The aim of this section is to show that each valid formula has a cut-free proof in $\mathbf{M_1^\infty}$, implying, of course, that cut-free $\mathbf{M_1^\infty}$ is complete. We establish this result by a canonical counter-model construction.

We first need a soundness result for the positive cut-free fragment of $\mathbf{M_1^\infty}$; the soundness of full $\mathbf{M_1^\infty}$ is addressed in the next section. If we have a cut-free proof of a positive sequent Γ, then this proof cannot employ the

Ω-rule. As a consequence of this observation, the following lemma is straightforward by induction on α.

Lemma 16. *For all ordinals α and all positive sequents Γ we have that*

$$\vdash_0^\alpha \Gamma \quad \Longrightarrow \quad \models \Gamma.$$

The crucial notions of this section are those of pre-saturated and saturated sequents. In particular, observe, for example by studying the proof of Lemma 22, how the Ω-rule is built into the saturation process.

Definition 17. A sequent Γ is called *pre-saturated* if for any formulae A and B and any operator form $\mathcal{A}$ the following conditions are satisfied :

(S.1) $\not\vdash_0 \Gamma$,
(S.2) if $A \vee B \in \Gamma$, then $A \in \Gamma$ and $B \in \Gamma$,
(S.3) if $A \wedge B \in \Gamma$, then $A \in \Gamma$ or $B \in \Gamma$,
(S.4) if $P_\mathcal{A} \in \Gamma$, then $\mathcal{A}(P_\mathcal{A}) \in \Gamma$.

Lemma 18. *If Γ is a sequent such that $\not\vdash_0 \Gamma$, then there exists a pre-saturated sequent Σ which contains Γ as a subset.*

Proof. We begin with fixing an arbitrary enumeration $A_0, A_1, A_2, \ldots$ of all formulae. The index of a formula A is the least number i for which A is identical to A_i. Depending on this enumeration we now define for each sequent Π with $\not\vdash_0 \Pi$ a new sequent Π^+:

1. If Π is pre-saturated, then $\Pi^+ := \Pi$.

2. If Π is not pre-saturated, then we choose the $A \in \Pi$ with smallest index for which one of the conditions (S.2), (S.3), or (S.4) of Definition 17 is violated with respect to Π and determine Π^+ by distinguishing between the possible forms of A.

(a) A is the formula $B \vee C$. Then we set

$$\Pi^+ \;:=\; \Pi,\, B,\, C.$$

(b) A is the formula $B \wedge C$. Since Π is not cut-free provable in $\mathbf{M}_1^\infty$, we know that

$$\not\vdash_0 \Pi,\, B \quad \text{or} \quad \not\vdash_0 \Pi,\, C,$$

and we set

$$\Pi^+ \;:=\; \begin{cases} \Pi,\, B & \text{if} \not\vdash_0 \Pi,\, B, \\ \Pi,\, C & \text{if} \not\vdash_0 \Pi,\, C \text{ and} \vdash_0 \Pi,\, B. \end{cases}$$

(c) A is the formula $P_\mathcal{A}$. Then we set

$$\Pi^+ \;:=\; \Pi,\, \mathcal{A}(P_\mathcal{A}).$$

By this construction it is guaranteed that

$$\not\vdash_0 \Pi^+ \quad \text{and} \quad \Pi \subseteq \Pi^+.$$

Now we take the given sequent Γ which does not have a cut-free proof and define a sequence $\Gamma_0, \Gamma_1, \Gamma_2, \ldots$ of sequents by

$$\Gamma_0 \;:=\; \Gamma \quad \text{and} \quad \Gamma_{n+1} \;:=\; \Gamma_n^+.$$

Accordingly, we have for all natural numbers n that

$$\not\vdash_0 \Gamma_n \quad \text{and} \quad \Gamma \subseteq \Gamma_n.$$

Write $\mathsf{Sufo}(A)$ for the collection of all subformulas of a given formula A and set

$$\Phi_\Gamma \;:=\; \bigcup\{\mathsf{Sufo}(A) : A \in \Gamma\} \quad \text{and} \quad \Psi_\Gamma \;:=\; \bigcup\{\mathsf{Sufo}(\mathcal{A}(P_\mathcal{A})) : P_\mathcal{A} \in \Phi_\Gamma\}.$$

Then $\Phi_\Gamma \cup \Psi_\Gamma$ is finite and $\Gamma_n \subseteq \Phi_\Gamma \cup \Psi_\Gamma$ for all natural numbers n. Hence we know that there has to be an m such that Γ_m is pre-saturated; we let Σ be this Γ_m. ☐

Carefully reading the previous construction also reveals the following specific variant of the extension of sequents to pre-saturated sequents which will be employed later in the proof of Lemma 22.

