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AMOEBA REALS

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Abstract. We define the ideal with the property that a real omits all Borel sets in the ideal which are coded in a transitive model if and only if it is an amoeba real over this model. We investigate some other properties of this ideal. Strolling through the "amoeba forest" we gain as an application a modification of the proof of the inequality between the additivities of Lebesgue measure and Baire category.

§0. Introduction. The concepts of Cohen and random real arose immediately after the discovery of the method of forcing because of their close relations to category and measure. These two kinds of generic reals are determined by meager Borel sets and Borel sets of Lebesgue measure zero, respectively. This basic property predestines the relevance of these reals in the investigation of properties of measure and category.

There are other kinds of generic reals having some influence on the behaviour of measure and category through intermediary phenomena. Though this influence is not straightforward, still it can shed some new light on the objects. In this paper we work with amoeba reals. We choose a way of access opposite to the one that was natural in the case of Cohen or random reals, and through an amoeba real we define a c.c.c. σ-ideal $\mathcal{I}_{\mathcal{C}}$ of sets of reals having Borel base such that r. o. $\mathbf{A} = \mathrm{Borel}/\mathcal{I}_{\mathcal{C}}$ and amoeba reals are just reals avoiding Borel members of the ideal $\mathcal{I}_{\mathcal{C}}$.

We can motivate our interest in amoeba reals also by asking how strong the following four assertions are:

(1) There is an amoeba real.
(2) The old amoeba forcing is σ-centered.
(3) There is a perfect set of random reals of positive measure.
(4) There is a perfect set of random reals.

For (1) $\rightarrow$ (2) see e.g. [1], and the implication (3) $\rightarrow$ (4) is trivial. To see the implication (2) $\rightarrow$ (3); if the old amoeba forcing is σ-centered, i.e. $\mathbf{A} \cap \mathbf{V} = \bigcup_{n\in\omega} \mathbf{A}_n$, with $\mathbf{A}_n$ centered (amoeba forcing $\mathbf{A}$ is the set of all closed sets of measure greater than $1/2$ ordered by inclusion). Then the sets $B_n = \bigcap \mathbf{A}_n$, $n \in \omega$, are closed sets of measure $\geq 1/2$ and $\bigcap_{n\in\omega}(B_n + \mathbb{Q})$ is a Borel set of full measure consisting of random reals over $\mathbf{V}$.

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The only result in the reverse direction known to us is that (4) does not imply (3). This follows from the observation that (3) implies the existence of a dominating real while by [4] the existence of a perfect set of random reals without the existence of a dominating real is possible.

Let us recall that $\mathscr{M}$, $\mathscr{N}$ denote the ideal of meager sets and the ideal of Lebesgue measure zero sets, respectively. The underlying set of these ideals is either an interval of reals or the Cantor space ${}^{\omega}2$; which case holds will be clear from the context and we do not distinguish them strictly. The Lebesgue measure on ${}^{\omega}2$ is denoted by μ.

Let $\mathscr{I}$ be an ideal on a set X and $(P, \leq)$ be a partially ordered set. The following cardinal invariants are commonly used:

$$\begin{aligned}
\operatorname{add}(\mathscr{I}) &= \min\{|\mathscr{X}| : \mathscr{X} \subseteq \mathscr{I} \ \& \ \bigcup \mathscr{X} \notin \mathscr{I}\},\\
\operatorname{cov}(\mathscr{I}) &= \min\{|\mathscr{X}| : \mathscr{X} \subseteq \mathscr{I} \ \& \ \bigcup \mathscr{X} = X\},\\
\operatorname{non}(\mathscr{I}) &= \min\{|A| : A \subseteq X \ \& \ A \notin \mathscr{I}\},\\
\operatorname{cof}(\mathscr{I}) &= \min\{|\mathscr{X}| : \mathscr{X} \subseteq \mathscr{I} \ \& \ (\forall A \in \mathscr{I})(\exists B \in \mathscr{X})\, A \subseteq B\},\\
\mathfrak{b}(P) &= \min\{|B| : B \subseteq P \ \& \ \neg(\exists p \in P)(\forall q \in B)\, q \leq p\},\\
\mathfrak{d}(P) &= \min\{|D| : D \subseteq P \ \& \ (\forall p \in P)(\exists q \in D)\, p \leq q\}.
\end{aligned}$$

§**1a. Borel representation of Boolean algebras.** It is a well-known fact that every σ-complete Boolean algebra with countably many σ-complete generators is a homomorphic image of the σ-complete Boolean algebra of Borel sets of reals, i.e. the σ-algebra of Borel sets is a free σ-complete Boolean algebra. This fact has a simple expression in terms of Boolean valued models when the generic extension is generated by a real: if $\mathscr{A}$ is a complete Boolean algebra and $\dot{x}$ is an $\mathscr{A}$-name of a real in $\mathbf{V}^{\mathscr{A}}$, then the mapping ϕ defined by

(1.1) $$\phi_{\mathscr{A}}(B) = \|\dot{x} \in B\|, \quad B \text{ is a Borel set,}$$

is a σ-complete homomorphism, $\phi_{\mathscr{A}} : \text{Borel} \to \mathscr{A}$. Moreover, if $\mathscr{A}$ is c.c.c., then the image of $\phi_{\mathscr{A}}$ is a complete subalgebra $\mathscr{A}_0 \subseteq \mathscr{A}$.

§**1b. Amoeba forcing.** For a tree $T \subseteq {}^{<\omega}2$, $[T] = \{x \in {}^{\omega}2 : (\forall n)\, x{\restriction}n \in T\}$. We choose the following form of amoeba forcing:

$$\mathbf{A} = \{T \subseteq {}^{<\omega}2 : T \text{ is a tree and } \mu([T]) > 1/2\},$$

$$T_1 \leq T_2 \quad \text{iff} \quad T_1 \subseteq T_2.$$

If $G \subseteq \mathbf{A}$ is a generic subset, then $P^* = \bigcap G$ is a perfect tree and $\mu([P^*]) = 1/2$. We want to express r.o.$\mathbf{A}$ in terms of §1a. Let us consider the following space (homeomorphic to a G_δ subset of the Cantor space via characteristic functions):

$$\mathbf{R} = \{P \subseteq {}^{<\omega}2 : P \text{ is a tree } \& \ \mu([P]) = 1/2\}.$$

So the homomorphism (1.1) will have the following form:

(1.2) $$\phi_{\mathbf{A}}(B) = \|P^* \in B\|, \quad \text{for Borel subsets } B \subseteq \mathbf{R}.$$

Note that although the generic tree is perfect we do not require all trees in $\mathbf{R}$ to be perfect. On the other hand, the set of nonperfect trees is small in the sense of category base for amoeba (see the definition in the next lemma), i.e. it is $\mathscr{C}$-meager.

§**1c. Category base for amoeba.** For $T \in \mathbf{A}$ let $\langle T\rangle = \{P \in \mathbf{R} : P \subseteq T\}$. We call the family $\mathscr{C} = \{\langle T\rangle : T \in \mathbf{A}\}$ a *family of regions.* Note that $\mathscr{C} \simeq \mathbf{A}$ and every disjoint family of regions is countable.

LEMMA 1.1. *The pair $\langle \mathbf{R}, \mathscr{C}\rangle$ is a category base, i.e. the following two conditions are satisfied* (*cf.* [9]):

(1) $\mathbf{R} = \bigcup \mathscr{C}$.

(2) *If $A \in \mathscr{C}$ and $\mathscr{D} \subseteq \mathscr{C}$ is a nonempty disjoint family of power less than $|\mathscr{C}|$, then*

(a) *if $A \cap \bigcup \mathscr{D}$ contains some region from $\mathscr{C}$, then there is a region $D \in \mathscr{D}$ such that $A \cap D$ contains a region, and*

(b) *if $A \cap \bigcup \mathscr{D}$ contains no region, then there is a region $B \subseteq A$ that is disjoint with $\bigcup \mathscr{D}$.*

PROOF. Standard proof using c.c.c. of amoeba.

(1) This is clear.

(2) Notice that $\mathscr{D}$ must be countable by c.c.c. of $\mathbf{A}$.

(a) If $\langle T\rangle \subseteq A \cap \bigcup \mathscr{D}$ is a region and $\langle T\rangle \cap \langle T'\rangle$ does not contain a region for any $\langle T'\rangle \in \mathscr{D}$, then for every $\langle T'\rangle \in \mathscr{D}$, $\mu([T] \cap [T']) \leq 1/2$. Hence by removing sets of small measure from $[T]$ (a small subset of $[T']$ for each $T' \in \mathscr{D}$) we manage to find $T_1 \subseteq T$ so that $T_1 \in \mathbf{A}$ and for all $\langle T'\rangle \in \mathscr{D}$, $\mu([T_1] \cap [T']) < 1/2$. This is possible since $\mathscr{D}$ is countable. Evidently $\langle T_1\rangle$ is disjoint with $\bigcup \mathscr{D}$. This is a contradiction, since $\langle T_1\rangle \subseteq \langle T\rangle$.

(b) If $A = \langle T\rangle$ and $A \cap \bigcup \mathscr{D}$ contains no region, then for every $\langle T'\rangle \in \mathscr{D}$, $\mu([T] \cap [T']) \leq 1/2$. As in the case of (a) we can find a region $A' \subseteq A$ disjoint with $\bigcup \mathscr{D}$. □

DEFINITION 1.2. (i) $X \subseteq \mathbf{R}$ is *$\mathscr{C}$-rare* (or *singular*, see [9]) if for every region $A \in \mathscr{C}$ there is a region $B \subseteq A$ such that $B \cap X = \emptyset$.

(ii) $X \subseteq \mathbf{R}$ is *$\mathscr{C}$-meager* if it is a countable union of $\mathscr{C}$-rare sets.

(iii) $X \subseteq \mathbf{R}$ is *$\mathscr{C}$-Baire* if for every region $A \in \mathscr{C}$ there is a region $B \subseteq A$ such that either $B - X$ is $\mathscr{C}$-meager or $B \cap X$ is $\mathscr{C}$-meager.

(iv) $\mathscr{I}_{\mathscr{C}}$ is the σ-ideal of $\mathscr{C}$-meager sets, and $\mathscr{C}$-Baire is the σ-algebra of $\mathscr{C}$-Baire sets in the category base $\langle \mathbf{R}, \mathscr{C}\rangle$.

LEMMA 1.3. (i) *$\mathscr{C}$-Baire is closed under Souslin $\mathscr{A}$-operation. In particular, it is a σ-algebra of sets.*

(ii) *The regions are not $\mathscr{C}$-meager.*

(iii) *Borel subsets of* $\mathbf{R}$ *are $\mathscr{C}$-Baire.*

(iv) *Analytic sets are $\mathscr{C}$-Baire.*

PROOF. The propositions (i) and (ii) hold true for any category base (see [9]), and (iv) follows easily from (ii) and (iii). Therefore it is enough to prove (iii).

