INTERNAL APPROACHABILITY AND REFLECTION

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ABSTRACT. We prove that the Weak Reflection Principle does not imply that every stationary set reflects to an internally approachable set. We show that several variants of internal approachability, namely internally unbounded, internally stationary, and internally club, are not provably equivalent.

Let $\lambda \geq \omega_2$ be a regular cardinal, and consider an elementary substructure $N \prec H(\lambda)$ with size $\aleph_1$. Then N is *internally approachable* if N is the union of an increasing and continuous chain $\langle a_i : i < \omega_1 \rangle$ of countable sets such that for all $\beta < \omega_1$, $\langle a_i : i < \beta \rangle$ is in N.

The *Weak Reflection Principle* or WRP is the statement that for all regular $\lambda \geq \omega_2$, for every stationary set $S \subseteq P_{\omega_1}(H(\lambda))$ there is a set $N \subseteq H(\lambda)$ with size $\aleph_1$ which contains $\aleph_1$ such that $S \cap P_{\omega_1}(N)$ is stationary in $P_{\omega_1}(N)$ (that is, S *reflects to* N). This principle follows from Martin's Maximum and captures some of its strength; for example, WRP implies Chang's Conjecture, the presaturation of the non-stationary ideal on ω_1, and the Singular Cardinal Hypothesis (see [3] and [6]).

An apparent strengthening of WRP is the statement that for all regular $\lambda \geq \omega_2$, every stationary subset of $P_{\omega_1}(H(\lambda))$ reflects to an internally approachable set with size $\aleph_1$. In practice it tends to be easier to draw strong consequences from this principle than from WRP. Foreman and Todorčević [4] asked whether the two reflection principles are equivalent. We answer this question in the negative.

Theorem 0.1. *Suppose κ is supercompact. Then there is a forcing poset which forces $\kappa = \omega_2$, WRP holds, and for all regular $\lambda \geq \omega_2$ there is a stationary subset of $P_{\omega_1}(H(\lambda))$ which does not reflect to any internally approachable set with size $\aleph_1$.*

Foreman and Todorčević [4] described several variations of the notion of internal approachability, and asked whether these properties are equivalent. Let $N \prec H(\lambda)$ be a set with size $\aleph_1$. Then N is *internally club* if $N \cap P_{\omega_1}(N)$ contains a club subset of $P_{\omega_1}(N)$. N is *internally stationary* if $N \cap P_{\omega_1}(N)$ is stationary in $P_{\omega_1}(N)$. N is *internally unbounded* if $N \cap P_{\omega_1}(N)$ is cofinal in $P_{\omega_1}(N)$. We prove that these properties are not equivalent.

Theorem 0.2. *MM implies that for all regular $\lambda \geq \omega_2$, there is a stationary set of N in $P_{\omega_2}(H(\lambda))$ which are internally unbounded but not internally stationary.*

Theorem 0.3. *PFA_2 implies that for all regular $\lambda \geq \omega_2$, there is a stationary set of N in $P_{\omega_2}(H(\lambda))$ which are internally stationary but not internally club.*

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1. Preliminaries

In this section we review the background material necessary for understanding the paper. Some basic familiarity with iterated forcing, proper forcing, and supercompact cardinals is required.

If κ is a regular uncountable cardinal and X is a set containing κ, $P_\kappa(X)$ denotes the set $\{a \subseteq X : |a| < \kappa\}$. A set $C \subseteq P_\kappa(X)$ is *club* if C is closed under unions of increasing sequences of length less than κ and C is cofinal in $P_\kappa(X)$. A set $S \subseteq P_\kappa(X)$ is *stationary* if it has non-empty intersection with each club. If C is a club then there is a function $F : X^{<\omega} \to P_\kappa(X)$ such that for all a in $P_\kappa(X)$, if $F(x_0, \ldots, x_n) \subseteq a$ for any $x_0, \ldots, x_n$ in a, then a is in C. In the case $\kappa = \omega_1$, if $C \subseteq P_{\omega_1}(X)$ is club then there is $H : X^{<\omega} \to X$ such that every a in $P_\kappa(X)$ which is closed under H is in C.

For a regular cardinal λ, $H(\lambda)$ denotes the collection of sets whose transitive closure has size less than λ. The set $H(\lambda)$ is determined in an absolute way by the bounded subsets of λ. So if W is an outer model of V with the same bounded subsets of λ, then $H(\lambda)^V = H(\lambda)^W$.

If $N \prec H(\lambda)$ for some regular $\lambda \geq \omega_2$, $\kappa + 1 \subseteq N$, $x \in N$, and $|x| \leq \kappa$, then $x \subseteq N$. For by elementarity there is a surjection $f : \kappa \to x$ in N, hence $x = f``\kappa \subseteq N$.

A sequence of sets $\langle N_i : i < \gamma \rangle$ is an *internally approachable sequence* if it is increasing and continuous and $\langle N_j : j \leq i \rangle \in N_{i+1}$ for $i < \gamma$.

Suppose $\mathbb{P}$ is a forcing poset and μ is a cardinal. We say that $\mathbb{P}$ is $< \mu$*-closed* if any descending sequence of conditions in $\mathbb{P}$ with length less than μ has a lower bound. We say $\mathbb{P}$ is $< \mu$*-distributive* if any family of fewer than μ many dense open subsets of $\mathbb{P}$ has dense open intersection. This is equivalent to the statement that forcing with $\mathbb{P}$ does not add any new sets of ordinals with order type less than μ.

If α is an ordinal, we say that $\mathbb{P}$ is α*-strategically closed* if Player II has a winning strategy in the following game. Player I begins the game with a condition p_0. After Player I has played a condition p_i where $i < \alpha$, Player II responds with $q_i \leq p_i$; then Player I responds with $p_{i+1} \leq q_i$. At a limit stage $\delta < \alpha$, Player II attempts to play a condition q_δ which is a lower bound of $\{q_i : i < \delta\}$. Player II wins iff he is able to play at all stages less than α.

For a cardinal μ, $< \mu$-closed implies μ-strategically closed, and μ-strategically closed implies $< \mu$-distributive.

A forcing poset $\mathbb{P}$ is *proper* if for any set X which contains ω_1, if S is a stationary subset of $P_{\omega_1}(X)$ then $\mathbb{P}$ forces that S is stationary in $P_{\omega_1}(X)$. A forcing poset $\mathbb{P}$ satisfies the *countable covering property* if $\mathbb{P}$ forces that whenever $x \subseteq V$ is countable, there is y in V which is countable in V such that $x \subseteq y$. Note that any proper forcing poset satisfies the countable covering property, since if $\mathbb{P}$ adds a countable set which cannot be covered by a countable set in the ground model, then for some X in V, $P_{\omega_1}^V(X)$ is not stationary in $P_{\omega_1}(X)$.

Lemma 1.1. *Let $\kappa \geq \omega_1$ be a regular cardinal and $\mathbb{P}$ a proper forcing poset. Then for any stationary set $T \subseteq \kappa \cap \mathrm{cof}(\omega)$, $\mathbb{P}$ forces T is stationary.*

Proof. Let $S = \{a \in P_{\omega_1}(\kappa) : \sup(a) \in T\}$. Since T is stationary in κ, S is stationary in $P_{\omega_1}(\kappa)$. For if $H : \kappa^{<\omega} \to \kappa$ is a function then there is a club set of $\alpha < \kappa$ closed under H. Choose such an α in T, fix $x \subseteq \alpha$ cofinal with order type ω, and let $a = cl_H(x)$. Then a is in S and closed under H. If G is generic for $\mathbb{P}$, S

remains stationary in $P_{\omega_1}(\kappa)$ in $V[G]$. Let $C \subseteq \kappa$ be club in $V[G]$. Then there is a in S closed under the map $\beta \mapsto \min(C \setminus \beta)$. But then $\sup(a) \in C \cap T$. $\square$

Any ω_1-c.c. or countably closed forcing poset is proper. A countable support iteration of proper forcing posets is proper.

If X is an uncountable set, $\text{Coll}(\omega_1, X)$ is a forcing poset for collapsing the size of X to be $\aleph_1$. A condition p is a countable partial function $p : \omega_1 \to X$, and the ordering is by extension of functions. This poset is countably closed hence proper. We write $\text{Add}(\omega)$ for the forcing to add a Cohen real. A condition is a finite partial function $p : \omega \to 2$, and the ordering is by extension of functions. This poset has size $\aleph_0$, hence is ω_1-c.c. and so proper.

