COMPUTABLE SHUFFLE SUMS OF ORDINALS

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ABSTRACT. The main result is that for sets $S \subseteq \omega + 1$, the following are equivalent:

1. The shuffle sum $\sigma(S)$ is computable.
2. The set S is a limit infimum set, i.e., there is a total computable function $f(x, s)$ such that the function $F(x) = \liminf_s f(x, s)$ enumerates S.
3. The set S is a limitwise monotonic set relative to $\mathbf{0}'$, i.e., there is a total $\mathbf{0}'$-computable function $g(x, t)$ satisfying $g(x, t) \leq g(x, t + 1)$ such that the function $G(x) = \lim_t g(x, t)$ enumerates S.

Other results discuss the relationship between these sets and the Σ^0_3 sets.

1. Introduction

A countable linear order is said to be computable if its universe can be identified with ω in such a way that the order is a computable relation on $\omega \times \omega$. The class of computable linear orders has been studied extensively; see [3] for an overview. In this paper we discuss the class of linear orders that are the shuffle sums of ordinals.

Definition 1.1. The *shuffle sum* of a countable set $S = \{\mathcal{L}_i\}_{i \in \omega}$ of linear orders, denoted $\sigma(S)$, is the (unique) linear order obtained by partitioning the rationals into dense sets $\{Q_i\}_{i \in \omega}$ and replacing each rational of Q_i by the linear order $\mathcal{L}_i$.

Equivalently, the shuffle sum of $S = \{\mathcal{L}_i\}_{i \in \omega}$ is the linear order obtained by interleaving copies of each $\mathcal{L}_i$ densely and unboundedly amongst each other.

The class of shuffle sums of ordinals has yielded various results in computable model theory. In [1], the authors use shuffle sums to produce, for each computable ordinal $\alpha \geq 2$, a linear order $\mathcal{A}_\alpha$ such that $\mathcal{A}_\alpha$ has αth jump degree but not βth jump degree for any $\beta < \alpha$. In [4], shuffle sums of ordinals are used to exhibit a linear order with both a computable model and a prime model, but no computable prime model.

In this paper, we characterize which shuffle sums of the finite order types and the order type ω are computable. In order to do so, we need the following notions.

Definition 1.2. A set $S \subseteq \omega + 1$ is a *limit infimum set*, written LimInf *set*, if there is a total computable function $f : \omega \times \omega \to \omega$ such that the function $F : \omega \to \omega$ given by

$$F(x) = \liminf_s f(x, s)$$

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enumerates S under the convention that $F(x) = \omega$ if $\liminf_s f(x,s) = \infty$. We say that f is a LimInf *witnessing function for* S.

Definition 1.3 ([5]). A set $S \subseteq \omega + 1$ is a *limitwise monotonic set relative to a degree* $\mathbf{a}$, written LimMon($\mathbf{a}$) set, if there is a total $\mathbf{a}$-computable function $g : \omega \times \omega \to \omega$ satisfying $g(x,t) \leq g(x,t+1)$ for all x and t such that the function $G : \omega \to \omega$ given by

$$G(x) = \lim_t g(x,t)$$

enumerates S under the convention that $G(x) = \omega$ if $\lim_t g(x,t) = \infty$. We say that g is a LimMon ($\mathbf{0}'$) *witnessing function for* S.

Although the notion of LimInf sets is new, LimMon ($\mathbf{0}'$) sets have been previously studied. Limitwise monotonic functions were first introduced and relativized in [5] and further studied in [2], [4] and [6]. Our definition departs slightly from the literature where $\lim_t g(x,t)$ is required to be finite. With the exception of the conclusion, we will only have need to consider limitwise monotonic sets relative to the degree $\mathbf{a} = \mathbf{0}'$.

Blurring the distinction between an ordinal α and the linear order of order type α (which we will do throughout the paper), we note that $\sigma(S) = \sigma(S \cup \{0\})$ for any set S of linear orders. In order to avoid complications in several of the proofs, we assume the following conventions.