For an integer $n \in \omega$ and a finite tree $\tau \subseteq {}^{\leq n}2$ whose every branch has length $n+1$, let

$$\mathrm{clopen}(\tau) = \{P \in \mathbf{R} : P \cap {}^{\leq n}2 = \tau\}.$$

Clearly, $\mathrm{clopen}(\tau)$ is a clopen subset of $\mathbf{R}$ and every clopen subset of $\mathbf{R}$ is the finite union of clopen sets of such form. Hence the sets clopen(τ), τ a finite tree, generate the σ-algebra of Borel subsets of $\mathbf{R}$, and so it is enough to prove that they are $\mathscr{C}$-Baire. Let us fix τ and let $n+1$ be the height of τ. It is enough to prove that for every region $\langle T \rangle$ there is a subregion $\langle T' \rangle \subseteq \langle T \rangle$ such that either $\langle T' \rangle \subseteq \text{clopen}(\tau)$ or $\langle T' \rangle \cap \text{clopen}(\tau) = \emptyset$.

For $s \in T \cap {}^n 2$, T_s denotes the subtree of T with stem s. Let $E \subseteq {}^n 2$ be a minimal subset of $T \cap {}^n 2$ such that $\mu(\bigcup_{s \in E}[T_s]) > 1/2$. Choose $T' \in \mathbf{A}$, $T' \subseteq T$, so that $0 < \mu([T'_s]) < \mu([T_s])$ for $s \in E$ and $[T'_s] = \emptyset$ for $s \in {}^n 2 - E$. Then $\mu([T'] - [T'_s]) < 1/2$ for every $s \in E$, and so if $P \in \langle T' \rangle$, then $P \cap {}^n 2 = E$. Hence either $\langle T' \rangle \subseteq \text{clopen}(\tau)$ or $\langle T' \rangle \cap \text{clopen}(\tau) = \emptyset$, according to whether E equals $\tau \cap {}^n 2$ or not. □

Let $e : \mathbf{A} \to \text{r.o.}\,\mathbf{A}$ denote the canonical completion of $\mathbf{A}$ (i.e. a compatibility-preserving mapping onto a dense subset of r.o. $\mathbf{A}$).

LEMMA 1.4. (i) *If $\langle T \rangle \in \mathscr{C}$, then $\phi_{\mathbf{A}}(\langle T \rangle) = e(T) \neq \mathbf{0}$ for $T \in \mathbf{A}$.*

(ii) *If $B \subseteq \mathbf{R}$ is a $\mathscr{C}$-rare Borel set, then $\phi_{\mathbf{A}}(B) = \mathbf{0}$.*

PROOF. It is trivial. □

COROLLARY 1.5. *For any Borel set $B \subseteq \mathbf{R}$, $\phi_{\mathbf{A}}(B) = \mathbf{0}$ iff B is $\mathscr{C}$-meager.*

PROOF. By the previous lemma, for $\mathscr{C}$-meager sets B we have $\phi_{\mathbf{A}}(B) = \mathbf{0}$ because $\phi_{\mathbf{A}}$ is σ-complete. The reverse implication follows from the fact that every Borel set is $\mathscr{C}$-Baire and so it can be represented as a countable union (using c.c.c.) of sets from $\mathscr{C}$ modulo a $\mathscr{C}$-meager set. □

COROLLARY 1.6. r.o. $\mathbf{A} = $ *Borel*$/\mathscr{I}_{\mathscr{C}} = \mathscr{C}$*-Baire*$/\mathscr{I}_{\mathscr{C}}$.

PROOF. r.o. $\mathbf{A}$ is a homomorphic image of the algebra of Borel sets by the homomorphism (1.2) with the kernel containing exactly $\mathscr{C}$-meager Borel sets. Every $\mathscr{C}$-Baire set is equal to a countable union of regions (hence to a Borel set) modulo a $\mathscr{C}$-meager set. □

§**2. The ideal $\mathscr{I}_{\mathscr{C}}$.** Note that if $\{T_n : n \in \omega\}$ is a predense subset of $\mathbf{A}$, then $\mathbf{R} - \bigcup_{n \in \omega} \langle T_n \rangle$ is a $\mathscr{C}$-rare set, and, conversely, every $\mathscr{C}$-rare set is contained in a $\mathscr{C}$-rare set of such form.

Let us define another version of amoeba forcing. For $\varepsilon > 0$,

$$\mathbf{A}_\varepsilon = \{T \subseteq {}^{<\omega}2 : T \text{ is a tree and } \mu([T]) > \varepsilon\} \tag{2.1}$$

is ordered by inclusion. In particular, $\mathbf{A} = \mathbf{A}_{1/2}$. Later we will use the fact that the complete Boolean algebras r.o. $\mathbf{A}$ and r.o. $\mathbf{A}_\varepsilon$ are isomorphic (see [12]).

LEMMA 2.1. *If $\varepsilon > 1/2$ and T_G is a generic tree in $\mathbf{A}_\varepsilon$, then $\mu([T_G]) = \varepsilon$ and the region $\langle T_G \rangle$ is disjoint with $\mathbf{R} \cap \mathbf{V}$.*

PROOF. For every $P \in \mathbf{R} \cap \mathbf{V}$, $D_P = \{T \in \mathbf{A}_\varepsilon : P \not\subseteq T\}$ is a dense subset of $\mathbf{A}_\varepsilon$, and every condition in D_P forces $P \not\subseteq T_G$. □

For $n \in \omega$ let $\Pi_n = \{\pi : \pi \text{ is a permutation of } {}^n 2\}$ and for $\pi \in \Pi_n$ and $T \in \mathbf{A}$ let $\pi(T) = \{s \in {}^{<\omega}2 : (\exists t \in T)\, |t| \geq n \;\&\; s{\restriction}n \subseteq \pi(t{\restriction}n) \;\&\; s{\restriction}\langle n, \infty) = t{\restriction}\langle n, \infty)\}$. Let $\Pi = \bigcup_{n \in \omega} \Pi_n$.

LEMMA 2.2. *Let $T \in \mathbf{A}$. Then $D_T = \{\pi(T) : \pi \in \Pi\}$ is a predense subset of $\mathbf{A}$.*

PROOF. Let $S \in \mathbf{A}$ be arbitrary. We show that there are $n \in \omega$ and $\pi \in \Pi_n$ such that S and $\pi(T)$ are compatible. Without loss of generality we can assume that $\mu([S]) \leq \mu([T])$. For some positive real δ, $\mu([S]) - \delta \geq 1/2$.

For $X \subseteq {}^n2$ let $U_X = \bigcup\{[s] : s \in X\}$ and for $n \in \omega$ let T_n, S_n denote the nth levels of the trees T, S, respectively.

There is $n \in \omega$ such that $\mu(U_{T_n} - [T]) < \delta$ and $|S_n| \leq |T_n|$. Let $\pi \in \Pi_n$ be such that $S_n \subseteq \pi(T_n)$. We prove that $\pi(T)$ and S are compatible.

Obviously $[S] \subseteq U_{S_n} \subseteq U_{\pi(T_n)}$ and so $[S] \cap [\pi(T)] = [S] - (U_{\pi(T_n)} - [\pi(T)])$. On the other hand $\mu(U_{\pi(T_n)} - [\pi(T)]) = \mu(U_{T_n} - [T]) < \delta$. Therefore $\mu([S] \cap [\pi(T)]) > \mu([S]) - \delta \geq 1/2$. □

Lemma 2.2 implies that after adding an amoeba real the old amoeba forcing becomes σ-centered (see also [1]). From the above two lemmata and the mentioned isomorphism it follows that

$$\Vdash_{\mathbf{A}} \text{“}\mathbf{R} \cap \mathbf{V} \text{ is a } \mathscr{C}\text{-rare set”}.$$

This fact distinguishes the amoeba reals from random and Cohen reals, since both random and Cohen reals preserve nonsmall sets with respect to their ideals. Let us recall that an amoeba real causes the set of old reals to be a measure zero set. Hence the last fact is a more straightforward consequence of the following lemma.

LEMMA 2.3. *If* $\mu({}^\omega 2 \cap \mathbf{V}) = 0$, *then* $\mathbf{R} \cap \mathbf{V}$ *is* $\mathscr{C}$*-rare.*

PROOF. Choose a sequence $T_n \in \mathbf{A}$, $n \in \omega$, such that $[T_n]$ is disjoint with ${}^\omega 2 \cap \mathbf{V}$ and $\mu([T_n]) \geq 1 - 1/2^n$. Then $\{T_n : n \in \omega\}$ is predense in $\mathbf{A}$ and $\bigcup_{n\in\omega}\langle T_n\rangle$ is disjoint with $\mathbf{R} \cap \mathbf{V}$. □

Let $\mathscr{D}(\mathbf{A})$ and $\mathscr{G}(\mathbf{A})$ denote the family of all maximal antichains and the family of all filters in $\mathbf{A}$, respectively. Let us introduce the following cardinal invariants for amoeba forcing:

$$\mathfrak{m}(\mathbf{A}) = \min\{\kappa : \neg \mathrm{MA}_\kappa(\mathbf{A})\},$$

$$\mathfrak{w}(\mathbf{A}) = \min\{|A| : A \subseteq \mathbf{R} \;\&\; (\forall \mathscr{D} \in [\mathscr{D}(\mathbf{A})]^{\leq\omega})(\exists G \in \mathscr{G}(\mathbf{A})) \; G \text{ is } \mathscr{D}\text{-generic} \;\&\; \textstyle\bigcap_{T\in G}\langle T\rangle \cap A \neq \emptyset\}.$$

LEMMA 2.4. (a) $\mathrm{cov}(\mathscr{I}_{\mathscr{C}}) = \mathfrak{m}(\mathbf{A})$.

(b) $\mathrm{non}(\mathscr{I}_{\mathscr{C}}) = \mathfrak{w}(\mathbf{A})$.

