



## Modeling and Simulation of a Horizontally Moving Suspended Mass Pendulum Base using $H^\infty$ Optimal Loop Shaping Controller with First and Second Order Desired Loop Shaping Functions

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**Abstract:** In this paper, a horizontally moving suspended mass pendulum base is designed and controlled using robust control theory.  $H^\infty$  optimal loop shaping with first and second order desired loop shaping function controllers are used to improve the performance of the system using Matlab/Simulink Toolbox. Comparison of the  $H^\infty$  optimal loop shaping with first and second order desired loop shaping function controllers for the proposed system have been done to track the desired angular position of the pendulum using step and sine wave input signals and a promising result has been obtained successfully.

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### 1. Introduction

A pendulum with suspended mass is a system that has a mass suspended in its base and a mass suspended from a pivot so that it can swing freely. When the suspended mass in the base of the pendulum forced to move horizontally by applying a force, the pendulum become displaced sideways from its resting, equilibrium position, then the pendulum will be subjected to a restoring force due to gravity that will accelerate it back toward the equilibrium position. The restoring force acting on the pendulum's mass causes it to oscillate about the equilibrium position, swinging back and forth. The time for one complete cycle, a left swing and a right swing, is called the period. The period depends on the length of the pendulum and also to a slight degree on the amplitude, the width of the pendulum's swing.

### 2. Mathematical Modeling of the System

A system consists of two point masses,  $m_1$  and  $m_2$ , connected with a weightless rigid rod of length  $l$  (Figure 1). The motion occurs in a gravity field and is considered to be in a plane, i.e. is considered in the coordinates  $x, y, t$ . The location of point of a mass  $m_1$  (suspension) is not fixed, and can move along the axis  $x$ . The mathematical model of the system will be as follow.

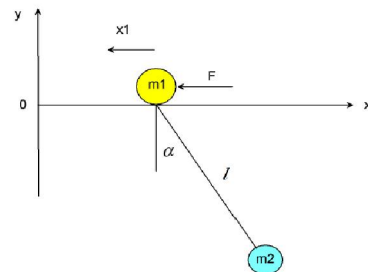


Figure 1 Pendulum with suspended mass

The four functions of time of the system are,  $x_1(t), y_1(t), x_2(t), y_2(t)$ , i.e. the Cartesian coordinates of the first and second points.

The suspension cannot move vertically  
 $y_1=0$

While the second is described by equation

$$(x_1 - x_2)^2 + y_2^2 = l^2$$

We choose the generalized coordinates as  $q_1(t) = x_1(t)$  and  $q_2(t) = \alpha(t)$  where  $\alpha$  is the angle between the vertical is and the axis of rod.

$$x_1 = q_1, x_2 = q_1 + l \sin q_2, y_2 = -l \cos q_2$$

The kinetic energy of system  $T = T_1 + T_2$  in coordinates  $q_1, q_2$ . For the suspension we have

$$T_1 = \frac{m_1 v_1^2}{2} = \frac{m_1 v_{1x}^2}{2} = \frac{m_1 \dot{x}_1^2}{2} \quad (1)$$

For the pendulum we obtain

$$T_2 = \frac{m_2 v_2^2}{2} = \frac{m_2}{2} (v_{2x}^2 + v_{2y}^2) \quad (2)$$

V2x and V2y simply simplified as

$$v_{2x} = \dot{x}_1 + l\dot{\alpha} \cos \alpha$$

$$v_{2y} = l\dot{\alpha} \sin \alpha$$

Rewrite T2 as function of  $\alpha$  and  $x_1$

$$T_2 = \frac{m_2 \dot{x}_1^2}{2} + \frac{m_2}{2} (2l\dot{\alpha}\dot{x}_1 \cos \alpha + l^2 \dot{\alpha}^2) \quad (3)$$

For the force of gravity F2 acting on the pendulum. For its projection we have

$$F_{gx} = 0$$

$$F_{gy} = -m_2 g = -m_2 \frac{\partial \Pi}{\partial y_2}$$

Where

$\Pi(y_2) = m_2 g y_2$  is the potential energy of the pendulum.

In coordinates  $q_1, q_2$   $\Pi(y_2)$  is expressed by the formula

$$V(q_2) = -m_2 l g \cos \alpha \quad (4)$$

In so far as the considered motion is potential, it is necessary to use the Lagrangian equations

$$L = T - V = T_1 + T_2 - V$$

Or

$$L = \frac{m_1 + m_2}{2} \dot{x}_1^2 + \frac{m_2 l}{2} (l\dot{\alpha}^2 + 2\dot{x}_1 \dot{\alpha} \cos \alpha) + m_2 l g \cos \alpha \quad (5)$$