Lemma 19. *Let Γ be any pre-saturated and Δ any positive sequent. If $\not\vdash_0 \Gamma, \Delta$, then there exists a pre-saturated sequent Π containing Γ and Δ such that for any operator form $\mathcal{A}$*

$$\sim\!P_\mathcal{A} \in \Pi \quad \Longrightarrow \quad \sim\!P_\mathcal{A} \in \Gamma.$$

Definition 20. A sequent Γ is called *saturated* if it is pre-saturated and if for any operator form $\mathcal{A}$ the following condition is satisfied:

(S.5) if $\sim\!P_\mathcal{A} \in \Gamma$, then there exists a positive $\Delta \subseteq \Gamma$ such that $\vdash_0^{<\omega} \Delta, P_\mathcal{A}$.

We aim at showing that any sequent without a cut-free proof in $\mathbf{M}_1^\infty$ is contained in a saturated sequent. For this proof we need a consequence of our Ω-rule.

Lemma 21. *Let $\mathcal{A}$ be an arbitrary operator form and Γ an arbitrary sequent. If $\not\vdash_0 \Gamma, \sim\!P_\mathcal{A}$, then there exist a positive sequent Δ such that*

$$\vdash_0^{<\omega} \Delta,\, P_\mathcal{A} \quad \textit{and} \quad \not\vdash_0 \Delta,\, \Gamma.$$

Proof. Let $\not\vdash_0 \Gamma, \sim\!P_\mathcal{A}$ and suppose, aiming at a contradiction, that for all positive sequents Δ it is not the case that

$$\vdash_0^{<\omega} \Delta,\, P_\mathcal{A} \quad \text{and} \quad \not\vdash_0 \Delta,\, \Gamma.$$

Then for any positive sequent Δ and natural number m there exists an ordinal α_m satisfying

$$\vdash_0^{m} \Delta,\, P_\mathcal{A} \quad \Longrightarrow \quad \vdash_0^{\alpha_m} \Gamma,\, \Delta.$$

Set $\beta := \sup(\alpha_m + 1 : m < \omega)$. Then the Ω-rule implies $\vdash_0^{\beta} \Gamma, \sim\!P_\mathcal{A}$, contradicting the fact that $\not\vdash_0 \Gamma, \sim\!P_\mathcal{A}$. Hence our lemma is proved. ☐

Lemma 22. *If Γ is a sequent such that $\not\vdash_0 \Gamma$, then there exists a saturated sequent Σ which contains Γ as a subset.*

Proof. We show that for every pre-saturated sequent Γ there exists a saturated sequent Σ which contains Γ. Then the claim follows immediately from Lemma 18. So assume that Γ is a pre-saturated. We prove the following auxiliary assertion by induction on n:

> If a pre-saturated sequent Γ has at most n elements of the form $\sim P_{\mathcal{A}}$ violating (S.5) with respect to Γ, then there exists a saturated Σ which contains Γ.

$n = 0$. Then let $\Sigma := \Gamma$, and we are done.

$n > 0$. We pick a $\sim P_{\mathcal{B}} \in \Gamma$ which violates (S.5) with respect to Γ. Because of $\not\vdash_0 \Gamma$, i.e. $\not\vdash_0 \Gamma, \sim P_{\mathcal{B}}$, and the previous lemma we know that there exists a positive sequent Δ such that

$$\vdash_0^{<\omega} \Delta, P_{\mathcal{B}}, \tag{1}$$

$$\not\vdash_0 \Delta, \Gamma. \tag{2}$$

From (2) and Lemma 19 we conclude that there exists a pre-saturated sequent Π with the properties

$$\Delta, \Gamma \subseteq \Pi, \tag{3}$$

$$\sim P_{\mathcal{A}} \in \Pi \quad \Longrightarrow \quad \sim P_{\mathcal{A}} \in \Gamma. \tag{4}$$

for all operator forms $\mathcal{A}$.