PROOF. For a $\mathscr{C}$-rare set $B \subseteq \mathbf{R}$ the set $D_B = \{T \in \mathbf{A} : \langle T\rangle \cap B = \emptyset\}$ is a dense subset of $\mathbf{A}$. Hence, whenever $\mathscr{A}$ is a family of $\mathscr{C}$-rare sets and G is a $\{D_B : B \in \mathscr{A}\}$-generic filter over $\mathbf{A}$, then $\bigcap_{T\in G}\langle T\rangle \cap \bigcup\mathscr{A} = \emptyset$, where the set $\bigcap_{T\in G}\langle T\rangle$ is nonempty. Hence, if $|\mathscr{A}| < \mathfrak{m}(\mathbf{A})$, then $\bigcup\mathscr{A} \neq \mathbf{R}$ and $\mathfrak{m}(\mathbf{A}) \leq \mathrm{cov}(\mathscr{I}_{\mathscr{C}})$. Similarly, if $A \subseteq \mathbf{R}$ witnesses the equality $|A| = \mathfrak{w}(\mathbf{A})$, then A cannot be covered by a countable family of $\mathscr{C}$-rare sets, and so $\mathrm{non}(\mathscr{I}_{\mathscr{C}}) \leq \mathfrak{w}(\mathbf{A})$

In the reverse direction: If $D \subseteq \mathbf{A}$ is a predense set, then the set $\mathbf{R} - \bigcup_{T\in D}\langle T\rangle$ is a $\mathscr{C}$-rare set. However, we need another observation.

We say that a maximal antichain $D \subseteq \mathbf{A}$ is *strict* if for every two different members $T, T' \in D$, $\mu([T] \cap [T']) < 1/2$. We say that a maximal antichain D' *strictly refines* a maximal antichain D if for every $T \in D$ and for every $T' \in D'$ either $T' \subseteq T$ or $\mu([T] \cap [T']) < 1/2$. We prove three simple facts about these notions.

CLAIM 2.4A. *For every maximal antichain* D *there is a strict maximal antichain* D' *which refines* D.

PROOF. Let $\{T_\xi : \xi < 2^\omega\}$ be an enumeration of $\mathbf{A}$. By induction on $\xi < 2^\omega$ define $T'_\xi \in \mathbf{A}$ as follows. If there is $\zeta < \xi$ such that T_ξ is compatible with T'_ζ, then set $T'_\xi = T'_\zeta$. Assume that T_ξ is not compatible with any T'_ζ, i.e. $\mu([T_\xi] \cap [T'_\zeta]) \leq 1/2$ for every $\zeta < \xi$. By c.c.c. of $\mathbf{A}$, the set $\{T'_\zeta : \zeta < \xi\}$ is countable; let T'_n, $n \in \omega$, be an enumeration of this set. Choose any sequence of positive reals ε_n, $n \in \omega$, such that $\mu([T_\xi]) - \sum_{n\in\omega} \varepsilon_n > 1/2$, and let $s_n \in T_\xi$, $n \in \omega$, be arbitrary with $\mu([s_n]) < \varepsilon_n$ and $\mu([T_\xi] \cap [T'_n] - [s_n]) < 1/2$ for all n. Now let $T'_\xi \in \mathbf{A}$ be such that $[T'_\xi] = [T_\xi] - \bigcup_{n\in\omega}[s_n]$. Clearly, the set $D' = \{T'_\xi : \xi < 2^\omega\}$ is a strict maximal antichain refining D. □

CLAIM 2.4B. *For every finite family $\mathscr{D}$ of maximal antichains there is a strict maximal antichain which strictly refines all antichains of the family.*

PROOF. For a antichain D the set

$$H_D = \{T' \in \mathbf{A} : (\forall T \in D)\, T' \leq T \text{ or } \mu([T] \cap [T']) < 1/2\}$$

is an open dense subset of $\mathbf{A}$. Let $D^* \subseteq \bigcap_{D'\in\mathscr{D}} H_{D'}$ be a maximal antichain. Now let D be a strict maximal antichain which refines D^*. Then clearly D strictly refines all antichains in $\mathscr{D}$. □

We say that a family of maximal antichains $\mathscr{D}$ is *closed under finite strict refinements* if for every finite subfamily $\mathscr{D}_0$ there is a strict maximal antichain $D \in \mathscr{D}$ which strictly refines all antichains in $\mathscr{D}_0$. Note that for an infinite family $\mathscr{D}$ of maximal antichains there is a family $\mathscr{D}'$ of the same size which is closed under finite strict refinements.

CLAIM 2.4C. *If $\mathscr{D}$ is a family of strict maximal antichains closed under finite strict refinements, then whenever $P \in \bigcap_{D\in\mathscr{D}} \bigcup_{T\in D}\langle T\rangle$, the set*

$$G = \{T \in \mathbf{A} : (\exists T' \in \bigcup \mathscr{D})\, P \subseteq T' \subseteq T\}$$

is a $\mathscr{D}$-generic filter.

PROOF. Notice that, as $P \in \bigcap_{D\in\mathscr{D}} \bigcup_{T\in D}\langle T\rangle$, for each $D \in \mathscr{D}$ there is a $T \in D$ such that $P \subseteq T$ (the presence of refining strict maximal antichains ensures that there is only one such T in each D). Let T_1, T_2 be any elements from $G \cap \bigcup \mathscr{D}$. There are $D_1, D_2 \in \mathscr{D}$ such that $T_i \in D_i$ for $i = 1, 2$. Let $D \in \mathscr{D}$ be a strict maximal antichain which strictly refines D_1, D_2 and let $T \in D \cap G$, i.e. $P \subseteq T$. Then, as $\mu([T] \cap [T_i]) \geq 1/2$, we have $T \leq T_i$ for $i = 1, 2$ and so G is a $\mathscr{D}$-generic filter. □

Now, if $\mathscr{D}$ is a family of maximal antichains of cardinality less than $\operatorname{cov}(\mathscr{I}_\mathscr{C})$, without loss of generality we can assume that $\mathscr{D}$ is closed under finite strict refinements, the set $\bigcup_{D\in\mathscr{D}}(\mathbf{R} - \bigcup_{T\in D}\langle T\rangle)$ is nonempty, and each element of this set determines a $\mathscr{D}$-generic filter. Consequently, $\operatorname{cov}(\mathscr{I}_\mathscr{C}) \leq \mathfrak{m}(\mathbf{A})$.

Similarly, let $A \subseteq \mathbf{R}$ have cardinality less than $\mathfrak{w}(\mathbf{A})$. So, there is a countable family $\mathscr{D}$ of maximal antichains such that, for each $\mathscr{D}$-generic filter G, $\bigcap_{T\in G}\langle T\rangle \cap A = \emptyset$. Without loss of generality we can assume that $\mathscr{D}$ is closed under finite refinements. Then, by Claim 3, for each $P \in \bigcap_{D\in\mathscr{D}} \bigcup_{T\in D}\langle T\rangle$, $P \notin A$. Hence A is $\mathscr{C}$-meager and $\mathfrak{w}(\mathbf{A}) \leq \operatorname{non}(\mathscr{I}_\mathscr{C})$. □

The same proof yields:

LEMMA 2.5. (a) *The $\mathscr{C}$-meager Borel sets coded in* $\mathbf{V}$ *cover* $\mathbf{R}$ *if and only if there is no amoeba real over* $\mathbf{V}$. *Moreover, a real is an amoeba real over* $\mathbf{V}$ *if and only if it omits all* $\mathscr{C}$*-meager Borel sets coded in* $\mathbf{V}$.

(b) *A set* $A \subseteq \mathbf{R}$ *is not* $\mathscr{C}$*-meager if and only if for every countable system* $\mathscr{D}$ *of dense subsets of* $\mathbf{A}$ *there is a* $\mathscr{D}$*-generic filter* G *such that* $\bigcap_{T\in G}\langle T\rangle \cap A \neq \emptyset$.

§3. **Special $\mathscr{C}$-rare sets.** Often when we construct a $\mathscr{C}$-meager set it has a simple definition.

DEFINITION 3.1. A $\mathscr{C}$-rare set A is *special* if there is $T \in \mathbf{A}$ such that $A \subseteq \mathbf{R} - \bigcup_{\pi\in\Pi}\langle\pi(T)\rangle$. A set A is a *special $\mathscr{C}$-meager set* if it is the countable union of special $\mathscr{C}$-rare sets, i.e. there are conditions $T_n \in \mathbf{A}$, $n \in \omega$, such that $A \cap \bigcap_{n\in\omega}\bigcup_{\pi\in\Pi}\langle\pi(T_n)\rangle = \emptyset$. We say that the sequence T_n, $n \in \omega$, *witnesses that A is a special $\mathscr{C}$-meager set.* We denote by $\mathscr{I}^*_{\mathscr{C}}$ the family of all special $\mathscr{C}$-meager sets.

By Lemma 2.3, if ${}^{\omega}2 \cap \mathbf{V}$ has measure zero, then $\mathbf{R} \cap \mathbf{V}$ is $\mathscr{C}$-rare. We show that it is even a special $\mathscr{C}$-rare set. Let us choose a perfect set $A \subseteq {}^{\omega}2$ consisting of new reals of measure greater than 1/2. Let $T \in \mathbf{A}$ be such that $A = [T]$. By Lemma 2.2, the set $\bigcup_{\pi\in\Pi}\langle\pi(T)\rangle$ is a complement of a $\mathscr{C}$-rare set, and because of new reals it is disjoint of $\mathbf{R} \cap \mathbf{V}$.

We can give another example of a special $\mathscr{C}$-rare set. If there is an unbounded real over $\mathbf{V}$, then $\mathbf{R} \cap \mathbf{V}$ is special $\mathscr{C}$-rare. To see this let us follow the notation of [8]: For every strictly increasing function $g \in {}^{\omega}\omega$, let $H_g = \{x \in {}^{\omega}2 : (\exists^{\infty} n)\ x\restriction\langle g(n), g(n)+n\rangle \equiv 0\}$. For $G \subseteq {}^{\omega}2$ an open set and $n \in \omega$ define $G_n = \bigcup\{[s] : s \in {}^{n}2\ \&\ [s] \subseteq G\}$. Define a sequence $\varepsilon_k > 0$ so that $\sum_{k\in\omega} 2^{k^2}\varepsilon_k < 1/2$ and for $G \subseteq {}^{\omega}2$ open define $f_G \in {}^{\omega}\omega$ so that for all n, $\mu(G - G_{f_G(n)}) < \varepsilon_n$. So the sets H_g are measure zero sets for suitable g. A. W. Miller has proved that if $H_g \subseteq G$, then f_G eventually dominates g.

Let us assume that there is an unbounded real g over $\mathbf{V}$. Let $[T]$ be a perfect set disjoint with H_g such that $T \in \mathbf{A}$ (i.e. $\mu([T]) > 1/2$). Then by Miller's result $\langle T\rangle \cap \mathbf{V} = \emptyset$, and consequently $\mathbf{R} \cap \mathbf{V}$ is a special $\mathscr{C}$-rare set.

