Cut-free common knowledge

Gerhard Jäger   Mathis Kretz*   Thomas Studer

Abstract

Starting off from the infinitary system for common knowledge over multi-modal epistemic logic presented in Alberucci and Jäger [1], we apply the finite model property to "finitize" this deductive system. The result is a cut-free, sound and complete sequent calculus for common knowledge.

1 Introduction

Common knowledge and *common belief* are important and interesting topics in areas such as computer science, logic, game theory, artificial intelligence, psychology and many other fields for which coordination among "agents" is of great importance. Formalizations of reasoning with and about common knowledge have been widely discussed in the literature, for example in Barwise [2, 3] and in the textbooks Fagin, Halpern, Moses and Vardi [4] as well as Meyer and van der Hoek [6], to give only a few examples.

In connection with calculi for common knowledge the question often arises whether there is a complete and cut-free system which has the subformula property and other desired structural properties. In the following we will show that such a cut-free sequent calculus for common knowledge indeed exists. In order to design it, we start off from the infinitary system $\mathbf{K}_n^\omega(\mathsf{C})$ for common knowledge over multi-modal epistemic logic presented in Alberucci and Jäger [1]. Then we recall that the finite model property is available for common knowledge and make use of this fact for restricting $\mathbf{K}_n^\omega(\mathsf{C})$ to a finite system $\mathbf{K}_n^{<\omega}(\mathsf{C})$. All we have to do is to change the ω-rule $(\omega\mathsf{C})$ for common knowledge (see below) in $\mathbf{K}_n^\omega(\mathsf{C})$ to the finite rule $(<\omega\mathsf{C})$ in which only a finite number of the infinitely many premises of $(\omega\mathsf{C})$ is used.

Obviously, every formula provable in $\mathbf{K}_n^\omega(\mathsf{C})$ is also provable in $\mathbf{K}_n^{<\omega}(\mathsf{C})$. Hence $\mathbf{K}_n^{<\omega}(\mathsf{C})$ is stronger than $\mathbf{K}_n^\omega(\mathsf{C})$, and thus the completeness of $\mathbf{K}_n^\omega(\mathsf{C})$

*Research partly supported by the Swiss National Science Foundation.

implies that of $\mathbf{K}_n^{<\omega}(\mathsf{C})$. On the other hand, the finite model property of common knowledge will guarantee the consistency of $\mathbf{K}_n^{<\omega}(\mathsf{C})$. However, $\mathbf{K}_n^{<\omega}(\mathsf{C})$ is a fairly "untypical" system in the sense that the number of premises needed in the rule ($<\omega\mathsf{C}$) depends on the complexity of the conclusion of ($<\omega\mathsf{C}$).

2 Basic semantic notions

Good expository introductions to and detailed motivations of an approach to common knowledge in the context of multi-modal propositional logics are presented, for example, in Fagin, Halpern, Moses and Vardi [4] and in Meyer and van der Hoek [6]. In the following we take up the syntactic and semantic notions of Alberucci and Jäger [1], which are based on these two textbooks, and refer the reader to this article for further details.

$L_n(\mathsf{C})$ is our standard language for multi-modal logic; it comprises a set PROP of *atomic propositions*, typically indicated by $P, Q, \dots$ (possibly with subscripts), the *propositional connectives* $\vee$ and $\wedge$, the *epistemic operators* $\mathsf{K}_1, \mathsf{K}_2, \dots, \mathsf{K}_n$ and the *common knowledge operator* C; in addition we assume that there is an auxiliary symbol $\sim$ for forming the complements of atomic propositions and dual epistemic operators.

The *formulas* $\alpha, \beta, \gamma, \dots$ (possibly with subscripts) of $L_n(\mathsf{C})$ and the *length* $\ell(\alpha)$ of each $L_n(\mathsf{C})$ formula α are inductively generated as follows:

1. All atomic propositions P and their complements $\widetilde{P}$ are $L_n(\mathsf{C})$ formulas;
$$\ell(P) \;:=\; \ell(\widetilde{P}) \;:=\; 1.$$
2. If α and β are $L_n(\mathsf{C})$ formulas, so are $(\alpha \vee \beta)$ and $(\alpha \wedge \beta)$;
$$\ell((\alpha \vee \beta)) \;:=\; \ell((\alpha \wedge \beta)) \;:=\; \ell(\alpha) + \ell(\beta).$$
3. If α is an $L_n(\mathsf{C})$ formula, so are $\mathsf{K}_i(\alpha)$ and $\widetilde{\mathsf{K}}_i(\alpha)$;
$$\ell(\mathsf{K}_i(\alpha)) \;:=\; \ell(\widetilde{\mathsf{K}}_i(\alpha)) \;:=\; \ell(\alpha) + 1.$$
4. If α is an $L_n(\mathsf{C})$ formula, so are $\mathsf{C}(\alpha)$ and $\widetilde{\mathsf{C}}(\alpha)$;
$$\ell(\mathsf{C}(\alpha)) \;:=\; \ell(\widetilde{\mathsf{C}}(\alpha)) \;:=\; \ell(\alpha) \cdot n + n + 1.$$

As usual we omit parentheses if there is no danger of confusion and abbreviate the remaining logical connectives as usual; in addition we set

$$\mathsf{E}(\alpha) \;:=\; \mathsf{K}_1(\alpha) \wedge \dots \wedge \mathsf{K}_n(\alpha) \quad \text{and} \quad \widetilde{\mathsf{E}}(\alpha) \;:=\; \widetilde{\mathsf{K}}_1(\alpha) \vee \dots \vee \widetilde{\mathsf{K}}_n(\alpha).$$

The factor and summand n, i.e. the number of agents, in the definition of $\ell(\mathsf{C}(\alpha))$ and $\ell(\widetilde{\mathsf{C}}(\alpha))$ ensure that we always have

$$\ell(\mathsf{E}(\alpha)) \;=\; \ell(\widetilde{\mathsf{E}}(\alpha)) \;<\; \ell(\mathsf{C}(\alpha)) \;=\; \ell(\widetilde{\mathsf{C}}(\alpha)).$$

Formulas $\mathsf{K}_i(\alpha)$ are typically interpreted – on the intuitive level – as *agent i knows (believes) that α* so that $\mathsf{E}(\alpha)$ means that *everybody knows α*. But observe that this does not mean that α is common knowledge.

