

Quantum logic as motivated by quantum computing

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1 Introduction

Our understanding of Nature comes in layers, so should the development of logic. Classic logic is an indispensable part of our knowledge, and its interactions with computer science have recently dramatically changed our life. A new layer of logic has been developing ever since the discovery of quantum mechanics. G. D. Birkhoff and von Neumann introduced quantum logic in a seminal paper in 1936 [BV]. But the definition of quantum logic varies among authors (see [CG]). How to capture the logic structure inherent in quantum mechanics is very interesting and challenging. Given the close connection between classical logic and theoretical computer science as exemplified by the coincidence of computable functions through Turing machines, recursive function theory, and λ -calculus, we are interested in how to gain some insights about quantum logic from quantum computing. In this note we make some observations about quantum logic as motivated by quantum computing (see [NC]) and hope more people will explore this connection.

The quantum logic as envisioned by Birkhoff and von Neumann is based on the lattice of closed subspaces of a Hilbert space, usually an infinite dimensional one. The quantum logic of a fixed Hilbert space \mathbb{H} in this note is the variety of all the true equations with finitely many variables using the connectives meet, join and negation. Quantum computing is theoretically based on quantum systems with finite dimensional Hilbert spaces, especially the states space of a qubit \mathbb{C}^2 . (Actually the qubit is merely a convenience. If \mathbb{C}^2 is replaced by any other $\mathbb{C}^n, n \geq 3$, then the same quantum computing theory will be obtained.) The n -qubit states space is \mathbb{C}^{2^n} . It is interesting to understand where the power of quantum computers could come from. One possible source is the exponential

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growth of the dimensions of the n -qubit states space. Another possibility is the entanglement of quantum states in \mathbb{C}^{2^n} when $n \geq 2$. Therefore, we ask for a logical framework to capture the above two sides of quantum computing. Those questions are only concerned with the static part of quantum computing. To include the quantum gates into a logical framework, we will need a temporal logic. Hence as a first approximation we ask whether or not the quantum logics of \mathbb{C}^n for all n 's are the same, and what are their relations with the quantum logic of infinite dimensional Hilbert spaces. The quantum logic of \mathbb{C} reduces to exactly the classical Boolean logic. Since the distribution law does not hold in the quantum logic of \mathbb{C}^2 , therefore, quantum logic of \mathbb{C}^2 is different from that of \mathbb{C} , hence different from classical propositional logic. In this note we show that the quantum logic of \mathbb{C}^{2^n} is always different from that of $\mathbb{C}^{2^{n+1}}$ for any $n \geq 0$. We also observe that quantum logic is not a finite-valued logic, and $\mathbf{QL}(\mathbb{C}^n)$ is decidable for any n . In the end, we discuss some open problems.

2 $\mathbf{QL}(\mathbb{C}^{2^n}) \neq \mathbf{QL}(\mathbb{C}^{2^{n+1}})$

Given a Hilbert space \mathbb{H} , let $L_c(\mathbb{H})$ be the lattice of all closed subspaces of \mathbb{H} with set inclusion as the partial order relation \leq , and for any two subspaces $p, q \in L_c(\mathbb{H})$, the meet $p \wedge q$ is the set intersection, and the join $p \vee q$ is the closure of the span of $p \cup q$. The closure is necessary in the definition of join when \mathbb{H} is infinitely dimensional. In this case, the span of two subspaces is not necessarily closed. For any closed subspace p , its negation \bar{p} is the orthogonal complement. With the above defined connectives $\wedge, \vee, \bar{}$ on $L_c(\mathbb{H})$, $L_c(\mathbb{H})$ becomes an orthomodular lattice. The maximum and minimum of $L_c(\mathbb{H})$ are \mathbb{H} and $\{0\}$, respectively. We will denote them by $\mathbf{1}$ and $\mathbf{0}$. Recall that an orthomodular lattice is not necessarily distributive. It is an ortholattice satisfying the following orthomodular law:

$$p \wedge [\bar{p} \vee (p \wedge q)] = p \wedge q. \quad (1)$$

The orthomodular law would follow from the distribution law if it holds, so the orthomodular law is a weakening of the distribution law. To see the failure of the distribution law $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ when $\dim(\mathbb{H}) \geq 2$, choose q and r to be two distinct lines, and p to be a different line in the plane spanned by q, r . Then $(p \vee q) \wedge (p \vee r)$ is the plane spanned by q, r , while $p \vee (q \wedge r)$ is just p . Note also that when $\dim(\mathbb{H}) = 1$, then $L_c(\mathbb{H})$ is just $\{\mathbf{0}, \mathbf{1}\}$.

