Clifford Algebraic Computational Fluid Dynamics: A New Class of Experiments.

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Abstract

Though some influentially critical objections have been raised during the 'classical' pre-computational simulation philosophy of science (PCSPS) tradition, suggesting a more nuanced methodological category for experiments[2], it safe to say such critical objections have greatly proliferated in philosophical studies dedicated to the role played by computational simulations in science. For instance, Eric Winsberg (1999-2003) suggests that computer simulations are methodologically unique in the development of a theory's models[3] suggesting new epistemic notions of application. This is also echoed in Jeffrey Ramsey's (1995) notions of "transformation reduction,"—i.e., a notion of reduction of a more highly constructive variety.[4] Computer simulations create a broadly continuous arena spanned by normative and descriptive aspects of theory-articulation, as entailed by the notion of transformation reductions occupying a continuous region demarcated by Ernest Nagel's (1974) logical-explanatory "domain-combining reduction" on the one hand, and Thomas Nickels' (1973) heuristic "domain-preserving reduction," on the other.

I extend Winsberg's and Ramsey's points here, by arguing that in the field of computational fluid dynamics (CFD) as well as in other branches of applied physics, the computer plays a constitutively *experimental* role—supplanting in many cases the more traditional experimental methods such as flow-visualization, etc. In this case, however CFD algorithms act as substitutes, not supplements (as the notions "simulation" suggests) when it comes to experimental practices. I bring up the constructive example involving the Clifford-Algebraic algorithms for modeling singular phenomena (i.e., vortex formation, etc.) in CFD by Gerik Scheuermann (2000) and Steven Mann & Alyn Rockwood (2003) who demonstrate that their algorithms offer greater descriptive and explanatory scope than the standard Navier-Stokes approaches. The mathematical distinction between Navier-Stokes-based and Clifford-Algebraic based CFD (i.e., NSCFD and CACFD) has essentially to do with the *regularization* features (i.e., overcoming and conditioning singularities) exhibited to a far greater extent by the latter, than the former.[5] Hence, CACFD indicate that the utilization of computational techniques can be based on *principled* reasons (i.e., the ability to characterize singular phenomena in ways that traditional

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[2] These criticisms were raised both by philosophers (Franklin, 1986) as well as historians of science (Galison, 1997; Friedel, 2001). Granted, the latter two published in the decades in which philosophical studies of computer simulations were already in full sway, nevertheless I deem them "traditional" since the topic of computer simulations does not occupy center stage, in their writings.

[3] Broadly construed in a heuristic sense, as those in the "cognitive turn" employ the term (Giere, 1988), as opposed to the narrower logical, model-theoretic sense.

[5] This reason, among others, goes a long way to account for why Clifford algebra comprises a venerable research tradition in applied, mathematical, and theoretical physics.

experimental methodologies are too coarse-grained to meet the explanatory demands suggested by CFD), as opposed to merely *practical* (i.e., that such computational procedures better fit the bill-literally!-in terms of contingent resource allocation). CACFD hence exhibit a new generative role in the field of fluid mechanics, by offering categories of experimental evidence that are optimally descriptive and explanatory—i.e., *pace* Batterman (2005) can be both ontologically and epistemically fundamental.

I. Introduction

In pre-computational simulation philosophy of science (PCSPS[6]), the analysis of scientific methodology subdivides itself into three natural kinds: the hypothesis/(es) *H* to be tested, with respect to the evidence *E* deemed relevant vis-à-vis *H*. Moreover, the relevance criteria characterizing the relationship of *H* with respect to *E* inevitably involves, to a certain extent, the "background" *B* consisting of the vast repository of epistemic, semantic, and ontological factors not directly entailed by *H*'s scope—whether logical or evidential. In this respect, experiments play a pivotal role in negotiating aspects of *H, E, B*—as the traditional accounts go—in serving an essential aim in (dis/)confirming *H*. For example, in this regard, perhaps Thomas Kuhn (1962) was the first to suggest such an essential methodological role entailed by experiments, since his notion of paradigm suggested the aspects of theory, applications, instrumentation, and nomology, all mutually dynamically interacting in a virtuously circular manner.

Nowadays, such facile global characterizations of the role played by experiments have been supplanted by particularly nuanced treatments, teeming with suggestions attempting to characterize their evidential roles in (more or less) semi-autonomous fashion. By and large, much of this appears to involve, in some irreducible fashion, closer scrutiny of the role played by *values* (as Kuhn (1977) already suggested in the case of theory-choice). For instance, Helen Longino (1992, 1998) attempts to characterize "scientific objectivity" in such a manner essentially involving aspects of socialized epistemology, out of which a objectivity would entail a potential "transformational criticism" including an irreducible admixture of *cognitive* and *contextual* values.[7] According to Longino, the complex interplay of cognitive and contextual

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[6] Although it may prove difficult to establish firm dates here—since such notions were already suggested by Pierre Duhem in the early twentieth century (Gillies, 1993). Nevertheless, the logical empiricists (e.g., Hempel, etc.) and their historically-motivated respondents (Hansen, Kuhn, etc.) as well as those in the "cognitive turn" (Giere) have presupposed the *H, E, B* subdivision by and large without question--suggesting a time frame spanning the early 1960s to the mid-1990s, at least in the Anglo-American tradition.

[7] I.e., epistemic and methodological values internal to a scientific disciplinary matrix or research program, versus the background epistemic values which are usually depicted in terms of what Manson & O'Neill (2008) refer to as *epistemic responsibility*—i.e., informing based on norms communicative transactions (like appropriate accuracy), as

values is evident, for instance, in four possible ways a hypothesis *H* may be evaluated in terms of its putative evidence *E*:

- Evidentially Criticizing the quality of *E*
- Conceptually (1): Questioning the nature of a *H's* "conceptual soundness"
- Conceptually (2): Questioning the nature of *H*'s "consistency with accepted body of theories."[8] (1998, 173)
- Conceptually (3): How *relevant* is *E* as a "support" to *H* ?

Indeed, it is the *third* conceptual kind of appraisal which "amounts to questioning the background beliefs [of scientists] …crucial for the problem of objectivity." (ibid.)

The layers of evaluation of *H* vis-à-vis *E*, though for Longino comprising a project of broader scope than that entailed by the study of experimentation per se, appear especially relevant in the case of some of the recent literature on computational simulation[9].

> In order to avoid the appearance of there being anything strange or paradoxical about a practice [e.g., computer simulations] that straddles the terrain between the theoretical and the experimental, we need to recognize that while simulation is, in the general sense, a form of what we once naively called theorizing, it is the kind of theorizing that has only recently begun to attract philosophical attention—construction of local, representative, models. (Winsberg (2003), 120)

> The history of a simulation technique is very much like the history of a scientific instrument. It begins with a relatively crude and simple technique for attacking a relatively small set of problems. Over time, the instrument or technique is called upon to attack a larger set of problems or to achieve a higher degree of accuracy. In order to achieve this, the technique needs to be improved upon, reconfigured, and ever radically revised. *In each case, the knowledge relied upon to devise and sanction the tool or method can come from a wide variety of domains*. (Winsberg (2003), 123-134, italics added)

The points discussed above by Winsberg lends credence to what he describes as simulations "having a life of their own" (1999, 2003) insofar as they (as in the general case of model-building) or "semiautonomous" insofar as although "[p]rima facie they [simulations] are nothing

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opposed to merely *disclosing* semantic content. O'Neill & Manson highlight such issues in the area of bioethics, regarding particular issues centering on notions of informed consent, etc.

[8] Note the analogy with Kuhn's (1977) "external consistency," (among the five proposed values of theory-choice: accuracy, broad explanatory scope, consistency (internal and external), fecundity, and accuracy.

[9] "I use the term 'simulation' to refer to comprehensive process of building, running, and inferring from computational models." (Winsberg, 2003, n. 3, p. 107)

but applications of scientific theories to systems under the theories' domain," (2003, 105) nevertheless in their 'theory-articulation'[10] "there is no algorithm for reading models off a theory." (106) Undoubtedly, among other factors, the role played by simulations is emblematic of the complex and multilayered means by which evidence may be evaluated in terms of its counterpart theory or hypothesis, as suggested by Longino's schema.

Reliability is the constitutive feature of simulations which is derived both from the credentials of its governing theory as well "the antecedently established credentials of the model building techniques developed over an extended tradition of employment," (122) of which no general algorithm can instantiate.[11] Winsberg sounds a generally skeptical note regarding making any inferences suggesting (in a fashion which would avoid begging the question) any realist interpretation of the simulation (insofar as them displaying any irreducible element of *bona fide* representational capacity), as levied in his criticisms of Hughes (1999) in his 2003 (113-116) as well as in his "counterexample that success implies truth" discussed in the case of artificial viscosity (2006).

This is a rendition of Larry Laudan's (1981, 1998) criticism against scientific realism applied specifically to the case of simulation.[12] Certainly attacks against realism are established in the canonical "classical" literature,[13] and little would be gained here from dredging up the debate *tout court*.[14] Nevertheless, as I argue below, there may be good reasons not to entirely

[10] I.e., Kuhn's view of the essence of "normal science:" Puzzle-solving inevitable involve theory articulation. Unificationists like Philip Kitcher (1989) consider this activity as the hallmark of scientific explanation. Oftentimes this is equivocated with "theorizing" per se, "in no small part because most commentators on science, especially philosophers, have woefully underestimated the importance of theory articulation, or model building." (Winsberg, 2003, 119).

[11] "Building testable models,…usually involves highly context-dependent idealizing and approximating assumptions, and often requires appealing to assumptions from…sometimes incompatible theories." (Frisch, 2005, p. 10) In this respect, high-level abstractions, or the theory's "laws," should be thought of as "tools for model-building, rather than as representative of structures of the world."(11)

[12] Characterized by Laudan as the fallacy of affirming the consequent: From the premise of a theory *T*'s truth entailing its success, obviously one cannot argue for *T*'s truth based on *T*'s success alone.

[13] Aside from the fallacy described in n. 12 above, many of the arguments against realism can also be generally be subdivided into the following (distinct but not disjunct) classes (the list is by no means exhaustive): i.) Pessimistic meta-induction arguments: If the historical arc of science bends toward truth, to paraphrase Martin Luther King's metaphor, how does this normative claim square with the obvious historically factual claim that the history of science is a vast graveyard of abandoned theories? ii.) Problem of induction: Past success is no guarantee of future success, etc., iii.) Methodological problems bedeviling verisimilitude, should some (like more recently in the case of Kitcher (2001)) suggest some kind of epistemic convergence.

[14] It may be worth mentioning, in passing, that some of the responses against the stock arguments (alluded to in n. 13) above include Richard Boyd's (1985) realism based on historical and methodological grounds (which may or may not, depending on one's views on the matter, provide a response to the pessimistic meta-induction claim). Other relatively more recent responses against the pessimistic meta-induction include Stathis Psillos' (1996) *divide*

abandon realist considerations, when considering the case of Clifford Algebraic computational fluid mechanics (CACFD)—in particular, a realism suggestive of *both* elements of Giere's (1988) and Hacking's (1982, 1998) separate uses (and senses) of term: "constructive realism."[15] At best (*a' la* Sellars (1962)) I can only do this via some general inference to the best explanation strategy which (at best) would offer reasons for such a nuanced realism. However, even if the reader remains unconvinced or agnostic, at the very least I aim to show herein that CACFD represent a new class of experiments.

II. Clifford Algebra: A Brief Overview

The Cambridge mathematician William Kingdon Clifford originally developed his algebra[16] in the years 1878-1882 as a means to systematically develop a matrix algebra representing rotations and spin, generalized in any *n*-dimensional space: $R^{n} = \{(x_1,..., x_n) \mid x_k \in R, 1 \leq k \leq n\}$ (where *R* are the real numbers). In keeping with Clifford's intentions, Hestenes (1984, 1986) and others ascribed the term 'geometric' to such classes of algebras to call attention to the primary feature of this mathematical system, portraying the class of all possible *rotations* (and spins) in *n*-dimensional space, which is an essentially geometrical dynamical property.

Geometric algebras can be fundamentally thought of as systematic collections of *directed* line segments (vectors), areas (bivectors), volumes (trivectors),…, *n*-dimensional hypervolumes (*n*-vectors or *n*-blades) as bounded above by the dimensionality *n* of the algebra's underlying

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at impera claims—that *components* of a preceding theory (having some irreducibly 'truth-tracking' or representational capacity) shall survive into superseding theory, even when the former is *tout court* falsified. (The analogy of 'cannibalizing' components of a junked car engine to be retro-fitted into another functioning car comes to mind.)

[15] For Giere, this has primarily to do with interpretations concerning the modal scope of models in a theory. For instance, a constructive realist (narrowly defined) would argue that a model *M* articulated by a theory *T* would agree with *the actual* history of *all* (or most) an experimental system *S*'s variables (1985, 83). Broadly defined, a constructive realist (narrowly defined) would argue that *M* articulated by *T* would agree with *all possible* histories of *all* (or most) an experimental system *S*'s variables. The constructive empiricist (*a' la* Bas Van Fraassen (1980)) on the other hand, according to Giere, would argue that at best only *the variables explicitly specified* in *M* (e.g., position *x* an momentum *p* for the case of an SHO –simple harmonic oscillator—model) would agree with those measure in actual system *S*. I take on the issue of Giere's (1988) constructive realism and the "modal rationalism" David Chalmers' (2002) particular version of 2D semantics in Kallfelz (2010). Whereas Ian Hacking's (1982) use centers itself specifically on the constructive characteristic of the "autonomous" role that experiments can play, in the epistemic and methodological aspects of representing and intervening. "We are completely convinced of the **reality of electrons** when we …build—and often succeed in **building—new devices that use various well understood causal properties of electrons** to **interfere** in other more hypothetical parts of nature." (Hacking (1982, 1998) 1158.)

[16] A vector space endowed with an associative product. For further technical details, see Appendix below.

vector space. While the concept of a directed line segment seems intuitive enough (partly due to the historical success of the 'rival' vector algebra of Gibbs), the concept of directed surfaces, volumes, and hypervolumes may seem less so. The concept of directed area however survives, for instance, in the geometric interpretation of a vector cross-product in R^3. As a further indication of its vestigial ancestry to Clifford, the cross-product is actually an example of a bivector, or *axial vector*, as it changes sign under reversal of parity of the coordinate system (from a left-handed to a right-handed system, and vice versa) while regular vectors do not.

Clifford algebras are *graded*: their generators form a basis of linearly independent *k*-vectors (where $0 \le k \le n$), where n is the dimensionality of the underlying vector space. For example, the Clifford algebra $G(R^3)$ over vector space R^3 is generated by a total of $2^3 = 8$ grade k elements (where $0 \le k \le 3$): 1 grade-0 element (the real scalars), 3 grade-1 elements (3 linearly independent vectors whose span is obviously R^3), 3 grade-2 elements (3 linearly independent bivectors), and 1 grade-3 (trivector) element. In general, for any vector space V of dimensionality n, its Clifford algebra is generated by a total of 2^n grade k elements (where $0 \le k \le n$), the dimensionality of each Clifford subspace of uniform grade k is: $C(n,k) = \frac{n!}{k!(n-k)!}$. That is to say, $C(n,k) = \frac{n!}{k!(n-k)!}$ linearly independent grade-*k* (or *k*-vector) elements generate the Clifford subspaces of uniform grade *k*. In addition, the (associative) Clifford product can be decomposed into a grade-lowering (inner) product and a grade-raising (outer) product, from which the notions of dot and cross products survive in the standard (Gibbs) vector algebra of R^3. For further details, see Appendix below.

