

Methodological Fundamentalism: or why Batterman's Different Notions of 'Fundamentalism' may not make a Difference

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June 19, 2006

Abstract

I argue that the distinctions Robert Batterman (2004) presents between 'epistemically fundamental' versus 'ontologically fundamental' theoretical approaches can be subsumed by *methodologically fundamental* procedures. I characterize precisely what is meant by a methodologically fundamental procedure, which involves, among other things, the use of multilinear graded algebras in a theory's formalism. For example, one such class of algebras I discuss are the Clifford (or Geometric) algebras. Aside from their being touted by many as a "unified mathematical language for physics," (Hestenes (1984, 1986) Lasenby, et. al. (2000)) Finkelstein (2001, 2004) and others have demonstrated that the techniques of multilinear algebraic 'expansion and contraction' exhibit a robust regularizability. That is to say, such regularization has been demonstrated to remove singularities, which would otherwise appear in standard field-theoretic, mathematical characterizations of a physical theory. I claim that the existence of such methodologically fundamental procedures calls into question one of Batterman's central points, that "our explanatory physical practice demands that we appeal essentially to (infinite) idealizations" (2003, 7) exhibited, for example, by singularities in the case of modeling critical phenomena, like fluid droplet formation. By way of counterexample, in the field of computational fluid dynamics (CFD), I discuss the work of Mann & Rockwood (2003) and Gerik Scheuermann, (2002). In the concluding section, I sketch a methodologically fundamental procedure potentially applicable to more general classes of critical phenomena appearing in fluid dynamics.

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I. Introduction

Robert Batterman (2005) distinguishes between “ontologically fundamental” and “epistemically fundamental” theories. The aim of former is to “get the metaphysical nature of the systems right,” (19) often at the expense of being explanatorily inadequate. Fundamentally explanatory issues involving the universal dynamical behavior of critical phenomena,² for instance, cannot be dealt with by the ontologically fundamental theory. Epistemologically fundamental theories, on the other hand, seek to achieve such an explanatory aim accounting for such universal behavior, at the expense of suppressing (if not outright misrepresenting) a physical system’s fundamentally ontological features.

In the case of critical phenomena such as drop formation,³ even in accounts of more fine-grained resolutions of the scaling similarity solution for the Navier-Stokes equations (which approximate a fluid as a continuum), “we must appeal to the non-Humean similarity solution (resulting from the singularity) of the *idealized* continuum Navier-Stokes theory.” (20) In a more general sense, though “nature abhors a singularity...without them one cannot characterize, describe, and explain the emergence of new universal phenomena at different scales.” (19)

In other words, we need the ontologically “false” but epistemically fundamental theory to account for the ontologically true but epistemically lacking fundamental theory. “[A] complete understanding (or at least an attempt) of the drop breakup problem

²Such critical phenomena exhibiting universal dynamical properties include, but are not limited to, examples including fluids undergoing phase transitions under certain conditions favorable for modeling their behavior using Renormalization Group methods, shock-wave propagation (phonons), caustic surfaces occurring under study in the field of catastrophe optics, quantum chaotic phenomena, etc.

³ applied to the nanoscale jets analyzed by Landman & Mosely.

requires essential use of a ‘nonfundamental’ [i.e. epistemically fundamental] theory...the continuum Navier-Stokes theory of fluid dynamics.” (18)

Batterman advocates this necessary coexistence of two kinds of fundamental theories, which in my opinion, can be viewed as a refinement of his more general themes presented in (2002). There, he argues that in the case of emergent phenomena, explanation and reduction part company: the superseded theory T can still play an essential role. That is to say, the superseding theory T' , though ‘deeply containing T ’ (in some non-reductive sense) cannot adequately account for emergent and critical phenomena alone, and thus enlists T in some essential manner. According to Batterman, this produces a rift between reduction and explanation, insofar as one is forced to accommodate an admixture of differing ontologies characterized by the respectively superseding and superseded theories. In his later work, Batterman (2005) seems to imply that epistemologically fundamental theories serve in a similarly necessary capacity in terms of what he explains the superseded theories do, in the case of emergent phenomena (2002).

I have critiqued (Kallfelz (2005b)) Batterman’s claims (2002, 2004) in a two-fold manner: Batterman confuses a theory’s (mathematical) topology with its (metaphysical) ontology. This confusion, in turn, causes him to reify unnecessarily certain notions of singularities, in the explanatory role they play in the superseded theory. I argue here that there exist methods of regularization in multilinear algebraic characterizations of microphysical phenomena employed by theoretical physicists (Finkelstein (2002-2005), Green (2000)) which seem to provide a truer ontological account for what goes on at the

microlevel, *and* bypass singularities that would otherwise occur in more conventional mathematical techniques (not based on multilinear algebras).

I characterize such a notion of ‘fundamental’ arising in algebraic expansion and contraction techniques as an example of a *methodological fundamentalism*: for it offers a means of intertheoretic reduction which overcomes the singular cases Batterman seems to reify (2002, 2004). In the case of fluid dynamics, multilinear algebras like Clifford algebras have been recently applied by Gerik Scheuermann (2000), Mann & Rockwood (2003), in their work on computational fluid dynamics (CFD). The authors show that CFD methods involving the Clifford algebraic techniques are often applicable in the same contexts as the Navier-Stokes treatment –minus the singularities. Such results imply that methodological fundamentalism can, in the cases Batterman investigates, provisionally sort out and reconcile epistemically and ontologically fundamental theories. Hence, *pace* Batterman, they need not act in cross purposes.

II. Epistemological Versus Ontological Fundamentalism (Batterman, 2005)

Robert Batterman explains the motivation for presenting a distinction between ontological versus epistemically fundamental theories:

I have tried to show that a complete understanding (or at least an attempt...) of the drop breakup problem requires essential use of a ‘nonfundamental’ theory...the continuum Navier Stokes theory of fluid dynamics...[But] how can a false (because idealized) theory such as continuum fluid dynamics be *essential* for understanding the behaviors of systems that fail completely to exhibit the principal feature of that idealized theory? Such systems [after all] are discrete in nature and not continuous...*I think the term ‘fundamental theory’ is ambiguous*...[An ontologically fundamental theory]...gets the metaphysical nature of the system right. On the other hand...ontologically fundamental theories are often explanatorily inadequate. Certain explanatory questions...about the emergence and reproducibility of patterns of behavior cannot be answered by

the ontologically fundamental theory. I think that this shows...there is an epistemological notion of ‘fundamental theory’ that fails to coincide with the ontological notion. (2005, 18-19, italics added)

On the other hand, epistemically fundamental theories aim at a more comprehensive explanatory account, often, however, at the price of introducing essential singularities.

