Mathematical Pluralism and Indispensability

forthcoming in *Erkenntnis*

Silvia Jonas

Abstract

Pluralist mathematical realism, the view that there exists more than one mathematical universe, has become an influential position in the philosophy of mathematics. I argue that, if mathematical pluralism is true (and we have good reason to believe that it is), then mathematical realism cannot (easily) be justified by arguments from the indispensability of mathematics to science. This is because any justificatory chain of inferences from mathematical applications in science to the total body of mathematical theorems can cover at most one mathematical universe. Indispensability arguments may thus lose their central role in the debate about mathematical ontology.[1]

Keywords: *indispensability; mathematical realism; mathematical pluralism; set-theoretic multiverse, Continuum Hypothesis.*

Contents

[1]Many thanks to Marianna Antonutti, Hazel Brickhill, Benedict Eastaugh, Claire Benn, Sharon Berry, Mark Colyvan, Erik Curiel, William D'Alessandro, Peter Gerdes, Hannes Leitgeb, Wolfgang Spohn, Casper Storm Hansen, Edward Zalta, and two anonymous referees for their very helpful comments. Work on this paper has been supported by the European Commission (Marie Skłodowska-Curie Action "Mathematics Analogies", grant number 846522).

0 Introduction

Pluralist conceptions of mathematics, according to which the mathematical cosmos comprises more than one mathematical universe, have been around at least since the beginning of the 19^{th} century. The discovery of non-Euclidean geometries by Lobachevsky and Bolyai in 1829 and 1831 respectively introduced the now firmly established distinction between applied and pure mathematics, and over time overturned the monistic picture of geometry that had been dominant since Antiquity. The introduction of intuitionistic and other non-classical logics, of various constructive mathematics, of category theory and homotopy type theory (HoTT) as potential alternative foundations for mathematics besides set theory finally turned the idea of a plurality of mutually irreducible mathematical enterprises into a fact of mathematical practice.[2] What is fairly new, however, is the view I henceforth refer to as 'pluralist mathematical realism,' i.e. the idea that a pluralist conception of mathematics can be squared with a realist commitment to mathematical ontology. Versions of this view have been defended by Bernard Linsky and Edward N. Zalta (1995), Mark Balaguer (1998b), and Joel David Hamkins (2012).

In section (1), I outline the key features and merits of pluralist mathematical realism, and I respond to three central objections that have been mounted against the view. In section (2), I show how the empiricist justification provided by indispensability arguments for applied mathematics can be extended to pure mathematics, all the way up to higher set theory, via an argument from transitivity. In section (3), I demonstrate that pluralism obstructs the argument from transitivity, so that there is no obvious way to construct an indispensability argument for pluralist mathematical realism. I conclude that going pluralist undercuts the most prominent empiricist argument in favour of mathematical realism. Thus, if pluralism is true, mathematical realism cannot (easily) be justified by arguments from the indispensability of mathematics to science. Indispensability arguments may thus lose their central role in the debate about mathematical ontology.

[2] Mathematical pluralism is the view that there is a plurality of equally legitimate *sui generis*, i.e. mutually irreducible mathematical investigations such as set theory, category theory, intuitionistic, or paraconsistent mathematics. The question where exactly to draw the line between legitimate and illegitimate mathematical investigations is a matter of debate (cf. Clarke-Doane (2022, Ch. 3.5) and Priest (2019)), but unlikely to be decided by philosophers (rather than mathematicians). Recent defences and cautiously nuanced sympathetic discussions of mathematical pluralism include Clarke-Doane (2022); Davies (2005); Hellman und Bell (2006); Koellner (2009) and Priest (2013). For an introduction to intuitionism, see Dummett (1977); to logical pluralism, see Beall und Restall (2006); to constructive mathematics, see Bridges und Richman (1987); to category theory, see Marquis (1995, 2019) and Linnebo und Pettigrew (2011), and to homotopy type theory (HoTT), see the Univalent Foundations Program (2013). It should be noted that the relative standing of different set theories (e.g. ZF vs. NF), mutually exclusive set-theoretic axioms (e.g. AD vs. Choice), of different logics (e.g. classical vs. intuitionistic), as well as of set theory vs. category theory vs. HoTT as alternative foundations for mathematics are all topics of research in their own right, which cannot be discussed in detail here. For an overview of alternative axiomatic set theories, see Holmes (2021); for a discussion of the different ways in which set theory, category theory, and univalent foundations might be regarded as playing a foundational role for mathematics, see Maddy (2019). For an overview of logical pluralism, see Russell (2021). For a discussion of the relationship between mathematical and logical pluralism, see Priest (2021).

1 Pluralist mathematical realism

Pluralist mathematical realism is the view that there exists a plurality of mathematical universes, none of which can be said to be the 'true' or the 'truest' one. The discovery of non-Euclidean geometries in the 19^{th} century made it clear that space could be described in more than one way, and introduced the idea that mathematics is not about finding the best axiomatic expression for human intuitions about space—the one true mathematical theory (Gray, 2019). Rather, the idea took root that mathematicians are free to study any mathematical structure they like, regardless of their usefulness for scientific applications beyond mathematics. As Georg Cantor put it, 'the essence of mathematics lies precisely in its freedom' (Cantor, 1882, §8).

Undecidable statements discovered in the 20^{th} century in higher-order arithmetic and set theory cast additional doubt on the view that there is exactly one uniquely true mathematical theory. Most famous among the undecidable statements is the Continuum Hypothesis (CH), formulated by Cantor (1878) and presented by David Hilbert at the conference of the International Congress of Mathematicians in Paris in 1900 as the first of 23 hitherto unsolved problems. CH states that there are no sets whose cardinality is strictly greater than that of the natural numbers, but strictly smaller than that of the reals, and neither the efforts of Gödel (1938), who showed that CH is consistent with the standard axioms of set theory, the Zermelo-Fraenkel axioms with the Choice axiom (ZFC), nor Cohen (1963), who showed that $\neg$CH is consistent with ZFC, yielded an answer to the question whether CH is true. Rather, the conjunction of their results showed that CH is independent of ZFC, and consequently, that the question whether the reals are the infinite set of second-smallest size cannot be settled in ZFC.[3] At least three different conclusions can be drawn from the fact that there exist mathematical statements whose truth-value cannot be determined in ZFC:

'Universe' view: The ZFC axioms provide only an incomplete description of mathematical reality, so the mathematician's task is to find well-justified extensions of ZFC that will settle all open questions ('Gödel's Program'; cf. Gödel, 1990, p. 163 and Woodin, 2017).

'Vagueness' view: Statements like CH are inherently vague[4] and consequently of no mathematical (if perhaps philosophical) interest; moreover, there is no evidence suggesting that anything beyond ZFC is needed to settle open combinatorial questions that *are* of genuine mathematical interest, such as those listed on the Millenium Prize list (Feferman, 2000, p. 405-407).

'Pluralist' view: The hope of settling open questions like CH in a conventional way, i.e. by finding new axioms that settle them, is misguided because there is no unique absolute theory of mathematics. In particular, there is no absolute background

[3]Note that the above statements of CH and independence assume AC and consistency, respectively.The Continuum Hypothesis is the most famous of the undecidable statements, but of course there are many more, for example whether inaccessible cardinals exist, whether all Whitehead groups are free, whether all projective point-sets are Lebesgue-measurable, etc.

[4]Note that there is an interpretation of this view that entails that not only undecidable statements like CH but also other statements in the language of third-order arithmetic must be considered vague as well; cf. Steel 2000, p. 432 and Potter 2004, p. 275.

> concept of set, and consequently also no unique set-theoretic truths. Rather, there are many different concepts of set, furnishing a 'multiverse', i.e. a plurality of set-theoretic universes, each exhibiting its own set of set-theoretic truths, such that a statement like CH can be true in one and false in another (Hamkins, 2012; Shelah, 2003).

Non-realist pluralists like Shelah do not believe in the actual existence of a multiverse. To them, the plurality of set-theoretic universes is a simple phenomenon of mathematical practice without ontological import (Shelah, 2003; Antos u. a., 2015, p. 2466). The realist pluralism under discussion in this paper, on the other hand, pairs a radically pluralistic view of mathematics with a realist commitment to mathematical ontology, focusing either solely on the set-theoretic realms of mathematics (Hamkins, 2012), or including all logically possible (Balaguer, 1998b; Linsky und Zalta, 1995) or even impossible (Beall, 1999) mathematical objects. In the remainder of this section, I outline the central features of pluralist mathematical realism and a number of reasons for defending the view. I also address three standard objections against it and respond to them.