After being eclipsed into relative obscurity for almost a century by Gibbs' vector notation,[17] the Clifford algebraic mathematical formalism (as well as its associated algebraic substructures like the Clifford groups) has enjoyed somewhat of a renaissance in the fields of physics (both purely theoretical as well as applied) and engineering in the last several decades. (Baugh 2003, Baylis 1995, Bolinder 1987, Conte 1993-2000, Finkelstein 1999-2004, Doren & Lasenby 2003, Gallier 2005, Hestenes 1984 -1986, Khrenikov 2005, Lansenby, et. al. 2000, Levine & Dannon 2000, Mann et. al. 2003, Nebe 1999-2000, Scheuermann 2000, Sloane 2001, Snygg 1997, Van den Nest, et. al. 2005, Vlasov 2000). All the authors listed above (who comprise just a miniscule sample of the enormous body of literature on the subject of applications of Clifford Algebra in

[17] As explained in the Appendix below, vestiges of Clifford's notation and algebra survive in the concept of Pauli and Dirac spin matrices, as well as the notion of a vector cross-product.

physics and engineering) either describe the mathematical formalism as especially appealing, due to its providing a 'unifying language' in the field of mathematical physics[18], or apply the formalism in key instances to make some interpretative point in the foundations of quantum theory, no matter how specific[19] or general.[20]

Clifford algebras can provide a complete notation for describing certain phenomena in physics that would otherwise require several different mathematical formalisms. For instance, in present-day quantum mechanics and field theory, a variety of different mathematical formalisms are often introduced: 3 dimensional vector algebra, Hilbert space methods, spinor algebra, diffeomorphism algebra on smooth manifolds, etc. This is due in part to the domain-specific nature of the aforementioned, all tailored to apply to a particularly specific context, but relatively restricted in their power of generalization. In contrast, as shall be shown below, Clifford Algebra provide a single and overarching formalism that can meet the needs of the mathematical physicist working in the applied as well as in the foundational domains. In Kallfelz (2009b) I argue that (*a' la* Kuhn (1977)) the comparatively broader scope and simplicity (aside from the technical consistency concerning semantic isomorphisms between algebraic and geometric concepts) yield non-trivial claims concerning issues in intertheoretic reduction, that are of philosophical interest.[21]

III. Navier Stokes and Clifford Algebraic Computational Fluid Dynamics

In the field of computational fluid dynamics (CFD) as well as in other branches of applied physics, the computer plays a constitutively *experimental* role—supplanting in many cases the more traditional experimental methods such as flow-visualization, etc. In this case, however CFD algorithms act as substitutes, not supplements (as the notions "simulation" suggests) when it comes to experimental practices. I bring up the constructive example below involving the Clifford-Algebraic algorithms for modeling singular phenomena (i.e., vortex formation, etc.) in CFD by Gerik Scheuermann (2000) and Steven Mann & Alyn Rockwood (2003) who demonstrate that their algorithms offer greater descriptive and explanatory scope

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[18] E.g., Finkelstein, Hestenes, Lasenby
[19] E.g., Conte, Hogreve, Snygg
[20] E.g., Hiley, Khrenikov, Vlasov
[21] Among other things, I seek to temper some of Robert Batterman's (2002, 2004, 2005) claims.

than the standard Navier-Stokes approaches. The mathematical distinction between Navier-Stokes-based and Clifford-Algebraic based CFD (i.e., NSCFD and CACFD) has essentially to do with the *regularization* features (i.e., overcoming and conditioning singularities) exhibited to a far greater extent by the latter, than the former.[22] Hence, CACFD indicate that the utilization of computational techniques can be based on *principled* reasons (i.e., the ability to characterize singular phenomena in ways that traditional experimental methodologies are too coarse-grained to meet the explanatory demands suggested by CFD), as opposed to merely *practical* (i.e., that such computational procedures better fit the bill-literally!-in terms of contingent resource allocation). CACFD hence exhibit a new generative role in the field of fluid mechanics, by offering categories of experimental evidence that are optimally descriptive and explanatory—i.e., *pace* Batterman (2005) can be both ontologically and epistemically fundamental.

Prior to introducing the work Mann & Rockwood and Scheuermann (in III.8 below), I make brief mention of some of the issues I discuss in a more detailed fashion in Kallfelz (2009b), bearing relevance to my claims concerning a tempered realist interpretation of these classes of simulations. Aside from some of the arcane aspects thereof, the reader is welcome to skip these (relatively free standing) sections below (III.1-III.7), should the topic of realism motivate not prove itself to be a motivating factor.

III.1: "Epistemic" and "Ontologically Fundamental" Aspects of Navier Stokes Simulations (Batterman (2005)).

Robert Batterman (2005) distinguishes between "ontologically fundamental" and "epistemically fundamental" theories. The aim of former is to "get the metaphysical nature of the systems right," (19) often at the expense of being explanatorily inadequate. Fundamentally explanatory issues involving the universal dynamical behavior of critical phenomena,[23] for instance, cannot be accounted for by the ontologically fundamental theory. The explanatory aim of epistemologically fundamental theories, on the other hand, is an account for such universal

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[22] This reason, among others, goes a long way to account for why Clifford algebra comprises a venerable research tradition in applied, mathematical, and theoretical physics.

[23] Such critical phenomena exhibiting universal dynamical properties include, but are not limited to, examples including fluids undergoing phase transitions under certain conditions favorable for modeling their behavior using Renormalization Group methods, shock-wave propagation (phonons), caustic surfaces occurring under study in the field of catastrophe optics, quantum chaotic phenomena, etc., some of which I reviewed in Chapter 2 above.

behavior at the expense of suppressing (if not outright misrepresenting) a physical system's fundamentally ontological features.

In the case of critical phenomena such as droplet formation, even in cases of more fine-grained resolutions of the scaling similarity solution for the Navier-Stokes equations (which approximate a fluid as a continuum), "we must appeal to the non-Humean similarity solution (resulting from the singularity) of the *idealized* continuum Navier-Stokes theory." (20) In a more general sense, though "nature abhors a singularity…without them one cannot characterize, describe, and explain the emergence of new universal phenomena at different scales." (19)

In other words, according to Batterman (2005) we need the ontologically "false" but epistemically fundamental theory to account for the ontologically true but epistemically lacking fundamental theory. "[A] complete understanding (or at least an attempt) of the drop breakup problem requires essential use of a 'nonfundamental' [i.e. epistemically fundamental] theory…the continuum Navier-Stokes theory of fluid dynamics." (18)

Batterman advocates this necessary coexistence of two kinds of fundamental theories can be viewed as a refinement of his more general themes presented in (2002). As I discussed in chapter 2 (Kallfelz (2009b)) he argues that in the case of emergent phenomena, explanation and reduction part company. The superseded theory T can still play an essential role. The superseding theory $T^{/}$, though 'deeply containing T' (in some non-reductive sense) cannot adequately account for emergent and critical phenomena alone, and thus enlists T in some essential manner. This produces a rift between reduction and explanation, and one is forced to accommodate an admixture of differing ontologies characterized by the respectively superseding and superseded theories. In his later work, Batterman (2005) seems to imply that epistemologically fundamental theories serve in a similarly necessary capacity in terms of what he explains the superseded theories do, in the case of emergent phenomena (2002).

I have critiqued Batterman's claims (2002, 2004) in my (2009b) in a two-fold manner: Batterman confuses a theory's mathematical content with its ontological content. This confusion, in turn, causes him to exaggerate the importance of certain notions of singularities in the explanatory role they play in the superseded theory. I argue here that there exist methods of regularization in geometric algebraic characterizations of microphysical phenomena, which can provide a more reliable ontological account for what goes on at the microlevel level *precisely* *because* they bypass singularities that would otherwise occur in more conventional mathematical techniques not based on geometric algebraic expansion and contraction.

III.2 Methodological Fundamentalism

I characterize such a notion of 'fundamental' arising in algebraic expansion and contraction techniques as an example of a *methodological fundamentalism*, which in principle can offers a means of intertheoretic reduction overcoming the singular cases Batterman discusses in (2002, 2004). In the case of fluid dynamics, mulitilinear algebras like Clifford algebras have been recently applied by Gerik Scheuermann (2000), and Mann & Rockwood (2003) in their work on computational fluid dynamics (CFD). The authors show that CFD methods imply that methodological fundamentalism can, in the cases Batterman investigates, provisionally sort out and reconcile epistemically and ontologically fundamental theories. Hence, *pace* Batterman, they need not act at cross purposes.

Robert Batterman explains the motivation for presenting a distinction between ontological versus epistemically fundamental theories:

> I have tried to show that a complete understanding (or at least an attempt…) of the drop breakup problem requires essential use of a 'nonfundamental' theory…the continuum Navier Stokes theory of fluid dynamics…[But] how can a false (because idealized) theory such as continuum fluid dynamics be *essential* for understanding the behaviors of systems that fail completely to exhibit the principal feature of that idealized theory? Such systems [after all] are discrete in nature and not continuous…*I think the term 'fundamental theory' is ambiguous…*[An ontologically fundamental theory]…gets the metaphysical nature of the system right. On the other hand…ontologically fundamental theories are often explanatorily inadequate. Certain explanatory questions…about the emergence and reproducibility of patterns of behavior cannot be answered by the ontologically fundamental theory. I think that this shows…there is an epistemological notion of 'fundamental theory' that fails to coincide with the ontological notion. (2005, 18-19, italics added)

On the other hand, epistemically fundamental theories aim at a more comprehensive explanatory account, often, however, at the price of introducing essential singularities. For example, in the case of 'universal classes' of behavior of fluid-dynamical phenomena exhibiting patterns like droplet formation:

> Explanation of [such] universal patterns of behavior require means for eliminating details that ontologically distinguish the different systems exhibiting the same behavior. *Such*

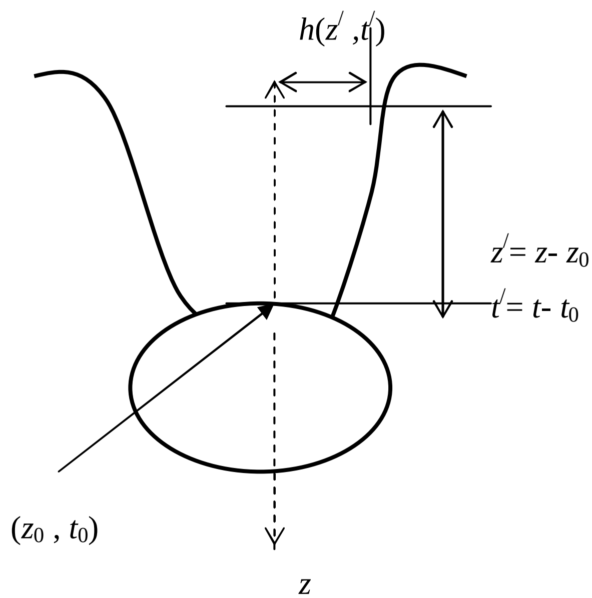
means are often provided by a blow-up or singularity in the epistemically more fundamental theory that is related to the ontologically fundamental theory by some limit. (ibid., italics added)

Obviously, any theory relying on a continuous topology[24] harbors the possibility of exhibiting singular behavior, depending on its domain of application.[25] In the case of droplet-formation, for example, the (renormalized) solutions to the continuous Navier-Stokes Equations (NSE) exhibit singular behavior. These singularities play an essentially explanatory role insofar as such solutions in the singular limit exhibit 'self-similar,' or universal behavior.[26] Only *one* parameter essentially governs the behavior of solutions to the NSEs in such a singular limit. Specifically, only the fluid's *thickness* parameter (neck radius *h*) governs the shape of the fluid near break-up,[27] in the asymptotic solution to the NSE (2004, 15)

Figure III.1 (*Representation of the parameters governing droplet formation*)

$$h(z',t') = f(t')^{\alpha} H(\zeta)$$

$$\zeta = \frac{z'}{f(t')^{\beta}}$$



where: $f(t')$ is a continuous (dimensionless) function expressing the time-dependence of the solution ($t' = t - t_0$ is the measured time after droplet breakup t_0).

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[24] In chapter 2 (Kallfelz 2009b), I borrow Bishop's (2002) usage, in which he distinguishes the *ontology*, i.e. the primitive entities stipulated by a physical theory, from its *topology*, or structure of its mathematical formalism.

[25] This is of course due to the rich structure of continuous sets themselves admitting such effects. Consider, for example, the paradigmatic example: $f \in (-\infty, \infty)^{(-\infty, \infty)}$ given by the rule: $f(x) = {}^{1}/_{x}$. This obviously produces an essential singularity at $x = 0$.

[26] "Batterman suggests that the similarities in behavior [i.e., the universality] may be explained as a consequence of the fact that the differences in realization at the physical level are irrelevant to the higher-order behavior, in the same way that the differences between diverse systems underlying phase transitions are irrelevant to the behavior near the critical temperature." (Strevens 2002, 655)

[27] For fluids of low viscosities see Batterman (2004), n 12, p.16.

α, β are phenomenological constants to be determined. *H* is a Haenkel function.[28]

One could understand the epistemically and ontologically fundamental theories as playing analogous roles to Batterman's (2002, 2003, 2004) previously characterized superseded and superseding theories (*T* and $T^{/}$, respectively). Analogous to the case of the superseded theory *T*, the epistemically fundamental theory offers crucial explanatory insight, at the expense of mischaracterizing the underlying ontology of the phenomena under study. Whereas, on the other hand, analogous to the case of the superseding theory $T^{/}$, the ontologically fundamental theory gives a more representative metaphysical characterization, at the expense of losing its explanatory efficacy.

For instance, in the case of the breaking water droplet, the ontologically fundamental theory would be the molecular-discrete one. But aside from practical limitations posed by the sheer intractability of the computational complexity of such a quantitative account, the discrete-molecular theory, precisely *because* it lacks the singular-asymptotic aspect, cannot depict the (relatively) universal character presented in the asymptotic limit of the (renormalized) solutions to the NSE.

However, I argue here that there are theoretical characterizations whose formalisms can regularize or remove singularities from some of the fluid-dynamical behavior in a sufficiently abstract and general manner, as to call into question the presumably essential distinctions between epistemological and ontological fundamentalism. I call such formal approaches "methodologically fundamental,"[29] because of the *general* strategy such approaches introduce, in terms of offering a regularizing procedure. Adopting such methodologically fundamental procedures, whenever it is possible to do so,[30] suggests that Batterman's distinctions may not be different theoretical *kinds*, but function at best as different *aspects* of a unified methodological strategy. This calls into question the explanatory pluralism Batterman appears to be advocating.

III.3: Belot's Critiques Revisited

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[28] I.e. belonging to a class of orthonormal special functions often appearing in solutions to PDEs describing dynamics of boundary-value problems.

[29] Recall my specification mentioned in n. 5 above.

[30] The generality of the methods do *not* imply that they are a panacea, ridding *any* theory's formalism of singularities.

Gordon Belot's (2003) criticism of Batterman (2002) consists of indicating that a more mathematically rigorous rendition of the superseding theory $T^{/}$ presumably eliminates the necessity of having to resort simultaneously to the superseded theory T to characterize some critical phenomenon (or class of phenomena) Φ. Like Belot, I also claim that geometric algebraic techniques abound which can regularize the singularities appearing in formalisms of T (or $T^{/}$). Conversely, when representing such critical phenomena Φ, singularities can occur in T (or $T^{/}$) when the latter are characterized by the more typically standard field-theoretic or phase space methods alone.

However, the mathematical content of the techniques I investigate differs significantly from those discussed by Belot (2003), who characterizes $T^{/}$ using the more general and abstract theory of partial differential equations on differentiable manifolds. He demonstrates that in principle, all of the necessary features of critical phenomena Φ can be so depicted by the mathematical formalism of superseding theory $T^{/}$ alone (2003, 23). Because the manifold structure is continuous, this can admit the possibility of depicting such critical phenomena Φ through complex and asymptotic singular behavior. In other words, Belot is *not* fundamentally questioning the underlying theoretical *topologies* typically associated with T and $T^{/}$.[31] Instead, he is questioning the need to bring the two different *ontologies* of the superseded and superseding theories together, to adequately account for Φ. Belot is questioning the presumed *ontological* pluralism that Batterman advanced in his notion of an 'asymptotic explanation'.

Batterman responds:

> I suspect that one intuition behind Belot's …objection is…I [appear to be] saying that for genuine explanation we need [to] appeal *essentially* to an idealization [i.e., the ontology of the superseded theory *T*.] …In speaking of this idealization as essential for explanation, they take me to be reifying [*T*'s ontology]…*It is this last claim only that I reject.* I believe that in many instances our explanatory physical practice demands that we appeal essentially to (infinite) idealizations. But I don't believe that this involves the reification of the idealized structures." (2003, 7)

It is, of course, precisely the latter claim "that we appeal essentially to (infinite) idealizations" that I take issue with here, according to what the regularization procedures indicate. Batterman, however, cryptically and subsequently remarks that: "In arguing that an account that appeals to the mathematical idealization is superior to a theory that does not invoke

[31] I.e., differential equations on phase space, characterizable through the theory of differential manifolds.

the idealizations, I am not reifying the mathematics…I am claiming that the 'fundamental' theory that fails to take seriously the idealized [asymptotic] 'boundary' is less explanatorily adequate." (8) In short, it seems that in his overarching emphasis in what he considers to be novel accounts of scientific explanation (namely, of the asymptotic variety) he often *blurs* the distinctions, and *shifts emphasis* between a theory's ontology and its topology. It is precisely this sort of equivocation, as I discussed in chapter 2 above, that causes him to inadvertently uphold mathematical notions like "infinite idealizations" as acting like some explanatory standard. To put it another way, since it is safe to assume that the actual critical phenomena Batterman discusses are ultimately metaphysically finite, precisely *how* can one 'appeal essentially to (infinite) idealizations' *without* inadvertently 'reifying the mathematics?'

I, on the other hand, *pace* Belot (2003) and Batterman (2002-2005) present an alternative to the mathematical formalisms that both authors appeal to, which rely so centrally on continuous topological structures.[32] I show how discretely graded, and ultimately finite-dimensional multi-linear geometric (Clifford) algebras can provide accounts for some of the *same* critical phenomena Φ in a *regularizable* or a singularity-free fashion.

III.4: Disclaimer Concerning the General Applicability of Clifford Algebra in Characterizing Critical Phenomena

Prior to describing the specific details of how to implement the strategy in the case of critical phenomena exhibited in fluid dynamics, however, I make the following disclaimer: I am definitely *not* arguing that the discrete, graded, multilinear Clifford-algebraic methods share such a degree of universal applicability that they should *supplant* the continuous, phase-space, infinite-dimensional differentiable manifold structure constituting the general formalism of the theory of differential equations, whether ordinary or partial. Nor do I *have* to make a general claim here in this chapter, but merely offer a counterexample for the case of the critical phenomenon of breaking droplets that Batterman (2005) analyzes. Research in geometric algebra is ongoing and burgeoning, both in the fields of fundamental as well as in applied physics. (Baugh et. al. (2003), Baylis (1995), Bolinder (1987), Conte (1993-2000), Finkelstein

[32] Of course, in the case of Batterman, continuous structures comprise as well the *ontology* of the epistemically fundamental theory: Navier-Stokes treats fluids as continua. In the case of Belot, the theory of partial differential equations he presents relies fundamentally on continuous, differentiable manifolds, characterizing the "formal ontology" of the theory of fluid mechanics (to use Rohrlich's notions, as discussed in I.2 above).