For example, in the case of ‘universal classes’ of behavior of fluid-dynamical phenomena exhibiting patterns like droplet formation:

Explanation of [such] universal patterns of behavior require means for eliminating details that ontologically distinguish the different systems exhibiting the same behavior. *Such means are often provided by a blow-up or singularity in the epistemically more fundamental theory that is related to the ontologically fundamental theory by some limit.* (ibid., italics added)

Obviously, any theory relying on a continuous topology⁴ harbors the possibility of exhibiting singular behavior, depending on its domain of application.⁵ In the case of droplet-formation, for example, the (renormalized) solutions to the (continuous) Navier-Stokes Equations (NSE) exhibit singular behavior. Such singularities play an essential explanatory role insofar as such solutions in the singular limit exhibit ‘self-similar,’ or universal behavior, to the extent that only *one* parameter essentially governs the behavior of the solutions to the NSEs in such a singular limit. Specifically, only the fluid’s *thickness* parameter (neck radius h) governs the shape of the fluid near break-up,⁶ in the asymptotic solution to the NSE (2004, 15):

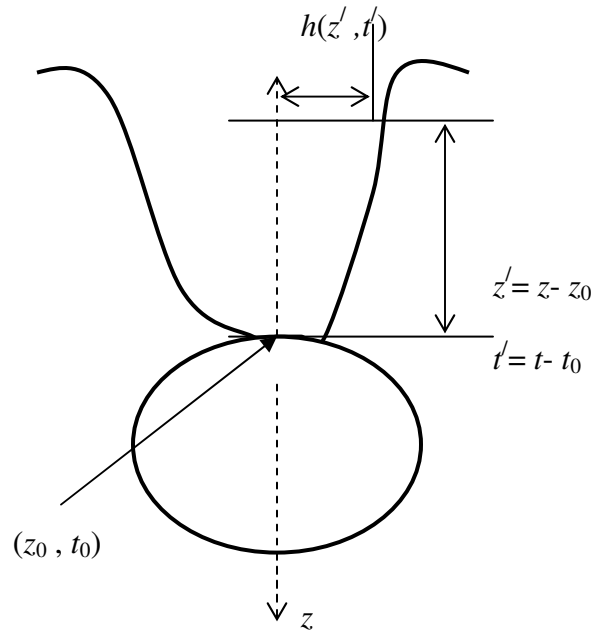
⁴ I am borrowing from Batterman’s (2002) usage, in which he distinguishes the *ontology*, i.e. the primitive entities stipulated by a physical theory, from its *topology*, or structure of its mathematical formalism.

⁵ This is of course due to the rich structure of continuous sets themselves admitting such effects. Consider, for example, the paradigmatic example: $f \in (-\infty, \infty)^{(-\infty, \infty)}$ given by the rule: $f(x) = 1/x$. This obviously produces an essential singularity at $x = 0$.

⁶ For fluids of low viscosities see Batterman (2004), n 12, p.16.

$$h(z', t') = f(t')^\alpha H(\zeta)$$

$$\zeta = \frac{z'}{f(t')^\beta}$$



where: $f(t')$ is a continuous (dimensionless) function expressing the time-dependence of the solution ($t' = t - t_0$ is the measured time after droplet breakup t_0).
 α, β are phenomenological constants to be determined.
 H is a Haenkel function.⁷

One could understand the epistemically and ontologically fundamental theories as playing analogous roles to Batterman's (2002, 2003, 2004) previously characterized superseded and superseding theories (T and T' , respectively). Like in the case of the superseded theory T , the epistemically fundamental theory offers crucial explanatory insight, at the expense of mischaracterizing the underlying ontology of the phenomena under study. Whereas, on the other hand, analogous to the case of the superseding theory T' , the ontologically fundamental theory gives a more representative metaphysical characterization, at the expense of losing its explanatory efficacy.⁸

⁷ I.e. belonging to a class of orthonormal special functions often appearing in solutions to PDEs describing dynamics of boundary-value problems.

⁸ For instance, in the case of the breaking water droplet, the ontologically fundamental theory would be the molecular-discrete one. But aside from practical limitations posed by the sheer intractability of the computational complexity of such a quantitative account, the discrete-molecular theory, precisely *because* it lacks the singular-asymptotic aspect, cannot depict the (relatively) universal character presented in the asymptotic limit of the (renormalized) solutions to the NSE.

However, I argue that there *are* theoretical characterizations whose formalisms can *regularize* or remove singularities from some of the fluid-dynamical behavior in a sufficiently abstract and general manner, as to call into question the presumably essential distinctions between epistemological and ontological fundamentalism. I call such formal approaches “methodologically fundamental,” because of the *general* strategy such approaches introduce, in terms of offering a regularizing procedure.⁹ Adopting such methodologically fundamental procedures, whenever it is possible to do so,¹⁰ suggests that Batterman’s distinctions may *not* be different theoretical *kinds*, but function at best as different *aspects* of a unified methodological strategy. This calls into question the explanatory pluralism Batterman appears to be advocating.

Similar to Gordon Belot’s (2003) criticism, I am also arguing that extending the breadth and scope of theories in mathematical physics applied to the domains of critical phenomena Batterman calls our attention to, goes a long way to qualify and diminish the distinctions he makes. Belot argues that a richer and more mathematically rigorous rendition of the superseding theory T' eliminates the necessity of one having to resort simultaneously to the superseded theory T to characterize some critical phenomenon (or class of phenomena) Φ . Like Belot, I also claim that multilinear algebraic techniques abound which can regularize the singularities appearing in formalisms of T (or T'). Conversely, when representing such critical phenomena Φ , singularities can occur in T

⁹ In other words, this strategy should *not* be conceived of as a merely souped-up version of an ontologically fundamental theory. The latter, according to Batterman, are stuck at the level of giving very detailed accounts involving the *particular features* of the phenomena at the expense of accounting for generally significant universal dynamical features shared, across the board, of many fundamentally distinct material *kinds* (like in the case of different *kinds* of fluids exhibiting universally self-similar behavior, during critical phase transitions.)

¹⁰ The generality of the methods do *not* imply that they are a panacea, ridding *any* theory’s formalism of singularities.

(or T') when the latter are characterized by the more typically standard field-theoretic or phase space methods alone.

However, the *mathematical content* of the techniques I investigate differs significantly from those discussed by Belot (2003), who characterizes T' using the more general and abstract theory of differentiable manifolds. He demonstrates that in principle, all of the necessary features of critical phenomena Φ can be so depicted by the mathematical formalism of superseding theory T' alone (2003, 23). Because the manifold structure is continuous, this can (and does) admit the possibility of depicting such critical phenomena Φ through complex and asymptotic singular behavior. In other words, Belot is *not* fundamentally questioning the underlying theoretical *topologies* typically associated with T and T' .¹¹ Instead, he is questioning the need to bring the two different *ontologies* of the superseded and superseding theories together, to adequately account for Φ . Belot is questioning the presumed *ontological* pluralism that Batterman advanced in his notion of an ‘asymptotic explanation’.¹²

¹¹ I.e., differential equations on phase space, characterizable through the theory of differential manifolds.