1.1 Key features

Hamkins (2012) argues that the existence of a set-theoretic multiverse best explains our 'experience' with the large range of alternative set-theoretic models ('universes') that can be constructed using set-theoretic tools like forcing, ultrapowers, and canonical inner models. He writes:

> 'A large part of set theory over the past half-century has been about constructing as many different models of set theory as possible, often to exhibit precise features or to have specific relationships with other models... As a result, the fundamental objects of study in set theory have become the models of set theory, and set theorists move with agility from one model to another... This abundance of set-theoretic possibilities poses a serious difficulty for the universe view, for if one holds that there is a single absolute background concept of set, then one must explain or explain away as imaginary all of the alternative universes that set theorists seem to have constructed. This seems a difficult task, for we have a robust experience in those worlds, and they appear fully set theoretic to us. The multiverse view, in contrast, explains this experience by embracing them as real, filling out the vision hinted at in our mathematical experience, that there is an abundance of set-theoretic worlds into which our mathematical tools have allowed us to glimpse.' (Hamkins, 2012, p. 418)

Hamkins explicitly asserts the actual existence of all those set-theoretic universes, thus committing himself to a realist—in fact, Platonist—view of the multiverse as opposed to non-realist forms of mathematical pluralism, such as formalist ones (Hamkins, 2012, pp. 417, 436). According to his view, there is no unique absolute background concept of set, instantiated in one uniquely true set-theoretic universe. Rather, there are as many different concepts of set as there are set-theoretic universes - multiple equally good interpretations of the notion of set as it appears in first-order ZFC, which, however, 'include' more than the ZFC axioms or anything derivable from some algorithmically recognisable collection of axioms in the language of first order logic.[5] Perhaps the most unorthodox consequence of this position is that the truth-values of set-theoretic statements like the Continuum Hypothesis can vary across the multiverse. However, instead of taking this as a point against the multiverse view, Hamkins suggests that the multiverse view settles CH:

> 'The answer to CH consists of the expansive, detailed knowledge set theorists have gained about the extent to which it holds and fails in the multiverse, about how to achieve it or its negation in combination with other diverse set-theoretic properties.' (Hamkins, 2012, p. 429)

He emphasises that his view neither compromises the status of set theory as an epistemological and ontological foundation for mathematics (because identical 'copies' of all familiar classical mathematical objects and structures, e.g. the natural numbers, can be found in any universe; Hamkins, 2012, p. 419), nor does it reduce to a kind of formalism according to which all universes of the multiverse are somehow equal, for there may be good mathematical reasons for considering some universes more interesting than others (Hamkins, 2012, p. 436).

Balaguer (1995, 1998a,b, 2017) argues, from a philosophical point of view, for 'Full-blooded Platonism' (FBP), a view akin to Hamkins' multiverse view, whose central tenet is a 'plenitude principle' according to which all logically possible[6] mathematical objects exist (Balaguer, 1998b, p. 5).[7] As he explains, the central advantage of FBP is that it offers solutions to two notorious epistemological problems for mathematical realism.[8]

[5] Cf. Soysal (2020) for a discussion of different conceptions of set. It should be noted here that, as Hamkins is well-aware, the question whether one can coherently describe the whole multiverse to state multiverse theory is in need of further discussion, given that the perspective from which one articulates the theory can seem to afford a privileged background set theory (Martin, 2001). Cf. also Clarke-Doane (2022) and Studd (2019) for discussions of issues related to (the formalisation of) generality statements in the foundations of mathematics.

[6] Beall (1999) is sympathetic to Balaguer's account but suggests that FBP should be expanded into a view he calls 'Really Full-blooded Platonism' (RFBP), according to which not only objects described by consistent, but also those described by inconsistent mathematical theories exist. For an extensive discussion of inconsistent mathematics and mathematical theories developed in the context of paraconsistent logic, see Mortensen (1995) and Priest (1997).

[7] To be more precise, Balaguer's argumentative goal in his (1998b) is to show that both mathematical Platonism, understood in the full-blooded, pluralistic way, and mathematical anti-Platonism are perfectly cogent philosophies of mathematics, and that there neither is nor could be a rational reason for us to decide between those two opposing views.

[8] In-passing suggestions to the effect that some kind of plenitude principle can solve the problem of epistemic access for mathematical realists can also be found in Anderson (1990) and Resnik (1989), and a parallel move is suggested by Deutsch (1991) for fictional objects. It should be noted that also Linsky und Zalta (1995) suggest an account of abstract objects intended to solve the two epistemological problems for mathematical realists. According to their view, 'Platonized naturalism', there exist 'as many abstract objects as there could possibly be' (p. 537)—numbers, sets, possible worlds, truth-values, extensions, fictional objects, etc. It is based on three central principles ('A!x' asserts that *x is abstract*):

> '(1) For every condition on properties, there is an abstract individual that *encodes* exactly the properties satisfying the condition. $\exists x(A!x \& \forall F(xF \equiv \phi))$, where s is not free in ϕ.
>
> (2) If x possibly encodes a property F, it does so necessarily. $\Diamond xF \rightarrow \Box xF$.
>
> (3) If x and y are abstract individuals, then they are identical if and only if they encode the same properties. $A!x \& A!y \rightarrow (x = y \equiv \forall F(xF \equiv yF))$.' (Linsky und Zalta, 1995, p. 536)

The first one is the problem of reductive uniqueness, addressed by Benacerraf in his (1965). As the group of French mathematicians working under the pen name 'Bourbaki' managed to demonstrate, all mathematical objects can be characterised, and all mathematical theorems reformulated, in the language of set theory (Bourbaki, 1968). Set theory thus constitutes a foundation for all mathematics. The problem Benacerraf addresses is that there are infinitely many, equally effective ways to reduce numbers to sets.[9] The number three, for example, can be expressed as $\{\{\{\emptyset\}\}\}$ (as suggested by Zermelo) or as $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ (as suggested by von Neumann), but given that von Neumann's reduction does, and Zermelo's reduction does not imply that the number one is a member of the number three, these two reductions are not equivalent. The point is that there are infinitely many sequences that satisfy the axioms of Peano Arithmetic (PA), whereas traditional (monist) mathematical realism seems to imply that our mathematical theories describe unique mathematical objects, and particularly a unique sequence that constitutes the natural numbers.[10]

Balaguer solves this problem by arguing that nothing in the idea of mathematical realism commits us to the belief that mathematical theories describe unique collections of mathematical objects. Consequently, he argues, mathematical realists can simply embrace the idea that all consistent mathematical theories, while truly describing collections of mathematical objects, do not describe unique collections of mathematical objects (Balaguer, 1998b, pp. 84-91). The problem of non-uniqueness thus turns out to be a 'non-problem' for pluralist mathematical realists.[11]

The second problem that receives an answer on the FBP-view is how to explain how epistemic access to non-spatiotemporal abstract objects is possible. According to FBP,

Their idea is to reconcile scientific naturalism with realism about mathematics and other abstract objects by introducing the general comprehension principle (1), which is characterised as being both synthetic and a priori, and by arguing that knowledge of abstract objects is linked to knowledge of this comprehension principle. A consequence of this view is that abstract objects (unlike concrete ones) are not subject to an appearance/reality distinction, not sparse, and not complete (p. 532).

[9]Though one might add here that only von Neumann's is extendible to the transfinite.

[10]Also structuralists like Resnik (1997) and Shapiro (1997) maintain that their view solves the problem of reductive uniqueness: they argue that mathematical theories pick out structural facts about the relations between mathematical objects, not facts about the internal properties of mathematical objects, so that the non-uniqueness problem does not arise for them. Balaguer points out that it is possible to reformulate the problem in such a way that it also applies to structuralism:

'(1') If there are any parts of the mathematical realm that satisfy the axioms of PA, then there are infinitely many such parts.

(2') There is nothing "metaphysically special" about any of these parts of the mathematical realm that makes it stand out from the others as *the* sequence of natural numbers (or natural-number positions or whatever).

Therefore,

(3') There is not a unique part of the mathematical realm that is the sequence of natural numbers (or natural-number positions or whatever).

But

(4') Platonism entails that there *is* a unique part of the mathematical realm that is the sequence of the natural numbers (or natural-number positions or whatever).

Therefore, (5') Platonism is false.'

He concludes: 'So platonists cannot solve the non-uniqueness problem by merely adopting structuralism and rejecting the thesis that mathematics is about objects, because the problem remains even after we make the switch to a structuralistic platonism' (Balaguer, 1998b, p. 81).

[11]Though Cheyne (1999) believes that there remains a worry about reference.

all logically possible objects exist, so

> "to acquire knowledge of mathematical objects, all we need to do is acquire knowledge that some purely mathematical theory is *consistent*... But knowledge of the consistency of a mathematical theory—or any other kind of theory, for that matter—does not require any sort of contact with, or access to, the objects that the theory is about. (Balaguer, 1998b, p. 48f; his italics)

In this way, the long-standing worry that mathematical realists may have no way to explain how we gain knowledge of mind-independent abstract objects receives an answer: all we need to do to acquire epistemic access to mathematical objects is to 'dream up' a consistent mathematical theory that describes them (Balaguer, 1998b, p. 49). Note that Balaguer's solution applies both to Benacerraf's original formulation of the access problem in terms of causal contact (Benacerraf, 1973, p. 673), and to Field's reformulation in terms of an unexplained striking match between our mathematical beliefs and the mathematical truths (Field, 1989, p. 26).[12]

Of course, many feel that some mathematical statements, for example arithmetical statements like '5+7=12', are true not only because their truth follows from the Peano axioms, but somehow absolutely. In order to account for this widely perceived difference between the truth of natural number statements and, say, statements like 'The size of the continuum is $\aleph_1$', Balaguer draws a distinction between 'true in a structure' and 'true simpliciter.' A mathematical statement m is true simpliciter iff there is an *intended* structure (a unique 'standard model') of the given branch of mathematics and m is true in the standard model. The natural numbers, for example, are the standard model of arithmetic. Statements like '5+7=12' are true in the structure of arithmetic as well as in the particular model of that structure constituted by the natural numbers. '5+7=12' is therefore true *simpliciter*, i.e. absolutely true.

It is not at all clear, however, that there exists a standard model for every part of the mathematical cosmos: for some mathematical objects, our full conception of the objects in question may not be precise enough to pick out one unique structure. To give an example, according to ZFC, sets are well-founded,[13] but it is not necessary to define them in this way (cf. Rieger, 2008, p. 175). In fact, there may be good mathematical reasons not to restrict the notion of sets so strongly, for instance the need to investigate 'exciting new ideas and intuitions [notably in the contexts of process algebra and non-standard analysis] that are in need of suitable mathematical expression' (Aczel, 1988, p. xix; my insertion). It may therefore be the case that there is not one uniquely true concept of set, but many subtly different ones, each one furnishing a different part of the mathematical cosmos. If this is the case, as both Balaguer and also Hamkins believe, then set-theoretic statements

[12]See Clarke-Doane (2019) for a critical discussion of set-theoretic pluralism and the access problem.