(1999-2004), Gallier (2005), Hestenes (1984, 1986), Khrenikov (2005), Lansenby, et. al. (2000), Levine & Dannon (2000), Mann et. al. (2003), Nebe (1999, 2000), Scheuermann (2000), Sloane (2001), Snygg (1997), Van den Nest, *et. al.* (2005), Vlasov (2000)). All the authors listed above (who comprise just a miniscule sample of the enormous body of literature on the subject of applications of Clifford Algebra in physics and engineering) either describe the mathematical formalism as especially appealing, due to its providing a 'unifying language' in the field of mathematical physics[33], or apply the formalism in key instances to make some interpretative point in the foundations of quantum theory, no matter how specific[34] or general.[35]

Certainly the empirical content of a specific problem domain determines which is the 'best' mathematical structure to implement in any theory of mathematical physics. By and large, such criteria are often determined essentially by practical limitations of computational complexity.

No danger of the aforementioned sort of equivocation that Batterman seems to commit, as I have argued above, is encountered so long as one can carefully distinguish the epistemological, ontological, and methodological issues vis-à-vis our choice of mathematical formalism(s) (i.e. distinguishing aspects $\mathcal{E}, \mathcal{O}, \mathcal{M}$. as discussed in chapter 1, Kallfelz 2009b). If the choice is primarily motivated by practical issues of computational facility, we can hopefully resist the temptation to reify our mathematical maneuvering, which would confuse the 'approximate' with the 'fundamental'— let alone confusing ontological, epistemological, and methodological *senses* of the latter notion.[36] Even Batterman admits that "nature abhors singularities." (2005, 20) So, I argue, should we. The entire paradigm behind regularization procedures is driven by the notion that a singularity, far from being an "infinite idealization we must appeal to" (Batterman 2003, 7), is a signal that the underlying formalism of theory is the pathological cause, resulting in the theory's failure to provide reliable information in certain critical cases.

Far from conceding to some class of "asymptotic-explanations," lending a picture of the world of critical phenomena as somehow carved at the joints of asymptotic singularities, we must instead search for regularizable procedures. This is precisely why such an approach is

[33] E.g., Finkelstein, Hestenes, Lasenby
[34] E.g., Conte, Hogreve, Snygg
[35] E.g., Hiley, Khrenikov, Vlasov
[36] I am, of course, *not* saying that there does not exist any connection whatsoever between a theory's computational efficacy and its ability to represent certain fundamentally ontological features of the phenomena of interest. *What* that connection ultimately *is* (whether empirical, or some complex and indirect logical blend thereof) I remain an agnostic. I do not take simplicity as evidence of a high degree of verisimilitude, in a manner similar to van Fraassen's (1980) "agnosticism" concerning the correct evidential consequences of a theory and its "truth."

methodologically fundamental: regularization implies some (weak) form of intertheoretic reduction, as I shall argue below.

III.5: Some Proposed Necessary Conditions for a Methodologically Fundamental Procedure

In this section, I summarize aspects of methods incorporating algebraic structures frequently used in mathematical physics, leading up to and including the regularization procedures latent in applications of Clifford Algebras. Because this material involves some technical notions of varying degrees of specialty, I have provided for the interested reader an Appendix at the end of this essay supplying all the necessary definitions and brief explanations thereon.

I review here a few basic techniques involving (abstract algebraic) *expansion* and *contraction*. Consider the situation in which the superceding theory $T^{/}$ is capable of being characterized, in principle, by an *algebra*.[37] Algebraic expansion denotes the process of extending out from algebraically characterized T' to some T'^* (denoted: $T' \xrightarrow{\lambda} T'^*$) where λ is some fundamental parameter characterizing the algebraic expansion. The inverse procedure: $\lim_{\lambda \to 0} T'^* = T'$ is *contraction.*

The question becomes: how to regularize? In other words, which T'^* should one choose to guarantee a regular (i.e., non-singular) limit for *any* λ in the greatest possible generality? Answer: expanding into an algebraic structure whose relativity group, i.e., the group of all its dynamical symmetries,[38] is *simple* implies that the Lie algebra depicting its infinitesimal transformations is *stable*.[39] This in turn entails greater reciprocity,[40] i.e., "reciprocal couplings in

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[37] That is to say, a vector space with an associative product. For further details, see Appendix A.2 below.

[38] Recall the discussion in chapter 2, §4 above. In other words, the group of all actions in leaving their *form* of dynamical laws invariant (in the active view) or the group of all 'coordinate transformations' preserving the tensor character of the dynamical laws (in the 'passive view.') Also, see Defn. A.2.2 in Appendix A.2 below for a description of simple groups.

[39] For a brief description of stable Lie algebras, see the discussion following Defn A.2.4, section A.2, Appedix.

[40] For example, in the case of the Lorenz group, which is simple, it is maximally reciprocal in terms of its fundamental parameters *x*, and *t*. That is to say, the form of Lorenz transformations (simplified in one dimensional motion along the *x*-axes of the inertial frame *F* and *F'*) become $x' = x'(x,t) = \gamma(x - Vt)$ and $t' = t'(x,t) = \gamma(t - Vx/c^2)$ (where $\gamma = (1-V^2/c^2)^{-1/2}$). Hence both space *x* and time *t* couple when transforming between inertial frames *F*, *F'*, as their respective transformations involve each other. On the other hand, the Galilean group is *not* simple, as it contains an invariant subgroup of boosts. The Galilean transformations are not maximally reciprocal, as $x' = x'(x,t) = x - Vt$ but $t' = t$. *x* is a cyclic coordinate with respect to transformation *t'*. Thus, when transforming between frames, *x* couples with respect to *t* but not vice versa.

the theory…reactions for every action." (Finkelstein, 2002,10, Baugh, et. al., 2003). This is an instance of a *methodologically fundamental* procedure, which I summarize by the following general necessary conditions:

- **Ansatz Ia:** If a procedure P for formulating a theory T in mathematical physics is *methodologically fundamental*, then there exists some algebraically characterized *expansion* T'^* of *T's* algebraic characterization (denoted by $T^{/}$) and some expansion parameter λ such that: $T' \xrightarrow{\lambda} T'^*$. Then, trivially, T'^* is regularizable with respect to $T^{/}$ since $\lim_{\lambda \to 0} T'^* = T'$ is well-defined (via the inverse procedure of algebraic contraction).
- **Ansatz Ib:** If T'^* is an expansion of $T^{/}$, then T'^*'s relativity group is *simple*, which results in a *stable* Lie algebra dT'^*, and whose set of observables in T'^* is maximally reciprocal.

The Segal Doctrine (Baugh, et. al. 2003) described any algebraic formalization of a theory obeying what I depict above, according to Ansatz Ib, as "fundamental." I insert here the adjective "methodological," since such a procedure comprises a method of regularization (viewed from the standpoint of the 'inverse' procedure of contraction) and so provides a formal, methodological means of reducing a superseding theory $T^{/}$ into its superseded theory *T*, when characterized by algebras.

In the following subsection, I summarize in detail how such a methodologically fundamental procedure, characterized by the Ansaetze above, has been developed by Baugh (2003), Finkelstein (2001-2004a) and Shiri-Garakhani (2004b) as a means to derive continuous structures, encountered in general relativity, from this discrete geometrical algebraic basis. Because of the specificity and technicality of some of the details, the reader may skip this section without loss of any of the conceptual insights presented in this chapter. I nevertheless present the section below as part of the chapter, rather than as a separate section to the Appendix below, to illustrate to the interested reader in a concrete fashion some of the successful developments of Clifford algebraic methods in some of the most daunting areas of theoretical physics involving the complex interplay between discrete-based and continuum-based theories as constitutive of quantum topology. Such applications in my opinion reinforce the claims made by numerous researchers regarding the promise of such method, in its specifically robust regularizability which presents itself as a viable alternative to the more common continuum-based methods typically constitutive of field theory (whether quantum or classical).[41]

III.6: Example: Deriving a (Continuous) Field Theory from a Discrete Graded Clifford Representation.

Baugh, et. al. (2003), Finkelstein (1996, 2001, 2004a-c) presents a unification of field theories (quantum and classical) and space-time theory based fundamentally on *finite* dimensional Clifford algebraic structures. The regularization procedure fundamentally involves group-theoretic simplification. The choice of the Clifford algebra[42] is motivated by two fundamental reasons:

1. The typically abstract (adjoint-based) algebraic characterizations of quantum dynamics (whether *C**, Heisenberg, etc.) represents how actions can be combined (in series, parallel, or reversed) but omits space-time fine structure.[43] On the other hand, a Clifford algebra can express a quantum space-time. (2001, 5)
2. Clifford statistics[44] for chronons adequately expresses the distinguishability of events as well as the existence of half-integer spin. (2001, 7)

The first reason entails that the prime variable is not the space-time field, as Einstein stipulated, but rather the dynamical law. That is to say, "the dynamical law [is] the only dependent variable, on which all others depend." (2001, 6) The "atomic" quantum dynamical unit (represented by a generator γ^{α} of a Clifford algebra) is the *chronon* χ, with the closest classical analogue being the tangent or cotangent vector (forming an 8-dimensional manifold) and *not* the space-time point (forming a 4-dimensional manifold).

Applying Clifford statistics to dynamics is achieved via the (category) functors[45] ENDO, SQ which map the mode space[46] X of the chronon χ, to its operator algebra (the algebra of

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[41] Of which renormalization group methods are the most notorious, as I explain in Kallfelz 2005a.

[42] The associated multiplicative groups embedded in Clifford algebras obey the simplicity criterion (Ansatz Ib, subsection 1 above). Hence Clifford algebras (or geometric algebras) remain an attractive candidate for algebraicizing any theory in mathematical physics (assuming the Clifford product and sum can be appropriately operationally interpreted in the theory *T*). For definitions and further discussion thereon, see Defn A.2.5, Appendix A.2.

[43] The space-time structure must are supplied by classical structures, prior to the definition of the dynamical algebra. (2001, 5)

[44] I.e., the simplest statistics supporting a 2-valued representation of S_N, the symmetry group on N objects.

[45] See Defn. A.1.2, Appendix A.1

endomorphisms[47] *A* on *X*) and to its spinor space *S* (the statistical composite of all chronons transpiring in some experimental region.) (2001, 10). The action of ENDO, SQ producing the Clifford algebra *CLIFF*, representing the global dynamics of the chronon ensemble is depicted in the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{ENDO} & A = \mathrm{ENDO}(X) \\ \downarrow{\scriptstyle SQ} & & \downarrow{\scriptstyle SQ} \\ S & \xrightarrow{ENDO} & CLIFF \end{array}$$

Fig. 3.2: Commutative diagram representing the action of deriving a statistics of quantum spacetime based on Clifford algebra

Analogous to H.S. Green's (2000) embedding of space-time geometry into a paraferminionic algebra of qubits, Finkelstein shows that a Clifford statistical ensemble of chronons can factor as a Maxwell-Boltzmann ensemble of Clifford subalgebras. This in turn becomes a Bose-Einstein aggregate in the $N \to \infty$ limit (where *N* is the number of factors). This Bose-Einstein aggregate condenses into an 8-dimensional manifold *M*, which is isomorphic to the tangent bundle of space-time. Moreover, *M* is a *Clifford manifold*, i.e. a manifold provided with a Clifford ring: $C(M) = C_0(M) \oplus C_1(M) \oplus \ldots \oplus C_N(M)$ (where: $C_0(M)$, $C_1(M)$,…,$C_N(M)$ represent the scalars, vectors,…, *N*-vectors on the manifold). For any tangent vectors $\gamma^{\mu}(x)$, $\gamma^{\nu}(x)$ on (Lie algebra *dM*) then:

$$\gamma^{\mu}(x) \circ \gamma^{\nu}(x) = g^{\mu\nu}(x)$$

where: $\circ$ is the scalar product. (2004a, 43) Hence the space-time manifold is a singular limit of the Clifford algebra representing the global dynamics of chronons in an experimental region.

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46 The mode space is a kinematic notion, describing the set of all possible modes for a chronon χ, the way a state space describe the set of all possible states for a state φ in ordinary quantum mechanics.

47 I.e, the set of surjective (onto) algebraic structure-preserving maps (those preserving the action of the algebraic 'product' or 'sum' between two algebras *A*, *A'*). In other words, Φ is an endomorphism on *X*, i.e. $\Phi: X \to X$ *iff*: $\forall x,y \in X$: $\Phi(x+y) = \Phi(x) + \Phi(y)$, where + is vector addition. Furthermore $\Phi(X) = X$: i.e. for any $z \in X$: $\exists\, x \in X$ such that $\Phi(x) = y$. For a more general discussion on the abstract algebraic notions, see A.2, Appendix.

Observable consequences of the theory are discussed in the model of the oscillator (2004c). Since the dynamical oscillator undergirds much of the framework of contemporary quantum theory, especially quantum field theory, the (generalized) model oscillator constructed via group simplification and regularization is isomorphic to a dipole rotator in the orthogonal group O(6*N*) (where: $N = l(l + 1) >> 1$). In other words, a *finite* quantum mechanical oscillator results, bypassing the ultraviolet and infrared divergences that occur in the case of the standard (infinite dimensional) oscillator applied to quantum field theory. In place of these divergences are "soft" and "hard" cases, respectively representing maximum potential energy unable to excite one quantum of momentum, and maximum kinetic energy being unable to excite one quantum of position. "These [cases]…resemble [and] extend the original ones by which Planck obtained a finite thermal distribution of cavity radiation. Even the 0-point energy of a similarly regularized field theory will be finite, and can therefore be physical." (2004c, 12)

In addition, such potentially observable extreme cases modify high and low energy physics, as "the simplest regularization leads to interactions between the previously uncoupled excitation quanta of the oscillator…strongly attractive for soft or hard quanta." (2004c, 19) Since the oscillator model quantizes and unifies time, energy, space, and momentum, on the scale of the Planck power (10^{51} W), time and energy can be interconverted.[48]

III.7: What Makes Multilinear Algebraic Expansion Methodologically Fundamental.

Before turning to the example involving applying Clifford algebraic characterization of critical phenomena in fluid mechanics, I shall give a final and brief recapitulation concerning the reasons why one should consider such methods described here as methodologically fundamental. For starters, the previous two Ansatze that I have proposed (in subsection 1 above) act as necessary conditions for what may constitute a methodologically fundamental procedure. Phrasing them in their contrapositive form (I.a*, I.b* below) also tell us what formalization

[48] In such extreme cases, equipartition and Heisenberg Uncertainty is violated. The uncertainty relation for the soft and hard oscillators read, respectively:

$$(\Delta L_1)^2(\Delta L_2)^2 \geq \frac{\hbar^2}{4}\langle L_3\rangle^2_{|L2\approx 0\rangle} \approx 0 \Rightarrow \Delta p\Delta q << \frac{\hbar}{2}$$

$$(\Delta L_1)^2(\Delta L_2)^2 \geq \frac{\hbar^2}{4}\langle L_3\rangle^2_{|L_1\approx 0\rangle} \approx 0 \Rightarrow \Delta p\Delta q << \frac{\hbar}{2}$$

schemes for theories in mathematical physics *cannot* be considered methodologically fundamental:

- **Ansatz (Ia*):** If T'^* is *singular* with respect to T', in the sense that the behavior of T'^* *in* the $\lambda \to 0$ limit does *not* converge to the theory T' *at* the $\lambda = 0$ limit (for any such contraction parameter λ), this entails that the procedure P for formulating a theory T in mathematical physics *cannot* be *methodologically fundamental*, and is therefore *methodologically approximate.*
- **Ansatz (Ib*):** If the relativity group of T'^* is not simple, its Lie algebra is subsequently unstable. Therefore T'^* *cannot* act as an effective algebraic expansion of T' in the sense of guaranteeing that the inverse contraction procedure is non-singular.

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Certainly Ansatz Ia* is just a re-statement (in algebraic terms) of Batterman's more general discussion (2002) of critical phenomena, evincing in his case-studies a singularity or inability for the superseding theory to reduce to the superseded theory. However this need not entail that we must preserve a notion of 'asymptotic explanations,' as Batterman would invite us to do, which would somehow inextricably involve the superseded and the superseding theories. Instead, as Ansatz I.a* states, this simply tells us that the mathematical scheme of the respective theory (or theories) is *not* methodologically fundamental, so we have a signal to search for methodologically fundamental procedures in the particular problem-domain, if they exist.[49]

Ansatz I.b* gives us further insight into criteria filtering out methodologically fundamental procedures. Finkelstein, et. al. (2001) demonstrate that *all* field theories exhibit, at root, an underlying fiber-bundle topology[50] and *cannot* have any relativity groups that are simple. This excludes a vast class of mathematical formalisms: *all*-field theoretic formalisms, whether classical or quantum.

However, as informally discussed in the preceding section, if any class of mathematical formalisms is methodologically approximate, this would not in itself entail that the computational efficacy or empirical adequacy of any theory T constituted by such a class is somehow diminished. If a formalism is found to be methodologically approximate, this should

[49] In a practical sense, of course, the existence of procedures entail staying within the strict bounds determined by what is computationally feasible.
[50] I.e., for Hausdorf (separable) spaces X, B, F, and map p: $X \to B$, defined as a bundle projection (with fiber F) if there exists a homeomorphism (topologically continuous map) defined on every neighborhood U for any point $b \in B$ such that: $\phi : p(\phi\langle b,f \rangle) = b$ for any $f \in F$. On $p^{-1}(U) = \{x \in X \mid p(x) \in U\}$, then p acts as a projection map on $U \times F \to F$. A fiber bundle consists is described by $B \times F$, (subject to other topological constraints (Brendon (2000), 106-107)) where B acts as the set of *base points* $\{b \mid b \in B \subseteq X\}$ and F the associated fibres $p^{-1}(b) = \{x \in X \mid p(x) = b\}$ at each b.

simply act as a caveat against laying excessive emphasis on the theory's ontology, until such a theory can be characterized by a methodologically fundamental procedure.