The *Proper Forcing Axiom* or PFA is the statement that for any proper forcing poset $\mathbb{P}$ and any family $\mathcal{D}$ of $\aleph_1$ many dense subsets of $\mathbb{P}$, there is a filter G on $\mathbb{P}$ which intersects each dense set in $\mathcal{D}$. *Martin's Maximum* or MM is the same statement with "proper forcing poset" replaced by "forcing poset which preserves stationary subsets of ω_1".

We review several basic facts used to extend elementary embeddings after forcing. Assume $M \subseteq N$ are inner models, λ is a regular cardinal in N, and $M^{<\lambda} \cap N \subseteq M$.

(1) If $\mathbb{P}$ is a forcing poset in M which is λ-c.c. in N and G is generic for $\mathbb{P}$ over N, then $M[G]^{<\lambda} \cap N[G] \subseteq M[G]$.

(2) If $\mathbb{P}$ is a forcing poset in M and $G \in N$ is generic for $\mathbb{P}$ over M, then $M[G]^{<\lambda} \cap N \subseteq M[G]$.

Suppose $j : M \to N$ is an elementary embedding between inner models. Assume $\mathbb{P}$ is a forcing poset, G is generic for $\mathbb{P}$ over M, and H is generic for $j(\mathbb{P})$ over N. Then $j``G \subseteq H$ iff there is an extension of j to an elementary embedding $j : M[G] \to N[H]$ such that $j(G) = H$. The extended map satisfies $j(\dot{x}^G) = j(\dot{x})^H$.

2. Forcing Axioms

Suppose Γ is a class of forcing posets and $\alpha \leq \omega_1$ is a cardinal. Then $\mathsf{MA}_\alpha(\Gamma)$ is the following forcing axiom. Suppose $\mathbb{P}$ is in Γ, $\mathcal{D}$ is a collection of $\aleph_1$ many dense subsets of $\mathbb{P}$, and $\{\dot{A}_i : i < \alpha\}$ is a collection of $\mathbb{P}$-names for stationary subsets of ω_1. Then there is a filter G on $\mathbb{P}$ which intersects each dense set in $\mathcal{D}$, and moreover for each $i < \alpha$, the interpretation $\dot{A}_i^G = \{\beta < \omega_1 : \exists p \in G \ \ p \Vdash \beta \in \dot{A}_i\}$ is stationary in ω_1.

The forcing axiom $\mathsf{MA}_1(\Gamma)$ is usually referred to as $\mathsf{MA}^+(\Gamma)$ in the literature. In general $\mathsf{MA}_\alpha(\Gamma)$ does not imply $\mathsf{MA}_{\alpha^+}(\Gamma)$. For more information see [7].

If G is a filter on a poset $\mathbb{P}$ and N is a set, then G is N*-generic* if for any dense set $D \subseteq \mathbb{P}$ in N, $G \cap D \cap N$ is non-empty.

We will use the following result of Woodin [8].

Proposition 2.1. *Assume $\mathsf{MA}_\alpha(\Gamma)$ holds for a class Γ of separative forcing posets. Let $\mathbb{P}$ be in Γ and suppose $\{\dot{A}_i : i < \alpha\}$ is a family of $\mathbb{P}$-names for stationary subsets of ω_1. Let $\theta \geq \omega_2$ be regular such that $\mathbb{P}$ is in $H(\theta)$.*

Then there is a stationary set of N in $P_{\omega_2}(H(\theta))$ for which there exists an N-generic filter G on $\mathbb{P}$ such that each $\dot{A}_i^G$ is stationary in ω_1.

Proof. Note that every $\mathbb{P}$ in Γ preserves ω_1.

We claim that for any function $H : H(\theta)^{<\omega} \to H(\theta)$ there is a $\mathbb{P}$-name $\dot{g} : \omega_1 \to H(\theta)^V$ such that $\mathbb{P}$ forces $M = \dot{g}``\omega_1$ is closed under H, M contains ω_1, and $\dot{G}$ is M-generic. Let G be generic for $\mathbb{P}$, and we construct g in $V[G]$ by induction.

Suppose $g \restriction (\omega \cdot \alpha)$ is defined. For each dense set $D \subseteq \mathbb{P}$ in $g``(\omega \cdot \alpha)$, choose p_D in $G \cap D$. Now extend g to have domain $\omega \cdot (\alpha + 1)$ by adding a bijection of $\omega \cdot (\alpha+1) \setminus \omega \cdot \alpha$ onto the H-closure of the set

$$g``(\omega \cdot \alpha) \cup \alpha \cup \{p_D : D \in g``(\omega \cdot \alpha), D \subseteq \mathbb{P} \text{ dense}\}.$$

This verifies the claim.

Let $F : H(\theta)^{<\omega} \to P_{\omega_2}(H(\theta))$ be given, and we find N as required which is closed under F. Let $H : H(\theta)^{<\omega} \to H(\theta)$ be a Skolem function for the structure $\langle H(\theta), \in, F \rangle$. Fix $\dot{g}$ for H as in the claim above. Let $\mathcal{D}$ be the collection of dense subsets of $\mathbb{P}$ which are definable in the structure $\mathcal{A} = \langle H(\theta), \in, \mathbb{P}, \dot{g}, H \rangle$ using countable ordinals as parameters. Apply $\mathsf{MA}_\alpha(\Gamma)$ and choose a filter G on $\mathbb{P}$ which intersects each dense set in $\mathcal{D}$ and such that each $\dot{A}_i^G$ is stationary in ω_1.

Now let $h = \dot{g}^G$; in other words, $h : \omega_1 \to H(\theta)$ and $h(\alpha) = x$ iff there is p in G which forces $\dot{g}(\alpha) = x$. Let $N = h``\omega_1$. We claim that N is closed under F and G is N-generic.

Suppose $D \subseteq \mathbb{P}$ is dense and is in N. Fix α such that $D = h(\alpha)$ and p in G which forces $\dot{g}(\alpha) = D$. Then there is a dense set of conditions q which either force $\dot{g}(\alpha)$ is not dense in $\mathbb{P}$, or there is β such that q forces $h(\beta) = \dot{g}(\beta) \in \dot{G} \cap \dot{g}(\alpha)$. This dense set is definable in $\mathcal{A}$ using α as a parameter. So choose $q \in G$ in this dense set. Since p and q are compatible, there is β such that $h(\beta)$ is in $D \cap N$. Since $\mathbb{P}$ is separative and q forces $h(\beta) \in \dot{G}$, $q \leq h(\beta)$ and $h(\beta)$ is in G. The proof that N is closed under H and contains the countable ordinals is similar.

Since N is closed under H, $N \prec \langle H(\theta), \in, F \rangle$. So for each $x_0, \ldots, x_n$ in N, $F(x_0, \ldots, x_n) \in N$. But N contains ω_1, so $F(x_0, \ldots, x_n) \subseteq N$. $\square$

Suppose $\mathbb{P}$ is a forcing poset which satisfies the countable covering property, and $\omega_2 \leq \lambda < \theta$ are regular cardinals with $\mathbb{P} \in H(\theta)$. Assume $\mathbb{P}$ collapses $H(\lambda)^V$ to have size $\aleph_1$, and $\langle \dot{a}_i : i < \omega_1 \rangle$ is a $\mathbb{P}$-name for an increasing and continuous sequence of countable sets with union $H(\lambda)^V$. Fix $N \prec \langle H(\theta), \in, \mathbb{P} \rangle$ with size $\aleph_1$ which contains the countable ordinals and has $\langle \dot{a}_i : i < \omega_1 \rangle$ as an element.

Lemma 2.2. *Suppose G is a filter on $\mathbb{P}$ which is N-generic. For each i let $a_i = \dot{a}_i^G = \{x \in H(\lambda) : \exists p \in G \; p \Vdash x \in \dot{a}_i\}$. Then $\langle a_i : i < \omega_1 \rangle$ is an increasing and continuous sequence of countable sets with union $N \cap H(\lambda)$.*

Proof. Let $i < \omega_1$. Since $\mathbb{P}$ satisfies the countable covering property and $\dot{a}_i$ is in N, there is a dense set in N of conditions which decide for some countable set $x \subseteq H(\lambda)$ that $\dot{a}_i \subseteq x$. So there is p in $G \cap N$ and a countable set $x \in N$ such that p forces $\dot{a}_i \subseteq x$. Clearly then $a_i \subseteq x$. But $x \subseteq N$ since x is countable. So each a_i is a countable subset of $N \cap H(\lambda)$.