[13]The Axiom of Regularity $\forall S(S \neq \emptyset \rightarrow (\exists x \in S) S \cap x = \emptyset)$ states that every non-empty set contains an element that is disjoint from S, so that the relation $\in$ on any family of sets is well-founded. Among other things, this axiom postulates the impossibility of infinitely descending sequences (such as $x_0 \ni x_1 \ni x_2 \ni ...$), sets that are members of themselves ($x \in x$), and cyclical chains of membership (such as $x_0 \in x_1 \in x_2 ... x_n \in x_0$). This restriction on the notion of set has no consequences for the development of ordinary mathematical objects like natural numbers, real numbers, etc. However, it has important consequences for the construction of set-theoretic models because it allows all sets to be assigned a rank and thus, to be arranged in a cumulative hierarchy; cf. Jech (1997, Ch. 6)

like CH are only ever 'true in a structure' (i.e. a particular set-theoretic model), never 'true simpliciter' (i.e. true in all set-theoretic models).[14]

To sum up: Pluralist mathematical realism is the view that there exists not only one uniquely true mathematical universe, but a multitude of mathematical universes, some of which are nested in one another, others mutually incompatible. Although mathematicians may have good reasons to investigate some of those universes and ignore others, there is no single mathematical universe that has a metaphysically privileged status.

There are strong mathematical as well as philosophical reasons to adopt pluralist mathematical realism. First, the view comes with all the different advantages attributed to more traditional, monist forms of mathematical realism, such as offering a uniform semantics for mathematical and non-mathematical statements (Benacerraf, 1973, p. 661), being able to account for the truth-aptitude (Dummett, 1978), objectivity (Putnam, 1979b), and irreducibility (Maddy, 1990; Bigelow, 1988) of mathematical truths, for mathematics' 'unreasonable effectiveness' in natural science (Wigner, 1960), its indispensability to the formulation of (Quine, 1981; Putnam, 1979b), and its explanatory contribution to (Baker, 2005, 2009; Colyvan, 2001, 2010), our best scientific theories. Second, pluralist mathematical realism (unlike monist mathematical realism) accurately reflects the fact that mathematicians have been exploring mutually incompatible mathematical theories for decades. Third, the view also accommodates the fact that not all mathematics 'fits' (Priest, 2013) into ZF(C) or any other overarching mathematical theory. Fourth, pluralist mathematical realism offers solutions to the problem of undecidable statements like CH (Hamkins), to the problem of squaring mathematical realism with scientific naturalism (Linsky and Zalta) and to the two most important epistemological problems for mathematical realism, the problem of epistemic access and the problem of reductive uniqueness (Balaguer). In short, the view features numerous merits traditional monist mathematical realism lacks. In the next section, I address the three most important objections against the view.

1.2 Central objections

Contradiction

One immediate worry is that pluralist mathematical realism might be contradictory. If all definable mathematical theories truly describe mathematical reality, then (say) both ZFC+CH and ZFC+¬CH truly describe mathematical reality. This is contradictory.

The pluralist's answer to this worry is that contradiction would only arise if both theories were taken to describe the same part of the mathematical realm. Yet that is not the case because both theories describe different mathematical universes instantiating different kinds of sets ($sets_{CH}$ vs. $sets_{\neg CH}$). Importantly, 'CH and ¬CH' will never be a theorem of any consistent formal mathematical theory, but will always be false. This forestalls the objection that 'CH and ¬CH' might truly describe mathematical reality while not being satisfiable (Balaguer, 1998b, p. 58f.).

[14] Hamkins is committed to the idea that there is no uniquely true concept of set, whereas Balaguer thinks it at least possible that mathematicians will one day agree on a standard model for set theory; cf. Balaguer (1998b, p. 64).

Truth

A second worry is that the notion of truth implicit in the characterisation of pluralist mathematical realism does not correspond to that of mathematical practice. If, as the pluralist holds, all definable mathematical theories truly describe mathematical reality, then consistency is sufficient for truth. But practising mathematicians treat only some consistent mathematical theories as true. So the pluralist's interpretation of 'true' seems to differ significantly from that of the practising mathematician.

Pluralists like Balaguer reject this concern as follows:

> What mathematicians standardly mean when they say that a sentence is true is that it is true in the *standard model*, or the *intended structure*... for the given branch of mathematics... The claim that a model (or class of models) is standard is a claim about *us* rather than the model; what is being claimed is that this is the model (or class of models) that is *intended*, that is, that we *have in mind* with respect to the given theory. Thus, for instance, a model of set theory is standard if and only if it jibes with *our notion of set*; and a model of arithmetic is standard if and only if it jibes with *our conception of the natural numbers*; and so on. (Balaguer, 1998b, p. 60; his italics)

The idea is to introduce the above-mentioned distinction between *truth simpliciter* and *truth in a structure*.[15] The latter is analogous to the notion of *truth in a language* L, with a language denoting an abstract object mapping sentence types onto truth conditions. Hence, to say that every definable mathematical theory truly describes a collection of mathematical objects is just a different way of saying that every such theory is true in some language L that is a mathematical structure and defines part of the mathematical realm.[16]

Objectivity

A third objection is that pluralist mathematical realism undermines the objectivity of mathematics. Even though the view does entail that mathematical statements are objectively true in the sense of 'true of a mind-independently existing mathematical realm,' one might worry that it also entails that undecidable statements like CH do not have determinate truth-values, and that this undermines the objective truth of mathematics in the sense of 'uniquely correct truth-values for every mathematical statement.'

However, pluralists can dismiss this objection by again pointing out that the notion of objectivity we often associate with mathematics hinges on the previously introduced notion of *truth simpliciter*. To say that a mathematical statement like '5+7=12' is objectively true in the sense of having a uniquely correct truth-value is just a different way of

[15] Note that the pluralist does not claim that every part of the mathematical realm has a standard model. There is wide agreement that the natural numbers constitute the standard model for the structure of arithmetic, but whether or not set theory has a standard model is a matter of debate (see, for example, Feferman u. a., 2000). Pluralist mathematical realism is neutral with regard to the question whether there exist, or will exist, standard models for every branch of mathematics.

[16] However, there are also multiversists who consider the main issue with multiversism to be the question how to justify the selection of certain universes as better than others (see for example the Hyperuniverse Program, discussed in Antos u. a. (2015)).

saying that there exists a standard model of arithmetic, i.e. the natural numbers, and that the statement '5+7=12' is true in that standard model. Conversely, to say that a mathematical statement like CH does not have a uniquely correct truth-value is just a different way of saying that there does not (yet) exist a standard model of set theory.[17] Thus, pluralist mathematical realism is in fact in a better position to interpret mathematical practice than traditional monist realism: the former, but not the latter, can account for the *de facto* existence of undecidable mathematical questions without uniquely correct answers (Balaguer, 1998b, p. 62ff). The view thus 'reconciles the objectivity of mathematics with the extreme freedom that mathematicians have' (Balaguer, 1998b, p. 69).

To sum up: Pluralist mathematical realists have straightforward responses to objections from contradiction, truth, and objectivity.[18] Together with the merits discussed above, this makes pluralist mathematical realism a plausible and attractive philosophy of mathematics.

Note, however, that none of the accounts discussed offers a reason to believe in mathematical realism in the first place; they merely show that pluralist mathematical realism is an *internally* coherent view. *External* justifications of realist commitments to mathematical ontology have long been rooted in empiricist considerations about the indispensable role mathematics plays in scientific practice. In the next section, I argue that no such empiricist justification is available for pluralist mathematical realism.

2 Indispensability: applied vs. unapplied mathematics

Empiricist justifications of mathematics go back at least to Mill, who considered the laws of arithmetic to be highly general laws of nature (Mill, 1947). Frege criticised Mill for confusing applications of arithmetic with arithmetic itself (Frege, 1988, § 23), but agreed that 'it is applicability alone that elevates arithmetic from a game to the rank of a science' (Frege, 1962, § 91). Several decades later, empiricist justifications of mathematics took a more precise form in the works of Quine and Putnam, who argued that, since scientific theories get empirically confirmed as wholes, we ought to be ontologically committed to all entities quantification over which is indispensable to our best empirical theories (Quine, 1981; Putnam, 1979a). Moreover, they argued, since we do assume, by inference to the best explanation,[19] the existence of scientific unobservables like quarks and electrons, rejecting ontological commitment to mathematical entities would amount to 'intellectual dishonesty' (Putnam, 1979a, p. 347). According to this argument, now known as (Quine's and Putnam's[20] version of) the 'indispensability argument,' mathematical statements are

[17] Remember that there is nothing metaphysically special about standard models—the question whether or not there exists a standard model for a given branch of mathematics calls for a sociological or psychological answer, not a metaphysical one (Balaguer, 1998b, p. 64f).

[18] For more detailed discussions of the three objections discussed here, as well as of some additional objections, see Balaguer (1998b, Chs. 3 and 4), as well as Cheyne (1999); Davies (2005); Colyvan und Zalta (1999); Hellman und Bell (2006), and Priest (2013).

[19] For discussions on the relation between indispensability arguments and inference to the best explanation, cf. the special issue of *Synthese* on 'Indispensability and Explanation' (2016).

[20] Although it has become customary in discussions of indispensability arguments, lumping together Quine's and Putnam's views on the matter is controversial, given that, beyond the basic idea of the

literal statements about entities like numbers or sets, whose existence we are justified in assuming because of the indispensable role they play in our best scientific theories.