Common knowledge of α is much stronger: it implies (i) that everybody knows α and, in addition, (ii) that everybody knows that everybody knows α, (iii) that everybody knows that everybody knows that every knows α plus (iv) all further iterations thereof. To make this precise, we inductively introduce for all natural numbers m the iterations $\mathsf{E}^m(\alpha)$ as

$$\mathsf{E}^0(\alpha) \;:=\; \alpha \quad \text{and} \quad \mathsf{E}^{m+1}(\alpha) \;:=\; \mathsf{E}(\mathsf{E}^m(\alpha))$$

and then represent common knowledge of α as the infinitary conjunction of all $\mathsf{E}^m(\alpha)$ for $m \geq 1$,

$$\mathsf{C}(\alpha) \;\approx\; \bigwedge_{i \geq 1} \mathsf{E}^m(\alpha).$$

The $L_n(\mathsf{C})$ formulas $\widetilde{P}$ act as negations of the atomic proposition P and are needed together with the duals $\widetilde{\mathsf{K}}_i$ and $\widetilde{\mathsf{C}}$ of the modal operators K_i and C, respectively, in forming the negations $\neg\alpha$ of general $L_n(\mathsf{C})$ formulas α (by making use of de Morgan's laws and the law of double negation):

1. If α is the atomic proposition P, then $\neg\alpha$ is $\widetilde{P}$; if α is the formula $\widetilde{P}$, then $\neg\alpha$ is P.

2. If α is the formula $(\beta \vee \gamma)$, then $\neg\alpha$ is $(\neg\beta \wedge \neg\gamma)$; if α is the formula $(\beta \wedge \gamma)$, then $\neg\alpha$ is $(\neg\beta \vee \neg\gamma)$.

3. If α is the formula $\mathsf{K}_i(\beta)$, then $\neg\alpha$ is $\widetilde{\mathsf{K}}_i(\neg\beta)$; if α is the formula $\widetilde{\mathsf{K}}_i(\beta)$, then $\neg\alpha$ is $\mathsf{K}_i(\neg\beta)$.

4. If α is the formula $\mathsf{C}(\beta)$, then $\neg\alpha$ is $\widetilde{\mathsf{C}}(\neg\beta)$; if α is the formula $\widetilde{\mathsf{C}}(\beta)$, then $\neg\alpha$ is $\mathsf{C}(\neg\beta)$.

We turn to the semantics of $L_n(\mathsf{C})$. As always, a *Kripke-frame* for $L_n(\mathsf{C})$ is a $(n{+}1)$-tuple

$$\mathfrak{M} \;=\; (W, \mathbb{K}_1, \ldots, \mathbb{K}_n)$$

consisting of a non-empty set W of *worlds* and n binary accessibility relations $\mathbb{K}_1, \ldots, \mathbb{K}_n$ on W; the set of worlds of a Kripke-frame $\mathfrak{M}$ is often denoted by

$|\mathfrak{M}|$. Besides that, a *valuation* in a Kripke-frame $\mathfrak{M}$ is a function $\mathcal{V}$ from the atomic propositions PROP to the power set $Pow(|\mathfrak{M}|)$ of $|\mathfrak{M}|$,

$$\mathcal{V} \;:\; \mathsf{PROP} \;\to\; Pow(|\mathfrak{M}|).$$

Finally, the *truth-set* $\|\alpha\|^{\mathfrak{M}}_{\mathcal{V}}$ of an $L_n(\mathsf{C})$ formula α with respect to a Kripke-frame $\mathfrak{M} = (W, \mathbb{K}_1, \ldots, \mathbb{K}_n)$ and a valuation $\mathcal{V}$ in $\mathfrak{M}$ is defined, as usual in multi-modal logics, by induction an the complexity of α with an additional clause for treating the operator C:

$$\begin{aligned}
\|P\|^{\mathfrak{M}}_{\mathcal{V}} &:= \mathcal{V}(P),\\
\|\widetilde{P}\|^{\mathfrak{M}}_{\mathcal{V}} &:= W \setminus \|P\|^{\mathfrak{M}}_{\mathcal{V}},\\
\|\alpha \vee \beta\|^{\mathfrak{M}}_{\mathcal{V}} &:= \|\alpha\|^{\mathfrak{M}}_{\mathcal{V}} \cup \|\beta\|^{\mathfrak{M}}_{\mathcal{V}},\\
\|\alpha \wedge \beta\|^{\mathfrak{M}}_{\mathcal{V}} &:= \|\alpha\|^{\mathfrak{M}}_{\mathcal{V}} \cap \|\beta\|^{\mathfrak{M}}_{\mathcal{V}},\\
\|\mathsf{K}_i(\alpha)\|^{\mathfrak{M}}_{\mathcal{V}} &:= \{\, v \in W : w \in \|\alpha\|^{\mathfrak{M}}_{\mathcal{V}} \text{ for all } w \text{ so that } (v,w) \in \mathbb{K}_i \,\},\\
\|\widetilde{\mathsf{K}}_i(\alpha)\|^{\mathfrak{M}}_{\mathcal{V}} &:= \{\, v \in W : w \in \|\alpha\|^{\mathfrak{M}}_{\mathcal{V}} \text{ for some } w \text{ so that } (v,w) \in \mathbb{K}_i \,\},\\
\|\mathsf{C}(\alpha)\|^{\mathfrak{M}}_{\mathcal{V}} &:= \bigcap \{\, \|\mathsf{E}^m(\alpha)\|^{\mathfrak{M}}_{\mathcal{V}} : m \geq 1 \,\},\\
\|\widetilde{\mathsf{C}}(\alpha)\|^{\mathfrak{M}}_{\mathcal{V}} &:= \bigcup \{\, \|\widetilde{\mathsf{E}}^m(\alpha)\|^{\mathfrak{M}}_{\mathcal{V}} : m \geq 1 \,\}.
\end{aligned}$$

Based on this notion, we say that an $L_n(\mathsf{C})$ formula α is *valid in the Kripke-frame* $\mathfrak{M}$, in symbols

$$\mathfrak{M} \models \alpha,$$

provided that for all worlds w from $|\mathfrak{M}|$ and all valuations $\mathcal{V}$ in $\mathfrak{M}$ we have $w \in \|\alpha\|^{\mathfrak{M}}_{\mathcal{V}}$.