Convention 1.4. *Any set* S *of ordinals is assumed to not contain* 0. *Any set* S *of linear orders is assumed to not contain the empty linear order.*

Any LimInf *witnessing function* $f(x,s)$ *is assumed to satisfy* $f(x,s) \neq 0$ *for all* x *and* s. *Any* LimMon ($\mathbf{0}'$) *witnessing function* $g(x,t)$ *is assumed to satisfy* $g(x,t) \neq 0$ *for all* x *and* t.

We briefly mention that the following facts justify that all the results in this paper are correct as stated, without needing to invoke Convention 1.4.

Fact 1.5. If S is a Σ^0_3 set, then $S \backslash \{0\}$ is a Σ^0_3 set. If S is a Σ^0_3 set, then $S \cup \{0\}$ is a Σ^0_3 set.

Fact 1.6. If S is a LimInf set, then $S \backslash \{0\}$ is a LimInf set. If S is a LimMon ($\mathbf{0}'$) set, then $S \backslash \{0\}$ is a LimMon ($\mathbf{0}'$) set.

If S is a LimInf set, then $S \cup \{0\}$ is a LimInf set. If S is a LimMon ($\mathbf{0}'$) set, then $S \cup \{0\}$ is a LimMon ($\mathbf{0}'$) set.

Fact 1.7. If S is a LimInf set not containing 0, then there is a LimInf witnessing function f for S satisfying $f(x,s) \neq 0$ for all x and s.

If S is a LimMon ($\mathbf{0}'$) set not containing 0, then there is a LimMon ($\mathbf{0}'$) witnessing function g for S satisfying $g(x,t) \neq 0$ for all x and t.

As the proofs of these facts are all straightforward, we leave them to the reader. Having introduced all the relevant notions, we are now in a position to state the main results of the paper. The first result is in computable model theory. In particular, it provides a necessary and sufficient condition for the shuffle sum $\sigma(S)$ to be computable in terms of the new computability-theoretic notion of LimInf sets.

Theorem 1.8. *For sets* $S \subseteq \omega + 1$, *the shuffle sum* $\sigma(S)$ *is computable if and only if* S *is a* LimInf *set.*

The second result is in classical computability theory. It provides an alternate characterization of the LimInf sets, showing their equivalence with the pre-existing notion of LimMon $(\mathbf{0}')$ sets.

Theorem 1.9. *A set $S \subseteq \omega+1$ is a* LimInf *set if and only if S is a* LimMon $(\mathbf{0}')$ *set.*

In Section 2 we prove Theorem 1.8, and in Section 3 we prove Theorem 1.9. In Section 4 we discuss the relationship between the LimInf and LimMon $(\mathbf{0}')$ sets and the Σ^0_3 sets, making use of previous work in [2] and [6]. We note that in [2] the authors show that for sets $S \subseteq \omega$, if the shuffle sum $\sigma(S)$ is computable, then S is a LimMon $(\mathbf{0}')$ set.

2. Proof of Theorem 1.8

We prove Theorem 1.8 by proving the forwards and backwards directions separately, making each a proposition.

Proposition 2.1. *If $S \subseteq \omega+1$ is a* LimInf *set, then the shuffle sum $\sigma(S)$ is computable.*

Proof. Let $f(x,s)$ be a LimInf witnessing function for S. Fix a uniformly computable partition of the rationals $\mathbb{Q}$ into dense sets $\{Q_x\}_{x\in\omega}$ with $Q_x = \{q_{x,y}\}_{y\in\omega}$. We build a computable copy of $\sigma(S)$ in ω many stages s using $f(x,s)$.

The basic idea is to build the finite linear order $f(x,s+1)$ at a rational $q_{x,y}$ at stage $s+1$. If $f(x,s+1)$ is larger than $f(x,s)$, then the appropriate number of points are added to the linear order already built for $q_{x,y}$. If $f(x,s+1)$ is smaller than $f(x,s)$, then the extra points already built for $q_{x,y}$ are no longer associated with $q_{x,y}$; instead they eventually become associated with some other rational at a later stage. In order to track whether a point is currently associated with some rational $q_{x,y}$, the states *associated* and *unassociated* will be used.