2.1 The indispensability argument

Quine's and Putnam's indispensability argument has been criticised for a variety of reasons. Field (1980) contends that it is the 'one and only one serious argument for the existence of mathematical entities' (p. 5), but then argues that mathematics is merely conservative over science, and mathematical entities consequently not necessary for drawing inferences about the physical world. Maddy (1992) points out that scientists accept only some parts of well-confirmed scientific theories as literally true, whereas other parts (e.g. idealisations like frictionless planes, infinitely deep oceans, or continuous space-time) are considered useful but false—yet it is precisely in those parts that the mathematics used in science typically appears. Sober (1993) points out that a hypothesis is supportable by observations iff there are observations that would count against it, but of course, mathematics features in all competing empirical theories, hence empirical evidence cannot confirm or disconfirm mathematical theories. Melia (2000, 2002) concedes that quantifying over mathematical entities in scientific theories may allow us to express possibilities about the concrete world that are inexpressible otherwise, but then argues that mathematical contributions to the formulation of empirical theories serve merely to index causally efficacious physical properties, and thus, have no ontological bearing on our 'picture of the world' (Melia, 2000, p. 474). There have also been worries that the indispensability argument can neither account for the self-evidence of basic mathematical statements (Parsons, 1980), nor explain why mathematics is indispensable to science (Kitcher, 1984), and that it may in fact be impossible for us to decide whether Quine's and Putnam's indispensability argument holds because there is no principled way to choose among competing criteria for evaluating the ontological commitments of a discourse (Azzouni, 1998).

In light of these criticisms, a new form of the indispensability argument for mathematical realism has emerged. The 'confirmational holism' (allegedly[21]) implicit in Quine's and Putnam's original argument, according to which all parts of a scientific theory get confirmed when the theory as a whole is confirmed, has been dropped. Instead, the new, 'enhanced' version of the indispensability argument assumes that we should be ontologically committed only to those parts of a scientific theory that are *explanatorily* indispensable, i.e. that contribute to the explanation of the physical phenomenon the scientific theory aims to explain. The crucial question for mathematical realists thus becomes whether there exist empirical phenomena that can be explained only with the contribution of mathematics, and it turns out that there are a number of examples that seem to fit the bill. The most widely discussed case is the *Magicicada*, a special type of North American cicada with a prime-numbered life cycle (Baker, 2005, 2009). *Magicicadas* emerge from the ground only every 13 and 17 years, most likely in order to avoid intersection with periodic predators (Goles u. a., 2001) and/or hybridisation with similar subspecies (Cox und Carlton, 1988; Yoshimura, 1997). Indispensable to both explanations are not only certain ecological facts and general biological laws, but also the number-theoretic fact that prime

argument, they disagree on a number of assumptions (cf. e.g. Putnam, 2012).

[21] See also previous footnote. It has been noted by many that confirmational holism is neither held by Putnam, nor necessary for formulating his argument; cf. e.g. Liggins (2008); Panza und Sereni (2015).

numbers maximise their lowest common multiple relative to all lower numbers (Landau, 1958). Since numbers are thus indispensable to explaining the life cycles of *Magicicadas*,[22] Baker (2005, p. 236) argues, we ought to conclude that numbers exist. More generally,

> If mathematics is contributing directly to explanations, it is hard to see how anyone who accepts inference to the best explanation can accept the explanations yet deny the truth of mathematics. (Colyvan, 2007, p. 120)

The debate about the role of mathematics in scientific explanations and the implications of that role for mathematical ontology is thus in full swing, and though the status of the indispensability argument is far from decided, two things have become clear. First, arguments for mathematical antirealism must address the question where, if at all, indispensability arguments go wrong (Baker, 2003; Field, 1980, 1989; Fine, 1984). Second, there seems to be wide agreement (between mathematical realists and antirealists alike) that questions concerning mathematical ontology, and consequently, the success of mathematical realism, are directly dependent on the role of mathematics in science.

2.2 Mathematics beyond scientific application: the transitivity argument

The plausibility of mathematical realism is thus widely believed to hang on an empiricist justification involving applications in science. However, only some parts of mathematics actually are 'applied'; other parts, often labelled 'pure' or 'recreational' (Quine) mathematics, investigate 'speculative and daring extensions of the basic mathematical apparatus of science' (Putnam, 1979a, p. 56)—mathematical objects, structures, and questions that have no bearing on scientific applications whatsoever. Do indispensability arguments justify ontological commitment to those parts of mathematics as well?

It is of course true that some theorems from pure mathematics once thought to be without physical relevance were later shown to be (indirectly) useful to scientific reasoning. However, it is also true that some parts of the mathematical cosmos, notably in the higher reaches of set theory, exhibit mathematical structures whose physical representation is categorically impossible. To see why, consider that space-time is usually construed as a continuum-sized set of $2^{\aleph_0}$ points (=cardinality of the real numbers 2^{ω}). The number of physically possible objects—objects representable in space-time—can therefore not exceed $2^{2^{\aleph_0}}$ (=cardinality of the power set of real numbers $2^{2^{\omega}}$). And in fact, there are good reasons to believe that the requirements of physically applied mathematics are well-covered by second-order real analysis (equivalently third-order number theory), the branch

[22] Other cases of allegedly indispensable mathematical explanations in science include the falling pattern of sticks thrown into the air (Lipton, 2004); the crossing of bridges at Königsberg (Pincock, 2007); the geometrical properties of curved vacuum space-times that manifest themselves in the bending of light rays (Colyvan, 2001, 2002); the Lorentz–Fitzgerald contraction of moving bodies in special relativity induced by the geometrical properties of the Minkowski metric (Colyvan, 2001, 2002); the location of the Kirkwood gaps (Colyvan, 2010); the hexagonal shape of honeycomb cells (Lyon und Colyvan, 2008; Lyon, 2012); the spiral arrangement of sunflower seeds (Lyon, 2012); and Plateau's laws for soap films (Lyon, 2012; Pincock, 2015).

of mathematics that studies the behaviour of sets of real numbers (RA^2).[23] Yet there are numerous set-theoretic structures whose existence is uncontroversial among mathematicians but whose cardinality exceeds $2^{2^{\aleph_0}}$, and if the physical representation of such entities in scientific applications is impossible, then we have no empirical justification to believe in their existence.[24]

Quine's answer to this worry is as follows:

> Pure mathematics, in my view, is firmly imbedded as an integral part of our system of the world. Thus my view of pure mathematics is oriented strictly to application in empirical science. Parsons has remarked, against this attitude, that pure mathematics extravagantly exceeds the needs of application. It does indeed, but I see these excesses as a simplistic matter of rounding out. . . I recognize indenumerable infinites only because they are forced on me by the simplest known systematizations of more welcome matters. Magnitudes in excess of such demands, e.g., $\beth_\omega$ or inaccessible numbers, I look upon only as mathematical recreation and without ontological rights. Sets that are compatible with 'V = L' in the sense of Gödel's monograph afford a convenient cut-off. (Quine, 1986a, p. 400)

Maddy (1992) criticises this response by pointing out that mathematicians don't decide questions of, say, new axioms for set theory, by checking if the axiom candidates are appropriately applicable to scientific theory. Rather, mathematics follows its own methodology, and if we are to respect mathematical practice and its inherent methods, then we ought to accept that applicability simply plays no role for the acceptance or rejection of new mathematical entities. Maddy then suggests a 'modified' indispensability argument, which

> first guarantees that mathematics has a proper ontology, then endorses (in a tentative, naturalistic spirit) its actual methods for investigating that ontology. For example, the calculus is indispensable in physics; the set-theoretic continuum provides our best account of the calculus; indispensability thus justifies our belief in the set-theoretic continuum, and so, in the set-theoretic methods that generate it; examined and extended in mathematically justifiable ways, this yields Zermelo-Fraenkel set theory. (Maddy, 1992, p. 280)

In a similar spirit, Colyvan (2007) argues for the 'transitivity of indispensability':

> If a nail gun is indispensable to building houses and building houses is indispensable to building suburbs, then a nail gun is indispensable to building suburbs. Similarly for mathematics. If transfinite set theory is indispensable

[23]Burgess (1984, p. 386) argues that it is 'probably sufficient to develop, making much use of coding devices, all the mathematics that has found scientific applications up to the present.' Also Hellman (1989, p. 105) grants that 'it defines. . . a limit to the mathematical richness of what can be conceived of as "concrete structures".'

[24]Parsons (1983) discusses this and other implications of higher set theory for Quine's philosophy of mathematics, specifically for his empiricist claim that there is no philosophically interesting distinction between mathematics and natural science, mathematical and natural necessity, and mathematical and physical existence (cf. Quine, 1969, 1986b).

> for analysis and analysis is indispensable for physics, then I say transfinite set theory is indispensable for physics. Perhaps this is what Quine had in mind with his notion of 'simplificatory rounding out'. In any case, this is the justification for the higher reaches of set theory that I endorse. Understood this way, there is only one mode of justification: playing an indispensable role (either directly or indirectly) in our best scientific theories. (Colyvan, 2007, p. 113)

One might object here that the idea of transitivity invoked in Maddy's and Colyvan's arguments involves an unjustified change in the meaning of 'indispensable'. The initial step of (enhanced) indispensability arguments justifies belief in the existence of mathematical entities via the indispensable role those entities play in *scientific* explanations. The second step suggested by Maddy and Colyvan then justifies belief in the existence of mathematical entities beyond RA^2 via the the indispensable role those entities play in *intra-mathematical* proofs and explanations. However, it is not clear that our pre-philosophical commitment to science (i.e. its methods, inference patterns, objectivity, etc.), on which indispensability arguments are based, extends also to mathematics itself. Indeed, there is reason to believe that it doesn't—otherwise any justification of mathematics via science would be superfluous. As a consequence, it is not clear that we can 'bootstrap' our way from individual mathematical *explanantia* (like the prime numbers in the *Magicicada* case) all the way up and beyond physically applied mathematics.