For notational simplicity we confine ourselves to accessibility relations without any specific properties (e.g. reflexivity, transitivity) with the consequence that only the K-axioms are satisfied with respect to our modalities $\mathsf{K}_1, \ldots, \mathsf{K}_n$; in particular, knowledge of a fact does not imply the truth of this fact. Extensions of the following approach to, for example, reflexive and transitive relations on the semantic side and inclusion of the T-axioms and the S4-axioms on the syntactic side would work without any problems.

Our semantics of C reflects the so-called *iterative interpretation* of common knowledge,

$$\mathfrak{M} \models \mathsf{C}(\alpha) \quad \Longleftrightarrow \quad \mathfrak{M} \models \bigwedge_{m \geq 1} \mathsf{E}^m(\alpha),$$

mentioned already above. Alternatively, we could also treat common knowledge in the sense of the *greatest fixed point interpretation* since

$$(\star) \qquad \|\mathsf{C}(\alpha)\|^{\mathfrak{M}}_{\mathcal{V}} \;=\; \bigcup\{\, X \subset |\mathfrak{M}| : X = \|\mathsf{E}(\alpha) \land \mathsf{E}(Q)\|^{\mathfrak{M}}_{\mathcal{V}[Q:=X]} \,\}$$

where Q is chosen to be an atomic proposition which does not occur in α and $\mathcal{V}[Q{:=}X]$ is the valuation which maps Q to X and otherwise agrees with $\mathcal{V}$. A proof of equation $(\star)$ can be found, for example, in Fagin, Halpern, Moses and Vardi [4].

Hilbert-style axiomatizations for common knowledge which are sound and complete with respect to this semantics are discussed in full detail in, for example, Fagin, Halpern, Moses and Vardi [4] and Meyer and van der Hoek [6]. However, this type of axiomatization is not needed for our following considerations, and so we turn to an infinitary Tait-style system immediately.

3 An infinitary Tait-style system for common knowledge

The iterative character of our approach to common knowledge lends itself to a formulation within infinitary deductive systems in which $\mathsf{C}(\alpha)$ can be derived by a kind of ω-rule from the infinitely many premises

$$\mathsf{E}^1(\alpha),\ \mathsf{E}^2(\alpha),\ \ldots,\ \mathsf{E}^m(\alpha),\ \ldots$$

for all natural numbers $m \geq 1$, just as in the semantic interpretation of $\mathsf{C}(\alpha)$, introduced in the previous section. In the following we take up the approach of Alberucci and Jäger [1] and reconsider the system $\mathbf{K}^{\omega}_n(\mathsf{C})$ introduced there.

$\mathbf{K}^{\omega}_n(\mathsf{C})$ is formulated as a Tait-style calculus which derives finite sets of $L_n(\mathsf{C})$ formulas rather than individual $L_n(\mathsf{C})$ formulas. These finite sets of $L_n(\mathsf{C})$ formulas are denoted by the capital Greek letters $\Gamma, \Delta, \Pi, \ldots$ (possibly with subscripts) and have to be interpreted disjunctively. We often write (for example) $\alpha, \beta, \Gamma, \Delta$ for the union $\{\alpha, \beta\} \cup \Gamma \cup \Delta$. In addition, if Γ is the set $\{\alpha_1, \ldots, \alpha_m\}$, we often use the following convenient abbreviations:

$$\begin{aligned}
\Gamma^{\vee} &:= \alpha_1 \vee \ldots \vee \alpha_m, \\
\widetilde{\mathsf{K}}_i(\Gamma) &:= \{\widetilde{\mathsf{K}}_i(\alpha_1), \ldots, \widetilde{\mathsf{K}}_i(\alpha_m)\}, \\
\widetilde{\mathsf{C}}(\Gamma) &:= \{\widetilde{\mathsf{C}}(\alpha_1), \ldots, \widetilde{\mathsf{C}}(\alpha_m)\}.
\end{aligned}$$

The axioms and rules of $\mathbf{K}^{\omega}_n(\mathsf{C})$ consist of the usual propositional axioms and rules of Tait-calculi, of rules for the epistemic operators K_i with incorporated formulas $\widetilde{\mathsf{C}}(\Delta)$ plus rules for introducing $\widetilde{\mathsf{C}}$ and ω-like rules for C. More precisely, $\mathbf{K}_n^\omega(\mathsf{C})$ contains the following collections of axioms and rules, with P being any atomic proposition, α and β any $L_n(\mathsf{C})$ formulas and Γ any finite set of $L_n(\mathsf{C})$ formulas.

I. Axioms of $\mathbf{K}_n^\omega(\mathsf{C})$

(ID) $$P,\, \widetilde{P},\, \Gamma$$

II. Propositional rules of $\mathbf{K}_n^\omega(\mathsf{C})$

($\vee$) $$\frac{\alpha,\, \beta,\, \Gamma}{\alpha \vee \beta,\, \Gamma}$$

($\wedge$) $$\frac{\alpha,\, \Gamma \qquad \beta,\, \Gamma}{\alpha \wedge \beta,\, \Gamma}$$

III. K_i-rules of $\mathbf{K}_n^\omega(\mathsf{C})$

(K_i) $$\frac{\alpha,\, \Gamma,\, \widetilde{\mathsf{C}}(\Delta)}{\mathsf{K}_i(\alpha),\, \widetilde{\mathsf{K}}_i(\Gamma),\, \widetilde{\mathsf{C}}(\Delta),\, \Pi}$$

III. $\widetilde{\mathsf{C}}$-rules of $\mathbf{K}_n^\omega(\mathsf{C})$

($\widetilde{\mathsf{C}}$) $$\frac{\widetilde{\mathsf{E}}(\alpha),\, \Gamma}{\widetilde{\mathsf{C}}(\alpha),\, \Gamma}$$