LEMMA 3.2. *For every special $\mathscr{C}$-meager set A there is a decreasing sequence of conditions $T_n \in \mathbf{A}$, $T_{n+1} \leq T_n$, witnessing that A is a special $\mathscr{C}$-meager set.*

PROOF. We can find such a sequence inductively using Lemma 2.2. □

LEMMA 3.3. *If* $\mathbf{R} \cap \mathbf{V}$ *is a $\mathscr{C}$-rare set, then it is a special $\mathscr{C}$-rare set.*

PROOF. If there is T such that $(\mathbf{R}\cap\mathbf{V}) \cap \langle T\rangle = \emptyset$, then also $(\mathbf{R}\cap\mathbf{V}) \cap \langle\pi(T)\rangle = \emptyset$, for all $\pi \in \Pi$. □

LEMMA 3.4. *Let* $\mathbf{B}$ *be random algebra. Then* $\mathbf{R} \cap \mathbf{V}$ *is a special $\mathscr{C}$-rare set if and only if* $\mathbf{B}^{\mathbf{V}}$ *is not a dense subset of* $\mathbf{B}$.

PROOF. If $\mathbf{B}^{\mathbf{V}}$ is not dense in $\mathbf{B}$, then there is a perfect set $F \subseteq {}^{\omega}2$ of positive measure such that no perfect subset of F of positive measure is coded in $\mathbf{V}$. Let $[s]$ be a basic clopen set for some $s \in {}^{<\omega}2$ such that $\mu([s] \cap F) > 1/2\mu([s])$. The set $F' = \{x \in {}^{\omega}2 : s^\frown x \in F\}$ has measure $> 1/2$ and does not contain perfect sets of positive measure coded in $\mathbf{V}$. Let $T \in \mathbf{A}$ be such that $[T] = F'$. Then obviously $\langle T\rangle \cap \mathbf{V} = \emptyset$, and so $\mathbf{R} \cap \mathbf{V}$ is special $\mathscr{C}$-rare.

Conversely, let $\langle T\rangle \cap \mathbf{V} = \emptyset$ for some $T \in \mathbf{A}$. Let $\mathscr{X}$ be a maximal family of disjoint perfect sets of positive measure coded in $\mathbf{V}$ which are subsets of $[T]$. The family $\mathscr{X}$ is countable, and so the set $X = \bigcup\mathscr{X}$ is measurable. Now $\mu(X) \leq 1/2$, since otherwise there would exist a finite system $\mathscr{X}_0 \subseteq \mathscr{X}$ such that $\mu(\bigcup \mathscr{X}_0) > 1/2$ while $\bigcup \mathscr{X}_0$ is a perfect set coded in $\mathbf{V}$. This leads to a contradiction, since by assumption $[T]$ does not contain any perfect set of measure $\geq 1/2$ coded in $\mathbf{V}$. Hence $\mu(\bigcup \mathscr{X}) < \mu([T])$. Let $F \subseteq [T] - \bigcup \mathscr{X}$ be any set of positive measure. Then no perfect subset of F of positive measure is coded in $\mathbf{V}$. □

LEMMA 3.5. *If special $\mathscr{C}$-rare sets with definition in* $\mathbf{V}$ *do not cover* $\mathbf{R}$, *then* $\mathbf{A} \cap \mathbf{V}$ *is σ-centered.*

PROOF. Let us assume that there is $P \in \mathbf{R}$ such that $P \in \bigcup_{\pi \in \Pi} \langle \pi(T) \rangle$ for every $T \in \mathbf{A} \cap \mathbf{V}$. Without loss of generality we can assume that, for every $s \in P$, $\mu([P_s]) > 0$ (otherwise take $P' = P - \{s : \mu([P_s]) = 0\}$). For every $T \in \mathbf{A} \cap \mathbf{V}$ there is π such that $\pi^{-1}(P) \in \langle T \rangle$. Evidently $P \notin \mathbf{V}$.

Let $\mathbf{A}_\pi = \{T \in \mathbf{A} \cap \mathbf{V} : \pi^{-1}(P) \in \langle T \rangle\}$. Since the sets $\mathbf{A}_\pi$ cover $\mathbf{A} \cap \mathbf{V}$, it is enough to prove that they are centered.

Let $T, T' \in \mathbf{A}_\pi$. We prove that $T \cap T' \in \mathbf{A}_\pi$, i.e. $\mu([T] \cap [T']) > 1/2$. If not and $\mu([T] \cap [T']) \leq 1/2$, then since $\pi^{-1}(P) \subseteq T \cap T'$, $\mu([T] \cap [T']) = 1/2$. Hence P is definable from T and T', since $\pi^{-1}(P) = T \cap T' - \{s : \mu([T_s] \cap [T'_s]) = 0\}$. Hence $P \in \mathbf{V}$, which is a contradiction. □

LEMMA 3.6. (a) $\operatorname{cov}(\mathscr{I}^*_{\mathscr{C}}) \leq \operatorname{add}(\mathscr{N})$.

(b) $\operatorname{cof}(\mathscr{N}) \leq \operatorname{non}(\textit{special } \mathscr{C}\textit{-rare set}) \leq \operatorname{non}(\mathscr{I}^*_{\mathscr{C}})$.

PROOF. For every measure zero set $A \subseteq {}^\omega 2$ we can find $T_A \in \mathbf{A}$ such that $(A + \mathbb{Q}) \cap [T_A] = \emptyset$. Since $[\pi(T_A)]$ contains only finite modifications of reals from $[T_A]$, it follows that $A \cap [\pi(T_A)] = \emptyset$ for every $\pi \in \Pi$, and $B_A = \mathbf{R} - \bigcup_{\pi \in \Pi} \langle \pi(T_A) \rangle$ is a special $\mathscr{C}$-rare set. For every $P \in \mathbf{R}$ the set $C_P = {}^\omega 2 - ([P] + \mathbb{Q})$ is a measure zero set, and whenever $P \notin B_A$, then $A \subseteq C_P$.

(a) Let $\mathscr{A}$ be a family of measure zero sets, $|\mathscr{A}| < \operatorname{cov}(\mathscr{I}^*_{\mathscr{C}})$. We prove that $\mu(\bigcup \mathscr{A}) = 0$. There is $P \in \mathbf{R} - \bigcup_{A \in \mathscr{A}} B_A$, and so for every $A \in \mathscr{A}$, $P \notin B_A$. Hence $\bigcup \mathscr{A} \subseteq C_P$, and so $\bigcup \mathscr{A}$ is a measure zero set.

(b) Let $\mathbf{R}_0 \subseteq \mathbf{R}$ not be a special $\mathscr{C}$-rare set. Then for every measure zero set $A \subseteq {}^\omega 2$, we have $\mathbf{R}_0 \nsubseteq B_A$, and so some $P \in \mathbf{R}_0$ is not in B_A. Consequently $A \subseteq C_B$ and $\{C_P : P \in P_0\}$ is cofinal in $\mathscr{N}$. □

In the next section we will see that the equalities in the above lemma hold, and we also give a characterization of the additivity and the cofinality of the ideal $\mathscr{I}^*_{\mathscr{C}}$ (Corollary 4.5 and Theorem 4.6).

§**4. The ideal $\mathscr{I}_{\mathscr{C}}$ and Lebesgue measure.** We prove the following theorem.

THEOREM 4.1. (a) $\operatorname{cov}(\mathscr{I}_{\mathscr{C}}) = \operatorname{add}(\mathscr{N})$

(b) $\operatorname{non}(\mathscr{I}_{\mathscr{C}}) = \operatorname{non}(\mathscr{C}\textit{-rare set}) = \operatorname{cof}(\mathscr{N})$.

In [1] the equality $\operatorname{add}(\mathscr{N}) = \mathfrak{m}(\mathbf{A})$ is proved, and (a) of Theorem 4.1 follows from this equality and Lemma 2.4. But there is a symmetry (or duality) in the present proof of the assertions (a) and (b), and this is why we prove them both here.

PROOF OF THEOREM 4.1. Since special $\mathscr{C}$-meager sets are $\mathscr{C}$-meager, the inequalities

$$\operatorname{cov}(\mathscr{I}_{\mathscr{C}}) \leq \operatorname{add}(\mathscr{N}), \qquad \operatorname{cof}(\mathscr{N}) \leq \operatorname{non}(\mathscr{C}\text{-rare set}) \leq \operatorname{non}(\mathscr{I}_{\mathscr{C}})$$

follow from Lemma 3.6. In the proof of the reverse inequalities we will use a characterization of add($\mathcal{N}$) and cof($\mathcal{N}$) using the following partial order. The set

$$\mathcal{S} = \left\{ f \in {}^{\omega}([\omega]^{<\omega}) : \sum_{n\in\omega} |f(n)|2^{-n} < \infty \right\}$$

is ordered by $f \leq^* g$ iff $(\exists m)(\forall n > m)\, f(n) \subseteq g(n)$. We will need just the inequalities proved in the next lemma, although e.g. by [2] it is evident that the reverse inequalities hold too. □

LEMMA 4.2 (J. Stern, J. Raisonnier). $\mathrm{add}(\mathcal{N}) \leq \mathfrak{b}(\mathcal{S})$, $\mathfrak{d}(\mathcal{S}) \leq \mathrm{cof}(\mathcal{N})$.

PROOF. We find two mappings $\alpha : \mathcal{S} \to \mathcal{N}$, $\beta : \mathcal{N} \to \mathcal{S}$ such that $\alpha(f) \subseteq A$ implies $f \leq^* \beta(A)$ for $f \in \mathcal{S}$ and $A \in \mathcal{N}$. From this property one can easily verify the inequalities in the lemma (see [6]).

Let $\{G_{i,j} : i, j \in \omega\}$ be a sequence of independent clopen sets in ${}^{\omega}2$ such that $\mu(G_{i,j}) = 2^{-i}$. For $f \in \mathcal{S}$ put

$$\alpha(f) = \bigcap_{m\in\omega} \bigcup_{i\geq m} \bigcup_{j\in f(i)} G_{i,j}.$$

Of course $\alpha(f) \in \mathcal{N}$.

Now we define the mapping β. Let $A \in \mathcal{N}$. Choose a closed set K of positive measure such that $K \cap A = \emptyset$ and $\mu(K \cap U) > 0$ iff $K \cap U \neq \emptyset$ for every open set U. Let $\{U_n : n \in \omega\}$ be an enumeration of all clopen sets U such that $K \cap U \neq \emptyset$. For every $n \in \omega$ let

$$F_n(i) = \{j : K \cap U_n \cap G_{i,j} = \emptyset\}.$$

Then $0 < \mu(K \cap U_n) \leq \mu(\bigcap_{i\in\omega} \bigcap_{j\in F_n(i)} ({}^{\omega}2 - G_{i,j})) = \prod_{i\in\omega} \prod_{j\in F_n(i)} (1 - 2^{-i})$. Therefore $\sum_{i\in\omega} |F_n(i)|2^{-i} < \infty$, and so $F_n \in \mathcal{S}$ for all $n \in \omega$. Let $\beta(A) \in \mathcal{S}$ be such that $F_n \leq^* \beta(A)$ for all $n \in \omega$.