Because of (1) and (3) the formula $\sim P_{\mathcal{B}}$ does not violate (S.5) with respect to Π. Also, any $\sim P_{\mathcal{A}} \in \Gamma$ which does not violate (S.5) with respect to Γ does not violate it with respect to Π. Thus, in view of (4), the number of formulae $\sim P_{\mathcal{A}} \in \Pi$ which violate (S.5) with respect to Π is smaller than n, so that we can apply the induction hypothesis. Hence we obtain a saturated sequent Σ which contains Π and therefore also Γ.

This finishes the proof of our auxiliary assertion which, in turn, immediately implies our saturation lemma. □

Our next step is to introduce the canonical counter-model $\mathbb{K}$, a Kripke structure built up from the saturated sequents. In the remainder of this section we show that for any formula A which does not possess a cut-free proof in $\mathbf{M}_1^\infty$ there exists a state of $\mathbb{K}$ not satisfying A.

Definition 23 (Canonical counter-model). Let $\mathbb{K}$ be the Kripke structure $(S_{\text{can}}, R_{\text{can}}, \pi_{\text{can}})$ where we define, for all $i \in \mathsf{M}$ and $P \in \textsc{Prop}$,

$$\begin{aligned} S_{\text{can}} &:= \{\Gamma : \Gamma \text{ a saturated sequent}\}, \\ R_{\text{can}}(i) &:= \{(\Gamma, \Delta) \in S_{\text{can}} \times S_{\text{can}} : \{B : \Diamond_i B \in \Gamma\} \subseteq \Delta\}, \\ \pi_{\text{can}}(P) &:= \{\Gamma \in S_{\text{can}} : P \notin \Gamma\}. \end{aligned}$$

Due to Lemmata 16 and 22, S_{can} is non-empty. The Kripke structure $\mathbb{K}$ has a series of important properties which finally lead to the so-called truth lemma. From that to the cut-free completeness of $\mathbf{M}_1^\infty$ it is a trivial step.

Lemma 24. *For all saturated sequents* Γ*, formulae* A*, and* $i \in \mathsf{M}$ *we have:*

1. *If* $\Box_i A \in \Gamma$*, then there exists a sequent* Δ *such that* $(\Gamma, \Delta) \in R_{can}(i)$ *and* $A \in \Delta$*.*
2. *If* $\Diamond_i A \in \Gamma$*, then* $A \in \Delta$ *for all sequents* Δ *which satisfy* $(\Gamma, \Delta) \in R_{can}(i)$*.*

Proof. To prove the first part of this lemma recall that, since Γ is saturated, Γ is not cut-free provable in $\mathbf{M}_1^\infty$. With $\Box_i A \in \Gamma$ we conclude

$$\not\vdash_0 \{B : \Diamond_i B \in \Gamma\}, A.$$

Hence Lemma 22 guarantees the existence of a saturated Δ such that

$$\{B : \Diamond_i B \in \Gamma\}, A \subseteq \Delta.$$

Hence we have $(\Gamma, \Delta) \in R_{\mathrm{can}}(i)$ and $A \in \Delta$. The second part of this lemma is obvious from the definition of $R_{\mathrm{can}}(i)$. □

Lemma 25. *Let* $\mathcal{A}$ *be any operator form. Then we have for all ordinals* α*, all saturated sequents* Γ*, and all formulae* B *of the basic language* $\mathcal{L}_0$ *not containing* $\sim\mathsf{X}$ *that*

$$B[\mathsf{X} := P_{\mathcal{A}}] \in \Gamma \quad \Longrightarrow \quad \Gamma \notin \|B\|_{\mathbb{K}[\mathsf{X}:=I^{<\alpha}_{\mathcal{A},\mathbb{K}}]}.$$

Proof. We show this implication by main induction on α and side induction on $\mathsf{rk}(B)$ and distinguish the following cases:

1. $B \in \textsc{Prop} \setminus \{\mathsf{X}\}$. Then the assertion is obvious by the definition of π_{can}.