IV. $\omega\mathsf{C}$-rules of $\mathbf{K}_n(\mathsf{C})$

($\omega\mathsf{C}$) $$\frac{\ldots \mathsf{E}^m(\alpha),\, \Gamma \ldots \quad \text{(for all } m \geq 1)}{\mathsf{C}(\alpha),\, \Gamma}$$

Observe that these axioms and rules of our Tait-calculus $\mathbf{K}_n^\omega(\mathsf{C})$ do not include the usual cut rule, i.e. the rule

(Cut) $$\frac{\alpha,\, \Gamma \qquad \neg\alpha,\, \Gamma}{\Gamma},$$

which will be shown to be admissible later. In fact, all rules of $\mathbf{K}_n^\omega(\mathsf{C})$ satisfy the so-called subformula property, provided that we regard all $\mathsf{E}^m(\alpha)$ as subformulas of $\mathsf{C}(\alpha)$.

The subformula property of a rule (R) means that all formulas in the premises of (R) are subformulas of the formulas in its conclusion. Clearly, the subformula property is a useful feature in the context of proof search since it restricts the search space for the reconstruction of proofs significantly.

Because of the rules ($\omega\mathsf{C}$), our system $\mathbf{K}_n^\omega(\mathsf{C})$ allows proof trees with infinitely many nodes. Thus we require ordinals, which are denoted by the small Greek letters $\sigma, \tau, \eta, \xi, \ldots$ (possibly with subscripts), to measure the length of proofs.

Starting from its axioms and rules of inference, derivability in $\mathbf{K}_n^\omega(\mathsf{C})$ is introduced as usual. For arbitrary ordinals σ and finite sets Γ of $L_n(\mathsf{C})$ formulas the notion $\mathbf{K}_n^\omega(\mathsf{C}) \vdash^\sigma \Gamma$ is defined by induction on σ as follows:

1. If Γ is an axiom of $\mathbf{K}_n^\omega(\mathsf{C})$, then we have $\mathbf{K}_n^\omega(\mathsf{C}) \vdash^\sigma \Gamma$ for all σ.
2. If $\mathbf{K}_n^\omega(\mathsf{C}) \vdash^{\sigma_i} \Gamma_i$ and $\sigma_i < \sigma$ for all premises Γ_i of a rule of $\mathbf{K}_n^\omega(\mathsf{C})$, then we have $\mathbf{K}_n^\omega(\mathsf{C}) \vdash^\sigma \Gamma$ for the conclusion Γ of this rule.

$\mathbf{K}_n^\omega(\mathsf{C}) \vdash^{<\sigma} \Gamma$ means $\mathbf{K}_n^\omega(\mathsf{C}) \vdash^\tau \Gamma$ for some ordinal $\tau < \sigma$, and $\mathbf{K}_n^\omega(\mathsf{C}) \vdash \Gamma$ means $\mathbf{K}_n^\omega(\mathsf{C}) \vdash^\tau \Gamma$ for some ordinal τ. Furthermore, $\mathbf{K}_n^\omega(\mathsf{C}) + (\mathrm{Cut}) \vdash^\sigma \alpha$ is defined analogously to $\mathbf{K}_n^\omega(\mathsf{C}) \vdash^\sigma \alpha$ with the rules (Cut) being admitted as additional rules of inference.

The system $\mathbf{K}_n^\omega(\mathsf{C})$ has a minor drawback: Suppose that α is provable in $\mathbf{K}_n^\omega(\mathsf{C})$, say

$$\mathbf{K}_n^\omega(\mathsf{C}) \vdash^\sigma \alpha$$

for some ordinal σ. Then we need something like $\sigma + m \cdot n$ steps to derive $\mathsf{E}^m(\alpha)$ from α, and afterwards ($\omega\mathsf{C}$) yields

$$\mathbf{K}_n^\omega(\mathsf{C}) \vdash^{\sigma+\omega} \mathsf{C}(\alpha).$$

By adding the auxiliary rules of the form (for all natural numbers $m \geq 1$)

$$(\mathsf{E}^m) \quad \frac{\alpha}{\mathsf{E}^m(\alpha),\, \Pi} \qquad \text{or} \qquad (0\mathsf{C}) \quad \frac{\alpha}{\mathsf{C}(\alpha),\, \Pi}$$

we could do much better and obtain $\mathsf{C}(\alpha)$ from α in only two or one additional step. Actually the formulation of $\mathbf{K}_n^\omega(\mathsf{C})$ in Alberucci and Jäger [1] includes the rules (E^m), but this difference is not important for the following.

Let us also mention some natural extensions of $\mathbf{K}_n^\omega(\mathsf{C})$: The system $\mathbf{T}_n^\omega(\mathsf{C})$ is obtained from $\mathbf{K}_n^\omega(\mathsf{C})$ by adding the rules

$$(\widetilde{\mathsf{K}}_i) \qquad \frac{\alpha,\, \Gamma}{\widetilde{\mathsf{K}}_i(\alpha),\, \Gamma}$$

which take care, in Tait-style systems, of the usual axiom (T), stating that $\mathsf{K}_i(\alpha)$ implies α. $\mathbf{S4}_n^\omega(\mathsf{C})$ stands for the infinitary Tait-style system of the multi-modal version of $\mathbf{S4}$ with common knowledge and extends $\mathbf{T}_n^\omega(\mathsf{C})$ by all rules

$$(4_i) \qquad \frac{\alpha,\, \widetilde{\mathsf{K}}_i(\Gamma),\, \widetilde{\mathsf{C}}(\Delta)}{\mathsf{K}_i(\alpha),\, \widetilde{\mathsf{K}}_i(\Gamma),\, \widetilde{\mathsf{C}}(\Delta),\, \Pi}$$

which then allow us to prove positive introspection, namely that $\mathsf{K}_i(\alpha)$ implies $\mathsf{K}_i(\mathsf{K}_i(\alpha))$.