A methodologically fundamental strategy does more than simply remove undesirable singularities. As discussed above in subsection 1, the finite number of degrees of freedom (represented by the maximum grade N of the particular Clifford algebra) positively informs certain ontologically fundamental notions regarding our metaphysical intuitions concerning the ultimately discrete characteristics of the entities fundamentally constituting the phenomenon of interest.[51] On the other hand, the regularization techniques have, *pace* Batterman, epistemically fundamental consequences that are positive.

In closing, one can ask how likely is it that methodologically fundamental multilinear algebraic strategies can be applied to any complex phenomena under study, such as critical behavior? The serious questions deal with practical limitations of computational complexity: asymptotic methods can yield simple and elegantly powerful results, which would undoubtedly otherwise prove far more laborious to establish by discrete multilinear structures, no matter how methodologically fundamental the latter turn out to be. Nevertheless, the ever-burgeoning field of computational physics gives us an extra degree of freedom to handle, to a certain extent, the risk of combinatorial explosion that such multilinear algebraic techniques may present, when applied to a given domain of complex phenomena.[52] I examine one case below, regarding the utilization of Clifford algebraic techniques in computational fluid dynamics (CFD), in modeling critical phenomena.

III.8 Summary of CACFD

Gerik Scheuermann (2000), as well as Mann & Rockwood (2003), employ Clifford algebras to develop topological vector field visualizations of critical phenomena in fluid

[51] Recall the discussion of ontological levels in I.3, I. 4 above. This is relative, of course, to the level of scale we wish to begin, in terms of characterizing the theories' ontological primitives. For instance, should one wish to begin at the level of quarks, the question of whether or not their fundamental properties are discrete or continuous becomes a murky issue. Though quantum mechanics is often understood as a fundamentally 'discrete' theory, the continuum nevertheless appears in a subtle manner, when considering entangled modes, which are based on particular superpositions of 'non-factorizable' products.

[52] To be precise, so long as the algorithms implementing such multilinear algebraic procedures are 'polytime,' i.e. grow in polynomial complexity, over time.

mechanics. Visualizations and CFD simulations form a respectable and epistemically robust way of characterizing critical phenomena, down to the nanoscale. (Lenhard (2004)) "The goal is not theory-based insight as it is [typically] elaborated in the philosophical literature about scientific explanation. Rather, *the goal is* [for instance] *to find stable design-rules that might even be sufficient to build a stable nano-device*." (2004, 99, italics added) Simulations offer potential for intervention, challenging the "received criteria for what may count as adequate quantitative understanding." (ibid.)

Thus, Lenhard's above remarks appear as a rather strong endorsement for an epistemically fundamental procedure: The heuristics of CFD-based phenomenogical approaches lend a quasi-empirical character to this kind of research.[53] CFD techniques can produce robust characterizations of critical phenomena where traditional, '[Navier-Stokes] theory-based insights' often cannot. Moreover, aside from their explanatory power, CFD visualizations can present more accurate depictions of what occurs at the microlevel, insofar as the numerical and modeling algorithms can support a more detailed depiction of dynamical processes occurring on the microlevel. Hence there appears to be no inherent tension here: Clifford-algebraic CFD procedures are epistemically as well ontologically fundamental.[54] Of course, I claim that what

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[53] The topic of computer simulations has received recent philosophical attention. Eric Winsberg (2003) makes the case that they enjoy 'a life of their own' (124) between the categories of activity such as theory-articulation on the one end, and laboratory experiments on the other. "[B]y the semiautonomy of a simulation model, one refers to the fact that it starts from theory but one modifies it with extensive approximations, idealizations, falsifications, auxiliary information, and the blood, sweat, and tears of much trial and error." (109) In other words, stated negatively, the simulation cannot be derived in any straightforward algorithmic procedure from its 'parent' theory. Stated positively, simulation activity inevitably involves an essential aspect of abductive reasoning. Though by the same token, argues Winsberg, to conflate computer simulation activity with standard laboratory activity would be to confuse paintings with mirrors, as being equally representative of human posture (borrowing from Wittgenstein's analogy used in a critique of Ramsey's theory of identity). (116)

> If in our analysis of simulation we take it to be a method that essentially begins with an algorithm antecedently taken to accurately mimic the system in question, then the question has been begged as to whether and how simulations can, and often do, provide us with genuinely new, previously unknown knowledge about the system being simulated. It would be as mysterious as if we could use portraits in order to learn new facts about the postures of our bodies in the way that Wittgenstein describes. (ibid.)

A fuller account of Clifford-algebraic CFD methods in the light of some of the recent philosophical work on computer simulations is a topic clearly worthy of another study, above and beyond the scope of this essay. I briefly remark on such implications in chapter 4 below.

[54] Which is not to say, of course, that the applications of Clifford algebras in CFD contain no inherent tensions. The trade-off, or tension, however, is of a *practical* nature: that between computational complexity and accurate representation of microlevel details. Lest this appears as though playing into the hands of Batterman's epistemically versus ontologically 'fundamental' distinctions, it is important to keep in mind that the trade-off is one of a practical and contingent issue involving computational resources. Indeed, in the ideal limit of unconstrained computational power and resources, the trade-off disappears: one can model the underlying microlevel phenomena

guarantees this reconciliation is precisely the underlying *methodologically fundamental* feature of applying Clifford algebras in these instances.

III.8.a: An Overview of Scheuermann's Results

Scheuermann, Mann & Rockwood are primarily motivated by the practical aim of achieving accurately representative (i.e. ontologically fundamental) CFD models of fluid singularities giving equally reliable (i.e. epistemically fundamental) predictions and visualizations covering all sorts of states of affairs.

For example, Scheuermann (2000) points out that standard topological methods in CFD, using bilinear and piecewise linear interpolation approximating solutions to the Navier-Stokes equation, fail to detect critical points or regions of higher order (i.e. order greater than 1). To spell this out, the following definitions are needed:

Definition 1 (Vector Field): A 2D or 3D *vector field* is a continuous function $V: M \rightarrow \boldsymbol{R}^n$ where M is a manifold[55] $M \subseteq \boldsymbol{R}^n$, where $n = 2$ or 3 (for the 2Dand 3D cases, respectively) and $\boldsymbol{R}^n = \boldsymbol{R}\times.(n \text{ times}).. \times\boldsymbol{R} = \{(x_1,\ldots, x_n| \; x_k \in \boldsymbol{R}, 1 \leq k \leq n\}$, i.e. n-dimensional Euclidean space (where $n = 2$ or 3.)[56]

Definition 2 (Critical points/region): A critical point[57] $x_c \in M \subseteq \boldsymbol{R}^n$ or region $U \subseteq M \subseteq \boldsymbol{R}^n$ for the vector field V is one in which $||V(x_c)|| = 0$ or $||V(x)|| = 0$ $\forall x \in U$, respectively.[58]

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to an arbitrary degree of accuracy. On the other hand, Batterman seems to be arguing that some philosophically important explanatory distinction exists between ontological and epistemic fundamentalism.

[55] A manifold (2D or 3D) is a Hausdorff (i.e. simply connected) space in which each neighborhood of each one of its points is homeomorphic (topologically continuous) with a region in the plane R^2 or space R^3, respectively. For more information concerning topological spaces, see Table A.1.1, Appendix A.1.

[56] I retain the characterization above to indicate that higher-dimensional generalizations are applicable. In fact, one of the chief advantages of the Clifford algebraic formulations is their automatic applicability and generalization to higher-dimensional spaces. This is in contrast to notions prevalent in vector algebra, in which some notions, like the case of the cross-product, are only definable for spaces of maximum dimension 3. See A.2 for further details.

[57] For simplicity, as long as no ambiguity appears, a point x in an n –dimensional manifold is depicted in the same manner as that of a scalar quantity x. However, it's important to keep in mind that x in the former case refers to an n –dimensional position vector.

[58] Note: $|| \; ||$ is simply the Euclidean norm. In the case of a 2D vector field, for example, $||V(x,y)|| = ||u(x,y)\boldsymbol{i} + v(x,y)\boldsymbol{j}|| = [u^2(x,y) + v^2(x,y)]^{1/2}$, where u and v are x and y are the x,y components of V, described as continuous functions, and $\boldsymbol{i}, \boldsymbol{j}$ are orthonormal vectors parallel to the x and y axis, respectively.

A higher-order critical point (or family of points) may signal, for instance, the presence of a saddle point (or suddle curve) in the case of the vector field being a gradient field of a scalar potential $\Phi(x)$ in $\boldsymbol{R}^{2(\text{or } 3)}$, i.e. $V(x) = \nabla\Phi(x)$. "Higher-order critical points cannot exist in piecewise linear or bilinear interpolations. This thesis presents an algorithm based on a new theoretical relation between analytical field description in Clifford Algebra and topology." (Scheuermann (2000), 1)

The essence of Scheuermann's approach, of which he works out in detail examples in $\boldsymbol{R}^2$ and its associated Clifford Algebra $CL(\boldsymbol{R}^2)$ of maximal grade $N = \dim\boldsymbol{R}^2 = 2$ consisting of $2^2 = 4$ fundamental generators,[59] involves constructing in $CL(\boldsymbol{R}^2)$ a coordinate-independent differential operator ∂: $\boldsymbol{R}^2 \rightarrow CL(\boldsymbol{R}^2)$. Here: $\partial V(x) = \sum_{k=1}^{2} g^k \frac{\partial V(x)}{\partial g^k}$, where g_k the grade-1 generators, or two (non-zero, non-collinear) vectors which hence span $\boldsymbol{R}^2$, and $\frac{\partial V}{\partial g^k}$ are the directional derivatives of V with respect to g^k. For example, if g^1, g^2 are orthonormal vectors $(\hat{e}_1, \hat{e}_2)$, then: $\partial V = (\nabla \bullet V)\boldsymbol{1} + (\nabla \wedge V)\boldsymbol{i}$, where $\boldsymbol{1}$, and $\boldsymbol{i}$ are the respective identity and unit pseudoscalars of $CL(\boldsymbol{R}^2)$.[60] For example, in the matrix algebra $M_2(\boldsymbol{R})$, i.e. the algebra of real-valued 2x2 matrices:

$$\mathbf{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \boldsymbol{i} = \hat{e}_1\hat{e}_2 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Armed with this analytical notion of a coordinate-free differential operator, as well as adopting conformal mappings from $\boldsymbol{R}^2$ into the space of Complex numbers (which the latter form a grade-1 Clifford algebra) Scheuermann develops a topological algorithm obtaining estimates for higher-order critical points as well as determining more efficient routines:

> We can simplify the structure of the vector field and simplify the analysis by the scientist and engineer…some topological features may be missed by a piecewise linear interpolation [i.e., in the standard approach]. This problem is successfully attacked by using locally higher-order polynomial approximations [of the vector field, using conformal maps]…[which] are based on the possible local topological structure of the vector field and the results of analyzing plane vector fields by Clifford algebra and analysis. (ibid (2000), 7)

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[59] For details concerning these features of Clifford algebras, see Defn A.2.5 and the brief ensuing discussions in A.2
[60] compare this expression with the Clifford product in Defn A.2.5, A.2

III.8.b: An Overview of Mann and Rockwood's Results

Mann and Rockwood (2003) show how adopting Clifford algebras greatly simplifies the procedure for calculating the index (or order) of critical points or curves in a 2D or 3D vector field. Normally (without Clifford algebra) the index is presented in terms of an unwieldy integral formula involving the necessity of evaluating normal curvature around a closed contour, as well the differential of an even more difficult term, known as the Gauss map, which acts as the measure of integration. In short, even obtaining a rough numerical estimate for the index using standard vector calculus and differential geometry is a computationally costly procedure.

On the other hand, the index formula takes on a far more elegant form when characterized in a Clifford algebra:

$$ind(x_c) = \frac{C}{I}\int_{B(x_c)} \frac{V \wedge dV}{\|V\|^n} \qquad \text{(IV.1)}$$

where: $n = \dim\boldsymbol{R}^n$ (where n = 2 or 3)

x_c : a critical point, or point in a critical region.

C :a normalization constant.

I : the unit pseudoscalar of $CL(\boldsymbol{R}^n)$.

$\wedge$: the exterior (Grassmann) product.[61]

The authors present various relatively straightforward algorithms for calculating the index of critical points using (IV.1) above. "[W]e found the use of Clifford algebra to be a straightforward blueprint in coding the algorithm…the…computations of Geometric [Clifford] algebra automatically handle some of the geometric details…simplifying the programming job." (ibid., 6)

The most significant geometric details here of course involve *critical surfaces* arising in droplet-formation, which produce singularities in the standard Navier-Stokes continuum-based theory. Though Mann and Rockwood (2003) do not handle the problem of modeling droplet-formation using Clifford-algebraic CFD per se, they do present an algorithm for the computation of surface singularities:

[61] For definitions and brief discussions of these terms, see DefnA.2.5, A.2

> To compute a surface singularity, we essentially use the same idea as for computing curve singularities…though the test for whether a surface singularity passes through the edge [of an idealized test cube used as the basis of 'octree' iterative algorithm, i.e. the 3D equivalent of a dichotomization procedure using squares that tile a plane] is simpler than in the case of curve singularities. No outer products are needed—if the projected vectors along an edge [of the cube] change orientation/sign, then there is a [surface] singularity in the projected vector field. (ibid., 4)

III.8.c: Assessment of Some Strengths and Shortcomings in the Approaches

Shortcomings, however, include the procedure's inability to determine the index for curve and surface singularities. "Our approach here should be considered a first attempt….in finding curve and surface singularities…[our] heuristics are simple, and more work remains to improve them." (7)

Nevertheless, what is of interest here is the means by which a Clifford algebraic CFD algorithm can *determine the existence* of curve and surface singularities, and track their location in $\boldsymbol{R}^3$ given a vector field $V: M \to \boldsymbol{R}^3$. The authors demonstrate their results using various constructed examples. Based on the fact that every element in a Clifford algebra is invertible,[62] the authors ran cases such as determining the line singularities for vector fields such as:

$$V(x, y, z) = (uw^{-1})u + z\hat{e}_3 \qquad \text{(IV.2)}$$

where: $u(x, y) = x\hat{e}_1 + y\hat{e}_2$

$w(x, y) = \sqrt{x^2 + y^2}\,\hat{e}_1$

and $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ are the unit orthonormal vectors spanning $\boldsymbol{R}^3$.

An example like this would prove impossible to construct using standard vector calculus on manifolds, since the 'inverse' or quotient operation is undefined in the case of ordinary vectors. Hence the rich geometric and algebraic structure of Clifford algebras admits constructions and cases for fields that would prove inadmissible using standard approaches. The algorithm works also for *sampled* vector fields. "Regardless of the interpolation method, our method would find the singularities within the interpolated sampled field." (ibid., 5)

[62] See A.2, in the discussion following Defn A.2.5, for further details.

The Clifford algebraic CFD algorithms developed by the authors yield some of the following results:

1. A means for determining higher-order singularities, otherwise off-limits in standard CFD topology.
2. A means for locating surface and curve singularities for computed as well as sampled vector fields. Moreover, in the former case, the invertibility of Clifford elements produces constructions of vector fields subject to analyses that would otherwise prove inadmissible in standard vector field based formalisms.
3. A far more elegant and computationally efficient means for calculating the indices of singularities.

Clifford algebraic CFD procedures that would refine Mann and Rockwood's algorithms (described in §2 in this chapter) by determining for instance the indices of surface singularities, as well as being computationally more efficient, are precisely the cases that will serve as effective responses against Batterman's claims. For there would exist formalisms rivaling, in their expressive power, the standard Navier-Stokes approach. But such CFD research relies exclusively on finite-dimensional Clifford algebraic techniques, and would not appeal to the asymptotic singularities in the standard Navier-Stokes formulation in any meaningful way. Certainly the "first attempt" by Mann and Rockwood in characterizing surface singularities is an impressive one, in what appears to be the onset of a very promising and compelling research program.

IV. Concluding Remarks: A Case for Constructive Realism?

In Kallfelz (2009b) I show how a Clifford Algebraic approach finds a natural home in elements of structuralist philosophy of science. In other words, as physicist and philosopher Fritz Rohrlich has demonstrated: it is perfectly consistent for a physicist to accommodate an ontological pluralism in a particular class of theory-formation, but at the same time remain methodologically monist. Moreover, such methodological monism is best characterized in a mode and manner that does not hearken back to renditions of logical empiricism based on strong claims of logical reductionism (shown, as I do below, to be largely irrelevant if not outright hindering the progress of the development of theory formation). The school of thought that

Rohrlich implicitly ascribes to as well as some of his associates like Diedrik Aertz and Juergen Ehlers explicitly advocate is *structuralism*, a highly mathematical version of the semantic view of theories enjoying ongoing and active innovation by European physicists and philosophers of the likes of Erhard Scheibe (1997-1999).[63] Structuralists like Rohrlich ascribe to realism, as it best underwrites notions such as domains of validity, ontological strata, etc., that any mature mathematically based theory exhibits. I summarize some of his claims below:

IV.1. Elements of Structuralism

The 'internal' structure of certain aspects of a non-developing (i.e., accepted, mature, or established) theory *T* includes:

- An *ontology* $\mathcal{O}(T)$
- A set of central *terms* $\tau(T)$, with an accompanying *semantics* $\sigma(T)$.
- Set(s) of *principles* $\Pi(T)$
- A *mathematical structure* $\mathcal{M}(T)$
- A *domain of validity* $\mathcal{D}(T)$

In Rohrlich (1988, 302) the list of aspects is presented in a subtly different manner.[64] In terms of theory-reduction, one should deal only with mature or established theories, whose characteristic components include:

[63] "Structuralism" is also a term that appears often in certain branches of the philosophy of mathematics (e.g., Charles Chihara (1990, 2003)). Certainly, structuralists in the philosophy of mathematics share metaphysically resonant themes with those mentioned above, as both schools of thought assent to a generally *constructivist* position (as opposed to a Platonic "essentialism") concerning the ontological status of theoretical entities. Nevertheless, the projects' motivations differ. Mathematical structuralists are primarily concerned with resolving issues centering on ontological status, while structuralists in the philosophy of science are typically motivated more by epistemic and methodological concerns. Aside from the issue of a "rapprochement" of methods in philosophy of science vis-à-vis philosophy of mathematics I briefly discuss, a larger comparative and contrastive analysis concerning these two structuralist traditions lies beyond the scope of this essay.