¹² Batterman (2003) responds:

I suspect that one intuition behind Belot’s ...objection is...I [appear to be] saying that for genuine explanation we need [to] appeal *essentially* to an idealization [i.e., the ontology of the superseded theory T .] ...In speaking of this idealization as essential for explanation, they take me to be reifying [T ’s ontology]...*It is this last claim only that I reject.* I believe that in many instances our explanatory physical practice demands that we appeal essentially to (infinite) idealizations. But I don’t believe that this involves the reification of the idealized structures.” (7)

It is, of course, precisely the latter claim “that we appeal essentially to (infinite) idealizations” that I take issue with here, according to what the regularization procedures indicate. Batterman, however, cryptically and subsequently remarks that: “In arguing that an account that appeals to the mathematical idealization is superior to a theory that does not invoke the idealizations, I am not reifying the mathematics...I am claiming that the ‘fundamental’ theory that fails to take seriously the idealized [asymptotic] ‘boundary’ is less explanatorily adequate.” (8) In short, it seems that in his overarching emphasis of his interest in what he considers to be novel accounts of scientific explanation (namely, of the asymptotic variety) he often *blurs* the distinctions, and *shifts emphasis* between a theory’s ontology and its topology. It is precisely this sort of equivocation, I maintain, that causes him to inadvertently reify mathematical notions like ‘infinite idealizations.’ To put it another way, since it is safe to assume that the actual critical phenomena

I, on the other hand, *pace* Belot (2003) and Batterman (2002-2005) present an alternative to the mathematical formalisms that both authors appeal to, which rely so centrally on continuous topological structures.¹³ I show how *discretely graded*, and ultimately *finite-dimensional* multi-linear geometric (Clifford) algebras can provide accounts for some of the *same* critical phenomena Φ in a *regularizable* (singularity-free) fashion.

Prior to describing the specific details of how to implement the strategy in the case of critical phenomena exhibited in fluid dynamics, however, I make the following disclaimer: I am definitely *not* arguing that the discrete, graded, multilinear Clifford-algebraic methods share such a degree of universal applicability that they should *supplant* the continuous, phase-space, infinite-dimensional differentiable manifold structure constituting the general formalism of the theory of differential equations, whether ordinary or partial. Certainly the empirical content of a specific problem domain determines which is the ‘best’ mathematical structure to implement in any theory of mathematical physics. By and large, such criteria are often determined essentially by practical limitations of computational complexity.

We run into no danger, so long as we can carefully distinguish the epistemological, ontological, and methodological issues vis-à-vis our choice of mathematical formalism(s). If the choice is primarily motivated by practical issues of computational facility, we can hopefully resist the temptation to reify our mathematical maneuvering which would confuse the ‘approximate’ with the ‘fundamental’-- let alone

Batterman discusses are ultimately metaphysically finite, precisely *how* can one ‘appeal essentially to (infinite) idealizations’ *without* inadvertently ‘reifying the mathematics?’

¹³ Of course, in the case of Batterman, continuous structures comprise as well the *ontology* of the epistemically fundamental theory: Navier-Stokes treats fluids as continua. In the case of Belot, the theory of partial differential equations he presents relies fundamentally on continuous, differentiable manifolds.

confusing ontological, epistemological, and methodological *senses* of the latter notion.¹⁴ Even Batterman admits that “nature abhors singularities.” (2005, 20) So, I argue, should we. The entire paradigm behind regularization procedures is driven by the notion that a singularity, far from being an “infinite idealization we must appeal to” (Batterman 2003, 7), is a signal that the underlying formalism of theory is the pathological cause, resulting in theory’s failure to provide information, in certain critical cases.

Far from conceding to some class of “asymptotic-explanations,” lending a picture of the world of critical phenomena as somehow carved at the joints of asymptotic singularities, we must instead search for regularizable procedures. This is precisely why such an approach is methodologically fundamental: regularization implies some (weak) form of intertheoretic reduction, as I shall argue below.

III. Clifford Algebraic Regularization Procedure: A Brief Overview

In this section, I summarize aspects of methods incorporating algebraic structures frequently used in mathematical physics, leading up to and including the regularization procedures latent in applications of Clifford Algebras. Because this material involves some technical notions of varying degrees of specialty, I have provided for the interested reader an Appendix at the end of this essay supplying all the necessary definitions and brief explanations thereon.

¹⁴ I am, of course, *not* saying that there does not exist any connection whatsoever between a theory’s computational efficacy and its ability to represent certain fundamentally ontological features of the phenomena of interest. *What* that connection ultimately *is* (whether empirical, or some complex and indirect logical blend thereof) I remain an agnostic.

I review here a few basic techniques involving (abstract algebraic) *expansion* and *contraction*.¹⁵ Consider the situation in which the superceding theory T' is capable of being characterized, in principle, by an *algebra*.¹⁶ Algebraic expansion denotes the process of extending out from algebraically characterized T' to some T'^* (denoted: $T' \xrightarrow{\lambda} T'^*$) where λ is some fundamental parameter characterizing the algebraic expansion (Finkelstein (2002) 4-8). The inverse procedure: $\lim_{\lambda \rightarrow 0} T'^* = T'$ is *contraction*.

The question becomes: how to regularize? In other words, which T'^* should one choose to guarantee a regular (i.e., non-singular) limit for *any* λ in the greatest possible generality? Answer: expanding into an algebraic structure whose relativity group, i.e., the group of all its dynamical symmetries,¹⁷ is *simple* implies the Lie algebra depicting its infinitesimal transformations is *stable*.¹⁸ This in turn entails greater reciprocity,¹⁹ i.e., “reciprocal couplings in the theory...reactions for every action.” (Finkelstein, 2002,10). This is an instance of a *methodologically fundamental* procedure, which I summarize by the following general necessary conditions:

¹⁵ For a concrete summary of Wigner’s (1952) analysis of algebraic expansion from the Galilean to the Lorentz groups, for example, see Kallfelz (2005b), 16-17.

¹⁶ That is to say, a vector space with an associative product. For further details, see A.2 of the Appendix

¹⁷ In other words, the group of all actions in leaving their *form* of dynamical laws invariant (in the active view) or the group of all ‘coordinate transformations’ preserving the tensor character of the dynamical laws (in the ‘passive view.’) Also, see Defn. A.2.2 (Appendix A.2) for a description of simple groups.

¹⁸ For a brief description of stable Lie algebras, see the discussion following Defn A.2.4, section A.2, Appedix.

¹⁹ For example, in the case of the Lorentz group, which is simple, it is maximally reciprocal in terms of its fundamental parameters x , and t . That is to say, the form of Lorentz transformations (simplified in one dimensional motion along the x -axes of the inertial frame F and F') become $x' = x'(x,t) = \gamma(x - Vt)$ and $t' = t'(x,t) = \gamma(t - Vx/c^2)$ (where $\gamma = (1 - V^2/c^2)^{-1/2}$). Hence both space x and time t couple when transforming between inertial frames F, F' , as their respective transformations involve each other. On the other hand, the Galilean group is *not* simple, as it contains an invariant subgroup of boosts. The Galilean transformations are not maximally reciprocal, as $x' = x'(x,t) = x - Vt$ but $t' = t$. x is a cyclic coordinate with respect to transformation t' . Thus, when transforming between frames, x couples with respect to t but not vice versa.

- **Ansatz IIIa:** If a procedure \mathbb{P} for formulating a theory T in mathematical physics is *methodologically fundamental*, then there exists some algebraically characterized *expansion* T'^* of T 's algebraic characterization (denoted by T') and some expansion parameter λ such that: $T' \xrightarrow{\lambda} T'^*$. Then, trivially, T'^* is regularizable with respect to T' since $\lim_{\lambda \rightarrow 0} T'^* = T'$ is well-defined (via the inverse procedure of algebraic contraction).
- **Ansatz IIIb:** If T'^* is an expansion of T' , then T'^* 's relativity group is *simple*, which results in a *stable* Lie algebra dT'^* , and whose set of observables in T'^* is maximally reciprocal.