Clearly $a_i \subseteq a_j$ for $i < j$. Consider a set z in $N \cap H(\lambda)$. Let i_z be a name for the least index such that z is in $\dot{a}_{i_z}$. Since $\mathbb{P}$ forces $\langle a_i : i < \omega_1 \rangle$ is continuous, i_z is forced to be a non-limit ordinal. Fix p in $N \cap G$ and a non-limit ordinal i such that p forces $i_z = i$. Then z is in a_i but not in a_j for $j < i$. It follows that $\langle a_i : i < \omega_1 \rangle$ is an increasing and continuous sequence with union $N \cap H(\lambda)$. $\square$

3. Distinguishing Variants of Internal Approachability

Suppose N is a set with size $\aleph_1$. Then N is *internally stationary* if $N \cap P_{\omega_1}(N)$ is stationary in $P_{\omega_1}(N)$. N is *internally club* if $N \cap P_{\omega_1}(N)$ contains a club subset of $P_{\omega_1}(N)$. This is equivalent to the existence of an increasing and continuous sequence $\langle a_i : i < \omega_1\rangle$ of countable sets with union N such that $a_i \in a_{i+1}$ for all $i < \omega_1$.

Lemma 3.1. *Suppose N is a set with size $\aleph_1$. Then the following are equivalent.*

(1) N is internally stationary.

(2) There exists an increasing and continuous sequence $\langle a_i : i < \omega_1\rangle$ of countable sets with union N and a stationary set $A \subseteq \omega_1$ such that $\{a_i : i \in A\} \subseteq N$.

(3) For every increasing and continuous sequence $\langle a_i : i < \omega_1\rangle$ of countable sets with union N, there exists a stationary set $A \subseteq \omega_1$ such that $\{a_i : i \in A\} \subseteq N$.

Proof. Since N has size $\aleph_1$, N can be written as the union of an increasing and continuous sequence $\langle a_i : i < \omega_1\rangle$ of countable sets. For any set $S \subseteq P_{\omega_1}(N)$, S is stationary in $P_{\omega_1}(N)$ iff $\{i < \omega_1 : a_i \in S\}$ is stationary in ω_1. If $\langle b_i : i < \omega_1\rangle$ is another increasing and continuous sequence of countable sets with union N, then there is a club set $C \subseteq \omega_1$ such that $a_i = b_i$ for i in C. The lemma follows easily from these facts. $\square$

Lemma 3.2. *Suppose N is a set with size $\aleph_1$. Then the following are equivalent.*

(1) N fails to be internally club.

(2) There is an increasing and continuous sequence $\langle a_i : i < \omega_1\rangle$ of countable sets with union N and a stationary set $B \subseteq \omega_1$ such that $\{a_i : i \in B\} \cap N = \emptyset$.

(3) For every increasing and continuous sequence $\langle a_i : i < \omega_1\rangle$ of countable sets with union N there is a stationary set $B \subseteq \omega_1$ such that $\{a_i : i \in B\} \cap N = \emptyset$.

The proof is similar to the proof of Lemma 3.1. The next lemma follows immediately.

Lemma 3.3. *Suppose N is a set with size $\aleph_1$. If $\mathbb{P}$ is a forcing poset which preserves stationary subsets of ω_1, then $\mathbb{P}$ preserves each of the following properties of N:*

(1) N is internally stationary.

(2) N is not internally club.

We will use the following variant of a theorem of Abraham and Shelah [1].

Theorem 3.4. *Suppose $\mathbb{P}$ is ω_1-c.c. and adds a new real. Then for all regular $\lambda \geq \omega_2$, $\mathbb{P}$ forces $P_{\omega_1}(H(\lambda)^V) \setminus V$ is stationary in $P_{\omega_1}(H(\lambda)^V)$.*

Proposition 3.5. *Let $\omega_2 \leq \lambda \leq \theta$ be regular cardinals, and let $\mathbb{P}$ denote the forcing poset $\textsc{Add}(\omega)*\textsc{Coll}(\omega_1, H(\theta)^V)$. Then $\mathbb{P}$ forces that $H(\lambda)^V$ is internally stationary but not internally club.*

Proof. Write $N = H(\lambda)$. Let $G * H$ be generic for $\mathbb{P}$. By Theorem 3.4, $S = P_{\omega_1}(N) \setminus V$ is stationary in $P_{\omega_1}(N)$ in $V[G]$. Since $\textsc{Coll}(\omega_1, N)$ is proper, S is stationary in $P_{\omega_1}(N)$ in $V[G * H]$. Now work in $V[G * H]$. Since $|N| = \omega_1$, fix an increasing and continuous chain $\langle a_i : i < \omega_1\rangle$ of countable sets with union N. As S is stationary in $P_{\omega_1}(N)$, there is $B \subseteq \omega_1$ stationary such that $\{a_i : i \in B\} \subseteq S$. But S is disjoint from V and hence disjoint from N, since $N \subseteq V$. So N is not internally club by Lemma 3.2.

On the other hand, $P_{\omega_1}(N) \cap V$ is stationary in $P_{\omega_1}(N)$ since $\mathbb{P}$ is proper. But $P_{\omega_1}(N) \cap V = P_{\omega_1}(N) \cap N$ since $N = H(\lambda)^V$. So N is internally stationary. $\square$

Theorem 3.6. *The forcing axiom PFA_2 implies that for every regular $\lambda \geq \omega_2$, there is a stationary set of N in $P_{\omega_2}(H(\lambda))$ such that N is internally stationary but not internally club.*

Proof. Let $\mathbb{P} = \text{Add}(\omega) * \text{Coll}(\omega_1, H(\lambda)^V)$. By Proposition 3.5, Lemma 3.1, and Lemma 3.2, there are $\mathbb{P}$-names $\dot{A}$, $\dot{B}$, and $\langle \dot{a}_i : i < \omega_1 \rangle$ such that $\mathbb{P}$ forces the following statements:

(1) $\dot{A}$ and $\dot{B}$ are stationary subsets of ω_1,

(2) $\langle \dot{a}_i : i < \omega_1 \rangle$ is an increasing and continuous sequence of countable sets with union $H(\lambda)^V$,

(3) $\{\dot{a}_i : i \in \dot{A}\} \subseteq H(\lambda)^V$,

(4) $\{\dot{a}_i : i \in \dot{B}\} \cap H(\lambda)^V = \emptyset$.

Let $F : H(\lambda)^{<\omega} \to P_{\omega_2}(H(\lambda))$ be given. Fix $\theta \gg \lambda$ regular with $\mathbb{P} \in H(\theta)$, and let $\mathcal{A}$ denote the structure $\langle H(\theta), \in, \mathbb{P}, \dot{A}, \dot{B}, F \rangle$. By Proposition 2.1 choose $N \prec \mathcal{A}$ with size $\aleph_1$ containing the countable ordinals and a filter G on $\mathbb{P}$ which is N-generic such that $A = \dot{A}^G$ and $B = \dot{B}^G$ are stationary subsets of ω_1. Let $a_i = \dot{a}_i^G$ for each $i < \omega_1$. Since $\mathbb{P}$ is proper, it satisfies the countable covering property; so Lemma 2.2 implies that $\langle a_i : i < \omega_1 \rangle$ is an increasing and continuous sequence of countable sets with union $M = N \cap H(\lambda)$.

Since $\omega_1 \subseteq M$ and $N \prec \mathcal{A}$, M is closed under F. By Lemmas 3.1 and 3.2 it suffices to show that $\{a_i : i \in A\} \subseteq M$ and $\{a_j : j \in B\} \cap M = \emptyset$. Let i be in A and fix p in G which forces $i \in \dot{A}$. Then p forces $\dot{a}_i \in H(\lambda)^V$. By elementarity there is a dense set in N of conditions which either force i is not in $\dot{A}$ or decide for some x that $\dot{a}_i = x$. Since G is N-generic, there is q in $G \cap N$ and x in N such that q forces $x = \dot{a}_i$. Then x is in $H(\lambda) \cap N = M$. Clearly $x = a_i$. Hence $\{a_i : i \in A\} \subseteq M$.