The auxiliary set of formulas Π in the conclusions of the rules (K_i), (E^m), $(0\mathsf{C})$ and (4_i) are added just in order to guarantee the weakening property of our calculi; that is, if Γ is provable and if Γ is a subset of Δ, then Δ is provable (with the same length) as well. Trivially, these auxiliary sets could be dropped in these rules and a general weakening rule added.

A Kripke-frame $\mathfrak{M}$ is a *model* of $\mathbf{K}_n^\omega(\mathsf{C})$ if all axioms of $\mathbf{K}_n^\omega(\mathsf{C})$ are valid in $\mathfrak{M}$ and if $\mathfrak{M}$ is closed under the rules of inference of $\mathbf{K}_n^\omega(\mathsf{C})$ with respect to validity. We call a formula α a *semantic consequence* of $\mathbf{K}_n^\omega(\mathsf{C})$, in symbols

$$\mathbf{K}_n^\omega(\mathsf{C}) \models \alpha,$$

if α is valid in all models of $\mathbf{K}_n^\omega(\mathsf{C})$. The following theorem states soundness and completeness of syntactic derivability in $\mathbf{K}_n^\omega(\mathsf{C})$ with respect to this notion of semantic consequence.

Theorem 1 (Soundness and completeness of $\mathbf{K}_n^\omega(\mathsf{C})$)
The two systems $\mathbf{K}_n^\omega(\mathsf{C})$ and $\mathbf{K}_n^\omega(\mathsf{C}) + (\mathrm{Cut})$ are sound and complete with respect to our semantics; i.e. for all finite sets Γ of $L_n(\mathsf{C})$ formulas we have

$$\mathbf{K}_n^\omega(\mathsf{C}) \vdash \Gamma \quad \Longleftrightarrow \quad \mathbf{K}_n(\mathsf{C}) \models \Gamma^\vee \quad \Longleftrightarrow \quad \mathbf{K}_n^\omega(\mathsf{C}) + (\mathrm{Cut}) \vdash \Gamma.$$

This theorem is proved in Alberucci and Jäger [1]. It is clear that only (i) the soundness of $\mathbf{K}_n^\omega(\mathsf{C}) + (\mathrm{Cut})$ and (ii) the completeness of $\mathbf{K}_n^\omega(\mathsf{C})$ have to be established. Assertion (i) is more or less obvious; assertion (ii) is obtained by a canonical model construction utilizing $\mathbf{K}_n^\omega(\mathsf{C})$ saturated sets of $L_n(\mathsf{C})$ formulas.

From Theorem 1 we also deduce that the two informal systems $\mathbf{K}_n^\omega(\mathsf{C})$ and $\mathbf{K}_n^\omega(\mathsf{C}) + (\mathrm{Cut})$ prove the same $L_n(\mathsf{C})$ formulas. As a further consequence this theorem also states admissibility of cuts in $\mathbf{K}_n^\omega(\mathsf{C})$. However, this is a form of semantic cut elimination which does not provide a method of how proofs in $\mathbf{K}_n^\omega(\mathsf{C}) + (\mathrm{Cut})$ can be transformed into proofs in $\mathbf{K}_n^\omega(\mathsf{C})$.

4 Finitizing $\mathbf{K}_n^\omega(\mathsf{C})$

Given a Kripke-frame $\mathfrak{M}$, we define $card(\mathfrak{M})$ to be the cardinality of its universe $|\mathfrak{M}|$; accordingly, $\mathfrak{M}$ is called *finite* if $card(\mathfrak{M}) < \omega$. As mentioned above, $\mathsf{C}(\alpha)$ is generally treated as the infinite conjunction of the $\mathsf{E}^{m+1}(\alpha)$ for all natural numbers m. But if we work over a finite Kripke-frame $\mathfrak{M}$, then $\mathsf{C}(\alpha)$ is reached after finitely many iteration steps.

Lemma 2 *Suppose that α is an $L_n(\mathsf{C})$ formula, $\mathfrak{M}$ a model of $\mathbf{K}_n(\mathsf{C})$, $\mathcal{V}$ a valuation in $\mathfrak{M}$ and m a natural number. Then we have:*

1. $\| \bigwedge_{i=1}^{m} \mathsf{E}^i(\alpha)\|_{\mathcal{V}}^{\mathfrak{M}} = \| \bigwedge_{i=1}^{m+1} \mathsf{E}^i(\alpha)\|_{\mathcal{V}}^{\mathfrak{M}} \quad \Longrightarrow \quad \|\mathsf{C}(\alpha)\|_{\mathcal{V}}^{\mathfrak{M}} = \| \bigwedge_{i=1}^{m} \mathsf{E}^i(\alpha)\|_{\mathcal{V}}^{\mathfrak{M}}.$

2. $card(\mathfrak{M}) \leq m \quad \Longrightarrow \quad \|\mathsf{C}(\alpha)\|_{\mathcal{V}}^{\mathfrak{M}} = \| \bigwedge_{i=1}^{m} \mathsf{E}^i(\alpha)\|_{\mathcal{V}}^{\mathfrak{M}}.$

Proof For the first assertion, simply show by induction on the natural number k that our assumption implies $\| \bigwedge_{i=1}^{m} \mathsf{E}^i(\alpha)\|_{\mathcal{V}}^{\mathfrak{M}} = \| \bigwedge_{i=1}^{m+k} \mathsf{E}^i(\alpha)\|_{\mathcal{V}}^{\mathfrak{M}}$. To establish the second part, consider the decreasing sequence

$$\| \bigwedge_{i=1}^{0} \mathsf{E}^i(\alpha)\|_{\mathcal{V}}^{\mathfrak{M}} \supset \| \bigwedge_{i=1}^{1} \mathsf{E}^i(\alpha)\|_{\mathcal{V}}^{\mathfrak{M}} \supset \ldots \supset \| \bigwedge_{i=1}^{m} \mathsf{E}^i(\alpha)\|_{\mathcal{V}}^{\mathfrak{M}} \supset \| \bigwedge_{i=1}^{m+1} \mathsf{E}^i(\alpha)\|_{\mathcal{V}}^{\mathfrak{M}}$$

of subsets of $|\mathfrak{M}|$. From $card(\mathfrak{M}) \leq m$ we conclude that not all of these $m+2$ sets can be different so that our assertion follows from the first part of this lemma. □

In a next step the finite model property of $\mathbf{K}_n(\mathsf{C})$ comes into play. It states that each satisfiable formula α is satisfied in a finite frame with at most $2^{\ell(\alpha)}$ worlds and will allow us to collapse the infinite derivations in $\mathbf{K}_n^\omega(\mathsf{C})$ to finite derivations. This is possible since only finitely many premises for an application of the infinitary rule $(\omega\mathsf{C})$ are really important.