Segal (1951) described any algebraic formalization of a theory obeying what I depict above according to Ansatz IIIb as “fundamental.” I insert here the adjective “methodological,” since such a procedure comprises a method of regularization (viewed from the standpoint of the ‘inverse’ procedure of contraction) and so a formal means of reducing a superseding theory T' into its superseded theory T , when characterized by algebras.

III.a) An Example of a Methodologically Fundamental Procedure: Deriving a Continuous Space-Time Field Theory as an Asymptotic Approximation of a Finite Dimensional Clifford Algebraic Characterization of Spatiotemporal Quantum Topology (Finkelstein (1996, 2001, 2002-2004))²⁰.

Motivated by the work of Inonou & Wigner (1952) and Segal (1951) on group regularization, Finkelstein (1996, 2001, 2004a-c) presents a unification of field theories (quantum and classical) and space-time theory based fundamentally on *finite* dimensional Clifford algebraic structures. The regularization procedure fundamentally involves group-theoretic simplification. The choice of the Clifford algebra²¹ is motivated by two fundamental reasons:

²⁰ This is somewhat of a more technical discussion and optional for the reader looking for a basic application of Clifford algebraic techniques in fluid mechanics alone.

²¹ The associated multiplicative groups embedded in Clifford algebras obey the simplicity criterion (Ansatz IIIb). Hence Clifford algebras (or geometric algebras) remain an attractive candidate for algebraicizing any theory in mathematical physics (assuming the Clifford product and sum can be appropriately operationally interpreted in the theory T). For definitions and further discussion thereon, see Defn A.2.5, Appendix A.2.

1. The typically abstract (adjoint-based) algebraic characterizations of quantum dynamics (whether C^* , Heisenberg, etc.) just represent how actions can be combined (in series, parallel, or reversed) but omit space-time fine structure.²² On the other hand, a Clifford algebra can express a quantum space-time. (2001, 5)
2. Clifford statistics²³ for chronons adequately expresses the distinguishability of events as well as the existence of half-integer spin. (2001, 7)

The first reason entails that the prime variable is not the space-time field, as Einstein stipulated, but rather the dynamical law. That is to say, “the dynamical law [is] the only dependent variable, on which all others depend.” (2001, 6) The “atomic” quantum dynamical unit (represented by a generator γ^α of a Clifford algebra) is the *chronon* χ , with a closest classical analogue being the tangent or cotangent vector, (forming an 8-dimensional manifold) and *not* the space-time point (forming a 4-dimensional manifold).

Applying Clifford statistics to dynamics is achieved via the (category) functors²⁴ ENDO, SQ which map the mode space²⁵ X of the chronon χ , to its operator algebra (the algebra of endomorphisms²⁶ A on X) and to its spinor space S (the statistical composite of all chronons transpiring in some experimental region.) (2001, 10). The action of ENDO, SQ producing the Clifford algebra *CLIFF*, representing the global dynamics of the chronon ensemble is depicted in the following commutative diagram:

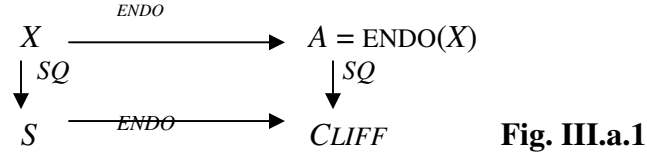
²² The space-time structure must be supplied by classical structures, prior to the definition of the dynamical algebra. (2001, 5)

²³ I.e., the simplest statistics supporting a 2-valued representation of S_N , the symmetry group on N objects.

²⁴ See Defn. A.1.2, Appendix A.1

²⁵ The mode space is a kinematic notion, describing the set of all possible modes for a chronon χ , the way a state space describes the set of all possible states for a state ϕ in ordinary quantum mechanics.

²⁶ I.e., the set of surjective (onto) algebraic structure-preserving maps (those preserving the action of the algebraic ‘product’ or ‘sum’ between two algebras A, A'). In other words, Φ is an endomorphism on X , i.e. $\Phi: X \rightarrow X$ iff: $\forall x, y \in X: \Phi(x+y) = \Phi(x) + \Phi(y)$, where $+$ is vector addition. Furthermore $\Phi(X) = X$: i.e. for any $z \in X: \exists x \in X$ such that $\Phi(x) = z$. For a more general discussion on the abstract algebraic notions, see A.2, Appendix.



Analogous to H.S. Green's (2000) embedding of the space-time geometry into a parafermionic algebra of qubits, Finkelstein shows that a Clifford statistical ensemble of chronons can factor as a Maxwell-Boltzmann ensemble of Clifford subalgebras. This in turn becomes a Bose-Einstein aggregate in the $N \rightarrow \infty$ limit (where N is the number of factors). This Bose-Einstein aggregate condenses into an 8-dimensional manifold M which is isomorphic to the tangent bundle of space-time. Moreover, M is a *Clifford manifold*, i.e. a manifold provided with a Clifford ring:

$C(M) = C_0(M) \oplus C_1(M) \oplus \dots \oplus C_N(M)$ (where: $C_0(M), C_1(M), \dots, C_N(M)$ represent the scalars, vectors, ..., N -vectors on the manifold). For any tangent vectors $\gamma^\mu(x), \gamma^\nu(x)$ on (Lie algebra dM) then:

$$\gamma^\mu(x) \circ \gamma^\nu(x) = g^{\mu\nu}(x) \tag{III.1}$$

where: \circ is the scalar product. (2004a, 43) Hence the space-time manifold is a singular limit of the Clifford algebra representing the global dynamics of the chronons in an experimental region.

Observable consequences of the theory are discussed in the model of the oscillator (2004c). Since the dynamical oscillator undergirds much of the framework of contemporary quantum theory, especially quantum field theory, the (generalized) model oscillator constructed via group simplification and regularization is isomorphic to a

dipole rotator in the orthogonal group $O(6N)$ (where: $N = l(l + 1) \gg 1$). In other words, a *finite* quantum mechanical oscillator results, bypassing the ultraviolet and infrared divergences that occur in the case of the standard (infinite dimensional) oscillator applied to quantum field theory. In place of these divergences, are “soft” and “hard” cases, respectively representing maximum potential energy unable to excite one quantum of momentum, and maximum kinetic energy being unable to excite one quantum of position. “These [cases]...resemble [and] extend the original ones by which Planck obtained a finite thermal distribution of cavity radiation. Even the 0-point energy of a similarly regularized field theory will be finite, and can therefore be physical.” (2004c, 12)

In addition, such potentially observable extreme cases modify high and low energy physics, as “the simplest regularization leads to interactions between the previously uncoupled excitation quanta of the oscillator...strongly attractive for soft or hard quanta.” (2004c, 19) Since the oscillator model quantizes and unifies time, energy, space, and momentum, on the scale of the Planck power (10^{51} W) time and energy can be interconverted.²⁷

III.b) Some General Remarks: What Makes Multilinear Algebraic Expansion Methodologically Fundamental

Before turning to the example involving applying Clifford algebraic characterization of critical phenomena in fluid mechanics, I shall give a final and brief