However, there is a more charitable interpretation of the notion of 'transitivity of indispensability' that builds on examples in which mathematics transcending RA^2 can be shown to be indirectly physically significant. One such example is Gleason's theorem (Gleason, 1957). The most general version of this theorem uses mathematics well beyond RA^2. Nevertheless, it has indirect physical significance insofar as it entails that there can be no dispersion-free measures on the sub-spaces of Hilbert space (equivalently, that under minimal, physically significant assumptions, the standard probability measure used in quantum theory to calculate the probabilities of experimental outcomes—the one defining the Born Rule—is the only one possible; Piron, 1976, pp. 73-81). The moral of such examples is that understanding physical theory properly will, at least in some cases, necessarily involve entertaining mathematical structures that take us beyond the physically representable structures of RA^2. The use of mathematics that is not physically representable is thus indispensable to our understanding of at least some parts of physics.

Now, this much 'non-representable' mathematics might be easy to accept. But what about those parts of mathematics, found, for example, in higher set theory, that have no obvious connection to physical theory whatsoever, that appear nowhere in a chain of applications that bottoms out in a scientific theory, and of which it is consequently utterly doubtful that they will ever be needed to formulate or understand scientific theory? Does the 'transitivity of indispensability' justify ontological commitment even to those parts of the mathematical realm?

In order to see that it does, we need to consider a second, slightly more speculative class of examples that can be derived from the fact that higher-order mathematical theories are non-conservative over lower-order ones. Gödel's incompleteness theorems famously demonstrate that mathematical theories involving strong existence axioms can deductively entail sentences expressible in a lower-level mathematical language that are undecidable at that level. For example, if first-order Peano arithmetic (PA^1) is consistent, then the number-theoretic assertion that this is the case is undecidable in PA^1 (though it is, of course, decidable in any set theory that proves the existence of a model of the natural numbers). More generally, mathematical systems featuring objects of higher type have consequences on lower levels, consequences that are not demonstrable (or refutable) in the formalism of the lower level. As Hellman puts it, "[b]ecause of this phenomenon of non-conservativeness of richer mathematical theories with respect to lower-level theories, there arises the prospect of justifying the richer theories indirectly in virtue of their power to decide questions at the lower (more "observational") level that otherwise would remain undecided (except, perhaps, in ad hoc extensions)" (Hellman, 1989, p. 121).

While most examples demonstrating this feature meta-mathematical rather than mathematical statements (and are thus of limited value for the project of providing an empiricist underpinning to realism about unapplied mathematics), there are a number of undecidable statements that are of direct mathematical concern. One such example is the Paris-Harrington theorem (Paris und Harrington, 1977), which proves that a number-theoretic statement of a cleverly-modified finite version of Ramsey's partition theorem is unprovable in Peano arithmetic. Friedman (1981) discusses a number of other cases, for example intuitive mathematical statements whose proof requires uncountable iterations of the power set operation (e.g. for statements like 'Every symmetric Borel subset of the unit square contains or is disjoint from the graph of a Borel function'), or statements that are provable in ZFC plus a large cardinal axiom, but not in ZFC. 'Thus,' Hellman summarises, 'even at the level of "large, large cardinals"... it is necessary to go even further to prove certain "natural mathematical statements"' (Hellman, 1989, p. 122, footnote 118).

To sum up: If the notion of 'transitivity of indispensability' is interpreted along the lines of the examples just discussed, then it becomes clear how the 'bootstrapping' from individual mathematical *explanantia* featuring in scientific explanations all the way up to physically unapplied mathematics and even higher set theory is supposed to work: higher-level theories are justified in virtue of the indispensable role they play in deciding, at the lower level, natural mathematical questions, which may, in turn, have some bearing on applied mathematics.[25] Call this combination of the (enhanced) indispensability argument and the argument from the transitivity of indispensability the *transitivity argument*. The transitivity argument thus justifies realist ontological commitment to (a) individual mathematical entities that contribute indispensably to scientific explanations of empirical phenomena (e.g. natural numbers), but also to (b) all entities that contribute indispensably to the mathematical explanations (proofs) of mathematical theorems that feature in scientific explanations, as well as to (c) all entities featuring in mathematical explanations of mathematical theorems, which in turn contribute indispensably to the mathematical explanation of mathematical theorems that contribute indispensably to scientific explanations of empirical phenomena, and so forth. In the next section, I argue that the transitivity argument fails to justify mathematical realism if mathematical pluralism is true, i.e. if there exists more than one mathematical universe.

[25] Gödel based some of his deliberations concerning the justification of strong axioms of infinity for set theory on this potential (cf. Gödel, 1990). Hellman (1989, Ch. 3) offers an extensive discussion of the question how much mathematics is needed for physics, which includes of all the examples mentioned above.

3 The transitivity argument and pluralist mathematical realism

We have seen that, according to the transitivity argument, justification of ontological commitment to unapplied mathematics follows abductively from the justification to believe in the existence of all mathematical *explanantia* featuring in confirmed scientific theories. The question now becomes whether there are conditions under which the transitivity argument does not work.

3.1 Pluralism obstructs transitivity

Recall that what guarantees that our justification of ontological commitment covers *all of* mathematics are the explanatory links between the entities of applied and unapplied mathematics. However, there is an additional assumption that needs to be made, namely that every part, or branch, of mathematics is connected by an explanatory link to some other part. In other words, for the transitivity argument to work, we need to assume that there are no parts of the mathematical realm that are explanatorily isolated from all other parts. And indeed, for the mathematical monist who believes that the mathematical cosmos consists of exactly one mathematical universe, it is reasonable to assume that all parts of contemporary applied and unapplied mathematics are thus connected.

To see why, consider that set theory constitutes a foundation for all mathematics. Every mathematical object can be expressed in its language: the natural numbers can be defined as the finite ordinal numbers such that $\mathbb{N} = \omega$, the integers $\mathbb{Z}$ as the set of equivalence classes of pairs of natural numbers such that $(n, m) \equiv (n', m')$ iff $n+m' = n'+m$, the rationals $\mathbb{Q}$ as the set of equivalence classes of pairs of integers (p, q) such that $p \neq 0$ and $(p, q) \equiv (p', q')$ iff $p \cdot q' = p' \cdot q$, the reals as Dedekind cuts of $\mathbb{Q}$, algebraic structures like groups, rings, or lattices as sets of n-tuples of elements of a set, and so on. Consequently, every mathematical statement can be reformulated in the language of set theory, and every mathematical theorem derived from ZFC (or some extension thereof) using the calculus of first-order logic (cf. Bourbaki, 1968; Bagaria, 2020, Section 5). All parts of contemporary mathematics are thus connected by their common set-theoretic foundation, which in turn justifies the assumption that there are no parts of the mathematical realm that are entirely isolated from all other parts.

However, things look different for the mathematical pluralist. On her view, the mathematical cosmos consists of a plenitude of mathematical universes, none of which enjoys a privileged metaphysical status, some of which are nested in others, others of which exist 'side by side' such that the truth-value of a mathematical statement like CH can differ from universe to universe. We noted that the pluralist's idea of a mathematical multiverse neither thwarts the meaning of 'truth,' nor the objectivity of mathematics (provided that we distinguish between *truth simpliciter* and truth in any definable model), and it also doesn't imply a contradiction because each mathematical universe is taken to exist in complete isolation from all other universes.

However, it is precisely this premise from isolation that simultaneously obstructs the transitivity argument. Of course, as demonstrated above, within each individual universe, there exist explanatory links between all individual mathematical entities contributing indispensably to scientific explanations up to higher set theory. However, there exist no 'horizontal' explanatory links between strictly independent mathematical universes, such as V_{CH} and $V_{\neg CH}$. As a consequence, proponents of pluralist mathematical realism cannot appeal to the indispensability of mathematics to science in order to support mathematical realism. The view thus loses its empiricist justification.

3.2 Monist justification, pluralist ontology?

The most straightforward way to resist the conclusion that belief in the existence of a mathematical multiverse cannot be empirically justified is to bite one bullet and spit out another. Pluralist mathematical realists could admit that an empiricist justification of realist belief through arguments from indispensability and transitivity can only ever cover one out of the many mathematical universes that constitute the mathematical cosmos. This is the biting part. They could then try to come up with some argument as to why the absence of empirical justification for realist belief in *all* mathematical universes does not harm the pluralist's overall ontological project. This is the spitting part. In other words, pluralists could try to argue that it is perfectly consistent to embrace a monist justification of realist belief while hanging on to a pluralist ontology.

No horizontal transitivity

There are at least two problems with this approach. One is that it is not clear what such an argument would look like. Ontological commitments require full, not merely partial justification, and nothing will serve to justify belief in a pluralist mathematical cosmos except an argument that covers *all* mathematical universes. One idea might be to try constructing a 'horizontal' transitivity argument that establishes explanatory links between mutually independent mathematical universes. However, given the very definition of these universes as strictly isolated from one another, such an argument would be hard pressed to defend itself against the charge of being *ad hoc*.

Another idea is to appeal to Maddy's line of reasoning: establish that 'mathematics has a proper ontology' (Maddy, 1992, p. 280), for example by arguing for the existence of natural numbers via an enhanced indispensability argument, and then let practising mathematicians fill us in on the details of this ontology. But, as was argued above, this line of reasoning either involves a change in the meaning of 'indispensable' (from 'indispensable to scientific explanations' to 'indispensable to intra-mathematical explanations'), which is unhelpful if the goal is to offer a *science*-based justification of ontological commitment. Or it must be interpreted along the lines sketched in section 2.2, i.e. as an argument for the (potential) rootedness of all mathematics in scientific theory, which, as was argued, covers one universe at most.