A term $T(p, q, \dots, r)$ is any formula with finitely many variables p, q, \dots, r using connectives $\wedge, \vee, \bar{}$. An equation $T_1(p, q, \dots, r) = T_2(p, q, \dots, r)$ in $L_c(\mathbb{H})$ is true if the values of the two terms T_1 and T_2 always agree when the variables p, q, \dots, r range over all $L_c(\mathbb{H})$.

Definition Given a Hilbert space \mathbb{H} , the quantum logic $\mathbf{QL}(\mathbb{H})$ is the variety of all true equations in $L_c(\mathbb{H})$.

In the remaining part of this section, \mathbb{H} will be always finitely dimensional. So it will be \mathbb{C}^n for some n , hence every subspace is closed.

Theorem 1 1) $\mathbf{QL}(\mathbb{C}^2) \neq \mathbf{QL}(\mathbb{C}^4)$.

2) For any $i \geq 0$, there exists an equation $\beta \in \mathbf{QL}(\mathbb{C}^{2^k})$ for any $k \leq i$, and β is not in any $\mathbf{QL}(\mathbb{C}^{2^l})$ if $l > i$.

We start with some lemmas.

Lemma 2 1) The negation $\bar{\cdot}$ is classical, i.e.

a) $\bar{\bar{p}} = p$.

b) $\overline{p \wedge q} = \bar{p} \vee \bar{q}$.

c) $\overline{p \vee q} = \bar{p} \wedge \bar{q}$.

d) $p \wedge \bar{p} = \mathbf{0}$.

2) Two subspaces $p = q$ if and only if $(p \vee q) \wedge (\bar{p} \vee \bar{q}) = \mathbf{0}$, or equivalently if and only if $(\bar{p} \wedge \bar{q}) \vee (p \wedge q) = \mathbf{1}$.

Proof: The proof of 1) is left to the readers.

To prove 2), denote by $d(s)$ the dimension of a subspace s . If $p = q$, it is a straightforward check. For the other direction, suppose $(p \vee q) \wedge (\bar{p} \vee \bar{q}) = \mathbf{0}$. Note that $\mathbf{1} = (p \vee q) \vee (\bar{p} \vee \bar{q}) \subseteq (p \vee q) \vee (\overline{p \wedge q}) = (p \vee q) \vee (\bar{p} \vee \bar{q})$. It follows that $\mathbf{1} = (p \vee q) \vee (\bar{p} \vee \bar{q})$. Computing the dimension of $(p \vee q) \vee (\bar{p} \vee \bar{q})$ using the formula $d(s \vee t) = d(s) + d(t) - d(s \wedge t)$, we obtain

$$n = d(p \vee q) + d(\bar{p} \vee \bar{q}) - 0 = d(p) + d(q) - d(p \wedge q) + n - d(p \wedge q).$$

Hence $d(p) + d(q) = 2d(p \wedge q)$. It follows that $d(p \wedge q) = d(p) = d(q)$. Since $p \wedge q$ is a subspace of p , and q , we have $p \wedge q = p = q$.

To prove Theorem 1, we will employ the failure of the distribution law using Lemma 2 (2). Given three variables p, q, r , we define the distributivity test formula $\alpha(p, q, r)$ as follows: let $a = p \vee (q \wedge r)$ and $b = (p \vee q) \wedge (p \vee r)$, and then define

$$\alpha(p, q, r) = (a \vee b) \wedge (\bar{a} \vee \bar{b}).$$

Note that $a \leq b$, it follows that $\alpha(p, q, r) = b \wedge \bar{a} = [(p \vee q) \wedge (p \vee r)] \wedge [\bar{p} \wedge (\bar{q} \vee \bar{r})] \subseteq \bar{p}$. The distribution law holds if and only if α is always $\mathbf{0}$. Therefore, if α does not vanish for some choice of p, q, r in a Hilbert space, then the distribution law fails for the quantum logic of this Hilbert space.

Lemma 3 Given any three subspaces p, q, r of \mathbb{C}^n , we have $\dim(\alpha(p, q, r)) \leq \frac{n}{2}$.