²⁷ In such extreme cases, equipartition and Heisenberg Uncertainty is violated. The uncertainty relation for the soft and hard oscillators read, respectively:

$$(\Delta L_1)^2 (\Delta L_2)^2 \geq \frac{\hbar^2}{4} \langle L_3 \rangle^2_{|L_2=0} \approx 0 \Rightarrow \Delta p \Delta q \ll \frac{\hbar}{2}$$

$$(\Delta L_1)^2 (\Delta L_2)^2 \geq \frac{\hbar^2}{4} \langle L_3 \rangle^2_{|L_1=0} \approx 0 \Rightarrow \Delta p \Delta q \ll \frac{\hbar}{2}$$

recapitulation concerning the reasons why one should consider such methods described here as being methodologically fundamental. For starters, the previous two Ansätze I proposed (in §III.a) act as necessary conditions for what may constitute a methodologically fundamental procedure. Phrasing them in their contrapositive form (III.a*, III.b* below) also tell us what formalization schemes for theories in mathematical physics *cannot* be considered methodologically fundamental:

- **Ansatz (IIIa*):** If T'^* is *singular* with respect to T' , in the sense that the behavior of T'^* in the $\lambda \rightarrow 0$ limit does *not* converge to the theory T' at the $\lambda = 0$ limit (for any such contraction parameter λ), this entails that the procedure \mathbb{P} for formulating a theory T in mathematical physics *cannot* be *methodologically fundamental*, and is therefore *methodologically approximate*.
- **Ansatz (IIIb*):** If the relativity group of T'^* is not simple, its Lie algebra is subsequently unstable. Therefore T'^* *cannot* act as an effective algebraic expansion of T' in the sense of guaranteeing the inverse contraction procedure is non-singular.

Certainly IIIa* is just a re-statement (in algebraic terms) of Batterman's more general discussion (2002) of critical phenomena, evincing in his case-studies a singularity or inability for the superseding theory to reduce to the superseded theory. However this need not entail that we must preserve a notion of 'asymptotic explanations,' as Batterman would invite us to do, which would somehow inextricably involve the superseded and the superseding theories. Instead, as III.a* glibly states, this simply tells us that mathematical scheme of the respective theory (or theories) is *not* methodologically fundamental, so we have a signal to search for methodologically fundamental procedures in the particular problem-domain, if they exist.²⁸

²⁸ In a practical sense, of course, the existence of procedures entail staying within the strict bounds determined by what is computationally feasible.

III.b* gives us further insight into criteria filtering out methodologically fundamental procedures. In fact, Finkelstein (2001) shows that *all* physical theories exhibiting, at root, an underlying fiber-bundle topology,²⁹ *cannot* have any relativity groups that are simple. This excludes a vast class of mathematical formalisms: *all*-field theoretic formalisms, whether classical or quantum.

However, as informally discussed in the preceding section (II), if any class of mathematical formalisms is methodologically approximate, this would not in itself entail that the computational efficacy or empirical adequacy of any theory *T* constituted by such a class is somehow diminished. If a formalism is found to be methodologically approximate, this should simply act as a caveat against reifying the theory's ontology, until such a theory can be characterized by a methodologically fundamental procedure.

A methodologically fundamental strategy does more than simply remove undesirable singularities. As discussed above in previous subsection, the finite number of degrees of freedom (represented by the maximum grade *N* of the particular Clifford algebra) positively informs certain ontologically fundamental notions regarding our metaphysical intuitions concerning the ultimately discrete characteristics of the entities fundamentally constituting the phenomenon of interest.³⁰ On the other hand, the

²⁹ I.e., for Hausdorff (separable) spaces X, B, F , and map $p: X \rightarrow B$, defined as a bundle projection (with fiber F) if there exists a homeomorphism (topologically continuous map) defined on every neighborhood U for any point $b \in B$ such that: $\phi: p(\langle b, f \rangle) = b$ for any $f \in F$. On $p^{-1}(U) = \{x \in X \mid p(x) \in U\}$, then p acts as a projection map on $U \times F \rightarrow F$. A fiber bundle consists is described by $B \times F$, (subject to other topological constraints (Brendon (2000), 106-107)) where B acts as the set of *base points* $\{b \mid b \in B \subseteq X\}$ and F the associated fibres $p^{-1}(b) = \{x \in X \mid p(x) = b\}$ at each b .

³⁰ Relative, of course, to the level of scale we wish to begin, in terms of characterizing the theories' ontological primitives. For instance, should one wish to begin at the level of quarks, the question of whether or not their fundamental properties are discrete or continuous becomes a murky issue. Though quantum mechanics is often understood as a fundamentally 'discrete' theory, the continuum nevertheless appears in a subtle manner, when considering entangled modes, which are based on particular superpositions of 'non-factorizable' products.

regularization techniques have, *pace* Batterman, epistemically fundamental consequences that are positive.

In closing, one can ask how likely is it that methodologically fundamental multilinear algebraic strategies can be applied to any complex phenomena under study, such as critical behavior? The serious questions deal with practical limitations of computational complexity: asymptotic methods can yield simple and elegantly powerful results, which would undoubtedly otherwise prove far more laborious to establish by discrete multilinear structures, no matter how methodologically fundamental the latter turn out to be. Nevertheless, the ever-burgeoning field of computational physics gives us an extra degree of freedom to handle, to a certain extent, the risk of combinatorial explosion that such multilinear algebraic techniques may present, when applied to a given domain of complex phenomena.³¹ I examine one case below, regarding utilizing Clifford algebraic techniques in computational fluid dynamics (CFD), in modeling critical phenomena.

IV. Clifford Algebraic Applications in CFD: An Alternative to Navier-Stokes in the Analysis of Critical Phenomena.

Gerik Scheuermann (2000), as well as Mann & Rockwood (2003) employ Clifford algebras to develop topological vector field visualizations of critical phenomena in fluid mechanics. Visualizations and CFD simulations form a respectable and epistemically robust way of characterizing critical phenomena, down to the nanoscale. (Lehner (2000)) “The goal is not theory-based insight as it is [typically] elaborated in the philosophical literature about scientific explanation. Rather, *the goal is* [for instance] to

³¹ To be precise, so long as the algorithms implementing such multilinear algebraic procedures are ‘polytime,’ i.e. grow in polynomial complexity, over time.

find stable design-rules that might even be sufficient to build a stable nano-device.”

(2000, 99, italics added) Simulations offer potential for intervention, challenging the “received criteria for what may count as adequate quantitative understanding.” (ibid.)

Thus, Lehner’s above remarks appear as a rather strong endorsement for an epistemically fundamental procedure: The heuristics of CFD-based phenomenological approaches lend a quasi-empirical character to this kind of research. CFD techniques can produce robust characterizations of critical phenomena where the traditional, ‘[Navier-Stokes] theory-based insights’ often cannot. Moreover, aside from their explanatory power, CFD visualizations can present more accurate depictions of what occurs at the microlevel, insofar as the numerical and modeling algorithms can support a more detailed depiction of dynamical processes occurring on the microlevel. Hence there appears to be no inherent tension here: Clifford-algebraic CFD procedures are epistemically as well ontologically fundamental.³² Of course, I claim that what guarantees this reconciliation is precisely the underlying *methodologically fundamental* feature of applying Clifford algebras in these instances.

Scheuermann, Mann & Rockwood are primarily motivated by the practical aim of achieving accurately representative (i.e. ontologically fundamental) CFD models of fluid singularities giving equally reliable (i.e. epistemically fundamental) predictions and visualizations covering all sorts of states of affairs. .