Theorem 3 (Finite model property)
If the $L_n(\mathsf{C})$ formula α is satisfiable with respect to $\mathbf{K}_n(\mathsf{C})$, then there exist a model $\mathfrak{M}$ of $\mathbf{K}_n(\mathsf{C})$, a valuation $\mathcal{V}$ in $\mathfrak{M}$ and an element w of $|\mathfrak{M}|$ so that

$$card(\mathfrak{M}) \leq 2^{\ell(\alpha)} \quad \textit{and} \quad (\mathfrak{M}, \mathcal{V}, w) \models \alpha.$$

The proof of this theorem follows immediately from the proof of the completeness of $\mathbf{K}_n(\mathsf{C})$. All details can be found again in Fagin, Halpern, Moses and Vardi [4] or, for example, in Halpern and Moses [5] and Meyer and van der Hoek [6].

The finite model property of $\mathbf{K}_n(\mathsf{C})$ is instrumental in designing a finitized version ($<\omega\mathsf{C}$) of the infinitary rule ($\omega\mathsf{C}$). For any $L_n(\mathsf{C})$ formula α and any finite set $\Gamma = \{\beta_1, \ldots, \beta_m\}$ of $L_n(\mathsf{C})$ formulas we first define a bounding function $bd(\alpha, \Gamma)$ by

$$bd(\alpha, \Gamma) \ := \ 2^{\ell(\mathsf{C}(\alpha))+\ell(\beta_1)+\ldots+\ell(\beta_m)}$$

which plays a crucial rôle in restricting ($\omega\mathsf{C}$) to a new finite subrule with only finitely many premises.

Lemma 4 *Let α be an $L_n(\mathsf{C})$ formula and Γ a finite set of $L_n(\mathsf{C})$ formulas. Suppose, in addition, that*

$$\mathbf{K}_n(\mathsf{C}) \models \mathsf{E}^m(\alpha) \vee \Gamma^\vee$$

for all $1 \leq m \leq bd(\alpha, \Gamma)$. Then we also have

$$\mathbf{K}_n(\mathsf{C}) \models \mathsf{C}(\alpha) \vee \Gamma^\vee.$$

Proof We proceed indirectly and assume that the formula $\mathsf{C}(\alpha) \vee \Gamma^\vee$ is not valid with respect to $\mathbf{K}_n(\mathsf{C})$. Hence $\neg(\mathsf{C}(\alpha) \vee \Gamma^\vee)$ is satisfiable with respect to $\mathbf{K}_n(\mathsf{C})$, and according to the previous theorem there exists a model $\mathfrak{M}$ of $\mathbf{K}_n(\mathsf{C})$ so that $card(\mathfrak{M}) \leq bd(\alpha, \Gamma)$ and $(\mathfrak{M}, \mathcal{V}, w) \not\models \mathsf{C}(\alpha) \vee \Gamma^\vee$ for some valuation $\mathcal{V}$ in $\mathfrak{M}$ and some element w of $\mathfrak{M}$. This implies that $\mathsf{E}^m(\alpha) \vee \Gamma^\vee$ is not valid with respect to $\mathbf{K}_n(\mathsf{C})$ for some m with $1 \leq m \leq bd(\alpha, \Gamma)$. By contraposition of this argument we have the assertion of our lemma. $\Box$

The following finite ω-rule for C is a restriction of the rule ($\omega\mathsf{C}$) to finitely many premises, the number of which depends on the length of the conclusion.

Finite ωC-rules For all $L_n(\mathsf{C})$ formulas α and all finite sets Γ, Π of $L_n(\mathsf{C})$ formulas:

$$(<\omega\mathsf{C}) \qquad \frac{\ldots \mathsf{E}^m(\alpha),\, \Gamma \ldots \quad (\text{for all } 1 \leq m \leq bd(\alpha, \Gamma))}{\mathsf{C}(\alpha),\, \Gamma,\, \Pi}$$

The addition of (possibly empty) sets Π of side formulas in the conclusions of these rules is necessary for making them stable under weakening.

The system $\mathbf{K}_n^{<\omega}(\mathsf{C})$ is $\mathbf{K}_n^{\omega}(\mathsf{C})$ with the ωC-rules ($\omega\mathsf{C}$) replaced by the finite ωC-rules ($<\omega\mathsf{C}$). Naturally, $\mathbf{K}_n^{<\omega}(\mathsf{C})$ is a finite system and all rules of $\mathbf{K}_n^{<\omega}(\mathsf{C})$ have the subformula property, again with the proviso that the $\mathsf{E}^m(\alpha)$ are regarded as subformulas of $\mathsf{C}(\alpha)$.

The notion $\mathbf{K}_n^{<\omega}(\mathsf{C}) \vdash^k \Gamma$ is introduced as $\mathbf{K}_n^{\omega}(\mathsf{C}) \vdash^\sigma \Gamma$, but with any ($\omega\mathsf{C}$) replaced by ($<\omega\mathsf{C}$). Since all rules have finitely many premises only, natural numbers k are sufficient to bound the depth of proof trees. As a consequence, we write $\mathbf{K}_n^{<\omega}(\mathsf{C}) \vdash \Gamma$ if $\mathbf{K}_n^{<\omega}(\mathsf{C}) \vdash^k \Gamma$ for some natural number k.