Now assume that $f \in \mathcal{S}$, $A \in \mathcal{N}$ and $\alpha(f) \subseteq A$. Then $K \cap \alpha(f) = \emptyset$, and by the Baire category theorem there are m, n such that $K \cap U_n \cap \bigcup_{i\geq m} \bigcup_{j\in f(i)} G_{i,j} = \emptyset$. Therefore $f(i) \subseteq F_n(i)$ for all $i \geq m$, and so $f \leq^* F_n \leq^* \beta(A)$. □

Let us denote

$$\mathcal{S}_0 = \left\{ f : (\forall n)\, f(n) \subseteq {}^{n}2 \ \& \sum_{n\in\omega} |f(n)|2^{-n} < 1/2 \right\}.$$

Since $\mathfrak{b}(\mathcal{S}) \leq \mathfrak{b}(\mathcal{S}_0)$, $\mathfrak{d}(\mathcal{S}_0) \leq \mathfrak{d}(\mathcal{S})$, it is obvious that Lemma 4.2 holds also for the family $\mathcal{S}_0$ instead of $\mathcal{S}$. We will use elements of $\mathcal{S}_0$ for a representation of elements of amoeba forcing in the following way.

For $f \in \mathcal{S}_0$, the set $U_f = \bigcup_{n\in\omega} \bigcup_{s\in f(n)} [s]$ is an open set of measure less than 1/2. Let $T_f \subseteq {}^{<\omega}2$ be the tree representing the closed set $[T_f] = {}^{\omega}2 - U_f$. Then $T_f \in \mathbf{A}$.

We write $f =^* g$ if $f \leq^* g$ and $g \leq^* f$. For $f \in \mathcal{S}_0$ let $\mathbf{A}_f = \{T_g : g \in \mathcal{S}_0 \ \& \ g =^* f\}$. $\mathbf{A}_f$ is a countable subset of $\mathbf{A}$, but, as we can see from the next lemma, $\mathbf{A}_f$ is a "good approximation" of the forcing $\mathbf{A}$.

LEMMA 4.3. *There is a mapping ψ which assigns to every open dense subset D of $\mathbf{A}$ a member $\psi(D) \in \mathcal{S}_0$ such that for every $f \in \mathcal{S}_0$, whenever $\psi(D) \leq^* f$, then $D \cap \mathbf{A}_f$ is dense in $\mathbf{A}_f$.*

To make the proof of Lemma 4.3 more lucid we prefer to work for a while with an isomorphic version of amoeba forcing:

$$\mathbf{A}' = \{U \subseteq {}^{\omega}2 : U \text{ is open } \& \ \mu(U) < 1/2\},$$

with ordering defined by $U \leq V$ iff $V \subseteq U$. In this context, $\mathbf{A}'_f = \{U_g : g \in \mathcal{S}_0 \ \& \ g =^* f\}$, and we prove

LEMMA 4.3′. *There is a mapping ψ' which assigns to every open dense subset D of $\mathbf{A}'$ a member $\psi'(D) \in \mathcal{S}_0$ such that for every $f \in \mathcal{S}_0$, whenever $\psi'(D) \leq^* f$, then $D \cap \mathbf{A}'_f$ is dense in $\mathbf{A}'_f$.*

PROOF. For an open set V let

$$f_V(n) = \{s \in {}^n2 : [s] \subseteq V \ \& \ [s\restriction(n-1)] \not\subseteq V\}.$$

Then obviously $V \subseteq U_{f_V}$ (actually $V = U_{f_V}$).

Let $\mathcal{X} \subseteq D$ be a maximal antichain. Since $\mathcal{X}$ is countable, there is $g \in \mathcal{S}_0$ such that $(\forall V \in \mathcal{X})(\forall^\infty n)\, f_V(n) \subseteq g(n)$. Set $\psi'(D) = g$. We prove that g has the property stated in the lemma.

Let $g \leq^* f$ and $U \in \mathbf{A}'_f$ be arbitrary. We have $U = U_h$ for some $h =^* f$. There is $V \in \mathcal{X}$ such that the set $W = U \cup V$ is in $\mathbf{A}'$. We show that $W \in \mathbf{A}'_f$, which finishes the proof since $W \in D$ and $W \in U$.

By the choice of g and since $g \leq^* f$, there is $n_0 \in \omega$ such that

$$(\forall n \geq n_0)\, f_V(n) \subseteq f(n) \ \& \ h(n) = f(n)$$

and $\sum_{n \geq n_0} |f(n)| 2^{-n} < 1/2 - \mu(W)$. Let us define

$$f'(n) = \begin{cases} f_W(n), & \text{for } n < n_0, \\ f(n), & \text{for } n \geq n_0. \end{cases}$$

First we note that $f' \in \mathcal{S}_0$. This is because $\bigcup\{f'(n) : n < n_0\}$ represents a finite subsystem of the partition of W into clopen sets, and so

$$\sum_{n<n_0} |f'(n)| 2^{-n} \leq \mu(W).$$

Hence, by the choice of n_0, $\sum_{n\in\omega} |f'(n)| 2^{-n} < \mu(W) + (1/2 - \mu(W)) = 1/2$. We show that $U_{f'} = W$. Let us consider the following cases.

(a) If $|s| < n_0$, then, by definition, $[s] \subseteq U_{f'}$ iff $[s] \subseteq W$.

(b) If $|s| = n \geq n_0$ and $s \in f'(n)$, then $[s] \subseteq U \subseteq W$ since $f'(n) = f(n)$.

These two cases prove $U_{f'} \subseteq W$.

(c) If $|s| = n \geq n_0$ and $s \in f_V(n)$, then $s \in f'(n)$ since $f_V(n) \subseteq f'(n)$.

Since $V \subseteq W$, (c) together with (a) proves $V \subseteq U_{f'}$. So to prove $W \subseteq U_{f'}$ it is enough to prove $U \subseteq U_{f'}$.

(d) If $|s| = n \geq n_0$ and $s \in h(n)$, then $h(n) = f'(n)$ and so $s \in f'(n)$.

Since $U \subseteq W$, (d) together with (a) proves $U \subseteq U_{f'}$, and the proof of the lemma is finished. □

The next lemma finishes the proof of Theorem 4.1.

LEMMA 4.4. (a) $\operatorname{add}(\mathcal{N}) \leq \operatorname{cov}(\mathcal{I}_{\mathscr{C}})$
(b) $\operatorname{non}(\mathcal{I}_{\mathscr{C}}) \leq \operatorname{cof}(\mathcal{N})$.

PROOF. Similar characterizations as in Lemma 2.4 are true also for the cardinal invariants $\operatorname{cov}(\mathcal{M})$ and $\operatorname{non}(\mathcal{M})$ (and any countable nonatomic forcing notion instead of $\mathbf{A}$; recall that $\mathcal{M}$ is the ideal of meager sets in ${}^{\omega}2$).

(a) Let $\mathcal{A}$ be a family of $\mathscr{C}$-rare sets, $|\mathcal{A}| < \operatorname{add}(\mathcal{N})$. For $A \in \mathcal{A}$, $D_A = \{T \in \mathbf{A} : \langle T \rangle \cap A = \emptyset\}$ is an open dense subset of $\mathbf{A}$. Since $\operatorname{add}(\mathcal{N}) \leq \mathfrak{b}(\mathcal{S}_0)$, there is $f \in \mathcal{S}_0$ such that $\psi(D_A) \leq^* f$ for all $A \in \mathcal{A}$ (ψ was defined in Lemma 4.3). Hence, by Lemma 4.3, $D_A \cap \mathbf{A}_f$ is dense in $\mathbf{A}_f$ for every $A \in \mathcal{A}$. Since $|\mathcal{A}| < \operatorname{cov}(\mathcal{M})$, there is a $\{D_A \cap \mathbf{A}_f : A \in \mathcal{A}\}$-generic filter $G \subseteq \mathbf{A}_f$. The set $F = \bigcap_{T \in G}[T]$ is closed, and $\mu(F) \geq 1/2$. There is $P \in \mathbf{R}$ such that $[P] \subseteq F$. Evidently, $P \in \bigcap_{T \in G}\langle T \rangle$ and so $P \notin \bigcup \mathcal{A}$, which proves $\operatorname{add}(\mathcal{N}) \leq \operatorname{cov}(\mathcal{I}_{\mathscr{C}})$.

(b) For every $\mathscr{C}$-meager set $A \subseteq \mathbf{R}$ there is a sequence of $\mathscr{C}$-rare sets A_n such that $A \subseteq \bigcup_{n \in \omega} A_n$. The sets $D_n = \{T \in \mathbf{A} : \langle T \rangle \cap A = \emptyset\}$ are open dense subsets of $\mathbf{A}$. Since $\mathfrak{b}(\mathcal{S}_0) > \omega$, there is $g \in \mathcal{S}_0$ such that $\psi(D_n) \leq^* g$ for all $n \in \omega$. Hence, by Lemma 4.3, $D_n \cap \mathbf{A}_f$ are dense in $\mathbf{A}_f$ whenever $g \leq^* f$. For every $f \in \mathcal{S}_0$ let $\mathcal{G}_f$ be a family of filters in $\mathbf{A}_f$ such that $|\mathcal{G}_f| = \operatorname{non}(\mathcal{M})$ and for every countable family $\mathcal{D}_0$ of dense subsets of $\mathbf{A}_f$ there is $G \in \mathcal{G}_f$ which is $\mathcal{D}_0$-generic. For every $G \in \mathcal{G}_f$ let us choose $P_G \in \bigcap_{T \in G}\langle T \rangle$ and let $A_f = \{P_G : G \in \mathcal{G}_f\}$. Hence $|A_f| \leq \operatorname{non}(\mathcal{M}) \leq \operatorname{cof}(\mathcal{N})$. Let $\mathcal{F} \subseteq \mathcal{S}_0$ be a dominating family in $\mathcal{S}_0$ of cardinality $\operatorname{cof}(\mathcal{N})$ (i.e. $\geq \mathfrak{d}(\mathcal{S}_0)$). Then the set $A = \bigcup_{f \in \mathcal{F}} A_f$ has cardinality $\operatorname{cof}(\mathcal{N})$ and it witnesses the inequality $\operatorname{non}(\mathcal{I}_{\mathscr{C}}) \leq \operatorname{cof}(\mathcal{N})$. $\square$

Since $\mathcal{I}^*_{\mathscr{C}} \subseteq \mathcal{I}_{\mathscr{C}}$, immediately from Theorem 4.1, Lemma 2.4 and Lemma 3.6 we get the following.