Differentiating L by  $q_1, \dot{q}_1, q_2, \dot{q}_2$ , (recall, that  $q_1 = x_1, q_2 = \alpha$ ) we obtain

$$\frac{\partial L}{\partial q_1} = 0, \quad \frac{\partial L}{\partial \dot{q}_1} = (m_1 + m_2) \dot{x}_1 + m_2 l \dot{\alpha} \cos \alpha$$

$$\frac{\partial L}{\partial q_2} = -m_2 l \sin \alpha (\dot{x}_1 \dot{\alpha} + g), \quad \frac{\partial L}{\partial \dot{q}_2} = m_2 l (l\dot{\alpha} + \dot{x}_1 \cos \alpha)$$

Then

$$\frac{d}{dx} \left( \frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} \right) = F$$

$$\frac{d}{dx} \left( \frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} \right) = 0$$

Substituting the obtained expressions into Lagrangian equations and differentiating them by t,

we come to two equations with respect to  $\dot{x}_1$  and  $\alpha$

$$(m_1 + m_2) \ddot{x}_1 + m_2 l \ddot{\alpha} \cos \alpha - m_2 l \dot{\alpha}^2 \sin \alpha = F \quad (6)$$

$$\ddot{x}_1 \cos \alpha + l \ddot{\alpha} + g \sin \alpha = 0 \quad (7)$$

Linearizing the above equation as

$$\cos \alpha = 1$$

$$\sin \alpha = \alpha$$

$$\dot{\alpha}^2 = 0$$

After linearization, Equation (6) and Equation (7) becomes

$$(m_1 + m_2) \ddot{x}_1 + m_2 l \ddot{\alpha} = F \quad (8)$$

$$\ddot{x}_1 + l \ddot{\alpha} + g \alpha = 0 \quad (9)$$

Let

$$z_1 = x_1, z_2 = \dot{x}_1, z_3 = \alpha, z_4 = \dot{\alpha}, z_5 = \ddot{\alpha}$$

The state space representation of the system becomes

$$\dot{z} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{m_2 l}{(m_1 + m_2)} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \frac{F}{(m_1 + m_2)}$$

$$y = (0 \ 0 \ 1 \ 0 \ 0) z$$

The parameters of the system are shown in Table 1 below.

Table 1 Parameters of the system

No	Parameter	Symbol	Value
1	Suspended mass	$m_1$	0.4 Kg
2	Pendulum mass	$m_2$	0.2 Kg
3	Rod length	$l$	0.4 m
4	Gravitational constant	$g$	10 m/s <sup>2</sup>

The state space representation of the system then becomes

$$\dot{z} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.33 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ 1.67 \\ 0 \\ 0 \\ 0 \end{pmatrix} \frac{F}{(m_1 + m_2)}$$

$$y = (0 \ 0 \ 1 \ 0 \ 0) z$$

### 3. Proposed Controllers Design

#### 3.1 H<sup>∞</sup> Optimal Loop Shaping Control

$H^\infty$  Optimal Loop Shaping Control computes a stabilizing  $H^\infty$  controller  $K$  for plant  $G$  to shape the sigma plot of the loop transfer function  $GK$  to have desired loop shape  $G_d$  with accuracy  $\gamma = \text{GAM}$  in the sense that if  $\omega_0$  is the 0 db crossover frequency of the sigma plot of  $G_d(j\omega)$ , then, roughly,

$$\sigma(G(j\omega)K(j\omega)) \geq \frac{1}{\gamma} \sigma(G_d(j\omega)) \text{ for all } \omega > \omega_0$$

$$\sigma(G(j\omega)K(j\omega)) \leq \gamma \sigma(G_d(j\omega)) \text{ for all } \omega > \omega_0$$

A MIMO stable min-phase shaping pre-filter  $W$ , the shaped plant  $G_s = GW$ , the controller for the shaped plant  $K_s = WK$ , as well as the frequency range  $\{\omega_{\min}, \omega_{\max}\}$  over which the loop shaping is achieved. The block diagram of the pendulum on the free suspension with  $H^\infty$  Optimal Loop Shaping Controller is shown in Figure 2 below.

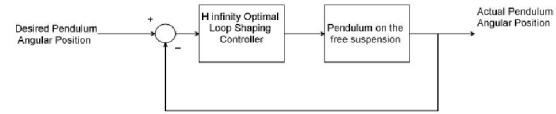


Figure 2 Block diagram of the pendulum on the free suspension with  $H^\infty$  Optimal Loop Shaping Controller

In this paper, the plant has been desired loop shaped with a first order and second order system.

For the first order, the desired loop shaping function is

$$G_{d1} = \frac{1}{s+1}$$

And the  $H^\infty$  Optimal Loop Shaping Controller becomes

$$K_1(s) = \frac{5.369s^5 + 4.409s^4 + 9.094s^3 + 1.764s^2 + 1.399s + 3.415}{s^6 + 1.639s^5 + 1.007s^4 + 2.751s^3 + 2.820s^2 + 2.818s + 6.955}$$

For the second order, the desired loop shaping function is

$$G_{d2} = \frac{1}{s^2 + 2s + 6}$$

And the  $H^\infty$  Optimal Loop Shaping Controller becomes

$$K_2(s) = \frac{7.314s^7 + 3.221s^6 + 2.645s^5 + 5.459s^4 + 1.123s^3 + 2.353s^2 + 6.077s + 1.222}{s^8 + 1.639s^7 + 1.007s^6 + 2.753s^5 + 2.826s^4 + 1.153s^3 + 2.867s^2 + 3.523s + 8.939}$$

#### 4. Result and Discussion

Here in this section, the investigation of the open loop response and the closed loop response with the proposed controller have been done. Finally the comparison of the system with the proposed controllers for a first order and a second order desired loop shaping design have been done.