2. B is the formula $C \vee D$. Since Γ is saturated, we know that $C[\mathsf{X} := P_{\mathcal{A}}] \in \Gamma$ and $D[\mathsf{X} := P_{\mathcal{A}}] \in \Gamma$. The induction hypothesis of the side induction therefore implies $\Gamma \notin \|C\|_{\mathbb{K}[\mathsf{X}:=I^{<\alpha}_{\mathcal{A},\mathbb{K}}]}$ and $\Gamma \notin \|D\|_{\mathbb{K}[\mathsf{X}:=I^{<\alpha}_{\mathcal{A},\mathbb{K}}]}$, hence $\Gamma \notin \|B\|_{\mathbb{K}[\mathsf{X}:=I^{<\alpha}_{\mathcal{A},\mathbb{K}}]}$.

3. B is the formula $C \wedge D$. Since Γ is saturated, we know that $C[\mathsf{X} := P_{\mathcal{A}}] \in \Gamma$ or $D[\mathsf{X} := P_{\mathcal{A}}] \in \Gamma$. The induction hypothesis of the side induction therefore implies $\Gamma \notin \|C\|_{\mathbb{K}[\mathsf{X}:=I^{<\alpha}_{\mathcal{A},\mathbb{K}}]}$ or $\Gamma \notin \|D\|_{\mathbb{K}[\mathsf{X}:=I^{<\alpha}_{\mathcal{A},\mathbb{K}}]}$, hence $\Gamma \notin \|B\|_{\mathbb{K}[\mathsf{X}:=I^{<\alpha}_{\mathcal{A},\mathbb{K}}]}$.

4. B is the formula $\Diamond_i C$. Then Lemma 24 tells us that $C[\mathsf{X} := P_{\mathcal{A}}] \in \Delta$ for all Δ such that $(\Gamma, \Delta) \in R_{\mathrm{can}}(i)$. By the induction hypothesis of the side induction we obtain $\Delta \notin \|C\|_{\mathbb{K}[\mathsf{X}:=I^{<\alpha}_{\mathcal{A},\mathbb{K}}]}$ for those sequents Δ. This implies $\Gamma \notin \|B\|_{\mathbb{K}[\mathsf{X}:=I^{<\alpha}_{\mathcal{A},\mathbb{K}}]}$.

5. B is the formula $\Box_i C$. Then the previous lemma tells us that there exists a sequent Δ such that $(\Gamma, \Delta) \in R_{\mathrm{can}}(i)$ and $C[\mathsf{X} := P_{\mathcal{A}}] \in \Delta$. By the induction hypothesis of the side induction we obtain $\Delta \notin \|C\|_{\mathbb{K}[\mathsf{X}:=I^{<\alpha}_{\mathcal{A},\mathbb{K}}]}$ for this sequent Δ. This implies $\Gamma \notin \|B\|_{\mathbb{K}[\mathsf{X}:=I^{<\alpha}_{\mathcal{A},\mathbb{K}}]}$.

6. B is the formula X. Then $B[\mathsf{X} := P_{\mathcal{A}}]$ is the formula $P_{\mathcal{A}}$, and we obtain $\mathcal{A}(P_{\mathcal{A}}) \in \Gamma$ by the saturation of Γ. Now we apply the induction hypothesis of the main induction and conclude

$$\Gamma \notin \|\mathcal{A}\|_{\mathbb{K}[\mathsf{X}:=I^{<\beta}_{\mathcal{A},\mathbb{K}}]}$$

for all $\beta < \alpha$. By the definition of the approximations of least fixed points this means that $\Gamma \notin I^{\beta}_{\mathcal{A},\mathbb{K}}$ for all $\beta < \alpha$. Consequently, $\Gamma \notin \|B\|_{\mathbb{K}[\mathsf{X}:=I^{<\alpha}_{\mathcal{A},\mathbb{K}}]}$, which concludes the proof of our lemma. □

Corollary 26. *For all operator forms $\mathcal{A}$ and all saturated sequents Γ we have that*

$$P_{\mathcal{A}} \in \Gamma \quad \Longrightarrow \quad \Gamma \notin \|P_{\mathcal{A}}\|_{\mathbb{K}}.$$

Proof. Assume $P_{\mathcal{A}} \in \Gamma$. Then the previous lemma implies $\Gamma \notin \|X\|_{\mathbb{K}[\mathsf{X}:=I^{<\alpha}_{\mathcal{A},\mathbb{K}}]}$ for all ordinals α. Hence $\Gamma \notin I_{\mathcal{A},\mathbb{K}}$ and, by Theorem 5, $\Gamma \notin \|P_{\mathcal{A}}\|_{\mathbb{K}}$. □