COROLLARY 4.5. (a) $\operatorname{cov}(\mathcal{I}^*_{\mathscr{C}}) = \mathfrak{m}(\mathbf{A}) = \operatorname{add}(\mathcal{N})$.
(b) $\operatorname{non}(\mathcal{I}^*_{\mathscr{C}}) = \operatorname{non}(\textit{special } \mathscr{C}\textit{-rare set}) = \mathfrak{w}(\mathbf{A}) = \operatorname{cof}(\mathcal{N})$.

We say that a tree $T \subseteq {}^{\omega}2$ is *regular* if for each $s \in T$, $\mu([T_s]) > 0$. Note that the family of all regular trees in $\mathbf{A}_\varepsilon$ (defined by (2.1)) is a dense (and even separative) subset of $\mathbf{A}_\varepsilon$. Let us recall some ideas of [12] which we will use in the proof of the next theorem.

Assume that $T_1 \in \mathbf{A}_{\varepsilon_1}$, $T_2 \in \mathbf{A}_{\varepsilon_2}$ are regular trees such that

$$\frac{\mu([T_1])}{\mu([T_1]) - \varepsilon_1} = \frac{\mu([T_2])}{\mu([T_2]) - \varepsilon_2}. \tag{4.1}$$

There are countable sets $Q_1 \subseteq [T_1]$, $Q_2 \subseteq [T_2]$ and a one-to-one continuous mapping m from $[T_1] - Q_1$ onto $[T_2] - Q_2$ which is measure preserving, i.e. $\mu(m(A)) = \mu(A)\mu([T_2])/\mu([T_1])$ for each measurable set $A \subseteq [T_1] - Q_1$. For $\varepsilon > 0$ and $T \in \mathbf{A}_\varepsilon$ let us denote $A_{\varepsilon,T} = \{S \in \mathbf{A}_\varepsilon : S \leq T\}$. The set $\mathcal{D} = \{T \leq T_1 : [T] \cap Q_1 = \emptyset\}$ is open dense in $\mathbf{A}_{\varepsilon_1,T_1}$, and for $T \in \mathcal{D}$ we can define $e(T) = T'$ iff $[T'] = m([T])$. Clearly, for any $T \in \mathcal{D}$, $e\restriction\mathbf{A}_{\varepsilon_1,T} : \mathbf{A}_{\varepsilon_1,T} \to \mathbf{A}_{\varepsilon_2,e(T)}$ is an isomorphism, and we say that this isomorphism is induced by the mapping m.

J. Truss [12] showed that there are maximal antichains $\{T'_{k,i} : k \in \omega\} \subseteq \mathbf{A}_{\varepsilon_i}$, $i = 1, 2$, such that each pair $T'_{k,1}, T'_{k,2}$ satisfies the equality (4.1), and the isomorphism from $\mathrm{r.o.}\,\mathbf{A}_{\varepsilon_1}$ onto $\mathrm{r.o.}\,\mathbf{A}_{\varepsilon_2}$ presented in [12, Theorem 3.3] can be described as follows: There are maximal antichains $\{T_{k,i} : k \in \omega\} \subseteq \mathbf{A}_{\varepsilon_i}$, $i = 1, 2$, and an isomorphism

$$e : \bigcup_{k\in\omega} \mathbf{A}_{\varepsilon_1,T_{k,1}} \to \bigcup_{k\in\omega} \mathbf{A}_{\varepsilon_2,T_{k,2}}$$

such that for each k the restriction $e\restriction\mathbf{A}_{\varepsilon_1,T_{k,1}}$ is induced by a one-to-one measure-preserving mapping. We will need the following particular property of the mapping e. For $\mathscr{A} \subseteq \mathbf{A}_{\varepsilon_1,T_{k,1}}$

$$\text{if } \mu\left(\bigcap_{T\in\mathscr{A}}[T]\right) \geq \varepsilon_1, \text{ then } \mu\left(\bigcap_{T\in\mathscr{A}}[e(T)]\right) \geq \varepsilon_2, \tag{4.2}$$

or, more generally,

$$\mu\left(\bigcap_{T\in\mathscr{A}}[e(T)]\right) = \mu\left(\bigcap_{T\in\mathscr{A}}[T]\right)\mu([T_{k,2}])/\mu([T_{k,1}]).$$

Now we prove the characterization promised at the end of the previous section.

THEOREM 4.6. (a) $\operatorname{add}(\mathscr{I}^*_{\mathscr{C}}) = \operatorname{add}(\mathscr{N})$.

(b) $\operatorname{cof}(\mathscr{I}^*_{\mathscr{C}}) = \operatorname{cof}(\mathscr{N})$.

PROOF. According to Corollary 4.5 it is enough to prove (a) $\mathfrak{m}(\mathbf{A}) \leq \operatorname{add}(\mathscr{I}^*_{\mathscr{C}})$ and (b) $\operatorname{cof}(\mathscr{I}^*_{\mathscr{C}}) \leq \mathfrak{w}(\mathbf{A})$.

Let us consider a fixed sequence of amoeba forcing notions $\mathbf{A}_{\varepsilon_n}$ with $1/2 < \varepsilon_{n+1} < \varepsilon_n$ and $\lim_{n\to\infty}\varepsilon_n = 1/2$. For each $n \in \omega$ let $D_n = \{T_{n,k} : k \in \omega\}$ be a maximal antichain in $\mathbf{A} = \mathbf{A}_{1/2}$ such that there is an isomorphism

$$e_n : \bigcup_{k\in\omega} \mathbf{A}_{1/2,T_{n,k}} \to \mathbf{A}_{\varepsilon_n}$$

such that each restriction $e_n\restriction\mathbf{A}_{1/2,T_{k,n}}$ is induced by a measure-preserving mapping. Without loss of generality we can assume that D_{n+1} refines D_n for each n and that the family $\mathscr{D}_0 = \{D_n : n \in \omega\}$ is closed under finite strict refinements (see the proof of Lemma 2.4).

We say that $P \in \mathbf{R}$ is *$\mathscr{D}_0$-generic* if the set $G_P = \{T \in \mathbf{A} : P \subseteq T\}$ chooses an element from each $D \in \mathscr{D}_0$. By Claim 3 in the proof of Lemma 2.4, if P is $\mathscr{D}_0$-generic, then G_P chooses exactly one element from each $D \in \mathscr{D}_0$. Moreover, by the property (4.2) of the mappings e_n, there are trees $T_n \subseteq {}^{<\omega}2$ such that $\mu([T_n]) \geq \varepsilon_n$ and $T_n \subseteq e_n(T)$ for each $T \in G_P \cap \operatorname{dom}(e_n)$. In particular, $T_n \in \mathbf{A}$ for each $n \in \omega$.

Now let $S \in \mathbf{A}$. There is $n \in \omega$ such that $S \in \mathbf{A}_{\varepsilon_n}$, and by Lemma 2.2 the set

$$D_{n,S} = \{T \in \operatorname{dom}(e_n) : (\exists\pi \in \Pi)\, e_n(T) \leq \pi(S)\}$$

is open dense in $\mathbf{A}$. Hence for any $\mathscr{D}_0$-generic point $P \in \mathbf{R}$ there is a sequence $\{T_n : n \in \omega\}$ of conditions in $\mathbf{A}$ such that whenever $G_P \cap D_{n,S} \neq \emptyset$, then there is $\pi \in \Pi$ with $T_n \subseteq \pi(S)$.

The above scheme describes two mappings. The first one finds for each element $S \in \mathbf{A}$ an integer $n = n(S)$, and the second one assigns to a $\mathscr{D}_0$-generic point $P \in \mathbf{R}$ a sequence of conditions $T_n \in \mathbf{A}$, $n \in \omega$, with the property that, for every $S \in \mathbf{A}$, if $n = n(S)$ and $G_P \cap D_{n,S} \neq \emptyset$, then there is $\pi \in \Pi$ such that $T_n \subseteq \pi(S)$.

Now let $\mathscr{A}$ be a family of special $\mathscr{C}$-rare sets. For every $A \in \mathscr{A}$ there is a tree $S_A \in \mathbf{A}$ such that $A \cap \bigcup_{\pi\in\Pi}\langle\pi(S_A)\rangle = \emptyset$. The family $\mathscr{D} = \{D_{n(S_A),S_A} : A \in \mathscr{A}\}$ of dense subsets of $\mathbf{A}$ has cardinality $|\mathscr{D}| \leq |\mathscr{A}|$. Hence, if $|\mathscr{A}| < \mathfrak{m}(\mathbf{A})$, there is a $(\mathscr{D} \cup \mathscr{D}_0)$-generic filter G, and clearly $G \subseteq G_P$ for some $\mathscr{D}_0$-generic point P. Hence there is a sequence $\{T_n : n \in \omega\}$ of conditions such that for every $A \in \mathscr{A}$ there is $n = n(S_A)$ and $\pi \in \Pi$ such that $T_n \subseteq \pi(S_A)$. Then $\bigcup_{\pi\in\Pi}\langle\pi(T_n)\rangle \subseteq \bigcup_{\pi\in\Pi}\langle\pi(S_A)\rangle$ and this set is disjoint with A. It follows that $\bigcup\mathscr{A} \cap \bigcap_{n\in\omega}\bigcup_{\pi\in\Pi}\langle\pi(T_n)\rangle = \emptyset$ and the union $\bigcup\mathscr{A}$ is a special $\mathscr{C}$-meager set. Therefore $\mathfrak{m}(\mathbf{A}) \leq \operatorname{add}(\mathscr{I}^*_{\mathscr{C}})$.

If $A \subseteq \mathbf{R}$ is a set that witnesses the equality $|A| = \mathfrak{w}(\mathbf{A})$, then for each countable family $\mathscr{D}$ of maximal antichains there is a $(\mathscr{D} \cup \mathscr{D}_0)$-generic point $P \in A$, and the same reasoning produces a cofinal family of special $\mathscr{C}$-meager sets (for each $P \in A$ the above procedure gives a special $\mathscr{C}$-meager set) of the same cardinality as the set A. Therefore $\operatorname{cof}(\mathscr{I}^*_{\mathscr{C}}) \leq \mathfrak{w}(\mathbf{A})$. □

§5. Meager and $\mathscr{C}$-meager sets.

Definition 5.1. Let $\mathscr{M}^*$ be a family of all subsets X of the interval $(0, 1/2)$ for which the set $A(X) = \{P \in \mathbf{R} : (\exists s \in {}^{<\omega}2)\, \mu([P_s]) \in X\}$ is a special $\mathscr{C}$-meager set, and let $\mathscr{M}^{**}$ be the family of all sets $X \subseteq (0, 1/2)$ for which $A(X)$ is $\mathscr{C}$-meager.

We prove that the both these families of sets coincide with the ideal of meager subsets of the interval $(0, 1/2)$.

Lemma 5.2. *The families $\mathscr{M}^* \subseteq \mathscr{M}^{**}$ are both σ-ideals extending the ideal of meager subsets of the interval* $(0, 1/2)$.