Construction: At each stage s we build a computable linear order $\mathcal{L}_s$ such that $\mathcal{L}_s \subseteq \mathcal{L}_{s+1}$ for all s. With $\mathcal{L} = \bigcup_s \mathcal{L}_s$, we aim for $\mathcal{L} \cong \sigma(S)$. At stage 0 we begin with the empty linear order, i.e., $\mathcal{L}_0$ is the empty linear order. At stage $s+1$ we work on behalf of all rationals $q_{x,y}$ with $x, y < s$. This work is done in s^2 substages, with a substage devoted to each such rational $q_{x,y}$ (in lexicographic order). Fixing a rational $q_{x,y}$ with $x, y < s$, we compare the value of $f(x,s+1)$ and $f(x,s)$; our action is determined by which is greater and whether or not work has already been done for the rational $q_{x,y}$.

If $f(x,s+1) > f(x,s)$ and work has already been done for $q_{x,y}$, then we insert the appropriate number of new points (namely $f(x,s+1) - f(x,s)$) at the right end of the linear order built at $q_{x,y}$ and give these inserted points the state associated.

If $f(x,s+1) < f(x,s)$ and work has already been done for $q_{x,y}$, then we split off the appropriate number of points (namely $f(x,s) - f(x,s+1)$) from the right end of the linear order built at $q_{x,y}$. The points split off have their state switched to unassociated and receive a priority amongst all points unassociated based first on the stage at which they became unassociated (lower stage, higher priority) and then their position in the linear order (further left, higher priority).

If no work has been done for $q_{x,y}$, then we insert the linear order $f(x,s+1)$ at $q_{x,y}$. In particular, we note whether or not there are any unassociated points greater than the greatest associated point to the left of $q_{x,y}$ and less than the least associated point to the right of $q_{x,y}$. If there are such unassociated points, we use the one with highest priority for the first point of the linear order built at $q_{x,y}$ and insert the appropriate number of new points (namely $f(x, s+1)-1$) immediately to the right of this first point. If there are no such unassociated points, we insert the appropriate number of new points (namely $f(x, s+1)$) at $q_{x,y}$. All points inserted at $q_{x,y}$ are given the state associated, including the previously unassociated point if one was used. This completes the construction.

Verification: Since the construction is computable, it suffices to show that $\mathcal{L} \cong \sigma(S)$. In order to demonstrate this equality, we verify the following two claims. The first implies that no extra points are built, and the second implies that enough points are built.

Claim 2.1.1. Every point has state unassociated for at most finitely many stages.

Proof. When a point changes its state to unassociated, there are at most finitely many unassociated points with higher priority. Moreover, as priority is determined first by stage, no later point will receive a higher priority.

As a consequence of the density of the rationals and that only those rationals $q_{x,y}$ with $x, y < s$ have had work done for them by stage s, at some later stage the point will meet the criterion for becoming the first point built for some rational which is having work done for it for the first time. When the point does meet this criterion, it will never again become unassociated as by convention $f(x, s) > 0$ for all s, and thus it will never be split off. Thus each point becomes unassociated at most once and eventually becomes associated permanently at some later stage. □

Claim 2.1.2. In $\mathcal{L}$, the linear order $F(x) = \liminf_s f(x, s)$ is built at the rational $q_{x,y}$.

Proof. Since $F(x) = \liminf_s f(x, s)$, there is a stage $\hat{s}$ such that $f(x, s) \geq F(x)$ for all $s > \hat{s}$. As a result, the rational $q_{x,y}$ will have at least $F(x)$ points built at it at every stage $s > \hat{s}$. On the other hand, no other points will remain permanently associated with $q_{x,y}$ as infinitely often the value of $f(x, s)$ will drop to $F(x)$, causing all other points to be split off from $q_{x,y}$.