Lemma 27. *For all positive formulae B and all saturated sequents Γ we have that*

$$B \in \Gamma \quad \Longrightarrow \quad \Gamma \notin \|B\|_{\mathbb{K}}.$$

Proof. This assertion is shown by induction on $\mathsf{rk}(B)$. If B is an element of Prop, then our claim is obvious from the definition of π_{can}, if B is a fixed point constant, then our claim follows from the previous corollary. In all other cases our arguments are analogous to those in the proof of Lemma 25. □

Lemma 28. *For all operator forms $\mathcal{A}$ and all saturated sequents Γ we have that*

$$\sim P_{\mathcal{A}} \in \Gamma \quad \Longrightarrow \quad \Gamma \notin \|\sim P_{\mathcal{A}}\|_{\mathbb{K}}.$$

Proof. Since Γ is saturated, $\sim P_{\mathcal{A}} \in \Gamma$ implies that there exists a positive sequent $\Delta \subseteq \Gamma$ such that

$$\vdash^{<\omega}_{0} \Delta, P_{\mathcal{A}}. \tag{5}$$

The positivity of Δ and the previous lemma give us

$$\Gamma \notin \|B\|_{\mathbb{K}} \tag{6}$$

for all $B \in \Delta$. Moreover, (5) and Lemma 16 tell us that the sequent $\Delta, P_{\mathcal{A}}$ is valid. Therefore, because of (6), we have $\Gamma \in \|P_{\mathcal{A}}\|_{\mathbb{K}}$, and this means that $\Gamma \notin \|\sim P_{\mathcal{A}}\|_{\mathbb{K}}$. □

Lemma 29 (Truth lemma). *For all formulae B and all saturated sequents Γ we have that*

$$B \in \Gamma \quad \Longrightarrow \quad \Gamma \notin \|B\|_{\mathbb{K}}.$$

Proof. Again, this is shown by induction on $\mathsf{rk}(B)$ and exactly as the proof of Lemma 27 with the additional case that B is a formula of the form $\sim P_{\mathcal{A}}$. However, this case is taken care of by the previous lemma. □

Theorem 30 (Cut-free completeness of $\mathbf{M}^{\infty}_{1}$). *For any valid formula A there exists an ordinal α such that $\vdash^{\alpha}_{0} A$.*

Proof. We show the contraposition of this assertion. So assume $\not\vdash_{0} A$. According to Lemma 22 there exists a saturated sequent Σ with A as an element. By the previous lemma we thus have $\Sigma \notin \|A\|_{\mathbb{K}}$. Therefore A cannot be valid. □

Corollary 31 (Positive completeness).

1. *For any valid and positive sequent* Γ *we have that* $\vdash_0^{<\omega} \Gamma$.
2. *For any positive and valid formula* A *we have that* $\mathbf{M_1} \vdash A$.

Proof. By induction on α we first show that for any positive sequent Γ

$$\vdash_0^{\alpha} \Gamma \quad \Longrightarrow \quad \vdash_0^{<\omega} \Gamma.$$

Since Γ is positive, $\vdash_0^{\alpha} \Gamma$ implies that no instance of the Ω-rule has been used. All other inference rules have finitely many premises only, and thus no problems arise in the induction step. Given this auxiliary consideration about the finitization of a cut-free proof in $\mathbf{M_1^\infty}$, the first part of our corollary is immediate from the cut-free completeness of $\mathbf{M_1^\infty}$.

Turning to the second assertion, take a positive and valid formula A. From what we have just proved, we conclude $\vdash_0^{<\omega} A$. Since A is positive, this means that there exists a finite proof of A in $\mathbf{M_1^\infty}$ not using the Ω-rule. According to the observation at the end of Section 3 we thus have $\mathbf{M_1} \vdash A$. □

6. Soundness of $\mathbf{M_1^\infty}$

The modal μ-calculus has the finite model property, see Kozen [8]. That yields the following theorem as a special case.

Theorem 32 (Finite model property). *If a formula* A *is satisfiable, then there exists a Kripke structure* K *such that* $|\mathsf{K}|$ *is finite and* $\|A\|_{\mathsf{K}} \neq \emptyset$.