Proof: We know that $\alpha(p, q, r)$ is a subset of \bar{p} . So if $\dim(p) > \frac{n}{2}$, the lemma holds. Now suppose $\dim(p) \leq \frac{n}{2}$. Since $a \subseteq b$, so a is perpendicular to \bar{b} . Hence $\bar{a} = a \vee \bar{b} = a \oplus \bar{b}$. Writing out \bar{a} , we have $\bar{a} = [\bar{p} \wedge \bar{q}] \vee (\bar{p} \wedge \bar{r}) \oplus [p \vee (q \wedge r)]$. Let $d(s)$ denote the dimension of any subspace s . A direct computation of the dimension of $\alpha(p, q, r)$ results in the following formulas:

$$\begin{aligned}
d(\bar{a}) &= d(p) + d(q \wedge r) - d(p \wedge q \wedge r) + d(\bar{p} \wedge \bar{q}) + d(\bar{p} \wedge \bar{r}) - d(\bar{p} \wedge \bar{q} \wedge \bar{r}) \\
&= d(p) + d(q \wedge r) - d(p \wedge q \wedge r) + d(p \vee q \vee r) + n - d(p \vee q) - d(p \vee r) \\
&= d(q \wedge r) - d(p \wedge q \wedge r) + d(p \vee q \vee r) + n - d(p) - d(q) + d(p \wedge q) - d(r) + d(p \wedge r) \\
&= n - d(p) + d(p \vee q \vee r) - d(q) - d(r) + d(q \wedge r) + d(p \wedge q) + d(p \wedge r) - d(p \wedge q \wedge r) \\
&= [n - d(p)] + [d(p \vee q \vee r) - d(q \vee r)] + [d(p \wedge q) + d(p \wedge r) - d(p \wedge q \wedge r)].
\end{aligned}$$

The second and third brackets are non-negative numbers. Since $d(p) \leq \frac{n}{2}$, so $n - d(p) \geq \frac{n}{2}$. Hence $d(\bar{a}) \geq \frac{n}{2}$, and the lemma follows.

Lemma 4 For \mathbb{C}^2 , $\alpha(p, q, r)$ is not $\mathbf{0}$ if and only if p, q, r are three distinct lines. Furthermore, $\alpha(p, q, r)$ is either $\mathbf{0}$ or \bar{p} .

Proof: Since q and r are symmetric, so we need to check only the following cases: $p = \mathbf{0}, \mathbf{1}, q = \mathbf{0}, \mathbf{1}, p = q, q = r$, and p, q, r are three distinct lines. Verifications are left as exercises. When $\alpha(p, q, r)$ is not $\mathbf{0}$, then it is a one-dimensional subspace of \bar{p} . Since \bar{p} is also one-dimensional, so $\alpha(p, q, r)$ equals \bar{p} in this case.

Lemma 5 Given two Hilbert spaces $V \subset W$, then $\mathbf{QL}(V) \supseteq \mathbf{QL}(W)$.

Proof: Suppose an equation $T_1 = T_2$ of k variables does not hold for subspaces p_1, \dots, p_k of V . Then this equation will not hold for the subspaces $p_i \oplus \bar{V}, i = 1, \dots, k$ of W , where \bar{V} is the orthogonal complement of V in W . This completes the proof.

Proof of Theorem 1:

(1): By Lemma 5, it suffices to find a true equation in $\mathbf{QL}(\mathbb{C}^2)$, but not in $\mathbf{QL}(\mathbb{C}^4)$. Let $\beta(p, q, r, s) = \alpha(\alpha(p, q, r), \overline{\alpha(p, q, r)} \wedge \bar{p}, s)$. We claim β is always $\mathbf{0}$ in \mathbb{C}^2 , but fails for a certain choice of p, q, r, s in \mathbb{C}^4 . First we verify that $\beta = \mathbf{0}$ in \mathbb{C}^2 . By Lemma 4, $\alpha(p, q, r)$ is either $\mathbf{0}$ or \bar{p} . So is $\overline{\alpha(p, q, r)} \wedge \bar{p}$. It follows from Lemma 4 that $\beta = \mathbf{0}$ since either at least one of $\alpha(p, q, r), \overline{\alpha(p, q, r)} \wedge \bar{p}$ is $\mathbf{0}$ or they are both \bar{p} . To show β is not always $\mathbf{0}$ in \mathbb{C}^4 , let $\{e_i, i = 1, 2, 3, 4\}$ be an orthonormal basis of \mathbb{C}^4 and $p = \text{span of } \{e_1, e_2\}$, $q = \bar{p}$, $r = \text{span of } \{e_1, e_2 + e_3\}$, and $s = \text{span of } \{e_1, e_3 + e_4\}$. Direct computation shows $\beta(p, q, r, s) = \text{span of } \{e_4\}$, which falsifies $\beta = \mathbf{0}$ in \mathbb{C}^4 .