Suppose j is in B and fix r in G which forces $j \in \dot{B}$. For each y in $N \cap H(\lambda)^V = M$, there is a dense set in N of conditions which either force j is not in $\dot{B}$ or decide for some z that z is in $\dot{a}_j \triangle y$. By N-genericity, a_j is not equal to y. So a_j is not in M. □

Let N be a set of size $\aleph_1$. Then N is *internally unbounded* if $N \cap P_{\omega_1}(N)$ is cofinal in $P_{\omega_1}(N)$. We prove that internally unbounded does not imply internally stationary, using the forcing poset $\text{Add}(\omega) * \mathbb{P}(\dot{S})$ described in [5].

Suppose $\lambda \geq \omega_2$ is regular and T is a stationary subset of $P_{\omega_1}(H(\lambda))$. Define $\mathbb{P}(T)$ as the poset consisting of countable increasing and continuous sequences $\langle b_i : i \leq \gamma \rangle$ contained in T, ordered by extension of sequences. The poset $\mathbb{P}(T)$ adds an increasing and continuous chain with order type ω_1 contained in T, collapsing $H(\lambda)$ to have size ω_1.

For proofs of the next two propositions see [5].

Proposition 3.7. *Let $\lambda \geq \omega_2$ be regular and let T be a stationary subset of $P_{\omega_1}(H(\lambda))$. Then $\mathbb{P}(T)$ is $< \omega_1$-distributive.*

Proposition 3.8. *Let $\lambda \geq \omega_2$ be regular and let $\dot{S}$ be an $\text{Add}(\omega)$-name for the set $P_{\omega_1}(H(\lambda)^V) \setminus V$. Then $\text{Add}(\omega) * \mathbb{P}(\dot{S})$ preserves stationary subsets of ω_1.*

Theorem 3.9. *The forcing axiom MM implies that for all regular $\lambda \geq \omega_2$, there is a stationary set of N in $P_{\omega_2}(H(\lambda))$ which are internally unbounded but not internally stationary.*

Proof. Let $\mathbb{P} = \text{Add}(\omega) * \mathbb{P}(\dot{S})$, where $\dot{S}$ is an $\text{Add}(\omega)$-name for $P_{\omega_1}(H(\lambda)^V) \setminus V$. Note that by Proposition 3.7, $\mathbb{P}$ satisfies the countable covering property (but it is clearly not proper).

Let $F : H(\lambda)^{<\omega} \to P_{\omega_2}(H(\lambda))$ be given. Fix $\theta \gg \lambda$ regular such that $\mathbb{P} \in H(\theta)$ and let $\mathcal{A} = \langle H(\theta), \in, \mathbb{P}, F \rangle$. By Proposition 2.1 choose $N \prec \mathcal{A}$ with size $\aleph_1$

containing the countable ordinals and a filter G on $\mathbb{P}$ which is N-generic. Let $M = N \cap H(\lambda)$. Note that since $\omega_1 \subseteq M$, M is closed under F.

Fix in N a $\mathbb{P}$-name $\langle \dot{a}_i : i < \omega_1 \rangle$ for an increasing and continuous sequence disjoint from V given by a generic for $\mathbb{P}(\dot{S})$. Let $a_i = \dot{a}_i^G$ for $i < \omega_1$. By Lemma 2.2, $\langle a_i : i < \omega_1 \rangle$ is an increasing and continuous sequence with union M.

To show that M is not internally stationary it suffices to show that $\{a_i : i < \omega_1\} \cap M = \emptyset$. Fix $i < \omega_1$. Then $\mathbb{P}$ forces $\dot{a}_i$ is not in $H(\lambda)^V$. So for each y in M there is a dense set in N of conditions which decide for some z that z is in $\dot{a}_i \triangle y$. By N-genericity it follows that a_i is not in M.

It remains to prove that M is internally unbounded. Let x be in $P_{\omega_1}(M)$. Fix $j < \omega_1$ with $x \subseteq a_j$. By the countable covering property and N-genericity of G, there is y in M countable and p in G which forces $\dot{a}_j \subseteq y$. Clearly then $a_j \subseteq y$. Hence $x \subseteq y \in M$. □

4. Approachability and Forcing Posets

In this section we present some background material for the consistency result of Section 5. We describe Shelah's approachability ideal $I[\lambda]$ and analyze some forcing posets. We will use $I[\lambda]$ as a tool for preserving the stationarity of certain stationary sets after forcing.

Suppose λ is a regular uncountable cardinal. For a sequence $\vec{a} = \langle a_i : i < \lambda \rangle$ of bounded subsets of λ, let $S(\vec{a})$ denote the set of limit ordinals $\beta < \lambda$ for which there exists an unbounded set $c_\beta \subseteq \beta$ with order type $\mathrm{cf}(\beta)$ such that each initial segment of c_β is enumerated in the sequence $\langle a_i : i < \beta \rangle$.

Define $I[\lambda]$ as the collection of sets S such that there is $\vec{a}$ and a club $C \subseteq \lambda$ with $S \cap C \subseteq S(\vec{a})$. Then $I[\lambda]$ is a normal ideal on λ which extends the non-stationary ideal.

Oftentimes $I[\lambda]$ will contain a maximal stationary set. For example, if $\lambda^{<\lambda} = \lambda$ and $\vec{a}$ enumerates all bounded subsets of λ, then $S(\vec{a})$ is stationary and for any set A in $I[\lambda]$ there is a club C such that $A \cap C \subseteq S(\vec{a})$. Suppose there is a maximal set S in $I[\lambda]$. Then $I[\lambda] = NS_\lambda \restriction (\lambda \setminus S)$. In this case we refer to S as the set of *approachable ordinals*.

Suppose λ is a Mahlo cardinal and $A \subseteq \lambda$ is a stationary set of regular cardinals. Define $\vec{a} = \langle a_i : i < \lambda \rangle$ by letting $a_i = i$ for $i < \lambda$. Then $A \subseteq S(\vec{a})$, as witnessed by letting $c_\alpha = \alpha$ for α in A. Hence A is in $I[\lambda]$.

See [2] for more information about $I[\lambda]$.

Now we discuss some forcing posets we will use in Section 5.

Let $\kappa > \omega_1$ be a regular cardinal and suppose $A \subseteq \kappa$. We define a forcing poset $\mathbb{P}(A)$ which adds a stationary subset of $\kappa \cap \mathrm{cof}(\omega)$ which does not reflect at any ordinal in A. A condition p in $\mathbb{P}(A)$ is a bounded subset of $\kappa \cap \mathrm{cof}(\omega)$ such that for all α in A, $p \cap \alpha$ is not stationary in α. The ordering is by end-extension.

Clearly $\mathbb{P}(A)$ is countably closed and hence proper, and has size $\kappa^{<\kappa}$. So by the next lemma, if $\kappa^{<\kappa} = \kappa$ then $\mathbb{P}(A)$ preserves all cardinals and cofinalities.

Lemma 4.1. *The forcing poset $\mathbb{P}(A)$ is κ-strategically closed.*

Proof. We describe a strategy for Player II. When Player I plays a condition p_i, Player II responds with $q_i = p_i \cup \{\sup(p_i) + \omega + \omega\}$. Also define $\gamma_i = \sup(p_i) + \omega$. If $\xi < \kappa$ is a limit ordinal, let $x = \bigcup\{q_i : i < \xi\}$, and define $q_\xi = x \cup \{\sup(x) + \omega\}$. Define $\gamma_\xi = \sup(x)$. It is easy to prove by induction that for each limit ordinal

$\xi < \kappa$, $\bigcup\{q_i : i < \xi\}$ is disjoint from $\{\gamma_i : i < \xi\}$, and the latter is a club subset of $\sup(\bigcup\{q_i : i < \xi\}) = \gamma_\xi$. Also q_ξ does not reflect at any ordinal in A which is below γ_ξ, since by the induction hypothesis q_i is a condition for $i < \xi$. So q_ξ is a condition, and the game continues. $\square$

Let $\dot{S}$ be a name for $\bigcup \dot{G}$.