[64] As mentioned above the list of aspects is by no means meant to be exhaustive, which reflects the anti-reductionism of structuralism in the sense of its repudiation of the attempt to reduce the semantic and syntactic content of scientific theories to formal axiomatic systems (recall n. 62 above). Hence no single list of structural aspects sufficiently constitutes a theory, let alone if such aspects were characterized in closed axiomatic form. Rorhlich and Hardin (1983) are even more explicitly adamant against axiomatic reductionism, which they are quick to mention is *not* what is implied by their model of inter-theoretic reduction. Scientists, they argue, should in general avoid axiomatization as the scheme "is difficult and in general equivocal." (605) (They proceed to mention the numerous schemes of attempts at axiomatizing quantum mechanics, all of which by nature are rather different, some even opposed). Instead they go on to say that scientists use *mathematical structures* of two or more theories, seeking to establish a 'conceptual dictionary' among notions conveyed by such mathematical structures which appear similar. (605) In yet another article, Aerts & Rohrlich (1998, 27) describe three kinds of reduction: a.) logical (i.e. reducing to some axiomatic framework), b.) theory reduction ('semantic reduction'), and c.) reductive explanation ('explanatory reduction'). They proceed to state that their paper will *not* cover logical reduction, since:

- An *ontic* component $\mathcal{O}$
- An *epistemic* component $\mathcal{E}$
- A *language and conceptual content* component $\mathcal{L}$, which includes formal and informal language, and a subset of central terms τ.[65]
- Set(s) of *principles* $\Pi(T)$
- A *mathematical-logical structure* component $\mathcal{M}$[66]
- A *domain of validity* $\mathcal{D}(T)$

Rohrlich succinctly states that Nagel's (1974) model of reduction (as mentioned briefly in the preceding subsection above) holds between (mature) theories T and $T^{/}$ whenever there exists a mapping Φ: $\tau(T^{/}) \rightarrow \tau(T)$, i.e. the central terms of T must be functions of those of $T^{/}$.[67] On the other hand (recalling Nickles (1975)) physicists are generally intuitive about the issue of inter-theoretic reduction, typically deriving just the *mathematical structures* from one theory to another. Moreover, in this more pedestrian but representative case, the physicists:

> …pay little attention to whether the concepts resulting from the physical interpretation of the symbols permit such a functional relation [a' la Nagel]…The mathematical structure or framework of the theory is considered to be primary, and the central terms (the meaning of certain central symbols) can be later derived from the applications of that framework to actual situations. (1988, 303)

"Logical reduction is a formal procedure that can be used in a scientific theory only *post facto*, after the theory has been formulated based on empirical information…*in no known case does axiomatization of a theory help to elucidate the scientific problems one encounters*." (Aerts & Rohrlich (1998) 28, italics added)

[65] One recognizes this as a slightly more refined description of the set of central *terms* $\tau(T)$, with an accompanying *semantics* $\sigma(T)$ mentioned in Rohrlich (1994).

[66] The essential importance of this component for mature scientific theories cannot be over-emphasized. Aside from its obvious feature including deriving the central equations of a theory, quantitative explanatory and predictive power:

> [$\mathcal{M}$ can probe where] human intuition fails…when the theory refers to those aspects of nature which lie outside our direct experience, the mathematical structure becomes the backbone of the scenario, [the model] which characterizes this indirect knowledge. [Moreover]…[t]he conceptual model associated with a theory is largely derived by confronting $\mathcal{M}$ with empirical evidence and with neighboring theories (testing and coherence)…involv[ing] informal language and is not the result of logical-mathematical deduction. (Rohrlich (1988), 301)

As mentioned in n. 80 above, so this above passage likewise distinguishes a structuralist's approach to mathematical structure and their use from a logical reductionist, as evidenced in the implication of abductive reasoning "involv[ing] informal language….not the result of …deduction."

[67] If the mapping is surjective (onto, i.e. $\Phi[\tau(T^{/})] = \tau(T)$) then the reduction is homogeneous. Otherwise (i.e., if the mapping is strictly *into*: $\Phi[\tau(T^{/})] \subset \tau(T)$) the reduction is heterogeneous.

The above point is used, for instance, to reconcile Feyerabend's (1963) theoretical pluralism (and its associated incommensurability issues) and at the same time ensuring a well-defined logical-mathematical linkage between two theories T and T' by recognizing that such two theories can refer to different cognitive (or epistemic) levels: In other words the fact that a reduction relation may hold between $\mathcal{M}(T)$ and $\mathcal{M}(T')$ does not guarantee that such a relation exists between $\mathcal{L}(T)$ and $\mathcal{L}(T')$, $\mathcal{O}(T)$ and $\mathcal{O}(T')$, or $\mathcal{E}(T)$ and $\mathcal{E}(T')$, etc.: "The mathematical framework of [T] is rigorously derived from that of [T'] (a derivation which involves limiting procedures); but the interpretation and the ensuing ontologies [of T and T'] are in general not so related."[68] (1988, 303)

a.) The epistemic Component $\mathcal{E}$

Recall the distinction between developing versus mature theories as discussed above. In an insightful commentary on Rohrlich, Ryszard Wojcicki (1998) writes:

> Rather than treating a theory which has reached the mature stage as a partially adequate description of the external world, Rohrlich (if I convey his position correctly) treats it as a cognitive counterpart of…*ontological levels*, or perhaps I should say 'ontological regions of reality.' (38)

In other words, what distinguishes a mature theory are distinctively stable reciprocal dynamics between its cognitive (or epistemic) and ontological levels. Such a stable correspondence implies (within its domain of validity) that one can associate a distinctive cognitive level associated with a robust ontological level:

> The existence of different concepts on different levels justifies one's talking about *qualitative* differences between levels…It thus follows that one level does not make another level superfluous. Both are needed; which theory is the suitable one depends on the domain of parameters…[t]he concepts we employ, the questions we ask, and the answers we are prepared to accept will be controlled by the domain of discourse—the ontological level—which we intend. (Rohrlich & Hardin (1983), 610)

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[68] Feyerabend of course may brush this aside as a red herring, or as just a re-statement of the problem of incommensurability. Recall in n. 75 Feyerabend's very point was just what Rohrlich (1988) seems to be re-iterating: a mathematical reduction will not guarantee a semantic one. However, if one accepts the structuralist maxim of a theory being composed of a plurality of *aspects* including semantic, mathematical, ontological components, then Rohrlich's points make good sense: one can guarantee reduction in one aspect but not in others. Only if one held fast to some reductionism claiming that the *semantic content* is what is essential to a theory (i.e., its $\mathcal{L}$, $\mathcal{O}$, $\mathcal{E}$ components) does the incommensurability issue then become a more serious concern.

So the ontological component, the epistemic component, and the validity domain of a mature theory ($\mathcal{O}, \mathcal{E}, \mathcal{D}$) all mutually co-refer in important ways. Yet each aspect or component has its distinct features as well, so they can (to a certain extent) be considered independently of each other. In the case of $\mathcal{O}$, I will mention in passing that it forms such a crucial tier of my discussion as to deserve its own major section (see §4 below), because it remains inextricably tied to notions like verisimilitude and representation. The validity domain $\mathcal{D}$, on the other hand, depends crucially on extensions of Nickel's (1975) 'domain preserving' reduction, that Batterman (2007) extended in his **Schema R** (discussed in the previous subsection above).

The issue of the epistemic aspect of a mature theory $\mathcal{E}$, as hooking into a coherent and consistent ontological aspect $\mathcal{O}$, is best illustrated by way of a counter-instance, as what would occur in the case of a *developing* or immature theory. Developing theories do not yet possess a stable ontological aspect $\mathcal{O}$, hence their epistemic component is volatile. To name one contemporary instance: consider the case of String Theory. This developing theory's greatest strength is also its chief weakness: String Theory possesses a richly mathematical component $\mathcal{M}$ at the expense of its epistemic and ontological components. Efforts to 'interpret' the theory range from some extremely dubious version of Platonism (Brian Greene) in which an ontology is imposed in a ham-fisted manner relegating most of the theory's essential terms to unobservable abstractions, devoid of any operational content.[69] Other interpretations verge on the instrumentalist, regarding some of its mathematical results as empirically adequate at best, but the essential terms are devoid of ontological content aside from predictive value. A similar case can be made for developing versions of Ptolemaic astronomy in Antiquity (as opposed to the late Middle Ages), despite its mathematical sophistication.

On the other hand, in the case of mature theories, cognitive levels occur in $\mathcal{E}$ due to "cognitive (or epistemic) emergence."[70] (Rohrlich (1988) 3) Rohrlich's notion of cognitive emergence is similar to the notion of 'epistemic emergence' discussed in Humphreys (1997), Silberstein & McGeever's (1999), and in Kronz & Tiehen (2002) in that the notion spells no ontological difficulties: Cognitive emergence is *contextual* insofar as it is entirely constituted by the relationship our cognitive apparatus has with its referent. An apparent emergence of new

[69] See Finkelstein (1996, 2001, 2004*a-c*, 2007) for criticism of this developing theory.
[70] I discuss the issue of emergence in greater detail in Chapter 2 below. See also Kallfelz (2009).

objects (atoms, stars, organisms, etc.) having certain unique properties identified by humans' cognitive apparatus:

> suggest…*something qualitatively new* has evolved…[only] because it differs perceptively from anything that there was at the earlier stages [of cosmic evolution]; there is a recognition of this fact that is sudden despite the realization that nothing discontinuous has happened. (Rohrlich (1988), 298)

In other words, such 'new' objects are characterized via an *idealization*: "their detailed structure has become unimportant. Characterizations are *approximations*…beyond a certain observational precision they become empirically inadequate." (298-299) It is a short step to realize the ubiquitous and unremarkable fact of epistemically emergent cognitive levels once one accepts the truism that "it is only through idealizations, and what…we can think of as their alter-ego—inexact truths—that we have access to the world."[71] (Paul Teller (2004b) 447)

b.) The Ontological Component $\mathcal{O}$

As described above, the epistemic component of mature theories corresponds with a robust ontology in a stably reciprocal manner, underwritten by the inevitably idealizing activity of both: For instance in the epistemic component of classical mechanics the emergent cognitive level of 'massive bodies subjected to macroscopic forces' corresponds to the ontological component of the theory containing 'fallible veracities'[72] like 'point mass,' 'frictionless planes,' etc., rendered possible only through an idealizing activity ignoring details of the massive bodies' constituents at the molecular, or atomic, or nuclear, or sub-nuclear, or Planck scales, etc.

A central metaphysical point that Rohrlich makes from the above is his advocating a *pluralist ontology*, <u>*constituted by*</u> *a substantial monism*:

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[71] Paul Teller's (2005) argument is certainly not some endorsement of idealism of sense-data characteristic of certain elements of British empiricism from the 17th to the 20th centuries: "The British empiricists thought that thinking consists in having a stream of 'ideas' [representations], and concluded mistakenly that all we ever think *about* are our own ideas." (Alan Musgrave (1985), in Curd & Cover (1998), n. 2, 1223-1224) Teller's claim comes as a concluding statement of his argument against quantitative verisimilitude, i.e., that there exists some context-independent way of determining 'closeness to truth' of our theories. Teller argues that 'closeness to truth' is an inevitably contextual notion and recognizing this entails that the distinction between a 'foundational theory' and 'phenomenological theory' is likewise context-relative: Foundational theories distort, approximate, and idealize as much as 'phenomenological' theories do. Conversely, "[Though] I accept that foundational theories do tell us a great deal about how the world really is. I note also that many 'phenomenological' theories [however] …tell us about the world in the same kind of way that the foundational theories do." (Teller (2004b), 446) I will discuss Teller's insights in greater detail in §4 below.

[72] A term Teller (2004b) suggests one should use in lieu of 'useful fictions.' "[I]mperfect characterizations [still] genuinely inform…just calling them 'fiction' thus misleads. But we do want to acknowledge that these characterizations are not simply true." (445)

> [I]t is our cognitive capacity, our ability to perceive, to recall, to recognize, and to draw analogies [all inevitably idealizing activities] that is…responsible for this pluralistic nature of our ontology. We…encounter it in the *cognitive emergence* of new objects…[nevertheless the standpoint of] cosmic evolution is in support of the notion of the unity of nature (substantive monism).[73] (1988, 297)

There is nevertheless a substantial monism as entities evolve continuously (or quasi-continuously in the case of quantum mechanics[74]) "unfold[ing] into increasing complexity." (298)

The idealization underwriting the conceptual levels of the epistemic as well as the associated ontological components of a mature theory corresponds to *a level of coarseness* (determined by the extent of the idealization and simplification) for the basic level of the domain of scientific inquiry. "I prefer the terms 'coarser' and 'finer' level of theory [rather than]…terms such as 'more fundamental', 'superseding', 'supervening', 'primary', etc. [as the latter notions] prejudice the case."[75] (299) Hence in this context, the convention I have been adopting for preceding and superseding theories (T and $T^{/}$, respectively) apply equally well to Rohrlich's 'coarser' and 'finer' theories; i.e., theories T and $T^{/}$, with the former whose ontological component $\mathcal{O}$ (T) is coarser relative to the latter's $\mathcal{O}$ ($T^{/}$). Moreover, though most physical theories have an ontological component at a certain level of coarseness, some 'framework theories' like mechanics (whether classical, statistical, or quantum) have ontological components containing several levels of coarseness.[76]

c.) The Validity Domain $\mathcal{D}$

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[73] The notion of cognitive emergence (vis-à-vis substantial unity in cosmic evolution) is resonant with some of A. N. Whitehead's (1929/1978) ideas: "*Process and Reality* divides actual entities/occasions into four grades of ascending complexity…[which] is not a fundamental division according to kind or essence, but a qualitative classification by complexity, and a coarse one at that." (Finkelstein & Kallfelz (1997), 289). For a review of certain contemporary notions of emergence with respect to the implied substantial monism of Whitehead, see also Kallfelz (2009)

[74] "The discontinuities in quantum mechanics do not prevent predictability but they restore it to a probabilistic one." (Rohrlich (1988), n.1 298)

[75] Note however, such terms apply just to the physical sciences, where the size of an object is a determining factor. "[F]or other scientific levels qualitative distinctions may dominate over quantitative ones." (299) It is this distinction of coarse versus fine that Batterman (2002) gave passing mention to.

[76] In the case of mechanics: the distinction between particle and rigid body dynamics. The latter corresponds to a finer ontological level relative to the former since accounting for torques, angular momenta and rotational inertia on the body necessitates that it *cannot* be modeled as a single point particle. In the case of statistical mechanics, the science "interpolates between levels of the microworld and the macroworld." (1988, n. 2, 299) Also in the case of quantum mechanics (non-relativistic and relativistic) its ontological component is not restricted to one level of coarseness either, ranging from the nucleonic to macroscopic in the case of Bose condensations.

The reduction of a coarser theory T to a finer theory $T^{/}$ requires $\mathcal{M}(T^{/})$ to converge to $\mathcal{M}(T)$ whenever the validity domain of $T^{/}$, i.e $\mathcal{D}(T^{/})$, is restricted to that of $\mathcal{D}(T)$. Echoing Nickles' (1975) domain preserving notion of intertheory reduction, the above necessarily involves a limiting process.(303) This limiting process involves a parameter p which must be dimensionless (recall n. 77 above) as well as have the functional form $p = f(x,x^{/})$ where: $x^{/}$ is a quantity or array of quantities in $\mathcal{M}(T^{/})$ and x is a quantity or an array of quantities in $\mathcal{M}(T)$. "Given the finer theory [alone], it is not obvious what the characteristic parameter p actually is. It becomes evident only when the coarser theory is known." (304) For example, in the previous example mentioned above involving momentum in finer theory of Special Relativistic Dynamics (SRD) vis-a-vis the coarser one of classical particle dynamics (CPD) a natural choice is $p = \frac{v^2}{c^2}$.