(2) This general argument will give a different proof for part (1). First we explain the restriction of a formula to a new variable. Suppose T is a formula for variables in $L_c(\mathbb{C}^n)$,

then the restriction of T to a new term α , denoted by $T|_\alpha$, is the following formula: first using the De Morgan law, we assume that all negations $\bar{}$ are applied to single variables, next we replace each variable p and \bar{p} by $p \wedge \alpha$ and $\bar{p} \wedge \alpha$, respectively. Inductively, we define

$$\alpha^m(p_m, q_m, r_m) = \alpha|_{\alpha^{m-1}}(p_m, q_m, r_m),$$

and $\alpha^1(p_1, q_1, r_1) = \alpha(p_1, q_1, r_1)$, $\alpha^{m-1} = \alpha^{m-1}(p_{m-1}, q_{m-1}, r_{m-1})$. Lemma 3 implies that

$$\dim(\alpha^m(p_m, q_m, r_m)) \leq \frac{\dim(\alpha^{m-1}(p_{m-1}, q_{m-1}, r_{m-1}))}{2} \leq \dots \leq \frac{n}{2^m}$$

if $\dim(\mathbb{H}) = n$. In $\mathbf{QL}(\mathbb{C}^{2^i})$, $\dim(\alpha^{i+1}) \leq \frac{2^i}{2^{i+1}} < 1$, so $\alpha^{i+1} = \mathbf{0}$ which gives a true equation in \mathbb{C}^{2^i} . By Lemma 5, this equation is also true for any $k \leq i$. To show it is not true for $\mathbb{C}^{2^{k+1}}$, we notice that if p, q, r are different subspaces of dimension $\frac{n}{2}$ and each pair has trivial intersection in \mathbb{C}^n , then $\dim \alpha(p, q, r) = \frac{n}{2}$ if n is even. By choosing subspaces in $\mathbb{C}^{2^{k+1}}$ this way, we have $\dim(\alpha^{i+1}) = \frac{2^{i+1}}{2^{i+1}} = 1$. Now (2) follows from Lemma 5.

3 Decidability of $\mathbf{QL}(\mathbb{C}^n)$

The modest observation here is that the first-order theory of each lattice $L(\mathbb{C}^n)$ may be reduced to the first-order theory of the reals. Hence, by Tarski's Theorem [T], we have the decidability of the first-order theory of each $L(\mathbb{C}^n)$. Moreover, we have a stronger result: the decidability is uniform in n .

Theorem 6 *There is an effective procedure which, given a natural number n and a sentence φ in the first-order language of complemented lattices, gives a sentence φ^* in the first-order theory of \mathbb{R} such that the following are equivalent:*

1. $L(\mathbb{C}^n) \models \varphi$.
2. $\mathbb{R} \models \varphi^*$.

Corollary 7 *The first-order theories of the lattices $L(\mathbb{C}^n)$ are decidable (uniformly).*

We sketch a proof of this result. The general result would be messy to write out in full, and so we content ourselves with a significantly complicated example and some general remarks.

Suppose the we want to know whether or not

$$L(\mathbb{C}^n) \models (\forall x, y, z) \overline{(x \wedge y)} \vee z = y \wedge (\bar{z} \vee x). \quad (2)$$

The first thing to do is to add new variables a, b, c, d, e, f for the subterms on both sides, and then write the following:

$$L(\mathbb{C}^n) \models (\forall x, y, z) (\forall a, b, c, d, e, f) \\ [(a = x \wedge y) \wedge (b = \bar{a}) \wedge (c = b \vee z) \\ \wedge (d = \bar{z}) \wedge (e = d \vee x) \wedge (f = y \wedge e)] \rightarrow (c = f) \quad (3)$$

The point of this is that the atomic formulas in (3) are very simple. In essence, we have taken the complex terms of (2) and “flattened” them out. This move has nothing to do with the subject at hand, and it is available (and useful) in other contexts. In any case, the reader should observe that (2) and (3) are equivalent.