As the rationals are dense, eventually the points split off will be separated from the $F(x)$ points permanently associated with $q_{x,y}$. Thus the linear order $F(x)$ is built at the rational $q_{x,y}$ in $\mathcal{L}$. □

It follows from the first claim that every point of the linear order eventually becomes associated permanently with some rational $q_{x,y}$. As each $q_{x,y}$ has the correct linear order built at it by the second claim, we conclude that $\mathcal{L} = \sigma(S)$. □

Before demonstrating the converse, we introduce some vocabulary and notation which will simplify the language in its proof.

Definition 2.2. A *maximal block* in a linear order is a maximal collection of points with respect to the property that for every pair of points a and b in the collection, the interval $[a, b]$ is finite.

The *block size* of an element x, denoted $\text{BlockSize}(x)$, is the number of points in the (unique) maximal block containing x.

Definition 2.3. If $A = \{a_x\}_{x\in\omega}$ is an enumeration of a linear order $\mathcal{A} = (A :\prec)$, define $|(a_i, a_j)|_s$ to be the number of points strictly between a_i and a_j amongst the first s points in the enumeration, i.e., the cardinality of the set $\{k : a_i \prec a_k \prec a_j, k < s\}$.

Proposition 2.4. *If the shuffle sum $\sigma(S)$ is computable with $S \subseteq \omega + 1$, then S is a* LimInf *set.*

Proof. Assume $\sigma(S)$ is computable and let $\mathcal{A} = (A :\prec)$ be a computable presentation of $\sigma(S)$ with universe $A = \{a_x\}_{x\in\omega}$. In order to show that S is a LimInf set, we define a LimInf witnessing function $f : \omega \times \omega \to \omega$ for S.

The idea will be to define auxiliary functions $\ell(x, s)$ and $r(x, s)$ that guess the number of points to the left and right of x in its maximal block. The difficulty is that all linear orders of a fixed finite cardinality are isomorphic. This obstacle is resolved by believing the left and right boundaries of the maximal block are determined by the most recently enumerated point on the left and on the right. Because of the dense nature of the maximal blocks, infinitely often $\ell(x, s)$ and $r(x, s)$ will be correct.

From the functions $\ell(x, s)$ and $r(x, s)$, we define the function $f(x, s)$. The idea will be to add $\ell(x, s)$ and $r(x, s)$ to obtain the value of $f(x, s)$, but we cannot do so directly as $\ell(x, s)$ and $r(x, s)$ may never be at their correct values simultaneously.

Construction: Before defining $f(x, s)$, we first define auxiliary functions $\ell(x, s) : \omega \times \omega \to \omega$ and $r(x, s) : \omega \times \omega \to \omega$ by

$$\ell(x, s) = |(a_i, a_x)|_s \qquad \text{and} \qquad r(x, s) = |(a_x, a_j)|_s$$

where i is the greatest index less than s such that $a_i \prec a_x$ and j is the greatest index less than s such that $a_x \prec a_j$. If no such index i exists, define $\ell(x, s) = |(-\infty, a_x)|_s$. Similarly, if no such index j exists, define $r(x, s) = |(a_x, +\infty)|_s$.

Fixing x and s, let v be the most recent time before s such that $r(x, \cdot)$ took the value $r(x, s)$. More formally, we define $v = v_{x,s}$ to be the greatest integer u less than s such that $\ell(x, u) = \ell(x, s)$ if one exists; otherwise we define $v = v_{x,s}$ to be s. We then define $f(x, s)$ by

$$f(x, s) = \ell(x, s) + 1 + \min_{z\in[v,s]} r(x, z).$$

Verification: Since $\mathcal{A}$ is a computable presentation of $\sigma(S)$, it is clear that $\ell(x, s)$ and $r(x, s)$ are computable, from which it follows that $f(x, s)$ is computable. We claim that the range of $F(x) = \liminf_s f(x, s)$ is exactly S, which we will show by demonstrating that $\liminf_s f(x, s) = \mathrm{BlockSize}(a_x)$. Fixing x, we consider the cases when $\mathrm{BlockSize}(a_x)$ is finite and infinite separately.