³² Which is not to say, of course, that the applications of Clifford algebras in CFD contain no inherent tensions. The trade-off, or tension, however, is of a *practical* nature: that between computational complexity and accurate representation of microlevel details. Lest this appears as though playing into the hands of Batterman’s epistemically versus ontologically ‘fundamental’ distinctions, it is important to keep in mind that the trade-off is one of a practical and contingent issue involving computational resources. Indeed, in the ideal limit of unconstrained computational power and resources, the trade-off disappears: one can model the underlying microlevel phenomena to an arbitrary degree of accuracy. On the other hand, Batterman seems to be arguing that some philosophically important explanatory distinction exists between ontological and epistemic fundamentalism.

For example, Scheuermann (2000) points out that standard topological methods in CFD, using bilinear and piecewise linear interpolation approximating solutions to the Navier-Stokes equation, fail to detect critical points or regions of higher order (i.e. order greater than 1). To spell this out, the following definitions are needed:

Defn IV.1 (Vector Field). A 2D or 3D *vector field* is a continuous function $V: M \rightarrow \mathbf{R}^n$ where M is a manifold³³ $M \subseteq \mathbf{R}^n$, where $n = 2$ or 3 (for the 2D and 3D cases, respectively) and $\mathbf{R}^n = \mathbf{R} \times (\mathbf{R} \text{ } n \text{ times})$. $\times \mathbf{R} = \{(x_1, \dots, x_n) \mid x_k \in \mathbf{R}, 1 \leq k \leq n\}$, i.e. n -dimensional Euclidean space (where $n = 2$ or 3).³⁴

Defn IV.2 (Critical points/region). A critical point³⁵ $x_c \in M \subseteq \mathbf{R}^n$ or region $U \subseteq M \subseteq \mathbf{R}^n$ for the vector field V is one in which $\|V(x_c)\| = 0$ or $\|V(x)\| = 0 \quad \forall x \in U$, respectively.³⁶

A higher-order critical point (or family of points) may signal, for instance, the presence of a saddle point (or saddle curve) in the case of the vector field being a gradient field of a scalar potential $\Phi(x)$ in $\mathbf{R}^{2(\text{or } 3)}$, i.e. $V(x) = \nabla \Phi(x)$. “Higher-order critical points cannot exist in piecewise linear or bilinear interpolations. This thesis presents an algorithm based on a new theoretical relation between analytical field description in Clifford Algebra and topology.” (Scheuermann (2000), 1)

The essence of Scheuermann’s approach, of which he works out in detail examples in \mathbf{R}^2 and its associated Clifford Algebra $CL(\mathbf{R}^2)$ of maximal grade $N = \dim \mathbf{R}^2 = 2$ consisting

³³ A manifold (2D or 3D) is a Hausdorff (i.e. simply connected) space in which each neighborhood of each one of its points is homeomorphic (topologically continuous) with a region in the plane \mathbf{R}^2 or space \mathbf{R}^3 , respectively. For more information concerning topological spaces, see Table A.1.1, Appendix A.1.

³⁴ I retain the characterization above to indicate that higher-dimensional generalizations are applicable. In fact, one of the chief advantages of the Clifford algebraic formulations include their automatic applicability and generalization to higher-dimensional spaces. This is in contrast to the notions prevalent in vector algebra, in which some notions, like the case of the cross-product, are only definable for spaces of maximum dimension 3. See A.2 for further details.

³⁵ For simplicity, as long as no ambiguity appears, in point x in an n -dimensional manifold is depicted in the same manner as that of a scalar quantity x . However, it’s important to keep in mind that x in the former case refers to an n -dimensional position vector.

³⁶ Note: $\| \cdot \|$ is simply the Euclidean norm. In the case of a 2D vector field, for example, $\|V(x,y)\| = \|u(x,y)\mathbf{i} + v(x,y)\mathbf{j}\| = [u^2(x,y) + v^2(x,y)]^{1/2}$, where u and v are x and y are the x, y components of V , described as continuous functions, and \mathbf{i}, \mathbf{j} are orthonormal vectors parallel to the x and y axis, respectively.

of $2^2 = 4$ fundamental generators,³⁷ involves constructing in $CL(\mathbf{R}^2)$ a coordinate-independent differential operator $\partial: \mathbf{R}^2 \rightarrow CL(\mathbf{R}^2)$. Here: $\partial V(x) = \sum_{k=1}^2 g^k \frac{\partial V(x)}{\partial g^k}$, where g_k are the grade-1 generators, or two (non-zero, non-collinear) vectors which hence span \mathbf{R}^2 , and $\frac{\partial V}{\partial g^k}$ are the directional derivatives of V with respect to g^k . For example, if g^1, g^2 are orthonormal vectors (\hat{e}_1, \hat{e}_2) , then: $\partial V = (\nabla \bullet V)I + (\nabla \wedge V)i$, where I , and i are the respective identity and unit pseudoscalars of $CL(\mathbf{R}^2)$.³⁸ For example, in the matrix algebra $M_2(\mathbf{R})$, i.e. the algebra of real-valued 2x2 matrices:

$$I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad i = \hat{e}_1 \hat{e}_2 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Armed with this analytical notion of a coordinate-free differential operator, as well as adopting conformal mappings from \mathbf{R}^2 into the space of Complex numbers (which latter form a grade-1 Clifford algebra) Scheuermann develops a topological algorithm obtaining estimates for higher-order critical points as well as determining more efficient routines:

We can simplify the structure of the vector field and simplify the analysis by the scientist and engineer...some topological features may be missed by a piecewise linear interpolation [i.e., in the standard approach]. This problem is successfully attacked by using locally higher-order polynomial approximations [of the vector field, using conformal maps]...[which] are based on the possible local topological structure of the vector field and the results of analyzing plane vector fields by Clifford algebra and analysis. (ibid (2000), 7)

Mann and Rockwood (2003) show how adopting Clifford algebras greatly simplifies the procedure for calculating the index (or order) of critical points or curves in a 2D or

³⁷ For details concerning these features of Clifford algebras, see Defn A.2.5 and the brief ensuing discussions in A.2

³⁸ compare this expression with the Clifford product in Defn A.2.5, A.2

3D vector field. Normally (without Clifford algebra) the index is presented in terms of an unwieldy integral formula involving the necessity of evaluating normal curvature around a closed contour, as well the differential of an even messier term, known as the Gauss map, which acts as the measure of integration. In short, even obtaining a rough numerical estimate for the index using standard vector calculus and differential geometry is a computationally costly procedure.