At this point, we recall that the subspaces of \mathbb{C}^n are the kernels of $n \times n$ complex matrices. Our main move in this note is to replace a quantifier $(\forall x)$ over subspaces of \mathbb{C}^n with quantification over n^2 variables $(\forall x_{11}, \dots, x_{nn})$ over \mathbb{C} .

In what follows, we shall use the notation \hat{x} for the (n^2) -tuple of variables x_{11}, \dots, x_{nn} . Further, we shall use the notation \vec{v} to denote an n -tuple v_1, \dots, v_n of variables. We write $\hat{a}\vec{v} = \vec{0}$, for example, to mean the following:

$$(a_{11}v_1 + \dots + a_{1n}v_n = 0) \wedge \dots \wedge (a_{n1}v_1 + \dots + a_{nn}v_n = 0)$$

Now we can render (3) as an assertion about \mathbb{C} alone.

$$\mathbb{C} \models (\forall \hat{x}, \hat{y}, \hat{z}, \hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{e}, \hat{f}) \\ [(\forall \vec{v})(\hat{a}\vec{v} = \vec{0} \leftrightarrow (\hat{x}\vec{v} = \vec{0} \wedge \hat{y}\vec{v} = \vec{0})) \\ \wedge (\forall \vec{v})(\hat{b}\vec{v} = \vec{0} \leftrightarrow (\forall \vec{w})(\hat{a}\vec{w} = \vec{0} \rightarrow \vec{v} \cdot \vec{w} = 0)) \\ \wedge (\forall \vec{v})(\hat{c}\vec{v} = \vec{0} \leftrightarrow (\exists \vec{w}^1, \dots, \vec{w}^n, r_1, \dots, r_n) \\ \bigwedge_i (\hat{b}\vec{w}^i = \vec{0} \vee \hat{z}\vec{w}^i = \vec{0}) \wedge \bigwedge_{j=1}^n (v_j = \sum_i r_i w_j^i)) \\ \wedge (\forall \vec{v})(\hat{d}\vec{v} = \vec{0} \leftrightarrow (\forall \vec{w})(\hat{z}\vec{w} = \vec{0} \rightarrow \vec{v} \cdot \vec{w} = 0)) \\ \wedge (\forall \vec{v})(\hat{e}\vec{v} = \vec{0} \leftrightarrow (\exists \vec{w}^1, \dots, \vec{w}^n, r_1, \dots, r_n) \\ \bigwedge_i (\hat{d}\vec{w}^i = \vec{0} \vee \hat{x}\vec{w}^i = \vec{0}) \wedge \bigwedge_{j=1}^n (v_j = \sum_i r_i w_j^i)) \\ \wedge (\forall \vec{v})(\hat{f}\vec{v} = \vec{0} \leftrightarrow (\hat{y}\vec{v} = \vec{0} \wedge \hat{e}\vec{v} = \vec{0}))] \\ \rightarrow (\forall \vec{v})(\hat{c}\vec{v} = \vec{0} \leftrightarrow \hat{f}\vec{v} = \vec{0}) \quad (4)$$

As one can see, the clauses of (3) have been replaced by the more complicated clauses of (4). We explain how this works in the hardest case, the one for the lattice join. One of the clauses of (3) is $c = b \vee z$. Recall that this has to do with subspaces of \mathbb{C}^n , and we want to change this to quantification over $(n \times n)$ -tuples over \mathbb{C} . One should think of \hat{c} as the matrix $C = (c_{ij})$. To say that $c = b \vee z$ is the same as saying that every \vec{v} such that $C\vec{v} = \vec{0}$ is in the span of

$$\{\vec{w} : B\vec{v} = \vec{0} \text{ or } Z\vec{v} = \vec{0}\}. \quad (5)$$

However, the fact that we are working in \mathbb{C}^n implies that if S is any set of vectors containing the zero vector, then the span of S of vectors is the same thing as the set of

linear combinations of sets of exactly n vectors from S . Getting back to our previous point, to say that $c = b \vee z$ is the same as saying that every \vec{v} such that $C\vec{v} = \vec{0}$ is a linear combination of n vectors from the set in (5). And this is

$$(\forall \vec{v})(\hat{c}\vec{v} = \vec{0} \leftrightarrow (\exists \vec{w}^1, \dots, \vec{w}^n, r_1, \dots, r_n) \bigwedge_{i=1}^n (\hat{b}\vec{w}^i = \vec{0} \vee \hat{z}\vec{w}^i = \vec{0}) \wedge \bigwedge_{j=1}^n (v_j = \sum_i r_i w_j^i)).$$

This is exactly the clause which we put into (4) on behalf of $c = b \vee z$ in (3).