Whenever $(\omega\mathsf{C})$ is applicable, the rule $(<\omega\mathsf{C})$ can be applied as well since only a finite number of the infinitely many premises are required. Therefore the following Lemma 5 is obvious. More problematic is the correctness of $(<\omega\mathsf{C})$; it is shown in Lemma 6 and follows from the finite model property of $\mathbf{K}_n(\mathsf{C})$ in disguise of Lemma 4 above.

Lemma 5 *For all finite sets Γ of $L_n(\mathsf{C})$ formulas and all ordinals σ, we have that*

$$\mathbf{K}_n^{\omega}(\mathsf{C}) \vdash^{\sigma} \Gamma \quad \Longrightarrow \quad \mathbf{K}_n^{<\omega}(\mathsf{C}) \vdash \Gamma.$$

Proof This assertion is proved by induction on σ. If Γ is an axiom of $\mathbf{K}_n^{\omega}(\mathsf{C})$, then it is also an axiom of $\mathbf{K}_n^{<\omega}(\mathsf{C})$. If Γ is the conclusion of a basic rule of $\mathbf{K}_n^{\omega}(\mathsf{C})$ or a $\widetilde{\mathsf{C}}$-rule of $\mathbf{K}_n^{\omega}(\mathsf{C})$, then $\mathbf{K}_n^{<\omega}(\mathsf{C}) \vdash \Gamma$ follows from the induction hypothesis.

It remains to consider the case that Γ is the conclusion of a rule $(\omega\mathsf{C})$. Then we have a set Δ, a formula α and ordinals $\sigma_1, \sigma_2, \ldots$ with the properties

$$\mathbf{K}_n^{\omega}(\mathsf{C}) \vdash^{\sigma_m} \mathsf{E}^m(\alpha), \Delta \qquad \text{and} \qquad \sigma_m < \sigma$$

for all natural numbers m greater than 0. Now we restrict our attention to the m between 1 and $bd(\alpha, \Delta)$, apply the induction hypothesis and obtain

$$\mathbf{K}_n^{<\omega}(\mathsf{C}) \vdash \mathsf{E}^m(\alpha), \Delta$$

for all m so that $1 \leq m \leq bd(\alpha, \Delta)$. Hence $(<\omega\mathsf{C})$ implies $\mathbf{K}_n^{<\omega}(\mathsf{C}) \vdash \Gamma$, and our lemma is proved. □

Lemma 6 *For all finite sets Γ of $L_n(\mathsf{C})$ formulas and all natural numbers k, we have that*

$$\mathbf{K}_n^{<\omega}(\mathsf{C}) \vdash^k \Gamma \quad \Longrightarrow \quad \mathbf{K}_n(\mathsf{C}) \models \Gamma^{\vee}.$$

Proof We proceed by induction on k. If Γ is an axiom of $\mathbf{K}_n^{<\omega}(\mathsf{C})$, then the assertion is obvious. If Γ is the conclusion of a basic rule of $\mathbf{K}_n^{<\omega}(\mathsf{C})$ or a $\widetilde{\mathsf{C}}$-rule of $\mathbf{K}_n^{<\omega}(\mathsf{C})$, we apply the induction hypothesis to the premise(s) of this rule and obtain $\mathbf{K}_n(\mathsf{C}) \models \Gamma$ immediately.

Finally, if Γ is the conclusion of a rule $(<\omega\mathsf{C})$, then there exist a set Δ, a formula α and natural numbers $k_1, k_2, \ldots$ so that Γ is of the form $\mathsf{C}(\alpha), \Delta$ and we have

$$\mathbf{K}_n^{<\omega}(\mathsf{C}) \vdash^{k_m} \mathsf{E}^m(\alpha), \Delta \qquad \text{and} \qquad k_m < k$$

for all natural numbers m, $1 \leq m \leq bd(\alpha, \Delta)$. By induction hypothesis we conclude

$$\mathbf{K}_n(\mathsf{C}) \models \mathsf{E}^m(\alpha) \vee \Delta^\vee$$

for all $1 \leq m \leq bd(\alpha, \Delta)$. It simply remains to utilize Lemma 4 which yields $\mathbf{K}_n(\mathsf{C}) \models \mathsf{C}(\alpha) \vee \Delta^\vee$ and completes the proof of our lemma. $\square$

Theorem 1 and Lemma 5 establish the completeness of our system $\mathbf{K}_n^{<\omega}(\mathsf{C})$; Lemma 6 states its soundness. All together, we obtain our main result about $\mathbf{K}_n^{<\omega}(\mathsf{C})$.

Theorem 7 (Soundness and completeness of $\mathbf{K}_n^{<\omega}(\mathsf{C})$)
The system $\mathbf{K}_n^{<\omega}(\mathsf{C})$ is sound and complete with respect to our semantics; i.e. for all finite sets Γ of $L_n(\mathsf{C})$ formulas, we have

$$\mathbf{K}_n^{<\omega}(\mathsf{C}) \vdash \Gamma \quad \Longleftrightarrow \quad \mathbf{K}_n(\mathsf{C}) \models \Gamma^\vee.$$

Notice that this theorem also implies the admissibility of cuts: adding cuts to $\mathbf{K}_n^{<\omega}(\mathsf{C})$ does not increase its proof-theoretic power.

5 Conclusion

The main achievement of this note is a positive answer to the old question whether a cut-free, sound and complete finite formalization of common knowledge does exist: $\mathbf{K}_n^{<\omega}(\mathsf{C})$ is a deductive system which satisfies all the required properties. What is conceptually interesting (and new) is the fact that a model-theoretic property, the finite model property, is directly integrated into a deductive system.

However, we also agree that this should not be the end of the story. As already mentioned, the inference rule ($<\omega\mathsf{C}$) of $\mathbf{K}_n^{<\omega}(\mathsf{C})$ – i.e. the finitary version of the very natural (infinitary) rule ($\omega\mathsf{C}$) of $\mathbf{K}_n(\mathsf{C})$ for introducing common knowledge – is somewhat "unusual" in the sense that the number of its premises depends on the complexity of its conclusion. This has some consequences with respect to the structural properties of $\mathbf{K}_n^{<\omega}(\mathsf{C})$.