On the other hand, the index formula takes on a far more elegant form when characterized in a Clifford algebra:

$$ind(x_c) = \frac{C}{I} \int_{B(x_c)} \frac{V \wedge dV}{\|V\|^n} \quad (IV.1)$$

- where: $n = \dim \mathbf{R}^n$ (where $n = 2$ or 3)
- x_c is a critical point, or point in a critical region
- C is a normalization constant
- I is the unit pseudoscalar of $CL(\mathbf{R}^n)$
- \wedge is exterior (Grassmann) product³⁹

The authors present various relatively straightforward algorithms for calculating the index of critical points using (IV.1) above. “[W]e found the use of Clifford algebra to be a straightforward blueprint in coding the algorithm...the...computations of Geometric [Clifford] algebra automatically handle some of the geometric details...simplifying the programming job.” (ibid., 6)

The most significant geometric details here of course involve *critical surfaces* arising in droplet-formation, which produce singularities in the standard Navier-Stokes continuum-based theory. Though Mann and Rockwood (2003) do not handle the problem of modeling droplet-formation using Clifford-algebraic CFD per se, they do present an algorithm for the computation of surface singularities:

³⁹ For definitions and brief discussions of these terms, see DefnA.2.5, A.2

To compute a surface singularity, we essentially use the same idea as for computing curve singularities...though the test for whether a surface singularity passes through the edge [of an idealized test cube used as the basis of ‘octree’ iterative algorithm, i.e. the 3D equivalent of a dichotomization procedure using squares that tile a plane] is simpler than in the case of curve singularities. No outer products are needed—if the projected vectors along an edge [of the cube] change orientation/sign, then there is a [surface] singularity in the projected vector field. (ibid., 4)

Shortcomings, however, include the procedure’s inability to determine the index for curve and surface singularities. “Our approach here should be considered a first attempt...in finding curve and surface singularities...[our] heuristics are simple, and more work remains to improve them.” (7)

Nevertheless, what is of interest here is the means by which a Clifford algebraic CFD algorithm can *determine the existence* of curve and surface singularities, and track their location in \mathbf{R}^3 given a vector field $V: M \rightarrow \mathbf{R}^3$. The authors demonstrate their results using various constructed examples. Based on the fact that every element in a Clifford algebra is invertible,⁴⁰ the authors ran cases such as determining the line singularities for vector fields such as:

$$V(x, y, z) = (uw^{-1})\mathbf{u} + z\hat{e}_3 \quad (\text{IV.2})$$

where:

$$u(x, y) = x\hat{e}_1 + y\hat{e}_2$$

$$w(x, y) = \sqrt{x^2 + y^2}\hat{e}_1$$

and $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ are the unit orthonormal vectors spanning \mathbf{R}^3

An example like this would prove impossible to construct using standard vector calculus on manifolds, since the ‘inverse’ or quotient operation is undefined in the case of ordinary vectors. Hence the rich geometric and algebraic structure of Clifford algebras admits constructions and cases for fields that would prove inadmissible using standard

⁴⁰ See A.2, in the discussion following Defn A.2.5, for further details.

approaches. The algorithm works also for *sampled* vector fields. “Regardless of the interpolation method, our method would find the singularities within the interpolated sampled field.” (ibid., 5)

The Clifford algebraic CFD algorithms developed by the authors yield some of the following results:

1. A means for determining higher-order singularities, otherwise off-limits in standard CFD topology.
2. A means for locating surface and curve singularities for computed as well as sampled vector fields. Moreover, in the former case, the invertability of Clifford elements produces constructions of vector fields subject to analyses that would otherwise prove inadmissible in standard vector field based formalisms.
3. A far more elegant and computationally efficient means for calculating the indices of singularities.

Clifford algebraic CFD procedures that would refine Mann and Rockwood’s algorithms described in 2., by determining for instance the indices of surface singularities, as well as being computationally more efficient, are precisely the cases I argue which will serve as effective responses against Batterman’s claims. For there would exist formalisms rivaling, in their expressive power, the standard Navier-Stokes approach. But such CFD research would relies exclusively on finite-dimensional Clifford algebraic techniques, and would not appeal to the asymptotic singularities in the standard Navier-Stokes formulation in any meaningful way. Certainly the “first attempt” by Mann and Rockwood in characterizing surface singularities is an impressive one, in what appears to be the onset of a very promising and compelling research program.

I have furthermore argued in this section that such Clifford algebraic CFD algorithms are both epistemically and ontologically fundamental. It remains to show how

these CFD algorithms are, in principle, methodologically fundamental. I sketch this in the conclusion.

V. Conclusion

To show how Clifford algebraic CFD algorithms in principle conform to a methodologically fundamental procedure, as defined in described in III in this essay, recall the (Category theoretic) commutative diagram (Fig. III.a.1):

$$\begin{array}{ccc}
 & \text{ENDO} & \\
 X & \longrightarrow & A = \text{ENDO}(X) \\
 \downarrow \text{SQ} & & \downarrow \text{SQ} \\
 S & \xrightarrow{\text{ENDO}} & CL
 \end{array}$$

Now, let X be the mode space of the eigenvectors of one particular fluid molecule. Then, the SQ functor acts on X to produce S : the statistical composite of the fluid's molecules. The ENDO functor acts on X to produce A : the algebra of endomorphism (operators) on the mode space of which represent intervention/transformations of the observables of the molecule's observables.

Acting on X either first with SQ and then with ENDO, or vice versa, will produce CL: the Clifford algebra representing the global dynamics of the fluid's molecules for some experimental region. Though the grade N of this algebra is obviously vast, N is *still finite*. Hence a Clifford algebraic characterization of fluid dynamics is, in principle, methodologically fundamental, for the same formal reasons as exhibited in the case of

deriving the space-time manifold limit of fundamental quantum processes, characterized by Clifford algebras and Clifford statistics. (Finkelstein (2001, 2004a-c)).

Robert Batterman is quite correct. Nature abhors singularities. So should we. The above procedure denoted as ‘methodological fundamentalism’ shows us how singularities, at least in principle, may be avoided. We need not accept some divergence between explanation and reduction (Batterman 2002), or between epistemological and ontological fundamentalism (Batterman 2004).

Appendix: A Brief Synopsis of the Relevant Algebraic Structures

A.1: Category Algebra and Category Theory

As authors like Hestenes (1984, 1986), Snygg (1997), Lasenby, et. al. (2000) promote Clifford Algebra as a unified mathematical language for physics, so Adamek (1990), Mikhalev & Pilz (2000) and many others similarly claim that Category Theory likewise forms a unifying basis for all branches of mathematics. There are also mathematical physicists like Robert Geroch (1985) who seem to bridge these two presumably unifying languages, by building up a mathematical toolchest comprising most of the salient algebraic and topological structures for the workaday mathematical physicist, from a Category-theoretic basis.

A *category* is defined as follows:

- **Defn. A1.1:** A category $C = \langle \Omega, \text{MOR}(\Omega), \circ \rangle$ is the ordered triple where:
 - a.) Ω is the class of C 's *objects*.
 - b.) $\text{MOR}(\Omega)$ is the set of *morphisms* defined on Ω . Graphically, this can be depicted (where $\varphi \in \text{MOR}(\Omega)$, $A \in \Omega$, $B \in \Omega$): $A \xrightarrow{\varphi} B$
 - c.) The elements of $\text{MOR}(\Omega)$ are connected by the *product* \circ which obeys the law of composition: For $A \in \Omega$, $B \in \Omega$, $C \in \Omega$: if φ is the morphism from A to B , and if ψ is a morphism from B to C , then $\psi \circ \varphi$ is a morphism from A to C , denoted graphically: $A \xrightarrow{\varphi} B \xrightarrow{\psi} C = A \xrightarrow{\psi \circ \varphi} C$. Furthermore:
 - c.1) \circ is *associative*: For any morphisms ϕ , φ , ψ with product defined in as in c.) above, then: $(\psi \circ \phi) \circ \varphi = \psi \circ (\phi \circ \varphi) \equiv \psi \circ \phi \circ \varphi$.

c.2) Every morphism is equipped with a left and a right identity. That is, if ψ is any morphism from A to B , (where A and B are any two objects) then there exists the (right) *identity* morphism on A (denoted ι_A) such that: $\psi \circ \iota_A = \psi$. Furthermore, for any object C , if φ is any morphism from C to A , then there exists the (left) *identity* morphism on A (ι_A) such that: $\iota_A \circ \varphi = \varphi$. Graphically, the left (or right) identity morphisms can be depicted as *loops*.