In the case of the reduction of electromagnetism (EM) to geometric optics (GO), $p = \frac{\lambda}{L}$, where λ is the wavelength of the EM wavefront, and L is the slit width. In the case of the Bohr Correspondence Principle between non-relativistic quantum mechanics (NRQM) and classical mechanics (CM), $p = \frac{f(\hbar)}{R}$, where f is some analytic function[77] of $\hbar$ with range values expressed in length dimension, and R is the average radius of the spatial region.[78]

Hence borrowing from Batterman's **Schema R** notation, one can characterize the reductions as: $\lim_{p \to 0} \mathcal{M}(T^{/}) = \mathcal{M}(T)$ whenever $\mathcal{D}(T^{/})$ is restricted to $\mathcal{D}(T)$. However, whenever such a reduction holds, it does *not* follow that there exists some mapping $\Phi: \mathcal{M}(T^{/}) \to \mathcal{M}(T)$, which would signal a stronger case of semantic reduction (a' la Nagel) (302). Also, the reduction need not be unique: There can exist several parameters p_1, p_2, …such that: $\lim_{p_1 \to 0}$ $\mathcal{M}(T^{/}) = \mathcal{M}(T_1)$,

$\lim_{p_2 \to 0} \mathcal{M}(T^{/}) = \mathcal{M}(T_2)$, etc.[79](305)

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[77] I.e. a 'smooth' or continuously differentiable (to all orders) function $f(x)$ (real or complex-valued. In the complex case, every differentiable function is automatically analytic. Every analytic function can be expressed as a convergent power series, hence its limit behavior is everywhere well-defined.

[78] Note however in other cases of reduction of NRQM to NM, one could also choose the more elementary p =(# of quanta) in the $p \to \infty$ limit.

[79] For an interesting case, see Finkelstein et. al. (2001) who develop several Clifford algebraic contraction parameters in their general Clifford algebraic quantum space-time formalism, and proceed to show how their Clifford commutation relations converge to the classical symplectic algebra in the limit of one of their contraction parameters, versus the former converging to the Heisenberg algebra for another contraction parameter.

The parameter p is naturally interpreted as establishing a validity domain $\mathcal{D}$ of a theory. "A *validity limit* is thus equivalent to a specification of the error made by using the lower level [coarser theory T] instead of the higher level theory [$T^{/}$]." (Rohrlich & Hardin (1983), 607) Hence in terms of $T^{/}$ any prediction made by T should be multiplied by the factor $(1 \pm p)$. For instance, in the case of NM predicting the motion of the Earth vis-à-vis SRD, the former is subject to measurement error $p = \frac{v^2}{c^2}$, where v is the average speed of the Earth relative to the Sun, hence the predictions of NM are accurate to within (1 ± 10^{-8}). This establishes of course a measure of the reliability of NM's predictions, hence its validity domain $\mathcal{D}$(NM). Validity limits characterize theories as approximate (in the light of their finer counterparts), however "[i]n most cases the approximation involved, is *extremely* good." (Rohrlich & Hardin (1983), 608)

The validity domain's connection with the ontological component $\mathcal{O}$ is apparent in the following sense: an ontological level naturally corresponds to a case in which p is negligible to a sufficiently good approximation.

> Since p either is or is not negligible, there is no intermediate situation. But what makes this definition of ontological level…is the large size of the domains of validity of theories: it spaces ontological levels far apart. (609)

Regarding the aforementioned issue of conceptual emergence:

> [T]here is in many cases no simple relation between the concepts of theories on two different [ontological] levels. The limiting procedure that relates
> [$T^{/}$]…to [T] can in fact create *new* concepts…not present in the higher level theory. (ibid)

By way of an elementary calculus example (reminiscent of Batterman's (2007) example of $\varepsilon x^2 + x - 9 = 0$) Rohrlich & Hardin demonstrate this in terms of an arclength of a circular sector $ds = rd\theta$, compared to the length of its inscribed secant dl:

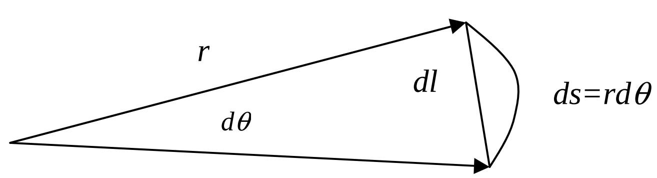

Fig. 4.1: Representation of the secant-tangent relation in Rohrlich's illustration concerning epistemic emergence and ontological levels of coarseness.

Now in the limit: $r \to \infty$, then $ds \to dl$, hence ds assumes the property 'straight'. "The property 'straight' did not exist on the circle but was produced by the limiting procedure." (609) In an emblematically physical example of the same conceptual kind, in the $N \to 0$ limit (where N is the number of bodies of appreciable mass) in a local space-time region M, $g^{\mu\nu}(x) \to \Lambda^{\mu\nu}$, where $g^{\mu\nu}(x)$ is the variable metric of general relativity (GR) at a space-time point $x \in M$, and $\Lambda^{\mu\nu}$ is the constant Lorentz metric characterizing flat Minkowski space-time in special relativity (SR). Various space-time symmetries occur in such a manner exhibiting Poincare' Group invariance in SR dynamics, but this property doesn't manifest in the curved and dynamical space-time of GR. (610)

Last of all, despite this seemingly facile characterization of the limit: $\lim_{p \to 0} \mathscr{M}(T') = \mathscr{M}(T)$ in *theory*, it remains a delicate and complicated procedure to attempt to carry it out in *practice*[80]:

> The limiting process involved can be very complicated as well as very subtle. Some of the limiting processes have so far not been carried out in a mathematically satisfactory way, but for enough to satisfy the intuitive expectations of the physicist. (Rohrlich & Hardin (1983), n.3, 605)

> The reduction of the symmetry properties of [$\mathscr{M}(T')$] to those of [$\mathscr{M}(T)$] plays a very significant role…[f]rom a technical point of view, it shows that the limiting process is highly nontrivial and must be carried out very carefully: the symmetry reduction may be the result of group contraction, and the limit can only be carried out in suitable group representations. But we shall not pursue these mathematical matters here. (Rohrlich (1988) 304)

IV.2: In Closing: A Case for a Realist Interpretation of CACFD

In "top-down" fashion I have summarized elements I discuss further in Kallfelz (2009b) concerning "methodological fundamentalism" and Rohrlich's notion echoed in his claims for an "ontological pluralism and methodological monism" within the framework of any mature mathematical theory. Certainly CFD is an example thereof: One can, depending on the relative coarse or fine-graining, move among the stable ontological strata characterized either by

[80] A noteworthy example is Ehlers (1986) who, inspired by Rohrlich & Hardin, constructed two concrete case studies rigorously demonstrating the reduction of Lorentz invariant scattering theory to Galilean invariant scattering theory (390-396), as well as a partial reduction of GR to Newtonian gravitation (396-400). The technical rigor and mathematical sophistication should prove itself to be convincing enough of the inherent challenges regarding the attempt to carry out the limiting procedure in practice.

continuous flow (Navier-Stokes) or by the relatively more finer grained and ontologically nuanced Clifford-Algebraic simulations.

As suggested above, a realist interpretation naturally suggests itself, insofar as it best underwrites in a rigorous and systematic manner a means of characterizing *verisimilitude*[81], as suggested above in the notion of a theory's "domains of validity." From the bottom-up, such a framework provides good reason, *pace* Winsberg (2003, 2006) of Hughes' realist interpretation of simulations: "[C]omputer 'experiments' reveal information about actual, possible, or impossible worlds." (Winsberg, 2003, 115). Indeed, this is a paraphrase of Giere's "constructive realism", as mentioned in n. 15 above. Winsberg persuasively argues that computer simulations are an instance of the model-building activity in theory-articulation, so the question of realism versus constructive empiricism (Van Frassen (1980)) and other instances of anti-realism reduces to how much stock one is willing to invest in terms of the breadth of a model's *modal* scope. Add to that, aspects of Hacking's notion suggest themselves as well, as the case can be plausibly advanced (within the general structuralist framework) that within the domain of validity as set by the "simulationists[‘] and experimenters[‘]…need to engage in error management," (Winsberg 2003, 120) a case can be made that this is an instance of *realization*[82] (Norton & Suppe (2000)) of CACFD vis-à-vis the states of the actual fluid.

Appendices

Appendix A.: A Brief Synopsis of the Relevant Algebraic Structures

A.1: Category Algebra and Category Theory

As authors like Hestenes (1984, 1986), Snygg (1997), Lasenby, et. al. (2000) promote Clifford Algebra as a unified mathematical language for physics, so Adamek (1990), Mikhalev & Pilz (2000) and many others similarly claim that Category Theory likewise forms a unifying basis for all branches of mathematics. There are also mathematical physicists like Robert Geroch (1985) who seem to bridge these two presumably unifying languages by building up a mathematical toolchest comprising most of the

[81] I expand upon these points in Kallfelz (2009b), pp. 30-40, along the lines of semantic issues as well, in which I discuss aspects of Yablo & Gallois (1998).

[82] Any system S_1 *realizes* system S_2 if there exists an *onto* mapping $\Phi: S_2 \rightarrow S_1$ which is behavior-preserving of the states of S_2 onto the states of S_1. (Winsberg, 2003, 114-115). One might add that this notion of "behavior-preservation" can be leant greater precision regarding the notion of an algebraic isomorphism (product structure preserving map) from the "product" (composition of states in S_2) to the (Clifford) product of CACFD S_1.

salient algebraic and topological structures for the workaday mathematical physicist from a Category-theoretic basis.

A *category* is defined as follows:

- **Defn. A1.1:** A category C = $\langle \Omega, \mathrm{MOR}(\Omega), \circ \rangle$ is the ordered triple where:

  a.) Ω is the class of C's *objects.*

  b.) MOR(Ω) is the set of *morphisms* defined on Ω. Graphically, this can be depicted (where $\varphi \in$ MOR(Ω), $A \in \Omega$, $B \in \Omega$): $A \xrightarrow{\varphi} B$

  c.) The elements of MOR(Ω) are connected by the *product* $\circ$ which obeys the law of composition: For $A \in \Omega$, $B \in \Omega$, $C \in \Omega$: if φ is the morphism from A to B, and if ψ is a morphism from B to C, then $\psi \circ \varphi$ is a morphism from A to C, denoted graphically: $A \xrightarrow{\varphi} B \circ B \xrightarrow{\psi} C = A \xrightarrow{\psi \circ \varphi} C$. Furthermore:

  c.1) $\circ$ is *associative*: For any morphisms ϕ, φ, ψ with product defined in as in c.) above, then: $(\psi \circ \phi) \circ \varphi = \psi \circ (\phi \circ \varphi) \equiv \psi \circ \phi \circ \varphi$.

  c.2) Every morphism is equipped with a left and a right identity. That is, if ψ is any morphism from A to B, (where A and B are any two objects) then there exists the (right) *identity* morphism on A (denoted ι_A) such that: $\psi \circ \iota_A = \psi$. Furthermore, for any object C, if φ is any morphism from C to A, then there exists the (left) *identity* morphism on A (ι_A) such that: $\iota_A \circ \varphi = \varphi$. Graphically, the left (or right) identity morphisms can be depicted as *loops.*

A simpler way to define a category is in terms of a special kind of *semigroup* (i.e. a set S closed under an associative product). Since identities are defined for every object, one can in principle identify each object with its associated (left/right) identity. That is to say, for any morphism φ from A to B, with associated left/right identities ι_B, ι_A, identify: $\iota_B = \lambda$, $\iota_A = \rho$. Hence condition c2) above can be re-stated as c2$'$): "For every φ there exist (λ, ρ) such that: $\lambda \circ \varphi = \varphi$, and $\varphi \circ \rho = \varphi$." With this apparent identification, DefnI.1 is coextensive with that of a "semigroup with enough identities.

Category theory provides a unique insight into the general nature, or universal features of the construction process that practically all mathematical systems share, in one way or another. Set theory can be embedded into category theory, but not vice versa. Such basic universal features involved in the construction of mathematical systems, which category theory generalizes and systematizes, include, at base, the following:

Feature	Underlying Notion
Objects	The collection of primitive, or stipulated, entities of the mathematical system.
Product	How to 'concatenate and combine,' in a natural manner, to form new objects or entities in the mathematical system respecting the properties of what are characterized by the system's stipulated objects.
Morphsim	How to 'morph' from one object to another.
Isomorphism (structural equivalence)	How all such objects, relative to the system, are understood to be equivalent.

Table A.1.1

For an informal demonstration of how such general aspects are abstracted from three different mathematical systems (sets, groups, and topological spaces[83]), for instance, see Table A.1.2 below.

I.a) Set	(by Principle of Extension) $S_{\Phi} = \{x \mid \Phi(x)\}$ for some property Φ
I.b) Cartesian Product	For any two sets X, Y : $X \times Y = \{(x,y) \mid x \in X,\ y \in Y\}$
I.c) Mapping	For any two sets X, Y, where $f \subseteq X \times Y$, f is a *mapping* from X to Y (denoted $f : X \to Y$) *iff* for $x_1 \in X$, $y_1 \in Y$,$y \in Y$, if $(x_1, y_1) \in f$ (denoted: $y_1 = f(x_1)$) $(x_1, y_2) \in f$ then: $y_1 = y_2$.
I.d) Bijection (set	For any two sets X, Y, where $f : X \to Y$ is a mapping, then f is a bijection

[83] Such systems, of course, are not conceptually disjunct: topological spaces and groups are of course defined in terms of sets. The additional element of structure comprising the concept of group includes the notion of a binary operation (which itself can be defined set-theoretically in terms of a *mapping*) sharing the algebraic property of associativity. The structural element distinguishing a topological space is also described set-theoretically by use of notions of 'open' sets. Moreover, groups and topological spaces can conceptually overlap as well in the notion of a *topological group*. So in an obvious sense, set theory remains a general classification language for mathematical systems as well. However, the *expressive power* of set theory pales in comparison to that of category theory. To put it another way, if category theory and set theory are conceived of as deductive systems (Lewis), it could be argued that category theory exhibits a better combination of "strength and simplicity" than does naïve set theory. Admittedly, however, this is not a point which can be easily resolved as far as the simplicity issue goes because the very concept of a category is usually cashed out in terms of three fundamental notions (objects, morphisms, associative composition), whereas, at least in the case of 'naïve' set theory (NST), we have fundamentally two notions: a) of membership $\in$ defined by extension, and b) the hierarchy of *types* (i.e., for any set X, $X \subseteq X$, but $X \notin X$. Or to put more generally, $Z \in W$ is a meaningful expression, though it may be false, provided, for any set, X: $Z \in \wp^{(k)}(X)$ and $W \in \wp^{(k+1)}(X)$, where k is any non-negative integer, and $\wp^{(k)}(X)$ defines the kth-level power-set operation, i.e.: $\wp^{(m)}(X) = \wp(\wp(\ldots k \text{ times} \ldots(X))$.)

equivalence)	*iff*: a) f is onto (surjective), i.e. $f(X) = Y$ (i.e., for any $y \in Y$ there exists a $x \in X$ such that: $f(x) = y$, b) f is 1-1 (injective) *iff* for $x_1 \in X$, $y_1 \in Y$, $y \in Y$, if $(x_1, y_1) \in f$ (denoted: $y_1 = f(x_1)$) $(x_1, y_2) \in f$ then: $y_1 = y_2$.
II.a) Group	I.e., a group $\langle G, \circ \rangle$ is a set G with a binary operation $\circ$ on G such that: a.) $\circ$ is *closed* with respect to G, i.e.: $\forall (x, y) \in G : (x \circ y) \equiv z \in G$ (i.e., $\circ$ is a *mapping into G* or $\circ : G \times G \to G$, or $\circ(G \times G) \subseteq G$)). b.) $\circ$ is *associative* with respect to G,: $\forall (x, y, z) \in G$: $(x \circ y) \circ z = x \circ (y \circ z) \equiv x \circ y \circ z$, c.) There (uniquely) exists a (left/right) identity element $e \in G : \forall (x \in G)$ $\exists! (e \in G) : x \circ e = x = e \circ x$. d.) For every x there exists an *inverse element* of x, i.e.: $\forall (x \in G) \exists (x^{/} \in G)$: $x \circ x^{/} = e = x^{/} \circ x$.
II.b) Direct product	For any two groups G, H, their *direct product* (denoted $G \otimes H$) is a group, with underlying set is $G \times H$ and whose binary operation * is defined as, for any $(g_1, h_1) \in G \times H$, $(g_2, h_2) \in G \times H$: $(g_1, h_1) * (g_2, h_2) = ((g_1 \circ h_1), (g_2 \bullet h_2))$, where $\circ$, $\bullet$ are the respective binary operations for G,and H.
II.c) Group homomorphism	Any *structure-preserving mapping* φ from two groups G and H. I.e. $\varphi : G \to H$ is a homomorphism *iff* for any $g_1 \in G$, $g_2 \in G$: $\varphi(g_1 \circ g_2) = \varphi(g_1) \bullet \varphi(g_2)$ where $\circ$, $\bullet$ are the respective binary operations for G,and H.
II.d) Group Isomorphism (group equivalence)	Any *structure-preserving bijection* ψ from two groups G and H. I.e. ψ : $G \to H$ is an isomorphism *iff* for any $g_1 \in G$, $g_2 \in G$: $\psi(g_1 \circ g_2) = \psi(g_1) \bullet \psi(g_2)$ (where $\circ$, $\bullet$ are the respective binary operations for G,and H) *and* ψ is a *bijection* (see I.d above) between group-elements G and H. Two groups are *isomorphic* (algebraically equivalent, denoted: $G \cong H$) *iff* there exists an isomorphism connecting them $\psi : G \to H$.)
III.a) Topological Space	Any set X endowed with a collection τ_X of its subsets (i.e. $\tau_X \subseteq \wp(X)$, where $\wp(X)$ is X's power-set, such that: 1) $\varnothing \in \tau_X$, $X \in \tau_X$ 2) For any U, $U^{/} \in \tau_X$, then: $U \cap U^{/} \in \tau_X$. 3) For any index (discrete or continuous) γ belonging to index-set Γ: if $U_\gamma \in \tau_X$, then: $\bigcup_{\gamma \in \Delta \subseteq \Gamma} U_\gamma \in \tau_X$. X is then denoted as a *topological space,* and τ_X is its *topology*. Elements U belonging to τ_X are denoted as *open* sets. Hence 1), 2), 3) say that the empty set and all of X are always open, and finite intersections of open sets are open,

	while arbitrary unions of open sets are always open. Moreover: 1) Any collection of subsets $\Im$ of X is a *basis* for *X's* topology *iff* for any $U\in \tau_X$, then for any index (discrete or continuous) γ belonging to index-set Γ: if $B_\gamma \in \Im$, then: $\bigcup_{\gamma\in\Delta\subseteq\Gamma} B_\gamma = U \in \tau_X$ (i.e., arbitrary unions of basis elements are open sets.) 2) Any collection of subsets Σ of X is a *subbasis* if for any $\{S_1,\ldots, S_N\}\subseteq \Sigma$, then $\bigcap_{k=1}^{N} S_k = B \in \Im$ (I.e. finite intersections of sub-basis elements are basis elements for *X*'s topology.)
III.b) Topological product	For any two topological spaces *X*, *Y*, their *topological product* (denoted $\tau_X \otimes \tau_Y$) is defined by taking, *as a sub-basis*, the collection: $\{(U,V)\mid U\in \tau_X, V\in \tau_Y\}$. I.e., $\tau_X \times \tau_Y$ is a subbasis for $\tau_X \otimes \tau_Y$. This is immediately apparent since, for U_1 and U_2 open in *X*, and V_1 and V_2 open in *Y* : since: $U_1 \times U_2 \cap V_1 \times V_2 = (U_1 \cap V_1)\times(U_2 \cap V_2)$ this indeed forms a basis.
III.c) Continuous mapping	Any mapping from two topological spaces *X* and *Y*, preserving openness. I.e. $f : X \to Y$ is continuous *iff* for any $U\in \tau_X$: $f(U) = V \in \tau_Y$
III.d) Homeomorphism (topological space equivalence)	Any *continous bijection h* from two topological spaces *X* and *Y*. I.e. $h : X \to Y$ is a homeomorphsim *iff* : a) *h* is continuous (see III.c), b) *h* is a bijection (See I.d). Two spaces X and *Y* are *topologically equivalent* (i.e., homeomorphic, denoted: $X \cong Y$) *iff* there exists a homeomorphism connecting them, i.e. $h : X \to Y$