Someone might try to salvage indispensability for multiversists by arguing that instead of a single universe V, the realist could analogously adopt a single multiverse MV, which is built in stages, much like V arises as the union of the V_α's through iteration of the powerset operation. MV arises as the union of the MV_α's, the multiverse of possible V_α's (MV_ω has just the single V_ω, but $MV_{\omega+1}$ already has many different $V_{\omega+1}$'s, arising from different powerset operations applied to V_ω). And as in the single universe view, where the higher V_α's are justified by their need to explain the lower V_α's, so too can the higher MV_α's be justified by their use in explaining the lower MV_α's.

The problem I see with this approach is that according to pluralists like Hamkins, each universe's 'notion of omega' turns out to be a nonstandard model from the point of view of some other universe. Furthermore, it is not clear that we can quantify over multiverses in the way suggested by this example. Thus, implementing this analogous indispensability approach to the multiverse seems problematic. One way for pluralists to salvage the indispensability argument would be to adopt a more restrictive multiverse, for example by imposing well-foundedness (and thus obtaining the same copy of the natural numbers in every universe) or by considering only a privileged sub-multiverse consisting of the 'better' universes (for example those characterised by the maximality criteria of the Hyperuniverse Programme). However, radical pluralists like Hamkins or Balaguer would probably see no compelling reason to restrict 'mathematical freedom' in this way.

Another way to salvage indispensability for multiversists might be to argue that there are in fact 'connections' between the different universes of the multiverse, which are the object of investigation of set-theoretic geology as conceived by Fuchs u. a. (2015). We standardly think of forcing as the technique used to construct outer models of set theory by taking a a model of set theory V and constructing a larger forcing extension V[G] through a V-generic filter G over some partial order P $\in$ V. Set-theoretic geology takes the opposite perspective by asking how the model V might have come about through forcing. The idea is to study the grounds of V (i.e. the transitive proper classes W $\subseteq$ V, such that W $\models$ ZFC and V is obtained by set forcing over W, so that V = W[G] for some W-generic filter G $\subseteq$ P $\in$ W) by investigating the classes over which V can be realised as a forcing extension (Fuchs u. a., 2015, p. 464f). Among the questions asked in set theoretic geology are whether the ground models of V are downward directed (p. 469f), whether the intersection of all grounds of V (the 'mantle' of V) necessarily satisfies ZF or ZFC (p. 475), and under what circumstances the mantle is also a ground model of V (p. 476).

Discussing these questions in detail here is well beyond the scope of this paper, but two things are worth noting. First, there is a difference between a mere connection and an explanatory connection. I have argued above that if a non-applied mathematical theory T can be used to prove anything about applied mathematics that cannot otherwise be proved, then the empirical justification for applied mathematics can be extended to T. I have not argued that any kind of connection between two theories suffices. Now, the motivations given by multiversists for positing a multiverse and studying the relations between given ground models do not involve a desire to better explain facts about one favoured and directly physically useful hierarchy of sets. For example, Hamkins argues that the phenomenology of forcing is such that it seems to afford contact with something real; he does not argue that it better explains some intrinsic feature of a universe of sets one is currently working in.

Second, and despite what was just said, it is of course possible that studying the connections between different universes might at some point be used to prove a theorem about applied mathematics that cannot otherwise be proved. However, the idea that set-theoretic geology might provide the relevant explanatory connections among different models that inhabit the multiverse is a very general and substantive thesis that would require an accordingly general and substantive argument to stand. Set-theoretic geology studies the relationship between a universe and its grounds and extensions obtained via set-forcing, but only very few universes are connected to each other in this way. Salvaging the indispensability argument would require connections between arbitrary universes. Again, it might be possible to show (perhaps using class-forcing and elementary embeddings) that such connections exist - but the burden is on the indispensability theorist to show this.

Indeterminacy of identity

A second problem with the attempt to combine a monist justification with a pluralist ontology concerns the identity of individual mathematical entities: far from being able to justify ontological commitment to the mathematical multiverse, pluralist realists cannot even justify ontological commitment to single mathematical entities. This is because, according to their view, there exist as many copies of mathematical entities featuring in standard models as there exist mathematical universes.

Consider the natural numbers, the standard model of that part of the mathematical cosmos referred to as 'arithmetic.' A monist realist can justify her belief in the existence of natural numbers via an enhanced indispensability argument, for instance citing the case of *Magicicadas*. The reason enhanced indispensability arguments are justification-conferring for monist realist belief is, first, that prime numbers figure indispensably in the explanation of the empirical phenomenon to be explained, and second, that the identity of the mathematical entities in question is determinate: it is 'the' number 13 and 'the' number 17 that explain the life cycles of *Magicicadas*. One might worry here that also on a monistic picture of mathematics, there are infinitely many omega-sequences to choose from when modelling $\mathbb{N}$, and that consequently the problem of indeterminacy is a problem for monists and pluralists alike. However, a monist can invoke 'Frege-arithmetic' as conceived of by Boolos (1987) to offer a way to conceive of numbers as metaphysically distinguished objects (cf. also Colyvan und Zalta (1999, p. 345) and Anderson und Zalta (2004)).

On the radically pluralist picture, on the other hand, it is entirely unclear which one of the countless copies of the numbers 13 and 17 are involved in the scientific explanation: those that exist in, and whose identities are determined by, V_{CH}? $V_{\neg CH}$? HOD? $L(\mathbb{R})$? Of course the different copies of the natural numbers that are instantiated in separate mathematical universes all 'behave' the same, i.e. they are characterised by the same mathematical axioms (i.e, the Peano-Dedekind axioms) and follow the same mathematical laws (i.e. the commutative, associative, and distributive laws). Metaphysically speaking, however, they are different entities: their identities are determined by the universes that instantiate them, and they are embedded in different universes. And the scientific applications of number theory only justify belief in one out of the many universes that contain a copy of the natural numbers.

One might think that the pluralist can solve this indeterminacy problem just as nonchalantly as she can solve Benacerraf's problem of non-uniqueness (discussed in section 1.1), i.e. by arguing that nothing in the idea of mathematical realism commits us to the belief that mathematical theories describe *unique* collections of mathematical objects. However, the function of mathematical entities in enhanced indispensability arguments is akin to that of physical unobservables like quarks and electrons in physical theories: they are individual entities, rather than structures or laws, that contribute to the explanation of a particular empirical phenomenon. Consequently, being able to pin down, in the metaphysical sense, exactly which mathematical entity is contributing to the explanation is crucial for the success of enhanced indispensability arguments. A pluralist realist, however, cannot do that.

4 Conclusion

If mathematical pluralism is true—and we have good reasons to believe that it is—, then there are very serious difficulties applying indispensability arguments to justify mathematical realism. This does not mean that we must reject either pluralism or mathematical realism, but that justifying their combination might require a shift from empiricist towards rationalist argumentation. An empiricist justification via indispensability may still be possible, but it would require substantial further work. I take the wider implication of my argument to be that, just like 'we should no longer expect science to provide the sort of methodological guidance for mathematics that it once did' (Maddy, 2011, p. 37), we should no longer expect it to guide our philosophy of mathematics.

References

[Aczel 1988] Aczel, Peter: *Non-well-founded sets*. Stanford, CA : Center for the Study of Language and Information, Lecture Notes, Vol. 14, 1988

[Anderson 1990] Anderson, C.A.: Logical Analysis and Natural Language: The Problem of Multiple Analyses. In: Klein, Peter (Hrsg.): *Praktische Logik*. Göttingen : Vandenhoeck and Ruprecht, 1990, S. 169–179

[Anderson und Zalta 2004] Anderson, David J. ; Zalta, Edward N.: Frege, Boolos, and Logical Objects. In: *Journal of Philosophical Logic* 33 (2004), Nr. 1, S. 1–26. – URL https://doi.org/10.1023/B:LOGI.0000019236.64896.fd. ISBN 1573-0433

[Antos u. a. 2015] Antos, Carolin ; Friedman, Sy-David ; Honzik, Radek ; Ternullo, Claudio: Multiverse conceptions in set theory. In: *Synthese* 192 (2015), S. 2463–2488

[Azzouni 1998] Azzouni, Jody: On "On what there is". In: *Pacific Philosophical Quarterly* 79 (1998), S. 1–18

[Bagaria 2020] Bagaria, Joan: Set Theory. In: Zalta, Edward N. (Hrsg.): *The Stanford Encyclopedia of Philosophy*. Spring 2020. Metaphysics Research Lab, Stanford University, 2020

[Baker 2003] Baker, Alan: The Indispensability Argument and Multiple Foundations for Mathematics. In: *The Philosophical Quarterly* 53 (2003), S. 49–67

[Baker 2005] Baker, Alan: Are there Genuine Mathematical Explanations of Physical Phenomena? In: *Mind* 114 (2005), S. 223–38

[Baker 2009] Baker, Alan: Mathematical Explanation in Science. In: *British Journal for the Philosophy of Science* 60 (2009), S. 611–633

[Balaguer 1995] Balaguer, Mark: A Platonist Epistemology. In: *Synthese* 103 (1995), S. 303–325

[Balaguer 1998a] Balaguer, Mark: Non-Uniqueness as a Non-Problem. In: *Philosophia Mathematica* 6 (1998), S. 63–84

[Balaguer 1998b] Balaguer, Mark: *Platonism and Anti-Platonism in Mathematics*. New York : Oxford University Press, 1998

[Balaguer 2017] Balaguer, Mark: Mathematical Pluralism and Platonism. In: *Journal of Indian Council of Philosophical Research* 34 (2017), Nr. 2, S. 63–84