Lemma 4.2. *The poset $\mathbb{P}(A)$ forces that $\dot{S}$ is stationary.*

Proof. Suppose p forces $\dot{C} \subseteq \kappa$ is club. Define by induction a descending sequence $\langle p_n : n < \omega \rangle$. Let $p_0 = p$. Given p_n, let $p_{n+1} \leq p_n$ be a condition such that for some α_n, $\max(p_{n+1}) > \alpha_n$ and p_{n+1} forces that $\alpha_n = \min(\dot{C} \setminus \max(p_n))$. Let $x = \bigcup\{p_n : n < \omega\}$ and let $q = x \cup \{\sup(x)\}$. Since $\mathrm{cf}(\sup(x)) = \omega$, q is a condition. Also $\sup(x) = \bigcup\{\alpha_n : n < \omega\}$, so q forces $\sup(x) \in \dot{C} \cap \dot{S}$. $\square$

Proposition 4.3. *If $B \subseteq \kappa$ is a stationary set in $I[\kappa]$, then $\mathbb{P}(A)$ preserves the stationarity of B. In particular, $\mathbb{P}(A)$ preserves the stationarity of any stationary set of regular cardinals in κ.*

Proof. Suppose p forces $\dot{C} \subseteq \kappa$ is a club. Fix $\theta \gg \kappa$ and let $\mathcal{A}$ denote the structure

$$\langle H(\theta), \in, <_\theta, \mathbb{P}(A), p, \dot{C}, \langle a_i : i < \kappa \rangle \rangle,$$

where $<_\theta$ is a well-ordering of $H(\theta)$ and $\langle a_i : i < \kappa \rangle$ witnesses that B is in $I[\kappa]$.

Since B is stationary, there is $N \prec \mathcal{A}$ such that $N \cap \kappa = \beta$ is in B. Note that for all $i < \beta$, a_i is in N. Let $x \subseteq \beta$ be cofinal with order type $\mathrm{cf}(\beta)$ such that every initial segment of x appears in $\langle a_i : i < \beta \rangle$.

Define by induction a sequence $\langle p_i, q_i : i < \mathrm{cf}(\beta) \rangle$ which is a run in the game in which Player II uses his winning strategy, and each initial segment of the sequence is definable in $\mathcal{A}$ from an initial segment of x. Let $p_0 = p$, and let q_0 be Player II's response. Suppose $\langle p_j, q_j : j < i \rangle$ is defined. If i is limit then let $p_i = q_i$ be Player II's play according to his strategy. If i is a successor ordinal, let p_i be the $<_\theta$-least refinement of q_{i-1} such that $p_i \setminus q_{i-1}$ has non-empty intersection with x and p_i forces that $\dot{C} \cap (\max(p_i) \setminus \max(q_{i-1}))$ is non-empty. Let q_i be Player II's response.

Since only an initial segment of x is needed at each stage in defining the sequence of conditions, each initial segment of the sequence is in N. Let $z = \bigcup\{q_i : i < \mathrm{cf}(\beta)\}$. Since each q_i is in N, $z \subseteq \beta$. On the other hand, $z \cap x$ has size $\mathrm{cf}(\beta)$; since $\mathrm{o.t.}(x) = \mathrm{cf}(\beta)$ this implies $z \cap x$ is unbounded in β. Let r be a lower bound of the sequence of conditions. Then r forces $\sup(z)$ is a limit point of $\dot{C}$, so r forces $\beta \in \dot{C} \cap B$. $\square$

Now we consider a forcing poset for adding a club subset to a given stationary set. Suppose $T \subseteq \omega_2 \cap \mathrm{cof}(\omega_1)$ is stationary. Define a forcing poset $\mathbb{P}_T$ which attempts to add a club subset to $T \cup \mathrm{cof}(\omega)$ as follows. A condition is a closed bounded subset of $T \cup \mathrm{cof}(\omega)$, and the ordering is by end-extension.

Note that $\mathbb{P}_T$ is countably closed and hence proper. If $2^{\omega_1} = \omega_2$ then $\mathbb{P}_T$ has size ω_2.

Proposition 4.4. *Suppose A is a stationary subset of T which is in $I[\omega_2]$. Then $\mathbb{P}_T$ is $< \omega_2$-distributive and preserves the stationarity of A.*

For example, if $2^{\omega_1} = \omega_2$, S is the maximal set in $I[\omega_2]$, and $S \cap T$ is stationary, then $\mathbb{P}_T$ preserves all cardinals and preserves stationary subsets of $S \cap T$. If T does not contain a stationary set which is in $I[\omega_2]$, it is possible that $\mathbb{P}_T$ will collapse ω_2; this was proved in [5].

Proof. Let $\langle D_i : i < \omega_1 \rangle$ be a sequence of dense open subsets of $\mathbb{P}_T$. Suppose p forces $\dot{C} \subseteq \omega_2$ is club. We find $r \leq p$ which is in $\bigcap \{D_i : i < \omega_1\}$ and an ordinal β such that r forces $\beta \in \dot{C} \cap A$.

Fix $\theta \gg \omega_2$ and let $\mathcal{A}$ denote the structure

$$\langle H(\theta), \in, <_\theta, T, \mathbb{P}_T, p, \dot{C}, \langle D_i : i < \omega_1 \rangle, \langle a_i : i < \omega_2 \rangle \rangle,$$

where $<_\theta$ is a well-ordering of $H(\theta)$ and $\langle a_i : i < \omega_2 \rangle$ witnesses that A is in $I[\omega_2]$.

Since A is stationary, there is $N \prec \mathcal{A}$ with size $\aleph_1$ such that $N \cap \omega_2 = \beta$ is in A. Note that $\{a_i : i < \beta\}$ and $\{D_i : i < \omega_1\}$ are both subsets of N. Fix $x \subseteq \beta$ unbounded with order type $\mathrm{cf}(\beta)$ such that every initial segment of x is enumerated in $\langle a_i : i < \beta \rangle$.

Define by induction a descending sequence $\langle p_i : i < \omega_1 \rangle$ of conditions in $\mathbb{P}_T$ whose initial segments are definable in $\mathcal{A}$ from initial segments of x. Let $p_0 = p$. Suppose $\langle p_i : i < \delta \rangle$ is defined where $\delta < \omega_1$ is limit. Let $y = \bigcup \{p_i : i < \omega_1\}$. Then $\sup(y)$ has cofinality ω, so $p_\delta = y \cup \{\sup(y)\}$ is in $\mathbb{P}_T$. Now assume p_i is defined for a fixed i. Let p_{i+1} be the $<_\theta$-least refinement of p_i in D_i such that $p_{i+1} \setminus p_i$ has non-empty intersection with x and p_{i+1} forces $\max(p_{i+1}) \setminus \max(p_i)$ contains an element of $\dot{C}$.

This completes the definition. Each initial segment of the sequence is in N because it is definable in $\mathcal{A}$ from an initial segment of x. Let $z = \bigcup \{p_i : i < \omega_1\}$. Then $z \subseteq \beta$. Since $z \cap x$ has size $\aleph_1$, $\sup(z) = \beta$. Therefore $r = z \cup \{\beta\}$ is a condition since β is in T. Also r is in $\bigcap \{D_i : i < \omega_1\}$ and r forces $\beta \in \dot{C} \cap A$. $\square$

We finish this section by proving two technical results we will need in the next section.

Proposition 4.5. *Suppose that κ is a Mahlo cardinal and $A \subseteq \kappa$ is a stationary set of strongly inaccessible cardinals. Let $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \kappa, j < \kappa \rangle$ be a countable support iteration of proper forcing posets. Assume that for each β in A the following statements hold:*

(1) for all $\gamma < \beta$, $|\mathbb{P}_\gamma| < \beta$,

(2) $\mathbb{P}_\beta$ forces $\beta = \omega_2$ and $2^\omega \leq \omega_2$,

(3) $\dot{\mathbb{Q}}_\beta$ is a name for $\mathrm{Coll}(\omega_1, \omega_2)$.

Then $\mathbb{P}_\kappa$ forces that $\kappa = \omega_2$, $A \subseteq \omega_2 \cap \mathrm{cof}(\omega_1)$, and A is in $I[\omega_2]$.

Proof. Let G_κ be generic for $\mathbb{P}_\kappa$. Clearly $\kappa = \omega_2$ in $V[G_\kappa]$ by (2) and the fact that $\mathbb{P}_\kappa$ is κ-c.c. Also $A \subseteq \omega_2 \cap \mathrm{cof}(\omega_1)$ because $\mathbb{P}_\kappa$ is proper and hence can only change the cofinality of ordinals in A to ω_1.

In $V[G_\kappa]$ construct a sequence $\langle a_i : i < \omega_2 \rangle$ by induction on ordinals in $A \cup \lim(A)$ such that for each β in A, $\langle a_i : i < \beta \rangle$ enumerates $[\beta]^\omega \cap V[G_\beta]$. Suppose $\alpha < \beta$ are successive ordinals in A and $\langle a_i : i < \alpha \rangle$ is defined. Since $\mathbb{P}_\beta$ forces $2^\omega \leq \omega_2 = \beta$, $[\beta]^\omega \cap V[G_\beta]$ has size $|\beta| = \omega_1$ in $V[G_\kappa]$. So we can extend the sequence to $\langle a_i : i < \beta \rangle$ as required.