Claim 2.4.1. If $\mathrm{BlockSize}(a_x)$ is finite, then $\liminf_s f(x, s) = \mathrm{BlockSize}(a_x)$.

Proof. If $\mathrm{BlockSize}(a_x) = n$, then there is a $\hat{s}$ such that $\{a_0, \dots, a_{\hat{s}}\}$ includes all of the elements of the maximal block of a_x. Moreover, we may assume that at stage $\hat{s}$, the points a_i and a_j (as in the definition of $\ell(x, s)$ and $r(x, s)$) are not part of the maximal block of a_x.

Denote the elements in a_x's maximal block by $\{a_{x_1} < \cdots < a_x = a_{x_k} < \cdots < a_{x_n}\}$. Then $\{a_{x_1}, \dots, a_{x_n}\} \subseteq \{a_0, \dots, a_{\hat{s}}\}$. Note that for any $s > \hat{s}$, we have $\ell(x, s) \geq k - 1$ and $r(x, s) \geq n - k$.

When a new element is enumerated directly to the left of a_{x_1}, we have $\ell(x,s) = k-1$; similarly, when a new element is enumerated directly to the right of a_{x_n}, we have $r(x,s) = n-k$. Because of the dense nature of shuffle sums, such points will be enumerated infinitely often. Thus $\liminf_s \ell(x,s) = k-1$ and $\liminf_s r(x,s) = n-k$, from which it follows that $\liminf_s f(x,s) = (k-1)+1+(n-k) = n$. □

Claim 2.4.2. If $\mathrm{BlockSize}(a_x)$ is infinite, then $\liminf_s f(x,s) = \mathrm{BlockSize}(a_x)$.

Proof. If $\mathrm{BlockSize}(a_x) = \infty$, then a_x belongs to a maximal block of order type ω. For every k, there is an $\hat{s} = \hat{s}_k$ such that $\{a_0, \ldots, a_{\hat{s}}\}$ includes the k points immediately to the right of a_x in $\sigma(S)$. Moreover, we may assume that at stage $\hat{s}$, the point a_j (as in the definition of $r(x,s)$) is not part of the maximal block of a_x. Then $r(x,s) \geq k$ for all $s > \hat{s}$. Since there is a stage $\hat{s} = \hat{s}_k$ for every k, it follows that $\lim_s r(x,s) = \infty$. We conclude that $\liminf_s f(x,s) = \infty$. □

As a consequence of $F(x) = \liminf_s f(x,s) = \mathrm{BlockSize}(a_x)$ for all x, we conclude that $f(x,s)$ is a LimInf witnessing function for S.

□

3. Proof of Theorem 1.9

We prove Theorem 1.9, again by proving the forwards and backwards directions separately as separate propositions.

Proposition 3.1. *If $S \subseteq \omega+1$ is a* LimInf *set, then S is a* LimMon $(\mathbf{0}')$ *set.*

Proof. Let $f(x,s)$ be a LimInf witnessing function for S. Define a function $g : \omega \times \omega \to \omega$ by setting $g(x,t)$ equal to the largest number n such that $f(x,s) \geq n$ for all $s \geq t$. Note that $g(x,t)$ is total, increasing in t, and computable in $\mathbf{0}'$. Moreover $\lim_t g(x,t) = \liminf_s f(x,s)$, so that the range of $G(x) = \lim_t g(x,t)$ is the same as the range of $F(x) = \liminf_s f(x,s)$. It follows that $g(x,t)$ is a LimMon $(\mathbf{0}')$ witnessing function for S. □

Proposition 3.2. *If $S \subseteq \omega+1$ is a* LimMon $(\mathbf{0}')$ *set, then S is a* LimInf *set.*

Proof. Let $g(x,t)$ be a LimMon $(\mathbf{0}')$ witnessing function for S. By the Limit Lemma, there is a total computable function $h : \omega \times \omega \times \omega \to \omega$ such that $\lim_k h(x,t,k) = g(x,t)$. Fixing x, for each s we define a natural number t_s by recursion. Let $t_0 = 0$ and let t_s for $s > 0$ be the least t not greater than t_{s-1} such that $h(x,t,s) \neq h(x,t,s-1)$ if such a t exists, and otherwise let t_s be $t_{s-1}+1$.