##### 4.1 Open Loop Response of the Pendulum

The open loop response of the system for a 0.1 Newton suspended mass force simulation is shown in Figure 3 below.

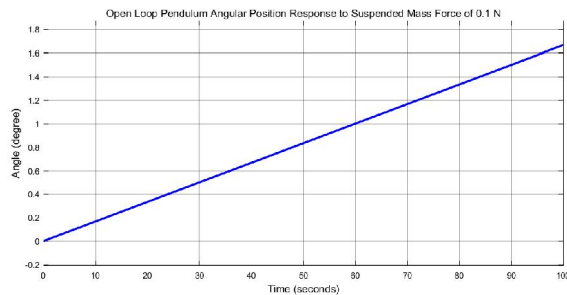


Figure 3. Open loop response

The pendulum angular position angle increases for the 0.1 N input.

##### 4.2 Comparison of the Step Response of Pendulum with Suspended Mass using $H^\infty$ Optimal Loop Shaping Controller with First and Second Order Desired Loop Shaping Function Controllers

The simulation result of the step response of pendulum with suspended mass using  $H^\infty$  optimal loop shaping controller with first and second order desired loop shaping function is shown in Figure 4 below.

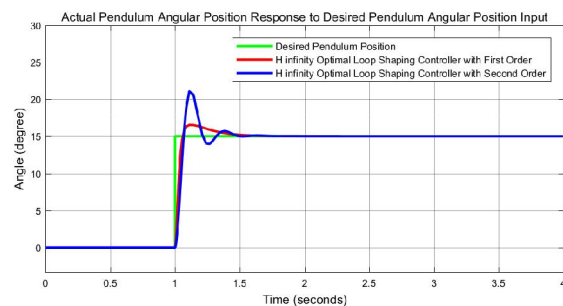


Figure 4. Step response

The data of the rise time, percentage overshoot, settling time and peak value is shown in Table 2.

Table 2. Step response data

No	Performance Data	First Order	Second Order
1	Rise time	1.05 sec	1.12 sec
2	Per. overshoot	13.3 %	40 %
3	Settling time	1.38 sec	1.45 sec
4	Peak value	17 Degree	21 Degree

As Table 2 shows that the pendulum with suspended mass using  $H^\infty$  optimal loop shaping controller with first order desired loop shaping function controller improves the performance of the system by minimizing the rise time, percentage overshoot and settling time.

### 4.3 Comparison of the Sine Wave Response of Pendulum with Suspended Mass using $H^\infty$ Optimal Loop Shaping Controller with First and Second Order Desired Loop Shaping Function Controllers

The simulation result of the sine wave response of pendulum with suspended mass using  $H^\infty$  optimal loop shaping controller with first and second order desired loop shaping function is shown in Figure 5 below.

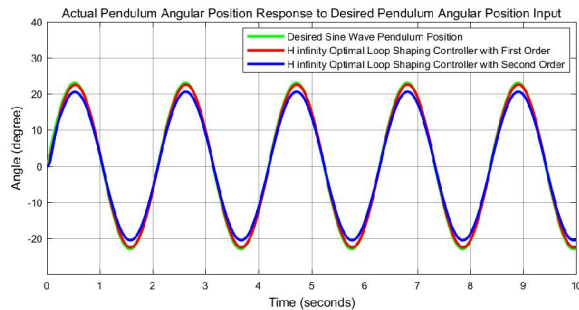


Figure 5. Sine wave response

As Figure 5 shows that the pendulum with suspended mass using  $H^\infty$  optimal loop shaping controller with first order desired loop shaping function controller improves the performance of tracking the set point input to the system.

## 5. Conclusion

In this paper, the design and simulation of a horizontally moving suspended mass pendulum base is done using  $H^\infty$  optimal loop shaping with first and second order desired loop shaping function controllers. Comparison of the proposed system with  $H^\infty$  optimal loop shaping with first and second

order desired loop shaping function controllers have been done to track the desired angular position of the pendulum using step and sine wave input signals. The step input signal response shows that the pendulum with suspended mass using  $H^\infty$  optimal loop shaping controller with first order desired loop shaping function controller improves the performance of the system by minimizing the rise time, percentage overshoot and settling time while the sine wave input signal response shows that the pendulum with suspended mass using  $H^\infty$  optimal loop shaping controller with first order desired loop shaping function controller improves the performance of tracking the set point input to the system. Finally the simulation comparison results prove that the system with  $H^\infty$  optimal loop shaping controller with first order desired loop shaping function controller improved the system performance better.

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