Proof. The fact that $\mathscr{M}^* \subseteq \mathscr{M}^{**}$ are σ-ideals is clear from the definition. We prove that for every meager set $X \subseteq (0, 1/2)$, the set $A(X)$ is a special $\mathscr{C}$-meager set. The idea of the proof can also be used to prove that, for a generic amoeba tree P^*, $\mu([P^*_s])$ is a Cohen real for every $s \in {}^{<\omega}2 - \{\emptyset\}$. Let us denote

$$A_s(X) = \{P : \mu([P_s]) \in X\} \quad \text{and} \quad A_n(X) = \bigcup_{s\in{}^n2} A_s(X).$$

We have $A(X) = \bigcup_{s\in{}^{<\omega}2} A_s(X) = \bigcup_{n\in\omega} A_n(X)$. First we show that for a nowhere dense set $X \subseteq (0, 1/2)$, the sets $A_n(X)$ are special $\mathscr{C}$-rare sets. Note that the sets $A_n(X)$ are closed under translations by permutations from Π, i.e. $\pi(A_n(X)) = A_n(X)$ for $\pi \in \Pi$. So it is enough to find $S \in \mathbf{A}$ such that $\langle S\rangle \cap A_n(X) = \emptyset$. We will need the following claim.

Claim 5.2a. *If $T \in \mathbf{A}$, $n > 0$, $s \in {}^n2$, then there is $S \leq T$, $S \in \mathbf{A}$ such that* $\langle S\rangle \cap A_s(X) = \emptyset$.

Proof. If $\mu([T_s]) = 0$, then take $S = T$ and we are done, since for every $P \in \langle S\rangle$ we have $\mu([P_s]) = 0 \notin X$. Similarly, if $\mu([T_s]) > 1/2$, put $S = T_s$. Then $\mu([P_s]) = 1/2 \notin X$ for $P \in \langle S\rangle$.

Let $\mu([T_s]) = a > 0$, $a \leq 1/2$, and $\mu([T]) - 1/2 = b > 0$. We can assume that $0 < b < a < 1/2$ (if not, then by shrinking T_s we can satisfy the requirement $a < 1/2$ and then by shrinking T_t, for $t \neq s$, $|t| = n$, we can satisfy the inequality $b < a$). Since X is nowhere dense, there is an interval $I = (c, d)$ such that $I \cap X = \emptyset$ and $\max\{a - b, b\} < c < d < a$. We can find $S \leq T$, $S \in \mathbf{A}$, such that $c < \mu([S_s]) < d$ (by shrinking T_s) and $\mu([S]) - 1/2 < \mu([S_s]) - c$ (by shrinking T_t for $t \neq s$, $|t| = n$).

Now, for every $P \in \langle S\rangle$, $d > \mu([S_s]) \geq \mu([P_s]) \geq \mu([S_s]) - (\mu([S]) - 1/2) > c$. So $\mu([P_s]) \in I$ and $\langle S\rangle \cap A_s(X) = \emptyset$. The proof of the claim is finished. □

Let us fix an enumeration s_n, $n = 1, 2, \dots, 2^n$, of the set ${}^n2$. Using Claim 5.2a iteratively 2^n times, we can find a decreasing sequence $T_{s_1} \geq T_{s_2} \geq \dots \geq T_{s_{2^n}}$ of conditions such that $T_{s_i} \cap A_{s_i}(X) = \emptyset$. Set $S = T_{s_{2^n}}$. Then $\langle S\rangle \cap A_n(X) = \emptyset$, and so the $A_n(X)$ are special $\mathscr{C}$-rare sets and $A(X)$ is a special $\mathscr{C}$-meager set.

If $X \subseteq (0, 1/2)$ is meager, then X is the union of countably many nowhere dense sets X_n, $n \in \omega$, and, by the previous part of the proof, $A(X) = \bigcup_{n\in\omega} A(X_n)$ is a special $\mathscr{C}$-meager set. □

LEMMA 5.3. *If $X \subseteq (0, 1/2)$ is open, $X \neq \emptyset$, then $A(X)$ is not $\mathscr{C}$-meager.*

PROOF. Let $n \in \omega$ and $c < d$ be such that $2^{-n-1} < c < d < 2^{-n}$ and $(c, d) \subseteq X$. Let us fix $s \in {}^n2$. As in the proof of Claim 5.2a, we can find $S \in \mathbf{A}$ such that $\mu([P_s]) \in (c, d)$ for every $P \in \langle S\rangle$. Hence $A(X)$ contains a region, and so it is not $\mathscr{C}$-meager. □

COROLLARY 5.4. *If $X \subseteq (0, 1/2)$ has the Baire property, then $X \in \mathscr{M}$ if and only if $X \in \mathscr{M}^*$ if and only if $X \in \mathscr{M}^{**}$*

PROOF. By Lemma 5.2 it is enough to show that whenever a set $X \in \mathscr{M}^{**}$ has the Baire property, then X is meager.

Let us assume that $X \in \mathscr{M}^{**}$ has the Baire property. So there is an open set U such that the symmetric difference $X \triangle U$ is meager. So there is a meager set X' such that $U \subseteq X \cup X'$. Since $A(U) \subseteq A(X) \cup A(X')$, $U \in \mathscr{M}^{**}$ and by Lemma 5.3, $U = \emptyset$. Therefore X is meager. □

Below we eliminate the assumption in the last corollary. For this reason we need the next lemma.

LEMMA 5.5. (a) *If $A(X)$ is a special $\mathscr{C}$-meager set, then there is $X^* \in \Pi^1_1$ such that $X \subseteq X^*$ and $A(X^*)$ is a special $\mathscr{C}$-meager set.*

(b) *If $A(X)$ is a $\mathscr{C}$-meager set, then there is $X^* \in \Pi^1_1$ such that $X \subseteq X^*$ and $A(X^*)$ is $\mathscr{C}$-meager.*

PROOF. (a) If $A(X)$ is a special $\mathscr{C}$-meager set, then there are $T_n \in \mathbf{A}$, $n \in \omega$, such that $A(X) \cap \bigcap_n \bigcup_\pi \langle \pi(T_n)\rangle = \emptyset$. Then for every $P \in \bigcap_n \bigcup_\pi \langle \pi(T_n)\rangle$ and for every $s \in {}^{<\omega}2$ we have $\mu([P_s]) \notin X$. Set

$$X^* = (0, 1/2) - \left\{ c \in \mathbb{R} : (\exists s) \left(\exists P \in \bigcap_n \bigcup_\pi \langle \pi(T_n)\rangle \right) \ c = \mu([P_s]) \right\}.$$

It is easy to see that $X^* \in \Pi^1_1$, $X \subseteq X^*$, and the same sequence T_n, $n \in \omega$, witnesses that $A(X^*)$ is a special $\mathscr{C}$-meager set.

(b) If $A(X)$ is $\mathscr{C}$-meager, then there are maximal antichains $\{T_{n,m} : m \in \omega\} \subseteq \mathbf{A}$, for $n \in \omega$, such that $A(X) \cap \bigcap_n \bigcup_m \langle T_{n,m}\rangle = \emptyset$. Then

$$X \subseteq X^* = (0, 1/2) - \left\{ c \in \mathbb{R} : (\exists s) \left(\exists P \in \bigcap_n \bigcup_m \langle T_{n,m}\rangle \right) \ c = \mu([P_s]) \right\}. \quad \square$$

COROLLARY 5.6. $\mathscr{M} = \mathscr{M}^* = \mathscr{M}^{**}$.

PROOF. It is enough to see that $\mathscr{M}^{**} \subseteq \mathscr{M}$.

Let $X \in \mathscr{M}^{**}$, i.e. $A(X)$ is $\mathscr{C}$-meager. By Lemma 5.5(b), there is $X^* \in \Pi^1_1$ such that $X \subseteq X^*$ and $A(X^*)$ is $\mathscr{C}$-meager. So $X^* \in \mathscr{M}^{**}$ and, by Corollary 5.4, since Π^1_1 sets have the Baire property, $X \subseteq X^* \in \mathscr{M}$. $\square$

COROLLARY 5.7. $\operatorname{add}(\mathscr{N}) \leq \operatorname{add}(\mathscr{M})$, *and* $\operatorname{cof}(\mathscr{M}) \leq \operatorname{cof}(\mathscr{N})$.

PROOF. Immediately from definition of the ideal $\mathscr{M}^*$ (the set mapping $A(\cdot)$ is additive) we can see that

$$\operatorname{add}(\mathscr{I}^*_{\mathscr{C}}) \leq \operatorname{add}(\mathscr{M}^*) \quad \text{and} \quad \operatorname{cof}(\mathscr{M}^*) \leq \operatorname{cof}(\mathscr{I}^*_{\mathscr{C}}).$$

But $\mathscr{M} = \mathscr{M}^*$, and the ideal $\mathscr{I}^*_{\mathscr{C}}$ has the same additivity and cofinality as the ideal $\mathscr{N}$ by Theorem 4.6. $\square$

§**6. Subsets of amoeba forcing.** We find characterizations for the following cardinal invariants related to subsets of amoeba forcing. Note that for the measure algebra the corresponding problem was studied in [5] and in [3]. Let us recall that $X \subseteq \mathbf{A}$ is separable in $\mathbf{A}$ if there is a countable set $X_0 \subseteq \mathbf{A}$ such that for every $p \in X$ there is $q \in X_0$ such that $q \leq p$. We define

$$\begin{aligned}
\operatorname{dense}(\mathbf{A}) &= \min\{|X| : X \subseteq \mathbf{A} \ \&\ X \text{ is a dense subset of } \mathbf{A}\},\\
\operatorname{centered}(\mathbf{A}) &= \min\{\kappa : \mathbf{A} \text{ is } \kappa\text{-centered}\},\\
\operatorname{nonseparable}(\mathbf{A}) &= \min\{|X| : X \subseteq \mathbf{A} \ \&\ X \text{ is not separable in } \mathbf{A}\},\\
\operatorname{noncentered}(\mathbf{A}) &= \min\{|X| : X \subseteq \mathbf{A} \ \&\ X \text{ is not } \sigma\text{-centered}\}.
\end{aligned}$$

One can easily verify that

$$\operatorname{centered}(\mathbf{A}) \leq \operatorname{dense}(\mathbf{A}) \quad \text{and} \quad \operatorname{nonseparable}(\mathbf{A}) \leq \operatorname{noncentered}(\mathbf{A}).$$

THEOREM 6.1. (a) $\operatorname{centered}(\mathbf{A}) = \operatorname{dense}(\mathbf{A}) = \operatorname{cof}(\mathscr{N})$,
(b) $\operatorname{nonseparable}(\mathbf{A}) = \operatorname{noncentered}(\mathbf{A}) = \operatorname{add}(\mathscr{N})$.