[Beall 1999] Beall, J. C.: From Full Blooded Platonism to Really Full Blooded Platonism. In: *Philosophia Mathematica* 7 (1999), Nr. 3, S. 322–325

[Beall und Restall 2006] Beall, J.C. ; Restall, Greg: *Logical Pluralism*. Oxford and New York : Oxford University Press, 2006

[Benacerraf 1965] Benacerraf, Paul: What Numbers Could not Be. In: *The Philosophical Review* 74 (1965), Nr. 1, S. 47–73

[Benacerraf 1973] Benacerraf, Paul: Mathematical Truth. In: *The Journal of Philosophy* 70 (1973), Nr. 19, S. 661–679

[Bigelow 1988] Bigelow, John: *The Reality of Numbers: A Physicalist's Philosophy of Mathematics*. Oxford : Clarendon, 1988

[Boolos 1987] Boolos, George: The Consistency of Frege's Foundations of Arithmetic. In: Thomson, J. (Hrsg.): *On Being and Saying: Essays in Honor of Richard Cartwright*. MIT Press, 1987, S. 3–20

[Bourbaki 1968] Bourbaki, Nicolas: *Theory of Sets.* Berlin : Springer, 1968

[Bridges und Richman 1987] Bridges, Douglas ; Richman, Fred: *Varieties of Constructive Mathematics.* Cambridge : Cambridge University Press, 1987

[Burgess 1984] Burgess, John: Synthetic Mechanics. In: *Journal of Philosophical Logic* 13 (1984), S. 379–395

[Cantor 1878] Cantor, Georg: Ein Beitrag zur Mannigfaltigkeitslehre. In: *Journal für die reine und angewandte Mathematik* 84 (1878), S. 242–258

[Cantor 1882] Cantor, Georg: Ueber unendliche, lineare Punktmannichfaltigkeiten. In: *Mathematische Annalen* 20 (1882), S. 113–121

[Cheyne 1999] Cheyne, Colin: Problems with Profligate Platonism. In: *Philosophia Mathematica (3)* 7 (1999), S. 164–177

[Clarke-Doane 2019] Clarke-Doane, Justin: Set-theoretic Pluralism and the Benacerraf Problem. In: *Philosophical Studies* tba (2019), S. tba

[Clarke-Doane 2022] Clarke-Doane, Justin: *Mathematics and Metaphilosophy.* Cambridge University Press, 2022 (Elements in the Philosophy of Mathematics)

[Cohen 1963] Cohen, Paul: The Independence of the Continuum Hypothesis. In: *Proceedings of the National Academy of Sciences of the United States of America* 50 (1963), Nr. 6, S. 1143–1148

[Colyvan 2001] Colyvan, Mark: *The Indispensability of Mathematics.* New York : Oxford University Press, 2001

[Colyvan 2002] Colyvan, Mark: Mathematics and Aesthetic Considerations in Science. In: *Mind* 111 (2002), Nr. 441, S. 69–74

[Colyvan 2007] Colyvan, Mark: Mathematical recreation versus mathematical knowledge. In: Leng, Mary (Hrsg.) ; Paseau, Alexander (Hrsg.) ; Potter, Michael (Hrsg.): *Mathematical Knowledge.* Oxford and New York : Oxford University Press, 2007, S. 109–122

[Colyvan 2010] Colyvan, Mark: There is No Easy Road to Nominalism. In: *Mind* 119 (2010), Nr. 474, S. 285–306

[Colyvan und Zalta 1999] Colyvan, Mark ; Zalta, Edward: Mathematics: Truth and Fiction? In: *Philosophia Mathematica* 7 (1999), Nr. 3, S. 336–349

[Cox und Carlton 1988] Cox, Randel ; Carlton, C.E.: Paleoclimatic Influences in the Evolution of Periodical Cicadas. In: *American Midland Naturalist* 120 (1988), S. 183–193

[Davies 2005] Davies, E.B.: A Defence of Mathematical Pluralism. In: *Philosophia Mathematica* 13 (2005), S. 252–276

[Deutsch 1991] Deutsch, Harry: The Creation Problem. In: *Topoi* 10 (1991), S. 209–225

[Dummett 1977] Dummett, Michael: *Elements of Intuitionism.* Oxford and New York : Oxford University Press, 1977

[Dummett 1978] Dummett, Michael: Realism. In: *Truth and Other Enigmas.* London : Duckworth, 1978, S. 145–165

[Feferman 2000] Feferman, Solomon: Why the programs for new axioms need to be questioned. In: *Bulletin of Symbolic Logic* 6 (2000), S. 401–413

[Feferman u. a. 2000] Feferman, Solomon ; Friedman, Harvey ; Maddy, Penelope ; Steel, John: Does Mathematics Need New Axioms? In: *The Bulletin of Symbolic Logic* 6 (2000), Nr. 4, S. 401–446

[Field 1980] Field, Hartry: *Science without Numbers.* New York : Oxford University Press, 1980

[Field 1989] Field, Hartry: *Realism, Mathematics, and Modality.* Oxford : Blackwell Publishing, 1989

[Fine 1984] Fine, Arthur: The Natural Ontological Attitude. In: Leplin, Jarrett (Hrsg.): *Scientific Realism.* Berkeley : University of California Press, 1984, S. 1–21

[Frege 1962] Frege, Gottlob: *Grundgesetze der Arithmetik, Begriffsschriftlich Abgeleitet (Volume II).* Hildesheim : Georg Olms, 1962

[Frege 1988] Frege, Gottlob: *Die Grundlagen der Arithmetik.* Hamburg : Meiner, 1988

[Friedman 1981] Friedman, Harvey: On the Necessary Uses of Abstract Set Theory. In: *Advances in Mathematics* 41 (1981), S. 209–280

[Fuchs u. a. 2015] Fuchs, Gunter ; Hamkins, Joel D. ; Reitz, Jonas: Set-theoretic geology. In: *Annals of Pure and Applied Logic* 166 (2015), Nr. 4, S. 464–501. – URL `https://www.sciencedirect.com/science/article/pii/S0168007214001225`. – ISSN 0168-0072

[Gleason 1957] Gleason, Andrew: Measures on the Closed Subspaces of a Hilbert Space. In: *Journal of Mathematics and Mechanics* 6 (1957), S. 885–893

[Goles u. a. 2001] Goles, Eric ; Schulz, Oliver ; Markus, Mario: Prime Number Selection of Cycles in a Predator-Prey Model. In: *Complexity* 6 (2001), Nr. 4, S. 33–38

[Gray 2019] GRAY, Jeremy: Epistemology of Geometry. In: ZALTA, Edward N. (Hrsg.): *The Stanford Encyclopedia of Philosophy*. Fall 2019. Metaphysics Research Lab, Stanford University, 2019

[Gödel 1938] GÖDEL, Kurt: The Consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis. In: *Proceedings of the National Academy of Sciences of the United States of America* 24 (1938), Nr. 12, S. 556–557

[Gödel 1990] GÖDEL, Kurt: What is Cantor's continuum problem? [1947]. In: FEFERMAN, Solomon (Hrsg.) ; DAWSON, John W. (Hrsg.) ; MOORE, Gregory H. (Hrsg.) ; SOLOVAY, Robert M. (Hrsg.) ; HEIJENOORT, Jean v. (Hrsg.): *Kurt Gödel: Collected Works, Vol. II, Publications 1938-1974*. New York : Oxford University Press, 1990, S. 176–187

[Hamkins 2012] HAMKINS, Joel D.: The Set-theoretic Multiverse. In: *The Review of Symbolic Logic* 5 (2012), Nr. 3, S. 416–449

[Hellman 1989] HELLMAN, Geoffrey: *Mathematics Without Numbers: Towards a Modal-Structural Interpretation*. New York : Oxford University Press, 1989

[Hellman und Bell 2006] HELLMAN, Geoffrey ; BELL, John: Pluralism and the Foundations of Mathematics. In: KELLERT, Stephen (Hrsg.) ; LONGINO, Helen (Hrsg.) ; WATERS, Kenneth (Hrsg.): *Scientific Pluralism. Minnesota Studies in the Philosophy of Science*. Minneapolis : University of Minnesota Press, 2006, S. 64–79

[Holmes 2021] HOLMES, M. R.: Alternative Axiomatic Set Theories. In: ZALTA, Edward N. (Hrsg.): *The Stanford Encyclopedia of Philosophy*. Winter 2021. Metaphysics Research Lab, Stanford University, 2021

[Jech 1997] JECH, Thomas: *Set Theory*. Berlin : Springer, 1997

[Kitcher 1984] KITCHER, Philipp: *The Nature of Mathematical Knowledge*. New York : Oxford University Press, 1984

[Koellner 2009] KOELLNER, Peter: Truth in Mathematics: The Question of Pluralism. In: BUENO, Otávio (Hrsg.) ; LINNEBO, Øystein (Hrsg.): *New Waves in Philosophy of Mathematics*. New York : Palgrave Macmillan, 2009, S. 80–116

[Landau 1958] LANDAU, Edmund: *Elementary Number Theory*. New York : Chelsea Publishing Company, 1958

[Liggins 2008] LIGGINS, David: Quine, Putnam, and the 'Quine-Putnam' Indispensability Argument. In: *Erkenntnis* 68 (2008), S. 113–127

[Linnebo und Pettigrew 2011] LINNEBO, Øystein ; PETTIGREW, Richard: Category Theory as an Autonomous Foundation. In: *Philosophia Mathematica* 19 (2011), Nr. 3, S. 227–254

[Linsky und Zalta 1995] LINSKY, Bernard ; ZALTA, Edward: Naturalized Platonism versus Platonized Naturalism. In: *The Journal of Philosophy* 92 (1995), Nr. 10, S. 525–555