The result about the positive completeness of $\mathbf{M_1^\infty}$ in the previous section together with this finite model property of $\mathbf{M_1}$ give us the soundness of $\mathbf{M_1^\infty}$.

Theorem 33 (Soundness of $\mathbf{M_1^\infty}$). *For all ordinals* α*, all natural numbers* k*, and all sequents* Γ *we have that*

$$\vdash_k^{\alpha} \Gamma \quad \Longrightarrow \quad \models \Gamma.$$

Proof. This is shown by induction on α, and only the case that Γ is the conclusion of an application of the Ω-rule requires special attention. So let Γ be a sequent of the form $\Sigma, \sim P_{\mathcal{A}}$ and let the premise of the Ω-rule be satisfied. Then

$$\vdash_0^{<\omega} \Delta, P_{\mathcal{A}} \quad \Longrightarrow \quad \vdash_k^{<\alpha} \Delta, \Sigma \tag{7}$$

for all positive sequents Δ. Now assume that $\not\models \Gamma$, i.e. $\not\models \Sigma, \sim P_{\mathcal{A}}$. According to the finite model property there exists a Kripke structure K such that $|\mathsf{K}|$ has finitely many elements and

$$\|P_{\mathcal{A}} \land \neg C\|_{\mathsf{K}} \neq \emptyset,$$

where C is the disjunction of the elements of Σ. With k being the cardinality of $|\mathsf{K}|$, we conclude with Theorem 5 that

$$\|\mathcal{A}^k(\bot) \land \neg C\|_{\mathsf{K}} \neq \emptyset. \tag{8}$$

On the other hand, Theorem 5 also implies the validity of $\neg\mathcal{A}^k(\bot), P_{\mathcal{A}}$ from which we deduce by Corollary 31 or prove directly by induction on k that

$$\vdash_0^{<\omega} \neg\mathcal{A}^k(\bot),\, P_{\mathcal{A}}.$$

Hence (7) yields

$$\vdash_k^{<\alpha} \neg\mathcal{A}^k(\bot),\, \Sigma.$$

Now we can apply the induction hypothesis and obtain

$$\models \neg\mathcal{A}^k(\bot) \vee C,$$

a contradiction to (8). Hence we have $\models \Gamma$, as desired. □

7. Cut elimination and collapsing

The proof-theoretic analysis of the theory $\mathbf{ID_1}$ by means of the infinitary system $\mathbf{ID_1^\infty}$ based on Buchholz's Ω_1-rule requires an intricate combination of cut elimination and collapsing; see Buchholz [3] for all details. In this section we present a corresponding approach for $\mathbf{M_1^\infty}$.

As a preparatory step, we eliminate all cuts whose cut formulae are different from fixed point constants and their complements. This is routine and creates no problems. Keep in mind that, according to Definition 7, fixed point constants and their complements have rank 0 and all other atomic formulae rank 1.

Lemma 34. *For all ordinals α, all natural numbers $k \geq 1$, and all sequents Γ we have that*

$$\vdash_{k+1}^{\alpha} \Gamma \quad \Longrightarrow \quad \vdash_k^{2^\alpha} \Gamma.$$

Now we must take care of cuts whose cut formulae are pairs of fixed point constants and their complements. They are eliminated in the context of positive sequents with the following lemma taking care of the critical case. In its formulation, if Σ is a set $\{P_{\mathcal{A}_1}, \dots, P_{\mathcal{A}_n}\}$ of finitely many fixed point constants, then $\sim\Sigma$ denotes $\{\sim P_{\mathcal{A}_1}, \dots, \sim P_{\mathcal{A}_n}\}$.

Lemma 35. *Let α be an ordinal, Γ and Π be positive sequents, and Σ a finite set of fixed point constants. Under the assumptions*

(A1) $\vdash_0^{<\omega} \Gamma, P_{\mathcal{A}}$ *for all $P_{\mathcal{A}} \in \Sigma$,*
(A2) $\vdash_1^{\alpha} \Pi, \sim\Sigma$

we obtain that $\vdash_0^{<\omega} \Gamma, \Pi$.

Proof. We show this lemma by induction on α and distinguish the following cases.

1. If $\Pi, \sim\Sigma$ is an axiom or the conclusion of a propositional, modal, or closure rule, then our assertion is obvious or an immediate consequence of the induction hypothesis.