Table A.1.2

Now the classes of mathematical objects exhibited in Table A.1.2 comprising sets, groups, and topological spaces, all exhibit certain common features:

- The concept of *product* (I.b, II.b, III.b) (or concatenating, in ‘natural manner’ property-preserving structures.) For instance, the Cartesian (I.b) product preserves the ‘set-ness’ property for chains of objects formed from the class of sets, the direct product (II.b) preserves the ‘group-ness’ property under concatenation, etc.
- The concept of ‘morphing’ (I.c, II.c, III.c) from one class of objects to another, in a property-preserving manner. For instance, the continuous map (III.c) respects what makes spaces *X* and *Y* ‘topological,’ when morphing from one to another. The homomorphism respects the group properties shared by *G* and *H*, when ‘morphing’ from one to another, etc.

- The concept of ‘equivalence in form’ (isomorphism) (I.d, II.d, III.d) defined via conditions placed on ‘how’ one should ‘morph,’ which fundmantally should be in an *invertible* manner. One universally necessary condition for this to hold, is that such a manner is modeled as a bijection. The other necessary conditions of course involve the particular property structure-respecting conditions placed on such morphisms.

Similar to naïve set theory (NST) Category theory also preserves its form and structure on any level or category ‘type.’ That is to say, any two (or more) categories C, D can be part of the set of structured objects of a *meta-category* **X** whose morphisms (functors) respect the categorical structure of its arguments C, D. That is to say:

- **Defn A1.2.** Given two categories C = $\langle \Omega, \mathrm{MOR}(\Omega), \circ \rangle$, D = $\langle \Omega', \mathrm{MOR}(\Omega'), \bullet \rangle$, a *categorical functor* **Φ** is a morphism in the *meta-category* **X** from objects C to D assigning each C-object (in Ω) a D-object (in Ω’) and each C-morphism (in MOR(Ω)) a D-morphism (in MOR(Ω’)) such that:
  a.) **Φ** preserves the ‘product’ (compositional) structure of the two categories, i.e., for any $\varphi \in$ MOR(Ω), $\psi \in$ MOR(Ω): $\mathbf{\Phi}(\varphi \circ \psi) = \mathbf{\Phi}(\varphi) \bullet \mathbf{\Phi}(\psi) \equiv \varphi' \bullet \psi'$ (where φ', ψ' are the **Φ-images** in D of the functors φ, ψ in C.
  b.) **Φ** preserves identity structure across all categories. That is to say, for any $A \in \Omega$, $\iota_A \in$ MOR(Ω), $\mathbf{\Phi}(\iota_A) = \iota_{\mathbf{\Phi}(A)} = \iota_{A'}$ where *A*’ is the D-object (in Ω’) assigned by **Φ**. (I.e., $A' = \mathbf{\Phi}(A)$)

Examples of functors include the ‘forgetful functor’ **FOR**: C→SET (where SET is the category of all sets) which has the effect of ‘stripping off’ any extra structure in a mathematical system C down to its ‘bare-bones’ set-structure only. That is to say, for any C-object $A \in \Omega$, $\mathbf{FOR}(A) = S_A$ (where S_A is *A*’s underlying set), and for any $\psi \in$ MOR(Ω): $\mathbf{FOR}(\psi) = f$ is just the mapping (or functional) property of ψ. Robert Geroch (1985, p. 132, p. 248), for example, builds up the toolchest of the most important mathematical structures applied in physics, via a combination of (partially forgetful[84]) and (free construction functors.) Part of this toolchest, for example, is suggested in the diagram below. The boxed items represent the categories (of sets, groups, Abelian or commutative groups, etc.), the solid arrows are the (partially) forgetful functors, and the dashed arrows represent the free construction functors.

[84] ‘Partially forgetful’ in the sense that the action o[…]llapse the structure entirely back to its set-base, just to the ‘nearmost’ (simpler) structure.

Abelian
(commutative)

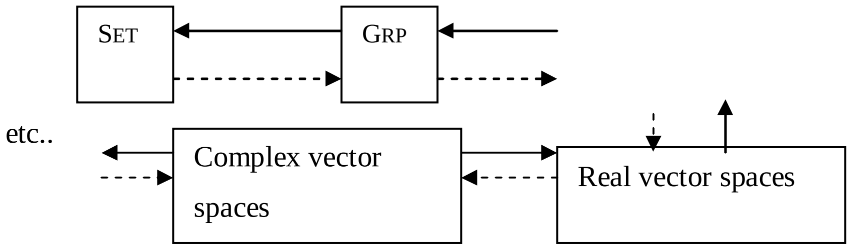

Figure A1.1: Hierarchy of Categories Bound by Free Construction Functors and Forgetful Functors

A.2 Clifford Algebras and Other Algebraic Structures

I proceed here by simply defining the necessary algebraic structures in an increasing hierarchy of complexity:

Defn A2.1: (Group) A group $\langle G, \circ \rangle$ is a set G with a binary operation $\circ$ on G such that:

a.) $\circ$ is *closed* with respect to G, i.e.: $\forall (x, y) \in G$: $(x \circ y) \equiv z \in G$ (i.e., $\circ$ is a *mapping into G* or $\circ$: $G \times G \rightarrow G$, or $\circ(G \times G) \subseteq G$)).

b.) $\circ$ is *associative* with respect to G,: $\forall (x, y, z) \in G$: $(x \circ y) \circ z = x \circ (y \circ z) \equiv x \circ y \circ z$,

c.) There (uniquely) exists a (left/right) identity element $e \in G$: $\forall$ $(x \in G)$ $\exists!$ $(e \in G)$: $x \circ e = x = e \circ x$.

d.) For every x there exists an *inverse element* of x, i.e.: $\forall$ $(x \in G)$ $\exists$ $(x' \in G)$: $x \circ x' = e = x' \circ x$.

In terms of categories, Defn A2.1 is coextensive with that of a monoid endowed with property A.2.1.d.). A monoid is a category in which all of its left and right identities coincide to one unique element. For example, the integers Z form a monoid under integer multiplication (since, $\forall n \in$ Z $\exists!$ $1 \in$ Z such that $n \cdot 1 = n = 1 \cdot n$), but not a group, since their multiplicative inverse can violate closure. Whereas, the non-zero rational numbers $Q^* = \{^n/_m \mid n \neq 0, m \neq 0\}$ form an Abelian (i.e. commutative) group under multiplication.

Defn A2.2: (Subgroups, Normal Subgroups, Simple Groups)

i.) Let $\langle G, \circ \rangle$ be a group. Then, for any $H \subseteq G$, H is a *subgroup* of G (denoted: $H \underline{\angle} G$) if for any $x, y \in H$, then $x \circ y' \in H$. In other words, H is closed under $\circ$, $e \in H$, and if $x \in H$ then $x' \in H$. If $H \underline{\angle} G$, and $H \subset G$, then H is a *proper subgroup*, denoted: $H \angle G$. Moreover, if denoted: $\varnothing \subset H$, then H is *non-trivial*.

ii.) H is a *normal* (or invariant) subgroup of G (denoted: $H \triangleleft G$) if its left and right cosets agree, for any $g \in G$. That is to say, $H \triangleleft G$ *iff* $\forall$ $g \in G$:

$gH = \{gh | h \in H\} = Hg = \{kg | k \in H\}$.

iii.) G is *simple* if G contains no proper, non-trivial, normal subgroups.

Defn A2.3: (Vector Space) A *vector space* is to a structure $\langle V, F, *, \cdot \rangle$ endowed with a (commutative) operation (i.e. $\forall(x,y) \in V : x*y = y*x$, denoted, by convention, by the "+" symbol, though not necessarily to be understood as addition on the real numbers) such that:

i) $\langle V, * \rangle$ is a commutative (or Abelian) group.

ii) Given a field[85] of scalars F the *scalar multiplication* mapping into $V \cdot : F \times V \rightarrow V$ obeys distributivity (in the following two senses):

iii) $\forall(\alpha,\beta) \in F \ \forall \varphi \in V : (\alpha+\beta)\cdot\varphi = (\alpha\cdot\varphi) + (\alpha\cdot\varphi)$

iv) $\forall(\varphi, \phi) \in V \ \forall\gamma\in F : \gamma\cdot(\varphi + \phi) = (\gamma\cdot\varphi) + (\gamma\cdot\phi)$.

Defn A2.4: (Algebra) An *algebra A*, then, is defined as a *vector space* $\langle V, F, *, \cdot, \bullet \rangle$ endowed with an associative binary mapping $\bullet$ into A (i.e., $\bullet : A\times A \rightarrow A$, such that $\forall(\psi, \varphi, \phi) \in G$: $(\psi \bullet \varphi) \bullet \phi = \psi \bullet(\varphi \bullet \phi) \equiv \psi \bullet \varphi \bullet \phi$ denoted, by convention, by the "×" symbol, though not necessarily to be understood as ordinary multiplication on the real numbers) This can be re-stated by saying that $\langle A, \bullet\rangle$ forms a *semigroup* (i.e. a set A closed under the binary associative product $\bullet$), while $\langle A, *\rangle$ forms an Abelian group.

Examples of algebras include the class of *Lie* algebras, i.e. an *algebra dA* whose 'product' $\bullet$ is defined by an (associative) Lie product (denoted [,])obeying the Jacobi Identity: $\forall(\varsigma,\xi,\zeta)\in dA : [[\varsigma,\xi],\zeta] + [[\xi,\zeta],\varsigma] + [[\zeta,\varsigma],\xi] = 0$. The structure of classes of infinitesimal generators in many applications often form a Lie algebra. Lie algebras, in addition, are often characterized by the behavior of their *structure constants C*. For any elements of a Lie algebra ς_μ, ξ_ν characterized by their covariant (or contravariant –if placed above) indices (μ, ν), then a *structure constant* is the indicial function $C(\lambda)^\sigma{}_{\mu\nu}$ such that, for any $\zeta_\rho \in dA$: $\left[\varsigma_\mu, \xi_\nu\right] = \sum_{\sigma=1}^{N} C^\sigma{}_{\mu\nu}(\lambda)\zeta_\sigma$, where N is the dimension of dA, and λ is the Lie Algebra's *contraction parameter*. A Lie algebra is *stable* whenever:

$\lim_{\lambda\rightarrow\infty\vee\lambda\rightarrow 0} C(\lambda)^\sigma{}_{\mu\nu}$ is well-defined for any structure constant $C(\lambda)^\sigma{}_{\mu\nu}$ and contraction parameter λ.

85 I.e. a an algebraic structure $\langle F, +, \times \rangle$ endowed with two binary operations such that $\langle F, +\rangle$ and $\langle F, \times\rangle$ form commutative groups and $+, \times$ are connected by left (and right, because of commutativity) distributivity, i.e., $\forall(\alpha,\beta,\gamma) \in F : \alpha\times(\beta + \gamma) = (\alpha\times\beta) + (\alpha\times\gamma)$.

Defn A2.5: (Clifford Algebra) . A *Clifford* Algebra is a *graded* algebra endowed with the (non-commutative) Clifford product. That is to say:

i.) For any two elements A, B in a Clifford algebra CL, their Clifford product is defined by: $AB = A \bullet B + A \wedge B$, where $A \bullet B$ is their (commutative and associative) *inner* product, and $A \wedge B$ is their anti-commutative, i.e. $A \wedge B = -B \wedge A$, and associative *exterior* (or Grassmann) product. This naturally makes the Clifford product associative: $A(BC) = (AB)C \equiv ABC$. Less obviously, however, for reasons that will be discussed below, is how the *existence of an inverse* A^{-1} for every (nonzero) Clifford element A arises from the Clifford product, i.e.: $A^{-1}A = I = AA^{-1}$, where I is the *unit pseudoscalar* of CL.

ii.) CL is equipped with an adjoint $^{\uparrow}$ and grade operator $< >_r$ (where $< >_r$ is defined as isolating the rth grade of a Clifford element A) such that, for any Clifford elements A, B: $<\mathrm{AB}>^{\uparrow}_r = (-1)^{C(r,2)} <\mathrm{B}^{\uparrow}\mathrm{A}^{\uparrow}>_r$ (where: $C(r,2) = {}^{r!}/_{(2!(r-2)!)} = {}^{r(r-1)}/_{2}$.)

Hence a general Clifford element (or multivector) A of Clifford algebra CL of maximal grade $N = \dim V$ (i.e the dimension of the underlying vector space structure of the Clifford algebra) is expressed by the linear combination:

$$A = \alpha^{(0)}A_0 + \alpha^{(1)}A_1 + \alpha^{(2)}A_2 + \ldots + \alpha^{(N)}A_N \qquad \text{(A.3.1)}$$

where: $\{\alpha^{(k)} \mid 1 \le k \le N\}$ are the elements of the scalar field (expansion coefficients) while $\{A_k \mid 1 \le k \le N\}$ are the *pure* Clifford elements, i.e. $<A_k>_l = A_k$ whenever $k = l$, and $<A_k>_l = 0$ otherwise, while for a general multivector (A.3.1), $<A>_l = \alpha^{(l)}A_l$, for

$1 \le l \le N$

Hence, the pure Clifford elements live in their associated closed Clifford subspaces $CL_{(k)}$ of grade k, i.e. $CL = CL_{(0)} \oplus CL_{(1)} \oplus \ldots \oplus CL_{(N)}$.

Consider the following example: Let $V = \boldsymbol{R}^3$, i.e. the underlying vector space for CL is a 3 dimensional Euclidean space $\boldsymbol{R}^3 = \{\vec{r} = (x,y,z) \mid x \in \boldsymbol{R},\ y \in \boldsymbol{R},\ z \in \boldsymbol{R}\}$. Then the maximum grade for Clifford Algebra over $\boldsymbol{R}^3$, i.e. $CL(\boldsymbol{R}^3)$ is $N = \dim\boldsymbol{R}^3 = 3$. Hence:

$CL(\boldsymbol{R}^3) = CL_{(0)} \oplus CL_{(1)} \oplus CL_{(2)} \oplus CL_{(3)}$ where: $CL_{(0)}$ (the Clifford subspace of grade 0) is (algebraically) isomorphic to the real numbers $\boldsymbol{R}$.[86] $CL_{(1)}$ (the Clifford subspace of grade 1) is algebraically isomorphic

[86] Since the real numbers are a *field*, they're obviously describable as an algebra, in which their underlying 'vector space' structure is identical to their field of scalars. In other words, scalar multiplication is the same as the 'vector' product $\bullet$.

to the Complex numbers $\boldsymbol{C}$. $CL_{(2)}$ (the Clifford subspace of grade 2) is algebraically isomorphic the Quaternions $\boldsymbol{H}$. $CL_{(3)}$ (the Clifford subspace of grade 3) is algebraically isomorphic to the Octonions $\boldsymbol{O}$.