If β is a limit point of A, then $\langle a_i : i < \beta \rangle$ is given by induction since the defined sequences are end-extensions of one another. If β is not in A then there are no requirements on this sequence. Suppose β is in A and let x be in $[\beta]^\omega \cap V[G_\beta]$. Since $\beta = \omega_2$ in $V[G_\beta]$, there is $\alpha < \beta$ such that $\sup(x) < \alpha$. As β is strongly inaccessible in V, there is γ in A with $\alpha < \gamma < \beta$ such that x is in $V[G_\gamma] \cap [\gamma]^\omega$. Then by induction x is in the sequence $\langle a_i : i < \gamma \rangle$. So $\langle a_i : i < \beta \rangle$ enumerates $[\beta]^\omega \cap V[G_\beta]$.

For each α in A fix c_α in $V[G_{\alpha+1}]$ which is cofinal in α with order type ω_1. Since $\textsc{Coll}(\omega_1, \omega_2) = \mathbb{Q}_\alpha$ is countably closed, every initial segment of c_α is in $V[G_\alpha]$, and hence is enumerated in $\langle a_i : i < \alpha \rangle$. So $A \subseteq S(\vec{a})$ and A is in $I[\omega_2]$. $\square$

Proposition 4.6. *Suppose N is a set with size $\aleph_1$.*

(1) N is internally stationary iff its transitive collapse is internally stationary.

(2) N is internally club iff its transitive collapse is internally club.

Proof. (1) Suppose N is internally stationary, and let $\pi : N \to M$ be its transitive collapse. Fix an increasing and continuous sequence $\langle N_i : i < \omega_1 \rangle$ of countable sets with union N and a stationary set $A \subseteq \omega_1$ such that $\{N_i : i \in A\} \subseteq N$. For each $i < \omega_1$ let $M_i = \pi``N_i$. Then $M$ is the union of the increasing and continuous sequence $\{M_i : i < \omega_1\}$ of countable sets. If $i$ is in $A$, then $N_i \in N$ so $M_i = \pi``N_i = \pi(N_i) \in M$ by the definition of the transitive collapse map. The converse and (2) are similar. $\square$

5. Internal Approachability and Reflection

Now we are ready to prove the following theorem.

Theorem 5.1. *Assume κ is a supercompact cardinal. Then there is a forcing poset which forces that $\kappa = \omega_2$ and for all regular $\lambda \geq \omega_2$:*

(1) for every family $\{U_i : i < \omega_1\}$ of stationary subsets of $P_{\omega_1}(H(\lambda))$, there is an internally stationary set M with size $\aleph_1$ such that each U_i reflects to M,

(2) there is a stationary set $S^* \subseteq P_{\omega_1}(H(\lambda))$ such that S^* does not reflect to any internally approachable set N with size $\aleph_1$.

Let V denote the ground model, and assume that in V the GCH holds and κ is supercompact. Let B denote the set of measurable cardinals less than κ and let A be the set of non-measurable strongly inaccessible cardinals less than κ. Note that for all α in $B \cup \{\kappa\}$, $A \cap \alpha$ is stationary in α.

First we define an Easton support iteration

$$\langle \mathbb{P}^S_i, \dot{\mathbb{Q}}^S_i : i \leq \kappa \rangle$$

by recursion. Afterwards we will define another iteration in a generic extension by $\mathbb{P}^S_\kappa * \dot{\mathbb{Q}}^S_\kappa$. Suppose $\mathbb{P}^S_\alpha$ is defined for some $\alpha \leq \kappa$. If α is not strongly inaccessible then let $\dot{\mathbb{Q}}^S_\alpha$ denote the trivial forcing. Suppose α is strongly inaccessible. Then let $\dot{\mathbb{Q}}^S_\alpha$ be a name for the forcing poset $\mathbb{P}(\alpha \setminus B)$ from Section 4. In other words, $\dot{\mathbb{Q}}^S_\alpha$ is a name for the α-strategically closed forcing poset for adding a stationary set $S_\alpha \subseteq \alpha \cap \mathrm{cof}(\omega)$ such that whenever $\beta < \alpha$ and $S_\alpha \cap \beta$ is stationary in β, then β is in B.

This completes the definition. Standard arguments show that $\mathbb{P}^S_\kappa$ preserves all cardinals, cofinalities, the function $\alpha \mapsto 2^\alpha$, and for any Mahlo cardinal $\alpha \leq \kappa$, $\mathbb{P}_\alpha$ is α-c.c.

Let $G * G_S$ be generic for $\mathbb{P}^S_\kappa * \mathbb{Q}^S_\kappa$ over V. Write $S = \bigcup G_S$, which is a stationary subset of $\kappa \cap \mathrm{cof}(\omega)$ such that for all $\beta < \kappa$, if $S \cap \beta$ is stationary in β then β is in B. Write $W = V[G * G_S]$.

Lemma 5.2. *In W, κ is a supercompact cardinal.*

Proof. Consider a regular cardinal $\lambda > \kappa$ in W. In V fix an elementary embedding $j : V \to M$ induced by a normal ultrafilter on $P_\kappa(\lambda)$. Then $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and M is closed under λ-sequences. Also $\lambda^+ < j(\kappa)$, and $|j(\kappa)| \leq \kappa^{|P_\kappa(\lambda)|} = \kappa^\lambda = \lambda^+$, so $|j(\kappa)| = \lambda^+$. Similarly, $|j(\kappa^+)| = \lambda^+$. Note that $\kappa \in j(B)$ since κ is measurable in M.

Write $j(\mathbb{P}^S_\kappa) = \mathbb{P}^S_\kappa * \dot{\mathbb{Q}}^S_\kappa * \mathbb{P}^S_{\text{tail}}$. The model $M[G * G_S]$ is closed under λ-sequences in W. So in W, $\mathbb{P}^S_{\text{tail}}$ is λ^+-strategically closed. There are $|j(\kappa)| = \lambda^+$ many maximal antichains of $\mathbb{P}^S_{\text{tail}}$ in $M[G * G_S]$. So we can construct in W a filter K on $\mathbb{P}^S_{\text{tail}}$ which is generic over $M[G * G_S]$. Since $j``G = G \subseteq G * G_S * K$, j extends to $j : V[G] \to M[G * G_S * K]$.

We claim that S is a condition in $j(\mathbb{Q}^S_\kappa)$. This is true since κ is in $j(B)$, so it does not matter that S is stationary in κ. The model $M' = M[G * G_S * K]$ is closed under λ sequences. So $j(\mathbb{Q}^S_\kappa)$ is λ^+-strategically closed and has λ^+ many maximal antichains in M'. Construct a filter h on $j(\mathbb{Q}^S_\kappa)$ which contains S and is generic over M'. Then $M'[h]$ is closed under λ-sequences. Since $j``G_S = G_S \subseteq h$, we can extend j to $j : W \to M'[h]$, which witnesses that κ is λ-supercompact in W. □

Lemma 5.3. *In W, if $j : W \to N$ is an elementary embedding induced by a normal ultrafilter on $P_\kappa(\lambda)$ for some $\lambda \geq \kappa$, then κ is in $j(B)$.*

Proof. In N, $j(S) \cap j(\kappa) = S$ is stationary in κ. By elementarity, any ordinal to which $j(S)$ reflects is in $j(B)$. □

Lemma 5.4. *In W, for all β in $B \cup \{\kappa\}$, $A \cap \beta$ is stationary in β.*

Proof. Write $\mathbb{P}^S_\kappa = \mathbb{P}^S_\beta * \dot{\mathbb{Q}}^S_\beta * \mathbb{P}^S_{\text{tail}}$. In V, $A \cap \beta$ is stationary in β, and it remains stationary after forcing with the β-c.c. poset $\mathbb{P}^S_\beta$. Since $A \cap \beta$ consists of regular cardinals, by Proposition 4.3, $A \cap \beta$ remains stationary after forcing with $\mathbb{Q}^S_\beta$. The poset $\mathbb{P}^S_{\text{tail}}$ does not add subsets to β, so $A \cap \beta$ is stationary in β in W. □

This completes our analysis of the model W.