The other steps in going from (3) to (4) are for the lattice meet and for orthogonal complements, and they are easier.

Now (4) is a sentence about \mathbb{C} . But first-order sentences about the arithmetic properties of \mathbb{C} are reducible to first-order sentence about \mathbb{R} . Taking all of these observations together, this gives a method of going from the sentence φ in (2) to a sentence φ^* in the theory of \mathbb{R} ; φ and φ^* have the property stated in Theorem 6.

Incidentally, our sentence in (2) is a universal sentence. We chose this because the sentences of greatest interest about the lattice $L(\mathbb{C}^n)$ are those universal sentences. But our method also would work if φ in (2) had existential quantifiers, or negation. The details are quite similar to what we have already done, and so we omit them here.

4 Open problems

The state space of n -qubits is the n -th tensor power $(\mathbb{C}^2)^{\otimes n}$ of \mathbb{C}^2 . Quantum computing suggests the relevance of $\mathbf{QL}((\mathbb{C}^2)^{\otimes n})$ for all n . We have the following inclusions of quantum logics:

$$\mathbf{QL}(\mathbb{C}) \supset \mathbf{QL}(\mathbb{C}^2) \supset \mathbf{QL}(\mathbb{C}^4) \supset \dots \supset \mathbf{QL}(\mathbb{C}^{2^n}) \supset \mathbf{QL}(\mathbb{C}^{2^{n+1}}) \supset \dots \supset \mathbf{QL}(\mathbb{C}^\infty).$$

We know that the quantum logics of n -qubit spaces are pair-wise distinct. It is also known that the intersection of all $\mathbf{QL}(\mathbb{C}^{2^n})$ is not $\mathbf{QL}(\mathbb{C}^\infty)$ as it contains the following true equation, which is one way to define the modular law (see [G]):

$$(p \wedge r) \vee (q \wedge r) = ((p \wedge r) \vee q) \wedge r. \quad (6)$$

It is known that the modular law holds in $\mathbf{QL}(\mathbb{H})$ iff \mathbb{H} is finitely dimensional (see [R]), but the orthomodular law (1) does hold in any Hilbert space. So we have

$$\bigcap_{i=0}^{\infty} \mathbf{QL}(\mathbb{C}^{2^i}) \supset \mathbf{QL}(\mathbb{C}^\infty).$$

Some open questions:

1. Is $\mathbf{QL}(\mathbb{C}^\infty)$ decidable?
2. How to characterize the difference between $\bigcap_{i=0}^{\infty} \mathbf{QL}(\mathbb{C}^{2^i})$ and $\mathbf{QL}(\mathbb{C}^\infty)$?

3. Are $\mathbf{QL}(\mathbb{C}^n)$ and $\mathbf{QL}(\mathbb{C}^m)$ always different for $n \neq m$?

It is interesting to characterize the quantum logics of finite dimensional Hilbert spaces. Modular lattices are a first approximation. But there are significant missing ingredients in the modular lattice formulation of $\mathbf{QL}(\mathbb{C}^n)$. It has been shown in [H] that the word problem for modular lattices is not decidable. On the other hand, $\mathbf{QL}(\mathbb{C}^n)$ is always decidable. Another interesting point is the following observation. A finite set of closed subspaces in \mathbb{C}^n is called *a universal test set* if the truth of any equation is determined by the evaluations of the subspaces in this set. It turns out there are no finite universal test sets for $\mathbf{QL}(\mathbb{C}^m)$, $m \geq 2$. To see this, consider the distributivity testing formula $\alpha(p, q, r)$. For simplicity, we will only give the details for $m = 2$. In order for the distributivity testing formula $\alpha(p, q, r)$ to fail, p, q, r must be three distinct lines. In order for $\alpha(\alpha(\alpha(\alpha(p, q, r), p, s), q, s), r, s)$ to fail, p, q, r, s must be distinct lines. Continuing in this manner, we can build a complicated formula γ , the failure of which means that the k subspaces p, q, \dots are distinct lines. Since k is arbitrary, no finite set of lines will falsify every invalid formula. This argument works for any \mathbb{C}^m , $m \geq 2$.

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