2. $\Pi, {\sim}\Sigma$ is the conclusion of a cut. Then there exists a fixed point constant $P_{\mathcal{B}}$ and ordinals $\alpha_0, \alpha_1 < \alpha$ such that

$$\vdash_1^{\alpha_0} \Pi,\ {\sim}\Sigma,\ P_{\mathcal{B}}, \tag{9}$$

$$\vdash_1^{\alpha_0} \Pi,\ {\sim}\Sigma,\ {\sim}P_{\mathcal{B}}. \tag{10}$$

By induction hypothesis, (A1) and (9) imply

$$\vdash_0^{<\omega} \Gamma,\ \Pi,\ P_{\mathcal{B}}. \tag{11}$$

Hence (A1), (11), (10) plus a further application of the induction hypothesis yield $\vdash_0^{<\omega} \Gamma, \Pi$.

3. $\Pi, {\sim}\Sigma$ is the conclusion of an application of the Ω-rule. Then we have a $P_{\mathcal{B}} \in \Sigma$ such that

$$\vdash_0^{<\omega} \Delta,\ P_{\mathcal{B}} \quad \Longrightarrow \quad \vdash_1^{<\alpha} \Delta,\ \Pi,\ {\sim}\Sigma$$

for all positive sequents Δ. Making use of this implication and assumption (A1) we obtain

$$\vdash_1^{<\alpha} \Gamma,\ \Pi,\ {\sim}\Sigma. \tag{12}$$

Therefore, the induction hypothesis applied to (A1) and (12) gives us again $\vdash_0^{<\omega} \Gamma, \Pi$. This was the last case to be considered, and thus our lemma is established. □

Theorem 36 (Collapsing + cut elimination). *For all ordinals α and all positive sequents Γ we have that*

$$\vdash_1^{\alpha} \Gamma \quad \Longrightarrow \quad \vdash_0^{<\omega} \Gamma.$$

Proof. Again, we proceed by induction on α. If Γ an axiom, then our claim is obvious; if Γ is the consequence of a inference rule different from a cut, then simply apply the induction hypothesis. So it remains the case that the last inference rule is a cut. Then there are $\alpha_0, \alpha_1 < \alpha$ and a fixed point constant $P_{\mathcal{A}}$ for which

$$\vdash_1^{\alpha_0} \Gamma,\ P_{\mathcal{A}}, \tag{13}$$

$$\vdash_1^{\alpha_1} \Gamma,\ {\sim}P_{\mathcal{A}}. \tag{14}$$

Hence the induction hypothesis applied to (13) immediately yields

$$\vdash_0^{<\omega} \Gamma,\ P_{\mathcal{A}}. \tag{15}$$

Fortunately, we can now apply the previous theorem to (14) and (15), implying that $\vdash_0^{<\omega} \Gamma$. This finishes the proof of our theorem. □

For the final result of this section it only remains to combine Lemma 34, which deals with the elimination of the ‘trivial’ cuts, with the previous collapsing + cut elimination lemma, eliminating fixed point constants and their complements from positive sequents.

Corollary 37. *For all ordinals α, all natural numbers k, and all positive sequents Γ we have that*

$$\vdash_k^\alpha \Gamma \quad \Longrightarrow \quad \vdash_0^{<\omega} \Gamma.$$

The previous corollary has been established by purely syntactic methods, neither the completeness nor the soundness of $\mathbf{M}_\mathbf{1}^\infty$ are needed. As presented, we go via the infinitary $\mathbf{M}_\mathbf{1}^\infty$; however, with methods from Buchholz [4] it should be possible to avoid an infinitary intermediate system. An alternative semantic proof is as follows:

(i) If $\vdash_k^\alpha \Gamma$, then the soundness of $\mathbf{M}_\mathbf{1}^\infty$ gives us the validity of Γ.

(ii) Hence our result about positive completeness implies $\vdash_0^{<\omega} \Gamma$.

As yet, we have full syntactic cut elimination for the positive fragment of $\mathbf{M}_\mathbf{1}^\infty$ only. On the other hand, combing Theorem 30 and Theorem 33 gives us a semantic proof of full cut elimination for all sequents provable in $\mathbf{M}_\mathbf{1}^\infty$.

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