[Lipton 2004] LIPTON, Peter: What Good is an Explanation? In: CORNWELL, John (Hrsg.): *Explanations. Styles of Explanation in Science*. Oxford : Oxford University Press, 2004, S. 1–21

[Lyon 2012] LYON, Aidan: Mathematical Explanations of Empirical Facts, and Mathematical Realism. In: *Australasian Journal of Philosophy* 90 (2012), Nr. 3, S. 559–578

[Lyon und Colyvan 2008] LYON, Aidan ; COLYVAN, Mark: The Explanatory Power of Phase Spaces. In: *Philosophia Mathematica* 16 (2008), Nr. 2, S. 227–243

[Maddy 1990] MADDY, Penelope: *Realism in Mathematics*. Oxford : Clarendon, 1990

[Maddy 1992] MADDY, Penelope: Indispensability and Practice. In: *The Journal of Philosophy* 89 (1992), Nr. 6, S. 275–289

[Maddy 2011] MADDY, Penelope: *Defending the Axioms. On the Philosophical Foundations of Set Theory*. New York : Oxford University Press, 2011

[Maddy 2019] MADDY, Penelope: *What Do We Want a Foundation to Do?* S. 293–311. In: CENTRONE, Stefania (Hrsg.) ; KANT, Deborah (Hrsg.) ; SARIKAYA, Deniz (Hrsg.): *Reflections on the Foundations of Mathematics: Univalent Foundations, Set Theory and General Thoughts*, Springer International Publishing, 2019. – URL `https://doi.org/10.1007/978-3-030-15655-8_13`. – ISBN 978-3-030-15655-8

[Marquis 1995] MARQUIS, Jean-Pierre: Category Theory and the Foundations of Mathematics: Philosophical Excavations. In: *Synthese* 103 (1995), S. 421–447

[Marquis 2019] MARQUIS, Jean-Pierre: Category Theory. In: ZALTA, Edward N. (Hrsg.): *The Stanford Encyclopedia of Philosophy*. Fall 2019. Metaphysics Research Lab, Stanford University, 2019

[Martin 2001] MARTIN, Donald A.: Multiple Universes of Sets and Indeterminate Truth Values. In:

Topoi 20 (2001), Nr. 1, S. 5–16. – URL https://doi.org/10.1023/A:1010600724850. ISBN 1572-8749

[Melia 2000] Melia, Joseph: Weaseling Away the Indispensability Argument. In: *Mind* 109 (2000), Nr. 435, S. 455–479

[Melia 2002] Melia, Joseph: Response to Colyvan. In: *Mind* 111 (2002), S. 75–79

[Mill 1947] Mill, John S.: *A System of Logic*. London : Longmans, Green and Co, 1947

[Mortensen 1995] Mortensen, Chris: *Inconsistent Mathematics*. Dordrecht : Kluwer, 1995

[Panza und Sereni 2015] Panza, Marco ; Sereni, Andrea: On the Indispensable Premises of the Indispensability Argument. In: Lolli, Gabriele (Hrsg.) ; Panza, Marco (Hrsg.) ; Venturi, Giorgio (Hrsg.): *From Logic to Practice*. Dordrecht : Springer, 2015, S. 241–276

[Paris und Harrington 1977] Paris, Jeff ; Harrington, Leo: A Mathematical Incompleteness in Peano Arithmetic? In: Barwise, Jon (Hrsg.): *Handbook of Mathematical Logic*. Amsterdam : North Holland, 1977, S. 1133–1142

[Parsons 1980] Parsons, Charles: Mathematical Intuition. In: *Proceedings of the Aristotelian Society* 80 (1980), S. 145–168

[Parsons 1983] Parsons, Charles: *Quine on the Philosophy of Mathematics*. S. 176–206. In: *Mathematics in Philosophy: Selected Essays*, Cornell University Press, 1983

[Pincock 2007] Pincock, Christopher: A Role for Mathematics in the Physical Sciences. In: *Noûs* 41 (2007), S. 253–275

[Pincock 2015] Pincock, Christopher: Abstract Explanations in Science. In: *British Journal for the Philosophy of Science* 66 (2015), S. 857–882

[Piron 1976] Piron, Constantin: *Foundations of Quantum Physics*. New York : W.A. Benjamin Inc., 1976

[Potter 2004] Potter, Michael: *Set Theory and its Philosophy*. Oxford : Oxford University Press, 2004

[Priest 1997] Priest, Graham: Inconsistent models of arithmetic, Part I: Finite Models. In: *Journal of Philosophical Logic* 26 (1997), S. 223–235

[Priest 2013] Priest, Graham: Mathematical Pluralism. In: *Logic Journal of the IGPL* 21 (2013), Nr. 1, S. 4–13

[Priest 2019] Priest, Graham: *From the Foundations of Mathematics to Mathematical Pluralism*. S. 363–380. In: Centrone, Stefania (Hrsg.) ; Kant, Deborah (Hrsg.) ; Sarikaya, Deniz (Hrsg.): *Reflections on the Foundations of Mathematics: Univalent Foundations, Set Theory and General Thoughts*. Cham : Springer International Publishing, 2019. – URL https://doi.org/10.1007/978-3-030-15655-8_16. – ISBN 978-3-030-15655-8

[Priest 2021] Priest, Graham: A note on mathematical pluralism and logical pluralism. In: *Synthese* 198 (2021), Nr. 20, S. 4937–4946. – URL https://doi.org/10.1007/s11229-019-02292-9. ISBN 1573-0964

[Putnam 1979a] Putnam, Hilary: Philosophy of Logic. In: *Mathematics, Matter and Method: Philosophical Papers*. Cambridge : Cambridge University Press, 1979, S. 323–357

[Putnam 1979b] Putnam, Hilary: What is Mathematical Truth? In: *Mathematics, Matter and Method: Philosophical Papers*. Cambridge : Cambridge University Press, 1979, S. 60–78

[Putnam 2012] Putnam, Hilary: Indispensability arguments in the philosophy of mathematics. In: Putnam, Hilary (Hrsg.) ; De Caro, Mario (Hrsg.) ; Macarthur, David (Hrsg.): *Philosophy in an Age of Science. Physics, Mathematics, and Skepticism*. Cambridge, MA : Harvard University Press, 2012, S. 181–201

[Quine 1986a] Quine, W. V. O.: Reply to Charles Parsons. In: Hahn, Lewis E. (Hrsg.) ; Schilpp, Paul A. (Hrsg.): *The Philosophy of W. V. Quine*. La Salle, Illinois : Open Court Publishing Company, 1986, S. 396–403

[Quine 1969] Quine, Willard Van O.: *Existence and quantification*. S. 91–113. In: *Ontological Relativity and Other Essays*. New York : Columbia University Press, 1969

[Quine 1986b] Quine, Willard Van O.: *Philosophy of Logic*. Cambridge, MA : Harvard University Press, 1986

[Quine 1981] Quine, W.V.O.: Success and Limits of Mathematization. In: *Theories and Things*. Cambridge, MA : Harvard University Press, 1981, S. 148–155

[Resnik 1989] Resnik, Michael: A Naturalized Epistemology for a Platonist Mathematical Ontology. In: *Philosophica* 43 (1989), S. 7–27

[Resnik 1997] Resnik, Michael: *Mathematics as a Science of Patterns.* New York : Oxford University Press, 1997

[Rieger 2008] Rieger, Adam: Paradox, ZF, and the Axiom of Foundation. In: DeVidi, David (Hrsg.) ; Hallett, Michael (Hrsg.) ; Clark, Peter (Hrsg.): *Logic, Mathematics, Philosophy: Vintage Enthusiasms.* New York : Springer, 2008, S. 171–188

[Russell 2021] Russell, Gillian: Logical Pluralism. In: Zalta, Edward N. (Hrsg.): *The Stanford Encyclopedia of Philosophy.* Summer 2021. Metaphysics Research Lab, Stanford University, 2021

[Shapiro 1997] Shapiro, Stewart: *Philosophy of Mathematics: Structure and Ontology.* New York : Oxford University Press, 1997

[Shelah 2003] Shelah, Saharon: Logical Dreams. In: *Bulletin of the American Mathematical Society* 40 (2003), Nr. 2, S. 203–228

[Sober 1993] Sober, Elliott: Mathematics and Indispensability. In: *The Philosophical Review* 102 (1993), Nr. 1, S. 35–57

[Soysal 2020] Soysal, Zeynep: Why is the universe of sets not a set? In: *Synthese* 197 (2020), Nr. 2, S. 575–597. – URL `https://doi.org/10.1007/s11229-017-1513-x`. ISBN 1573-0964

[Steel 2000] Steel, John R.: Mathematics needs new axioms. In: *Bulletin of Symbolic Logic* 6 (2000), S. 422–433

[Studd 2019] Studd, J. P.: *Everything, more or less. A defence of generality relativism.* Oxford : Oxford University Press, 2019

[Univalent Foundations Program 2013] Univalent Foundations Program, The: *Homotopy Type Theory: Univalent Foundations of Mathematics.* Institute for Advanced Study : `https://homotopytypetheory.org/book`, 2013

[Wigner 1960] Wigner, Eugene: The Unreasonable Effectiveness of Mathematics in the Natural Sciences. In: *Communications on Pure and Applied Mathematics* 13 (1960), S. 1–14

[Woodin 2017] Woodin, Hugh: In search of ultimate-L: The 19th Midrasha Mathematical Lectures. In: *The Bulletin of Symbolic Logic* 23 (2017), Nr. 1, S. 1–109

[Yoshimura 1997] Yoshimura, Jin: The Evolutionary Origins of Periodical Cicadas during Ice Ages. In: *American Naturalist* 149 (1997), S. 112–124