We define a function $f : \omega \times \omega \to \omega$ by

$$f(x,s) = \max\left\{\tilde{h}(x,i,s) : 0 \leq i \leq t_s\right\}.$$

As $f(x,s)$ is clearly total and computable, it suffices to show that $\liminf_s f(x,s) = \lim_t g(x,t)$ for all x. We begin with a combinatorial claim about the sequence $\{t_s\}_{s\in\omega}$.

Claim 3.2.1. For every t, there are at most finitely many s with $t_s = t$. In particular, for every t, there are at most finitely many s with $t_s \leq t$.

Proof. We prove the claim by induction on t. For $t = 0$, we have $t_s = 0$ when $s = 0$ and when $h(x,0,s) \neq h(x,0,s-1)$. Since $\lim_k h(x,0,k)$ exists, the latter condition occurs at most finitely often. Thus $t_s = 0$ for only finitely many s.

For $t+1$, we have $t_s = t+1$ possibly when $h(x,t+1,s) \neq h(x,t+1,s-1)$ and possibly when $t_{s-1} = t$. Since $\lim_k h(x,t+1,k)$ exists, the former condition happens at most finitely often. The inductive hypothesis assures that the latter condition happens at most finitely often. Thus $t_s = t+1$ for only finitely many s. □

In order to show that $\liminf_s f(x,s) \geq \lim_t g(x,t)$, we argue that for every t, there is an $\hat{s}$ such that $f(x,s) \geq g(x,t)$ for all $s \geq \hat{s}$. Fixing t, let $\hat{s}$ be such that $t_s \geq t$ for all $s \geq \hat{s}$, which is possible by the claim. As $\lim_k h(x,t,k)$ exists, we may assume that $\hat{s}$ satisfies $h(x,t,s) = g(x,t)$ for all $s \geq \hat{s}$. Then for $s \geq \hat{s}$ we have

$$\begin{aligned} f(x,s) &= \max\{h(x,i,s) : 0 \leq i \leq t_s\} \\ &\geq \max\{h(x,i,s) : 0 \leq i \leq t\} \\ &\geq h(x,t,s) \;=\; g(x,t). \end{aligned}$$

Thus for every t there is an $\hat{s}$ such that $f(x,s) \geq g(x,t)$ for all $s \geq \hat{s}$, from which the inequality $\liminf_s f(x,s) \geq \lim_t g(x,t)$ follows.

In order to show that $\liminf_s f(x,s) \leq g(x,t)$, we argue that for every t there is an s such that $f(x,s) = g(x,t)$ (with $s \neq s'$ if $t \neq t'$). Fixing t, let $\hat{s}$ be minimal such that $h(x,i,s) = g(x,i)$ for all $i \leq t$ and $s \geq \hat{s}$. Let s be the least number greater than or equal to $\hat{s}$ such that $t_s = t$, which is possible since $\hat{s}$ was chosen to satisfy $h(x,i,\hat{s}) = g(x,i) \neq h(x,i,\hat{s}-1)$ for some $i \leq t$. Then

$$\begin{aligned} f(x,s) &= \max\{h(x,i,s) : 0 \leq i \leq t_s\} \\ &= \max\{h(x,i,s) : 0 \leq i \leq t\} \\ &= \max\{g(x,i) : 0 \leq i \leq t\} \\ &= \tilde{g}(x,t). \end{aligned}$$

Moreover, the value of s will be distinct for distinct values of t as s satisfies $t_s = t$. Thus for every t there is an s such that $f(x,s) = g(x,t)$, with $s \neq s'$ if $t \neq t'$, from which the inequality $\liminf_s f(x,s) \leq \lim_t g(x,t)$ follows.