To understand *why* the Clifford algebra over $\boldsymbol{R}^3$ would invariably involve closed subspaces with elements related to the unit imaginary $i = \sqrt{-1}$ (and some of its derivative notions thereon, in the case of the Quaternions and Octonions) entails a closer study of the nature of the Clifford product. Defn. A.2.4 i) deliberately leaves the Grassman product under-specified. I now fill in the details here. First, it is important to note that $\wedge$ is a *grade-raising* operation: for any pure Clifford element A_k (where $k < N = \dim V$) and B_1, then $<A_kB_1> = k + 1$. It is for this reason that pure Clifford elements of grade k are often called *multivectors*. Conversely, the inner product $\bullet$ is a *grade-lowering* operation: for any pure Clifford element A_k (where $k < N = \dim V$) and B_1, then $<A_k \bullet B_1> = k - 1$. (Hence the inner product is often referred to as a *contraction*).

The reason for the grade-raising, anti-commutative nature of the Grassman product is historically attributed to Grassman's geometric notions of (directed) line segments, (rays) areas, volumes, hypervolumes, etc. For example, in the case of two vectors $\vec{A}, \vec{B}$, their associated directed area segments $\vec{A} \wedge \vec{B}, \vec{B} \wedge \vec{A}$ are illustrated below:

Fig. A.2.1: Directed Areas

The notion of directed area, volume, hypervolume segments indeed survives, to a certain limited sense, in the vector-algebraic notion of 'cross-product.' For example, the magnitude of the cross-product $\vec{A} \times \vec{B}$ is precisely the area of the parallelogram spanned by $\vec{A}, \vec{B}$ as depicted in Fig. A.2.1. The difference, however, lies in the fixity of grade in the case of $\vec{A} \times \vec{B}$, in the sense that the anti-commutativity is geometrically attributed to the directionality of the *vector* $\vec{A} \times \vec{B}$ (of positive sign in the case of right-handed coordinate system) perpendicular to the plane spanned by $\vec{A}, \vec{B}$. This limits the notion of the vector cross-product, as it can only be defined for spaces of maximum dimensionality 3.[87] On the other hand, the Grassmann product of multivectors interpreted as directed areas, volumes, and hypervolumes is unrestricted by the dimensionality of the vector space.

[87] "[T]he *vector algebra* of Gibbs…was effectively the end of the search for a unifying mathematical language and the beginning of a proliferation of novel algebraic systems, created as and when they were needed; for example, spinor algebra, matrix and tensor algebra, differential forms, etc." (Lansenby, et. al. (2000), 21)

The connection with the algebraic behavior of $i = \sqrt{-1}$ lies in the inherently anti-commutative aspect (i.e. the Grassmann component) of the Clifford product, as discussed above. To see this, consider the even simpler case of $V = \boldsymbol{R}^2$ (as discussed, for example, in Lasenby, et. al. (2000), 26-29). Then; $N = \dim\boldsymbol{R}^2 = 2$. Moreover, $\boldsymbol{R}^2 = \langle(\hat{e}_1, \hat{e}_2)\rangle$, where $\langle\ldots\rangle$ denotes the *span* and $(\hat{e}_1, \hat{e}_2)$ are the ordered pair of orthonormal vectors (parallel, for example, to the x and y axes.) Hence: $\hat{e}_1^{\,2} = \hat{e}_2^{\,2} = 1$, and $\hat{e}_1 \bullet \hat{e}_2 = \hat{e}_2 \bullet \hat{e}_1 = 0$. So: $\hat{e}_1\hat{e}_2 = \hat{e}_2 \bullet \hat{e}_1 + \hat{e}_1 \wedge \hat{e}_2 = \hat{e}_1 \wedge \hat{e}_2 = -\hat{e}_2 \wedge \hat{e}_1 = -\hat{e}_2\hat{e}_1$. Hence: $(\hat{e}_1\hat{e}_2)^2 = (\hat{e}_1\hat{e}_2)(\hat{e}_1\hat{e}_2) = \hat{e}_1(\hat{e}_2\hat{e}_1)\hat{e}_2 = -\hat{e}_1(\hat{e}_1\hat{e}_2)\hat{e}_2 = -(\hat{e}_1\hat{e}_1)(\hat{e}_2\hat{e}_2) = -(\hat{e}_1^2)(\hat{e}_2^2) = -1$ (using the anti-commutativity and associativity of the Clifford product.) Hence, the multivector $\hat{e}_1\hat{e}_2$ is algebraically isomorphic to $i = \sqrt{-1}$. Moreover, $(\hat{e}_1\hat{e}_2)\hat{e}_1 = -\hat{e}_2$ and $(\hat{e}_1\hat{e}_2)\hat{e}_2 = \hat{e}_1$, by the same simple algebraic maneuvering. Geometrically, then, the multivector $\hat{e}_1\hat{e}_2$ when multiplying on the left has the effect of a *clockwise* $^{\pi}/_{2}$ –rotation. Represented then in the matrix algebra $M_2(\boldsymbol{R})$ (the algebra of real-valued 2x2 matrices):

$$\hat{e}_1\hat{e}_2 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{where: } \hat{e}_1 \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \hat{e}_2 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Moreover, for $CL(\boldsymbol{R}^2)$ the multivector $\hat{e}_1\hat{e}_2$ is the *unit pseudoscalar*, i.e. the element of maximal grade. In general, for any Clifford Algebra $CL(V)$, where $\dim V = N$, and $V = \langle(\gamma_1, \gamma_2, \ldots, \gamma_N)\rangle$, where the basis elements aren't necessarily orthonormal, the unit pseudoscalar I of $CL(V)$ is: $I = \gamma_1\gamma_2\ldots\gamma_N$. In general, for grade k (where $1 \le k \le N$) the closed subspaces $CL_{(k)}$ of grade k in $CL(V) = CL_{(0)} \oplus CL_{(1)} \oplus\ldots\oplus CL_{(N)}$ have dimensionality $C(N,k) = {}^{N!}/_{[k!(N-k)!]}$, i.e are spanned by $C(N,k) = {}^{N!}/_{[k!(N-k)!]}$ multivectors of degree k. Hence the total number of Clifford basis elements generated by the Clifford product acting on the basis elements of the underlying vector space is: $2^N = \sum_{k=0}^{N} C(N,k)$. The unit pseudoscalar is therefore the (one) multivector (only one there are $C(N,N) = 1$ of them, modulo sign or order of mutliplication) spanning the closed Clifford subspace of maximal grade N.

For example, in the case of $CL(\boldsymbol{R}^3) = CL_{(0)} \oplus CL_{(1)} \oplus CL_{(2)} \oplus CL_{(3)}$, where:

$\boldsymbol{R}^3 = \langle(\hat{e}_1, \hat{e}_2, \hat{e}_3)\rangle$: $CL_{(0)} = \langle 1\rangle \cong R, CL_{(1)} = \langle(e_1, e_2, e_3)\rangle, CL_{(2)} = \langle(e_{12}, e_{13}, e_{23})\rangle, CL_{(3)} = \langle I\rangle = \langle e_{123}\rangle$

(where the abbreviation $e_{i\ldots k} = \hat{e}_i\ldots\hat{e}_k$ is adopted). As demonstrated in the case of $CL(\boldsymbol{R}^2)$ the multivector, the unit psuedoscalar I should *not* be interpreted as a multiplicative identity, i.e. it is certainly *not* the case that for any $A \in CL(V)$, $AI = A = IA$. Rather, the unit pseudoscalar is adopted to define an element of dual grade A^* : for any pure Clifford element A_k, (where $0 \le k < N$) : the grade of AI (or A^*) is $N - k$, and vice versa. Thus an inverse element A^{-1} can in principle be constructed, for every nonzero $A \in CL(V)$. So the linear equation $AX = B$ has the formal solution $X = A^{-1}B$ in $CL(V)$. "Much of the power of geometric (Clifford) algebra lies in this property of invertibility." (Lasenby, et. al. (2000), 25)

Appendix B: The Case of Spinors (Hestenes)

Clifford algebras can provide a complete notation for describing certain phenomena in physics that would otherwise require several different mathematical formalisms. For instance, in present-day quantum mechanics and field theory, a variety of different mathematical formalisms are often introduced: 3 dimensional vector algebra, Hilbert space methods, spinor algebra, diffeomorphism algebra on smooth manifolds, etc. This is due in part to the domain-specific nature of the aforementioned, all tailored to apply to a particularly specific context, but relatively restricted in their power of generalization.

For example, consider the simplest case of the three-dimensional vector algebra originally developed by Gibbs. The notion of cross-product cannot be generalized to spaces above a dimensionality 3, yet the Clifford multivector describing directed area, volumes, and hypervolumes applies to any *n*-dimensional space.[88] According to Hestenes the aforementioned restriction of the cross-product to 3 dimensions introduces unnecessary redundancies in the depiction of spinors in standard quantum mechanics:

> Physicists generally regard the σ_k [Pauli spin matrices] as three components of a single vector, instead of an orthonormal frame of three vectors…Consequently, they write: $\vec{\sigma} \bullet \vec{v} = \sum_{k=1}^{3} \sigma_k v_k$ …and to facilitate manipulation they employ the identity:
> $(\vec{\sigma} \bullet \vec{v})(\vec{\sigma} \bullet \vec{w}) = \vec{v} \bullet \vec{w} + i\vec{\sigma} \bullet (\vec{v} \times \vec{w})$… a good example of the redundancy in the language of physics which complicates the manipulations and obscures the meanings unnecessarily. (1986, 323)

The redundancy in the above identity $(\vec{\sigma} \bullet \vec{v})(\vec{\sigma} \bullet \vec{w}) = \vec{v} \bullet \vec{w} + i\vec{\sigma} \bullet (\vec{v} \times \vec{w})$ is due to its 'overlapping geometric content': The (vector) dot and cross products of course comprise the binary operations of standard (Gibbs') vector algebra in R^3, while the Pauli spin matrices ($\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$) acting as the 'vector coefficients' belong to the spinor algebra C^2, i.e. the 2×2 matrix algebra consisting of complex-valued entries.[89] The geometric contents of R^3 and C^2 can be unified, however, when one instead

[88] For further details, see Appendix
[89] As shown in the Appendix, C^2 is algebraically isomorphic to a Clifford Algebra of grade 2.

considers σ_k as generators of a Clifford algebra, thereby "eliminat[ing] all redundancy incorporating both languages into a single coherent language."[90]

So in a narrow sense, Clifford or geometric algebras unify the geometric content of mathematical formulations of physical theories, as Hestenes (1984, 1986) and others have demonstrated in the case of quantum theory and field theory. But this also implies a broader notion of unification. Using the same reasoning as demonstrated above in the case of the Pauli spin algebra, Hestenes (1986) likewise shows how the 4×4 matrix algebra C^4 of Dirac spinors γ_μ[91] is algebraically isomorphic to the Clifford algebra C_4, or the Clifford algebra generated by 4-dimensional vectors with complex coefficients. In several steps he proceeds to show how this Clifford algebra is projectively isomorphic to the Minkowski spacetime algebra $R_{1,3}$, or the Clifford algebra generated by four linearly independent rotation matrices in the Minkowski spacetime $R_{1,3}$.[92] Writes Hestenes:

> The relation of the Dirac theory [of spinors] to classical electrodynamics is not well understood…[with the projective extension into $R_{1,3}$ however] it is more intimate than originally thought. This intimate relation between …the Dirac theory and trajectories of the classical theory [shown in the $R_{1,3}$ reformulation] provides a much more detailed correspondence between the classical and quantum theories than the conventional approach using expectation values and Ehrenfest's theorem…[T]he basic idea…we have been exploiting provides a general geometrical approach to the interpretation of the Dirac theory as follows…any solution $\psi = \psi(x)$ of the Dirac Equation of form $\psi = (\rho e^{i\beta})^{1/2}R$ [where ρ, is a probability density, R is a spinor representation of a Lorentz transformation Λ, and β is an arbitrary phase factor] determines a field of orthonormal frames $e_\mu = e_\mu(x)$…at each spacetime point there's a streamline $x = x(\tau)$" with tangent $v = v(x(\tau))$. [Then] $e_\mu = e_\mu(x(\tau))$ is to be regarded as a 'comoving frame', on the streamline, where e_1 , e_2 rotate about the 'spin axis' e_3 . (332-333)

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[90] Ibid. To see this, simply write for any 3-vector $\vec{v} = \sum_{k=1}^{3} v_k \sigma_k$, then the above identity with its (otherwise geometrically overlapping content) now simply is represented as: $\vec{v}\vec{w} = \vec{v} \bullet \vec{w} + \vec{v} \wedge \vec{w}$. But this is just precisely the definition of the Clifford product of two 3-vectors! For further details, see Appendix.

[91] Dirac introduced such 4×4 matrices as the 'coefficients' of his equation $\gamma^\mu(\underline{i}\partial_\mu - eA_\mu)\Psi = m\Psi$ which linearizes the Klein-Gordon equation (KGE). The latter was the first attempt to make the Schroedinger equation Lorentz-covariant, though its non-linearity (being, as in the case of the Schroedinger equation, a 2nd order differential equation) introduced solutions with indefinite (negative valued) probabilities in Minkowski spacetime $R_{1,3}$ (the 4 dimensional spacetime with metric signature (1,3)). This is remedied by linearizing the KGE in the case of developing the Dirac equation, but only at the expense of introducing such 4x4 complex valued into the equation's coefficient ring.

[92] Hestenes (1986), 325-326. Borrowing from Luonesto (1981, 721), who shows that every complex Clifford algebra of dimension (or grade) $2n$ is algebraically isomorphic to the Clifford algebra generated by the $p+q$ dimensional (real) space $R_{p,q}$ of signature (p,q), where: $p+q = 2n+1$. Hence: $C_4 \cong R_3 \otimes R_2 \cong R_{4,1}$ (where '≅' means: 'is algebraic isomorphic to'). One can then set up a projective map identifying the Clifford algebra over Minkowski spacetime $R_{1,3}$ with the even subalgebra $R_{4,1}^{+}$ of $R_{4,1}$.

From the classical solution of the Dirac Equation, Hestenes derives the result: $\dot{R} = \frac{d}{d\tau} R = \frac{e}{2m} FR$, or the (electron's proper time) rate of change of R (the spinor representation of the Lorentz transformation Λ in the solution of canonical form $\psi = (\rho e^{i\beta})^{1/2} R$) as proportional to the product of the electron's charge with R and the magnitude of the electromagnetic field F.[93] Hestenes interprets $\dot{R} = \frac{d}{d\tau} R = \frac{e}{2m} FR$ as an expression of the *precession* of the comoving frame $e_\mu = e_\mu(x(\tau))$, with an additional rotation determined by a gauge factor. "It should be of genuine physical interest to identify and analyze any deviations from this classical rotation which QM might imply." (333) Hestenes (1985, 13) for instance suggests that such expressions of precession provide adequate models for the supposed *Zitterbewegung* mechanism of a free electron.

The upshot of all this, concludes Hestenes, runs as follows:

> My objective…has been to explicate the geometric structure of the Dirac theory and its physical significance. My approach may seem radical at first sight, but…it [is] ultimately conservative…by restricting my mathematical language to spacetime algebra [i.e. the Clifford algebra over Minkowski space $R_{1,3}$] I allow nothing in my formulation of physical theory without an interpretation of spacetime geometry…[though] I am not opposed to investigating possibilities for unifying physical theory by extending spacetime geometry to higher dimensions…we still have a lot to learn about the physical implications of conventional spacetime structure (346)

Hence, in the broader sense of unification, Clifford or geometric algebras can unify the *ontological* content of mathematical formulations of physical theories. David Hestenes suggests this in so many words, in his Clifford-algebraic characterization of Pauli and Dirac spinors, as briefly summarized above, as well as in his Clifford algebraic characterization of the Weinberg-Salam model which generalizes the electromagnetic gauge group to include the theory of weak interactions. (1986, 334-342) Now in the standard (non-Clifford) formulations of quantum theory, the interpretation of Dirac spinors in quantum field theory remains obscure. Hestenes' Clifford algebraic reformulation of the Dirac theory simplified and clarified its ontology, by indicating its interconnection with classical EM theory, vis-à-vis the intricate algebra of spins and rotations in Minkowski spacetime, represented by the Clifford algebra over $R_{1,3}$. Hestenes concludes:

> The most important thing…from the [Clifford algebraic] reformulation [of the Dirac theory] is that the imaginary $i = \sqrt{-1} Id$ [where *Id* is the identity operator] has definite geometrical and physical meaning…represent[ing] the generator of rotations in a spacelike plane associated with spin… $i = \sqrt{-1} Id$ can be identified with the spin bivector $\hat{S} = i\hbar\vec{\sigma}$ …[This identification] has far reaching consequences…[for instance] when the Schroedinger equation is derived as an approximation to the Dirac equation…[this] implies that a degenerate representation of the spin direction by the unit imaginary has been implicit in Schroedinger equation all along. (331)

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[93] In 4-vector notation: $F = \nabla \wedge A$, where ∇ is the D'Alembertian, and A is the 4-vector potential.

He is proposing a reduction involving the conceptually problematic nature of the role played by the unit imaginary *i* in quantum theory to ontological claims concerning *rotations* in R^3 (in the case of Pauli spinors) and in Minkowski spacetime $R_{1,3}$ (in the case of the Dirac theory). As mentioned in the previous sections, geometric algebras present a far more nuanced and systematic way of mathematically representing all possible rotational transformations in spaces involving any dimensionality than standard vector methods.[94] Such physical implications include the ontology of all possible rotational dynamics in spacetime. Clifford algebras provide a natural means of mathematically representing such transformations. That was none other than W. K. Clifford's original intention, when he developed his geometric algebras.

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[94] In fact, the unit imaginary scalar is automatically geometrically interpreted as the generator of rotations for any Clifford algebra of dimensionality 2 or greater. For details, see Appendix.

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