In W fix a Laver function $l : \kappa \to V^W_\kappa$. So for every x and every $\lambda \geq \kappa$, there is an elementary embedding $j : W \to M$ with critical point κ such that $j(\kappa) > \lambda$, M is closed under λ-sequences, and $j(l)(\kappa) = x$.

In W we define by recursion a countable support iteration

$$\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \kappa, j < \kappa \rangle$$

of proper forcing posets. Suppose $\mathbb{P}_\alpha$ is defined. Let $\dot{\mathbb{Q}}_\alpha$ be a name for $\textsc{Coll}(\omega_1, \omega_2)$, unless each of the following conditions are satisfied:

(1) α is in B,

(2) for all $\beta < \alpha$, $|\mathbb{P}_\beta| < \alpha$,

(3) $l(\alpha)$ is a pair $\langle \dot{f}, \dot{\lambda} \rangle$ of $\mathbb{P}_\alpha$-names and $\mathbb{P}_\alpha$ forces that $\alpha = \omega_2$, $\dot{f} : \omega_2 \to H(\omega_2)$ is a bijection, and $\dot{\lambda} \geq \omega_2$ is a regular cardinal.

Assume these conditions are satisfied.

Note that $\mathbb{P}_\alpha$ is α-c.c. Let G_α be generic for $\mathbb{P}_\alpha$ over W, and work in $W[G_\alpha]$. Let $f = \dot{f}^G$ and $\lambda = \dot{\lambda}^G$. Since f is a bijection, if $N \prec \langle H(\omega_2), \in, f \rangle$ then $f``(N \cap \omega_2) = N$. So for each $\beta < \omega_2$, there is at most one such elementary substructure N with $N \cap \omega_2 = \beta$. Let F denote the set of $\beta < \omega_2$ for which there exists a unique $N(\beta) \prec \langle H(\omega_2), \in, f \rangle$ such that $N(\beta) \cap \omega_2 = \beta$. Then F contains a club.

Define $T \subseteq \omega_2$ by letting β be in T iff (a) β is in F, (b) $\mathrm{cf}(\beta) = \omega_1$, and (c) if β is in B then $N(\beta)$ is not internally club.

Lemma 5.5. *In $W[G_\alpha]$, the set $A \cap T$ is stationary and is in $I[\omega_2]$.*

Proof. In W let $X \subseteq \alpha$ be club such that for all β in X, if $\gamma < \beta$ then $|\mathbb{P}_\gamma| < \beta$. If β is in $A \cap X$, then $\mathbb{P}_\beta$ forces $\beta = \omega_2$ and $\mathbb{Q}_\beta = \mathrm{Coll}(\omega_1, \beta)$. By Proposition 4.5, $A \cap X$ is in $I[\omega_2]$ and $A \cap X \subseteq \mathrm{cof}(\omega_1) \cap \omega_2$. Since A and B are disjoint, $A \cap X \cap F \subseteq T$. $\square$

Let $\mathbb{P}_T$ be the proper forcing poset from Section 4 for adding a club subset to $T \cup \mathrm{cof}(\omega)$. By Lemma 5.5 and Proposition 4.4, $\mathbb{P}_T$ is $< \omega_2$-distributive. Now let $\dot{\mathbb{Q}}_\alpha$ be a name for $\mathbb{P}_T * \mathrm{Add}(\omega) * \mathrm{Coll}(\omega_1, N)$, where N is a name for the set $H(\lambda)$ as computed after forcing with $\mathbb{P}_\alpha * \mathbb{P}_T$.

This completes the definition of $\mathbb{P}_\kappa$. Let G_κ be generic for $\mathbb{P}_\kappa$ over W. Standard arguments show that in $W[G_\kappa]$, $\kappa = \omega_2$ and $2^\omega = 2^{\omega_1} = \omega_2$.

Working in $W[G_\kappa]$ fix a bijection $f : \omega_2 \to H(\omega_2)$. Let F be the set of $\beta < \kappa$ for which there exists a unique $N(\beta) \prec H(\omega_2)$ closed under f. Then F contains a club. Define T by letting $\beta < \omega_2$ be in T iff (a) $\beta \in F$, (b) $\mathrm{cf}(\beta) = \omega_1$, and (c) if β is in B then $N(\beta)$ is not internally club. Let G_T be generic for $\mathbb{P}_T$ over $W[G_\kappa]$. Our final model is $W' = W[G_\kappa * G_T]$.

By Lemma 5.5 and Proposition 4.4, $\mathbb{P}_T$ is $< \omega_2$-distributive. Since $H(\omega_2)$ is determined by $\mathcal{P}(\omega_1)$, $H(\omega_2)$ is the same in $W[G_\kappa]$ and W'. Also f is a bijection in W', and F satisfies the same definition in W' as it does in $W[G_\kappa]$.

Let E denote the club set $\bigcup G_T$. Then $E \cap \mathrm{cof}(\omega_1) \subseteq T \subseteq F$.

Lemma 5.6. *In W', $A \subseteq \omega_2 \cap \mathrm{cof}(\omega_1)$ is stationary, $S \subseteq \omega_2 \cap \mathrm{cof}(\omega)$ is stationary, and for all $\beta < \omega_2$, if $S \cap \beta$ is stationary in β then β is in B.*

Proof. The poset $\mathbb{P}_\kappa$ is κ-c.c., so A and S are stationary in $W[G_\kappa]$. Since $\mathbb{P}_T$ is proper, $\mathbb{P}_T$ preserves the stationarity of S by Lemma 1.1. By Lemma 5.5, in $W[G_\kappa]$, $A \cap T$ is stationary and is in $I[\omega_2]$. So Proposition 4.4 implies $\mathbb{P}_T$ preserves the stationarity of A. The poset $\mathbb{P}_\kappa * \mathbb{P}_T$ is proper, so it can only change the cofinality of ordinals less than κ to ω_1. Suppose $S \cap \beta$ is stationary in β for some $\beta < \omega_2$. If β is not in B, then $S \cap \beta$ is not stationary in β in W and hence also not in W'. So β is in B. $\square$

Lemma 5.7. *If β is in $E \cap B$ and $\mathrm{cf}(\beta) = \omega_1$, then $N(\beta)$ is not internally club.*

Proof. By the definition of T, $N(\beta)$ is not internally club in $W[G_\kappa]$. Since $\mathbb{P}_T$ is proper, $N(\beta)$ is not internally club in W' by Lemma 3.3. $\square$

We now verify the main claims of Theorem 5.1.

Proposition 5.8. *In W', for all regular $\lambda \geq \omega_2$, there is a stationary set $S^* \subseteq P_{\omega_1}(H(\lambda))$ such that S^* does not reflect to any internally approachable set with size $\aleph_1$.*

Proof. Define S^* as the collection of a in $P_{\omega_1}(H(\lambda))$ such that $a \prec H(\lambda)$, $a \cap H(\omega_2) \prec \langle H(\omega_2), \in, f \rangle$, and $\sup(a \cap \omega_2) \in S \cap E$.

To show S^* is stationary, consider a function $H : H(\lambda)^{<\omega} \to H(\lambda)$. Then there are club many $\beta < \omega_2$ such that for all $\alpha_0 < \ldots < \alpha_n < \beta$, $cl_H(\{\alpha_0, \ldots, \alpha_n\}) \cap \omega_2 \subseteq \beta$. Choose such an ordinal β which is in $S \cap E$. Let $x \subseteq \beta$ be cofinal with order type ω and let $a = cl_H(x)$. Then $\sup(a \cap \omega_2) = \beta$. This argument shows there are stationarily many a in $P_{\omega_1}(H(\lambda))$ with $\sup(a \cap \omega_2) \in S \cap E$. The other two conditions in the definition of S^* are true for club many sets a. So S^* is stationary.

Suppose for a contradiction that $S^* \cap P_{\omega_1}(N)$ is stationary in $P_{\omega_1}(N)$ for an internally approachable set N with size $\aleph_1$. Fix an increasing and continuous chain $\langle N_i : i < \omega_1 \rangle$ of countable sets with union N such that for all $i < \omega_1$, $\langle N_j : j < i \rangle \in N$. Then there is a stationary set $A \subseteq \omega_1$ such that $\{N_i : i \in A\} \subseteq S^*$. For each i in A, $\sup(N_i \cap \omega_2) \in S \cap E$. Let $\beta = N \cap \omega_2$. Clearly β is in $E \cap \mathrm{cof}(\omega_1)$.