Inversion with respect to propositional conjunction and disjunction is a simple matter; more interesting is *inversion with respect to the modal operator* C. In the infinitary calculus $\mathbf{K}_n^{\omega}(\mathsf{C})$ it is evident that

$$\mathbf{K}_n^{\omega}(\mathsf{C}) \vdash^\sigma \mathsf{C}(\alpha), \Gamma \quad \Longrightarrow \quad \mathbf{K}_n^{\omega}(\mathsf{C}) \vdash^\sigma \mathsf{E}^m(\alpha), \Gamma$$

for all natural numbers $m \geq 1$, all ordinals σ, all $L_n(\mathsf{C})$ formulas α and all finite sets Γ of $L_n(\mathsf{C})$ formulas. Turning to the finite system $\mathbf{K}_n^{<\omega}(\mathsf{C})$, definitely a weak form of C-inversion is available.

Weak C-inversion. For any $L_n(\mathsf{C})$ formula α, any finite set Γ of $L_n(\mathsf{C})$ formulas and any natural numbers k and m with $m \geq 1$ we have that

$$\mathbf{K}_n^{<\omega}(\mathsf{C}) \vdash^k \mathsf{C}(\alpha),\, \Gamma \quad \Longrightarrow \quad \mathbf{K}_n^{<\omega}(\mathsf{C}) \vdash \mathsf{E}^m(\alpha),\, \Gamma.$$

However, so far we have no information about the complexities of the derivations of $\mathsf{E}^m(\alpha), \Gamma$ in relationship to k, and we do not even know whether a natural relationship of this sort exists at all. This gives rise to the following question:

Question. Assume the left hand side of the previous implication. Is it then the case that $\mathbf{K}_n^{<\omega}(\mathsf{C}) \vdash^k \mathsf{E}^m(\alpha),\, \Gamma$ for all m where $1 \leq m \leq bd(\alpha, \Gamma)$?

To see why this is rather intricate, assume that the answer to this question is "yes" and try to prove it by induction on k. If $\mathsf{C}(\alpha), \Gamma$ has been inferred in in the last step by an application of ($<\omega\mathsf{C}$) with main formula $\mathsf{C}(\alpha)$, then we have no problems. The critical case is a last step of the form

$$\frac{\ldots \mathsf{E}^i(\beta),\, \mathsf{C}(\alpha),\, \Delta \ldots \quad (\text{for all } 1 \leq i \leq bd(\beta, \{C(\alpha)\} \cup \Delta))}{\mathsf{C}(\beta),\, \mathsf{C}(\alpha),\, \Delta,\, \Pi}$$

with a different main formula $\mathsf{C}(\beta)$. Then for any natural number i so that $1 \leq i \leq bd(\beta, \{\mathsf{C}(\alpha)\} \cup \Delta)$ there exists a $k_i < k$ for which

$$\mathbf{K}_n^{<\omega}(\mathsf{C}) \vdash^{k_i} \mathsf{E}^i(\beta),\, \mathsf{C}(\alpha),\, \Delta.$$

Hence, by induction hypothesis, it is also the case that

$$\mathbf{K}_n^{<\omega}(\mathsf{C}) \vdash^{k_i} \mathsf{E}^i(\beta),\, \mathsf{E}^m(\alpha),\, \Delta,$$

and now the complications begin. Since, in general, $bd(\beta, \{\mathsf{C}(\alpha)\} \cup \Delta)$ is smaller than $bd(\beta, \{\mathsf{E}^m(\alpha)\} \cup \Delta)$ we do not have enough premises to derive the desired $\mathsf{C}(\beta), \mathsf{E}^m(\alpha), \Delta, \Pi$ by an application of ($<\omega\mathsf{C}$).

Another open problem in connection with $\mathbf{K}_n^{<\omega}(\mathsf{C})$ is syntactic cut elimination. Let $\mathbf{K}_n^{<\omega}(\mathsf{C})$ + (Cut) denote the extension of $\mathbf{K}_n^{<\omega}(\mathsf{C})$ which permits cuts as further rules of inference. In the face of Theorem 1 and Theorem 7 we know that semantic cut elimination is available. However, the proof of this result does not give any information about the relationship between the $\mathbf{K}_n^{<\omega}(\mathsf{C})$-proofs with and without cuts.

Question. Is there a syntactic procedure for transforming proofs in the system $\mathbf{K}_n^{<\omega}(\mathsf{C})$+(Cut) into proofs in $\mathbf{K}_n^{<\omega}(\mathsf{C})$? If so, what are the complexity bounds?

We do not expect an easy answer to this question. As a preparatory step it might be reasonable to study the related question first for infinitary $\mathbf{K}_n^{\omega}(\mathsf{C})$+ (Cut) and $\mathbf{K}_n^{\omega}(\mathsf{C})$. But even for this presumably much simpler system only semantic cut elimination is at our disposal as yet.

References

[1] L. Alberucci and G. Jäger, *About cut elimination for logics of common knowledge*, Annals of Pure and Applied Logic **133** (2005), 73–99.

[2] K. J. Barwise, *Three views of common knowledge*, 2nd Conference on Theoretical Aspects of Reasoning about Knowledge (Moshe Y. Vardi, ed.), Morgan-Kaufmann, 1988, pp. 365–379.

[3] ______, *The Situation in Logic*, CSLI Lecture Notes, vol. 17, Stanford, 1989.

[4] R. Fagin, J. Y. Halpern, Y. Moses, and M. Y. Vardi, *Reasoning about Knowledge*, MIT Press, 1995.

[5] J. Y. Halpern and Y. Moses, *A guide to completeness and complexity for modal logics of knowledge and belief*, Artificial Intelligence **54** (1992), 319–379.

[6] J.-J. Meyer and W. van der Hoek, *Epistemic Logic for AI and Computer Science*, Cambridge Tracts in Theoretical Computer Science, vol. 41, Cambridge University Press, 1995.

Address
Gerhard Jäger, Mathis Kretz, Thomas Studer
Institut für Informatik und angewandte Mathematik, Universität Bern
Neubrückstrasse 10, CH-3012 Bern, Switzerland
`{jaeger,kretz,tstuder}@iam.unibe.ch`