PROOF. (a) By previous remark and by Theorem 4.1 it is enough to prove

$$\operatorname{non}(\mathscr{C}\text{-rare}) \leq \operatorname{centered}(\mathbf{A}), \qquad \operatorname{dense}(\mathbf{A}) \leq \operatorname{non}(\mathscr{C}\text{-rare}).$$

Let $\mathbf{A}$ be κ-centered and let $\{\mathbf{A}_\xi : \xi \in \kappa\}$ be a partition of $\mathbf{A}$ into centered sets. For every ξ choose $P_\xi \in \mathbf{R}$ such that $P_\xi \subseteq T$ for every $T \in \mathbf{A}_\xi$. Then $\{P_\xi : \xi \in \kappa\}$ is not $\mathscr{C}$-rare, and so $\kappa \geq \operatorname{non}(\mathscr{C}\text{-rare})$.

Let $X \subseteq \mathbf{R}$ be a non-$\mathscr{C}$-rare set. Then the set $X' = \{\pi(P) : P \in X \ \&\ \pi \in \Pi\}$ has the same cardinality as X and X' is everywhere (in every region) non-$\mathscr{C}$-rare. Without loss of generality we can assume that $X = X'$. We prove that the family

$$\{P_1 \cup P_2 : P_1, P_2 \in X \ \&\ \mu([P_1 \cup P_2]) > 1/2\}$$

is a dense subset of $\mathbf{A}$, and so this family witnesses the inequality $\operatorname{dense}(\mathbf{A}) \leq \operatorname{non}(\mathscr{C}\text{-rare})$.

Let $T \in \mathbf{A}$ be arbitrary, $\mu([T]) = 1/2 + \varepsilon$. Since X is everywhere non-$\mathscr{C}$-rare, there is $P_1 \in X \cap \langle T \rangle$. Let $U \subseteq {}^\omega 2$ be an open set with $\mu(U) < \varepsilon$ such that

$\mu(U \cap [P_1]) > 0$. Let $T' \in \mathbf{A}$ be such that $[T'] = [T] - U$. Again, there is $P_2 \in X \cap \langle T' \rangle$. Since

$$\mu([P_1 \cup P_2]) \geq \mu([P_1]) + \mu([P_1] \cap U) > 1/2,$$

it follows that the tree $P_1 \cup P_2$ is a condition in $\mathbf{A}$ and $P_1 \cup P_2 \subseteq T$.

(b) By Theorem 4.1 it is enough to prove

$$\mathrm{cov}(\mathcal{J}_{\mathscr{C}}) \leq \mathrm{nonseparable}(\mathbf{A}), \qquad \mathrm{noncentered}(\mathbf{A}) \leq \mathrm{add}(\mathcal{J}^*_{\mathscr{C}}).$$

Let $\mathbf{R}_0$ be the set of all $P \in \mathbf{R}$ such that, for every open set $U \subseteq {}^{\omega}2$, $[P] \cap U \neq \emptyset$ iff $\mu([P] \cap U) > 0$. The set $\mathbf{R}_0$ is the complement of a special $\mathscr{C}$-meager set. To see this, for every $n \in \omega$ find $T_n \in \mathbf{A}$ so that for every $s \in {}^n2$, if $s \in T_n$, then $\mu([T_n] - [s]) < 1/2$. Obviously, $\bigcap_n \bigcup_{\pi \in \Pi} \langle \pi(T_n) \rangle \subseteq \mathbf{R}_0$.

Note that $\mathbf{R}_0$ is invariant under permutations from Π. We will use this property: whenever $P_1, P_2 \in \mathbf{R}_0$ and $P_1 \neq P_2$, then $P_1 \cup P_2 \in \mathbf{A}$ (i.e. $\mu([P_1 \cup P_2]) > 1/2$).

Let $X \subseteq \mathbf{A}$, $|X| < \mathrm{cov}(\mathcal{J}_{\mathscr{C}})$. The set $A = \bigcap_{T \in X} \bigcup_{\pi \in \Pi} \langle \pi(T) \rangle$ is not $\mathscr{C}$-meager, and so there are $P_1, P_2 \in A \cap \mathbf{R}_0$ such that for every $\pi \in \Pi$, $\pi(P_1) \neq P_2$. So the set $\{\pi_1(P_1) \cup \pi_2(P_2) : \pi_1, \pi_2 \in \Pi\} \subseteq \mathbf{A}$ witnesses the separability of X, and so $\mathrm{cov}(\mathcal{J}_{\mathscr{C}}) \leq \mathrm{nonseparable}(\mathbf{A})$.

To prove the last inequality, note that $\mathrm{noncentered}(\mathbf{A}) = \mathrm{noncentered}(\mathbf{A}_\varepsilon)$ for $\varepsilon > 0$. This fact follows from the isomorphism r. o. $\mathbf{A} \simeq$ r. o. $\mathbf{A}_\varepsilon$ (see [12]).

Let $\mathscr{A}$ be a family of special $\mathscr{C}$-rare sets, $|\mathscr{A}| < \mathrm{noncentered}(\mathbf{A})$. For every $A \in \mathscr{A}$, let us fix $T_A \in \mathbf{A}$ such that $A \cap \bigcup_{\pi \in \Pi} \langle \pi(T_A) \rangle = \emptyset$. Let us fix a sequence $\varepsilon_n > 1/2$, $n \in \omega$, of reals with $\lim_{n \in \omega} \varepsilon_n = 1/2$, and let $\mathscr{A}_n = \{A \in \mathscr{A} : \mu([T_A]) > \varepsilon_n\}$. Since $|\mathscr{A}_n| < \mathrm{noncentered}(\mathbf{A}_{\varepsilon_n})$, the family $X_n = \{T_A : A \in \mathscr{A}_n\}$ is σ-centered in $\mathbf{A}_{\varepsilon_n}$. Let $T_{n,m}$, $m \in \omega$, be a sequence of trees which are the intersections of countably many centered subsets of $X_n \subseteq \mathbf{A}_{\varepsilon_n}$. Hence, $\mu([T_{n,m}]) \geq \varepsilon_n$ and for every $A \in \mathscr{A}_n$ there is an m such that $T_{n,m} \subseteq T_A$, and consequently $\bigcup_{\pi \in \Pi} \langle \pi(T_{n,m}) \rangle \cap A = \emptyset$. Hence

$$\bigcap_{n,m \in \omega} \bigcup \pi \in \Pi \langle \pi(T_{n,m}) \rangle \cap \bigcup \mathscr{A} = \emptyset,$$

which means that $\bigcup \mathscr{A} \in \mathcal{J}^*_{\mathscr{C}}$, and so $\mathrm{noncentered}(\mathbf{A}) \leq \mathrm{add}(\mathcal{J}^*_{\mathscr{C}})$. □

LEMMA 6.2. *The forcing* $\mathbf{A}$ *adds a perfect set of amoeba reals.*

PROOF. Let $\varepsilon > 1/2$. By the already mentioned result of Truss, r. o. $\mathbf{A}_\varepsilon \simeq$ r. o. $\mathbf{A}$. Moreover, r. o.$(\mathbf{A} \times \mathbf{C})$ can be completely embedded into r. o. $\mathbf{A}$. Hence it is enough to prove the conclusion of the lemma for $\mathbf{A}_\varepsilon \times \mathbf{C}$.

Let P_ε be an $\mathbf{A}_\varepsilon$-generic tree, i.e. $\mu([P_\varepsilon]) = \varepsilon$. In $\mathbf{V}[P_\varepsilon]$, consider the following notion of forcing (conditions are subtrees of P_ε):

$$Q = \{T \subseteq P_\varepsilon : [T] \text{ is relatively clopen in } [P_\varepsilon] \ \& \ \mu([T]) > 1/2 \& \},$$

$$T_1 \leq T_2 \quad \text{iff} \quad T_1 \subseteq T_2.$$

Obviously Q is Cohen forcing, and it is well known that the existence of a Cohen real implies the existence of a perfect set of Cohen reals. So it can easily be seen that the existence of a single Cohen real implies the existence of a perfect set of generic trees in Q (i.e. a perfect set in the product topology of $\mathscr{P}({}^{<\omega}2)$). To finish the proof it is enough to note that every $\mathbf{V}[P_\varepsilon]$-generic tree in Q is an amoeba generic real over $\mathbf{V}$. The essential part of the proof of this fact is the following: if $D \subseteq \mathbf{A}$ is dense open and $D \in \mathbf{V}$, then the set

$$D_1 = \{T \in Q : (\exists T_1 \in D)\ T \subseteq T_1\}$$

is a dense open subset of Q in $\mathbf{V}[P_\varepsilon]$. The proof of the particulars we leave to the reader, since it is very similar to the proof of [11, Lemma 6.3]. □

§7. **Questions.** (1) Does there exist a c.c.c. forcing notion P killing $\mathscr{C}$-meager sets, i.e. $\Vdash_P$ "$\bigcup\{B \subseteq \mathbf{R} : B$ is a $\mathscr{C}$-meager Borel set coded in $\mathbf{V}\}$ is $\mathscr{C}$-meager"?

(2) Can we get a model with $\mathrm{add}(\mathscr{I}_{\mathscr{C}}) > \aleph_1$? Prove in ZFC that $\mathrm{add}(\mathscr{I}_{\mathscr{C}}) = \aleph_1$.

(3) Prove that each assertion in the following list is a consequence of the previous one.

(i) $\bigcup(\mathscr{I}_{\mathscr{C}} \cap \mathbf{V}) \neq \mathbf{R}$ (i.e. there is an amoeba real).
(ii) $\bigcup(\mathscr{I}^*_{\mathscr{C}} \cap \mathbf{V}) \neq \mathbf{R}$.
(iii) $\mathbf{A} \cap \mathbf{V}$ is σ-centered.
(iv) There is a perfect set of random reals of positive measure.
(v) $\mu({}^{\omega}2 \cap \mathbf{V}) = 0$.
(vi) $\mathbf{R} \cap \mathbf{V}$ is a special $\mathscr{C}$-rare set.
(vii) $\mathbf{R} \cap \mathbf{V} \in \mathscr{I}^*_{\mathscr{C}}$.
(viii) $\mathbf{R} \cap \mathbf{V} \in \mathscr{I}_{\mathscr{C}}$.

Which of these implications can be reversed?

Let us note that the implications (iv) $\rightarrow$ (v) $\rightarrow$ (vi) cannot be reversed. For the first implication consider a single Cohen extension. For the second one a counterexample is provided by the extension obtained by adding a single Laver real. In this extension there is an unbounded real (even a dominating one), so, by the note after Definition 3.1, $\mathbf{R} \cap \mathbf{V}$ is special $\mathscr{C}$-rare while the condition (v) does not hold in the extension (see [7]).

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