We conclude that $\liminf_s f(x,s) = \lim_t g(x,t)$ for all x, so that $f(x,s)$ is a LimInf witnessing function for S. □

4. LimInf and LimMon ($\mathbf{0'}$) Sets

With the characterization of the computable shuffle sums of subsets $S \subseteq \omega+1$ in terms of LimInf and LimMon ($\mathbf{0'}$) sets completed, it is natural to ask which subsets of $\omega+1$ are LimInf and LimMon ($\mathbf{0'}$) sets. We note that a LimInf set (and thus a LimMon ($\mathbf{0'}$) set) can be no more complicated than a Σ^0_3 set. For if $f(x,s)$ is a LimInf witnessing function for S, then

$$\begin{aligned} n \in S \quad &\text{iff} \quad \exists x\,[\liminf_s f(x,s) = n] \\ &\text{iff} \quad \exists x\big[\exists \hat{s}\forall s > \hat{s}\,[f(x,s) \geq n] \;\&\; \forall s \exists s' > s\,[f(x,s') = n]\big]. \end{aligned}$$

As the last predicate is clearly Σ^0_3, membership in S cannot be more complicated than Σ^0_3. We state this as a proposition.

Proposition 4.1. *If S is a* LimInf *and* LimMon ($\mathbf{0'}$) *set, then S is a Σ^0_3 set.*

We next show that for sets S with $\omega \in S$, the LimInf sets (and thus LimMon ($\mathbf{0'}$) sets) coincide exactly with the Σ^0_3 sets.

Proposition 4.2. *If $S \subseteq \omega$ is a Σ^0_3 set, then $S \cup \{\omega\}$ is a* LimInf *and* LimMon ($\mathbf{0'}$) *set.*

Proof. Let S be a Σ^0_3 set witnessed by the predicate $\exists m \exists^\infty s\, R(n,m,s)$, where R is a computable relation. Define a function $f : \omega \times \omega \to \omega$ by

$$f(x,s) = f(\langle n,m \rangle, s) = \begin{cases} n & \text{if } R(n,m,s), \\ s & \text{otherwise.} \end{cases}$$

Note that $f(x,s)$ is computable as R is computable.

If $n \in S$, we have $\exists m \exists^\infty s\, R(n,m,s)$. Letting m_0 witness this, we have $f(\langle n,m_0\rangle, s) = n$ for infinitely many s. As s will be less than n only a finite number of times, it follows that $\liminf_s f(\langle n,m_0\rangle, s) = n$. Thus n is in the range of $F(x) = \liminf_s f(x,s)$.

If instead $n \notin S$, we have $\forall m \exists^{<\infty} s\, R(n,m,s)$. For any $x = \langle n,m\rangle$, it follows that $f(x,s) = n$ for only finitely many s. Thus $\liminf_s f(x,s) = \infty$, and so ω is in the range of $F(x)$ and n is not in the range of $F(x)$.

In the extreme case when $S = \omega$, we can (non-uniformly) arrange to have the range of $F(x) = \liminf_s f(x,s)$ be $\omega \cup \{\omega\}$ if ω would otherwise not be in the range. □

It follows immediately from Theorem 1.8 and Proposition 4.2 that $\sigma(S \cup \{\omega\})$ is computable for every Σ^0_3 set S, a result shown in [1]. However $\sigma(S)$ is not computable for every Σ^0_3 set S, a corollary of our results and the following result found in [2] (which is a relativization of a result in [6]).

Proposition 4.3 (Coles, Downey, and Khoussainov)**.** *There is a Σ^0_3 set S that is not a* LimMon $(\mathbf{0}')$ *set.*

We conclude by leaving open two questions.

Question 4.4. For which subsets $S \subseteq \omega_1^{\mathrm{CK}} + 1$ is $\sigma(S)$ computable?

Question 4.5. Which subsets $S \subseteq \omega + 1$ are Σ^0_3 sets but not LimInf sets, or equivalently LimMon $(\mathbf{0}')$ sets?

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