For each i in A, $N_i \cap H(\omega_2) \prec \langle H(\omega_2), \in, f \rangle$, hence $N \cap H(\omega_2) \prec \langle H(\omega_2), \in, f \rangle$. So $N \cap H(\omega_2) = N(\beta)$. But $H(\omega_2)$ is a definable class of $H(\lambda)$, so it is easy to see that $N \cap H(\omega_2) = N(\beta)$ is internally approachable by the sequence $\langle N_i \cap H(\omega_2) : i < \omega_1 \rangle$. Lemma 5.7 implies that β is not in B.

Since β is not in B, S does not reflect to β. Choose a club $c \subseteq \beta$ disjoint from S. Since S^* is stationary in $P_{\omega_1}(N)$, there is a in S^* closed under the map $\alpha \mapsto \min(c \setminus \alpha)$. But then $\sup(a \cap \omega_2) \in c \cap S$, which is a contradiction. $\square$

We complete the proof by verifying the following claim in W': For all regular $\lambda \geq \omega_2$, for every family $\{U_i : i < \omega_1\}$ of stationary subsets of $P_{\omega_1}(H(\lambda))$, there is an internally stationary set M with size $\aleph_1$ such that U_i reflects to M for all $i < \omega_1$.

Let $\lambda \geq \omega_2$ be regular and assume $\{U_i : i < \omega_1\}$ is a family of stationary subsets of $P_{\omega_1}(H(\lambda))$. Then λ is regular in $W[G_\kappa]$. Fix $\mathbb{P}_\kappa$-names $\dot{f}$ and $\dot{\lambda}$ for f and λ. Back in W let $j : W \to M$ be an elementary embedding with critical point κ such that $j(\kappa) > \lambda$, M is closed under λ-sequences, and $j(l)(\kappa) = \langle \dot{f}, \dot{\lambda} \rangle$.

Note that $M[G_\kappa * G_T]$ is closed under λ-sequences in W'.

By the choice of j, we have

$$j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \mathbb{P}_T * \mathrm{A{\scriptsize DD}}(\omega) * \mathrm{C{\scriptsize OLL}}(\omega_1, N) * \mathbb{P}_{\mathrm{tail}}$$

where N is a name for $H(\lambda)$ as computed after forcing with $\mathbb{P}_\kappa * \mathbb{P}_T$, and $\mathbb{P}_{\mathrm{tail}}$ is a proper forcing iteration collapsing $j(\kappa)$ to become ω_2.

We would like to extend j to have domain W' in some outer model of W'. Let K be generic for $\mathrm{A{\scriptsize DD}}(\omega) * \mathrm{C{\scriptsize OLL}}(\omega_1, N) * \mathbb{P}_{\mathrm{tail}}$ over W'. Since $j``G_\kappa = G_\kappa \subseteq G_\kappa * G_T * K$, we can extend j to $j : W[G_\kappa] \to M[G_\kappa * G_T * K]$ in $W'[K]$. Write $M' = M[G_\kappa * G_T * K]$.

To further extend j to have domain W', we construct a master condition for $j(\mathbb{P}_T)$.

Lemma 5.9. *The set $s = E \cup \{\kappa\}$ is a condition in $j(\mathbb{P}_T)$.*

Proof. Since $M[G_\kappa * G_T]$ is closed under λ-sequences, it contains the set s, so M' does as well. As $E \cap \mathrm{cof}(\omega_1) \subseteq T$, $j``(E \cap \mathrm{cof}(\omega_1)) = E \cap \mathrm{cof}(\omega_1) \subseteq j(T)$. So it suffices to prove that κ is in $j(T)$.

In M', $\kappa \in j(B)$ by Lemma 5.3 and $\mathrm{cf}(\kappa) = \omega_1$. Also κ is in $j(F)$, so $N(\kappa)$ exists. By the definition of T, κ is in $j(T)$ iff $N(\kappa)$ is not internally club in M'.

At stage κ, $j(\mathbb{P}_\kappa)$ forces with $\mathbb{P}_T * \mathrm{A{\scriptsize DD}}(\omega) * \mathrm{C{\scriptsize OLL}}(\omega_1, N)$. By the closure of $M[G_\kappa * G_T]$, N is equal to $H(\lambda)^{W'}$. Now here is the reason we forced with $\mathrm{A{\scriptsize DD}}(\omega)$: by Proposition 3.5 and Lemma 3.3, $H(\omega_2)^{W'} = H(\omega_2)^{W[G_\kappa]}$ is not internally club in M'. Now the transitive collapse of $j``(H(\omega_2)^{W[G_\kappa]})$ is equal to $H(\omega_2)^{W[G_\kappa]}$, so $j``(H(\omega_2)^{W[G_\kappa]})$ is not internally club by Proposition 4.6.

We claim $j``(H(\omega_2)^{W[G_\kappa]})$ is in $M'$ and is equal to $N(\kappa)$, which completes the proof. Recall that $N(\kappa) = j(f)``(N(\kappa) \cap j(\kappa)) = j(f)``\kappa$. For each $\xi < \kappa$, $j(f)(\xi) = j(f)(j(\xi)) = j(f(\xi))$. So $N(\kappa) = j(f)``\kappa = j``(f``\kappa)) = j``(H(\omega_2)^{W[G_\kappa]})$. $\square$

Choose h generic for $j(\mathbb{P}_T)$ over $W'[K]$ containing the condition s. Now extend j in $W'[K*h]$ to $j : W' \to M'[h]$.

Recall that $N = H(\lambda)^{W'}$.

Lemma 5.10. *The set $j``N$ is in $M'[h]$.*

Proof. The model $M'[h]$ is not closed under λ-sequences in $W'[K*h]$. However $j``(H(\lambda)^W)$ is in M. Fix a surjection $g : [\lambda]^{<\lambda} \to N$ in W'. Every set in $[\lambda]^{<\lambda}$ is the interpretation of a $\mathbb{P}_\kappa * \mathbb{P}_T$-name in $H(\lambda)^W$. Also for any $\mathbb{P}_\kappa * \mathbb{P}_T$-name $\dot{x}$, $j(\dot{x}^{G_\kappa * G_T}) = j(\dot{x})^{G_\kappa * G_T * K * h}$. So

$$j``N = \{j(g)(y^{G_\kappa * G_T * K * h}) : y \in j``(H(\lambda)^W)\}$$

and this set is definable in $M'[h]$. $\square$

By Proposition 3.5 and Lemma 3.3, N is internally stationary in $M'[h]$. So by Proposition 4.6, $j``N$ is internally stationary in $M'[h]$.

Proposition 5.11. *In $M'[h]$, for all $i < \omega_1$, $j(U_i)$ reflects to $j``N$.*

Proof. Fix $i < \omega_1$. By the closure of $M[G_\kappa * G_T]$, U_i is in $M[G_\kappa * G_T]$ and is stationary in $P_{\omega_1}(N)$. Since $K * h$ is a generic filter for a proper forcing poset over $M[G_\kappa * G_T]$, U_i is stationary in $P_{\omega_1}(N)$ in $M'[h]$. Now $j \restriction N$ is the inverse of the transitive collapse of $j``N$, therefore is in $M'[h]$ by Lemma 5.10. For each $a$ in $U_i$, $a \subseteq N$ and $j(a) = j``a$. So $j``U_i$ is definable in $M'[h]$ from $j \restriction N$ and $U_i$, hence is in $M'[h]$. It is easy to check that since $U_i$ is stationary in $P_{\omega_1}(N)$ in $M'[h]$, $j``U_i$ is stationary in $P_{\omega_1}(j``N)$ in $M'[h]$. But $j``U_i \subseteq j(U_i) \cap P_{\omega_1}(j``N)$, therefore $j(U_i)$ reflects to $j``N$. $\square$

Since $j(\{U_i : i < \omega_1\}) = \{j(U_i) : i < \omega_1\}$, by elementarity there is an internally stationary set with size $\aleph_1$ to which each U_i reflects.

This completes the proof.

We were able to verify stationary reflection in the preceding argument because the iteration $\mathbb{P}_\kappa * \mathbb{P}_T$ is proper, and so preserves the stationarity of a given stationary set after forcing with a tail of $j(\mathbb{P}_\kappa * \mathbb{P}_T)$. On the other hand, the properness of the iteration guaranteed that the set to which the given stationary set reflects is internally stationary. This raises the following question.

Question 5.12. *Does WRP imply that every stationary set $S \subseteq P_{\omega_1}(H(\lambda))$ reflects to an internally stationary set N with size $\aleph_1$?*

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