A simpler way to define a category is in terms of a special kind of a *semigroup* (i.e. a set S closed under an associative product). Since identities are defined for every object, one can in principle identify each object with its associated (left/right) identity. That is to say, for any morphism φ from A to B , with associated left/right identities ι_B, ι_A , identify: $\iota_B = \lambda, \iota_A = \rho$. Hence condition c2) above can be re-stated as c2'): "For every φ there exist (λ, ρ) such that: $\lambda \circ \varphi = \varphi$, and $\varphi \circ \rho = \varphi$." With this apparent identification, DefnI.1 is coextensive with that of a "semigroup with enough identities.

Category theory provides a unique insight into the general nature, or universal features of the construction process that practically all mathematical systems share, in one way or another. Set theory can be embedded into category theory, but not vice versa. Such basic universal features involved in the construction of mathematical systems, which category theory generalizes and systematizes, include, at base, the following:

Feature	Underlying Notion
Objects	The collection of primitive, or stipulated, entities of the mathematical system.
Product	How to 'concatenate and combine,' in a natural manner, to form new objects or entities in the mathematical system respecting the properties of what are characterized by the system's stipulated objects.
Morphsim	How to 'morph' from one object to another.
Isomorphism (structural equivalence)	How all such objects, relative to the system, are understood to be equivalent.

Table A.1.1

For an informal demonstration of how such general aspects are abstracted from three different mathematical systems (sets, groups, and topological spaces⁴¹), for instance, see Table A.1.2 below.

⁴¹ Such systems, of course, are not conceptually disjunct: topological spaces and groups are of course defined in terms of sets. The additional element of structure comprising the concept of group includes the notion of a binary operation (which itself can be defined set-theoretically in terms of a *mapping*) sharing the algebraic property of associativity. The structural element distinguishing a topological space is also

I.a) Set	(by Principle of Extension) $S_\Phi = \{x \mid \Phi(x)\}$ for some property Φ
I.b) Cartesian Product	For any two sets $X, Y : X \times Y = \{(x,y) \mid x \in X, y \in Y\}$
I.c) Mapping	For any two sets X, Y , where $f \subseteq X \times Y$, f is a <i>mapping</i> from X to Y (denoted $f : X \rightarrow Y$) iff for $x_1 \in X, y_1 \in Y, y_2 \in Y$, if $(x_1, y_1) \in f$ (denoted: $y_1 = f(x_1)$) $(x_1, y_2) \in f$ then: $y_1 = y_2$.
I.d) Bijection (set equivalence)	For any two sets X, Y , where $f : X \rightarrow Y$ is a mapping, then f is a bijection iff: a) f is onto (surjective), i.e. $f(X) = Y$ (i.e., for any $y \in Y$ there exists a $x \in X$ such that: $f(x) = y$, b) f is 1-1 (injective) iff for $x_1 \in X, y_1 \in Y, y_2 \in Y$, if $(x_1, y_1) \in f$ (denoted: $y_1 = f(x_1)$) $(x_1, y_2) \in f$ then: $y_1 = y_2$.
II.a) Group	I.e., a group $\langle G, \circ \rangle$ is a set G with a binary operation \circ on G such that: a.) \circ is closed with respect to G , i.e.: $\forall(x, y) \in G : (x \circ y) \equiv z \in G$ (i.e., \circ is a <i>mapping into</i> G or $\circ : G \times G \rightarrow G$, or $\circ(G \times G) \subseteq G$). b.) \circ is <i>associative</i> with respect to G ; $\forall(x, y, z) \in G : (x \circ y) \circ z \equiv x \circ (y \circ z) \equiv x \circ y \circ z$, c.) There (uniquely) exists a (left/right) identity element $e \in G : \forall(x \in G) \exists!(e \in G) : x \circ e = x = e \circ x$. d.) For every x there exists an <i>inverse element</i> of x , i.e.: $\forall(x \in G) \exists(x' \in G) : x \circ x' = e = x' \circ x$.
II.b) Direct product	For any two groups G, H , their <i>direct product</i> (denoted $G \otimes H$) is a group, with underlying set is $G \times H$ and whose binary operation $*$ is defined as, for any $(g_1, h_1) \in G \times H, (g_2, h_2) \in G \times H$: $(g_1, h_1) * (g_2, h_2) = ((g_1 \circ h_1), (g_2 \bullet h_2))$, where \circ, \bullet are the respective binary operations for G , and H .
II.c) Group homomorphism	Any <i>structure-preserving mapping</i> ϕ from two groups G and H . I.e. $\phi : G \rightarrow H$ is a homomorphism iff for any $g_1 \in G, g_2 \in G : \phi(g_1 \circ g_2) = \phi(g_1) \bullet \phi(g_2)$ where \circ, \bullet are the respective binary operations for G , and H .
II.d) Group Isomorphism (group equivalence)	Any <i>structure-preserving bijection</i> ψ from two groups G and H . I.e. $\psi : G \rightarrow H$ is an isomorphism iff for any $g_1 \in G, g_2 \in G : \psi(g_1 \circ g_2) = \psi(g_1) \bullet \psi(g_2)$ (where \circ, \bullet are the respective binary operations for G , and H) and ψ is a <i>bijection</i> (see I.d above) between group-elements G and H . Two groups are <i>isomorphic</i> (algebraically equivalent, denoted: $G \cong H$) iff there exists an isomorphism connecting them $\psi : G \rightarrow H$.)
III.a) Topological Space	Any set X endowed with a collection τ_X of its subsets (i.e. $\tau_X \subseteq \wp(X)$, where $\wp(X)$ is X 's power-set, such that: 1) $\emptyset \in \tau_X, X \in \tau_X$ 2) For any $U, U' \in \tau_X$, then: $U \cap U' \in \tau_X$. 3) For any index (discrete or continuous) γ belonging to index-set Γ : if $U_\gamma \in \tau_X$, then: $\bigcup_{\gamma \in \Delta \subseteq \Gamma} U_\gamma \in \tau_X$. X is then denoted as a <i>topological space</i> , and τ_X is its <i>topology</i> . Elements U belonging to τ_X are denoted as <i>open</i>

described, set-theoretically by use of notions of ‘open’ sets. Moreover, groups and topological spaces can conceptually overlap as well, in the notion of a *topological group*. So in an obvious sense, set theory remains a general classification language for mathematical systems as well. However, the *expressive power* of set theory pales in comparison to that of category theory. To put it another way, if category theory and set theory are conceived of as deductive systems (Lewis), it could be argued that category theory exhibits a better combination of “strength and simplicity” than does naïve set theory. Admittedly, however, this is not a point which can be easily resolved, as far as the simplicity issue goes, since the very concept of a category is usually cashed out in terms three fundamental notions (objects, morphisms, associative composition), whereas, at least in the case of ‘naïve’ set theory (NST), we have fundamentally the two notions: a) of membership \in defined by extension, and b) the hierarchy of *types* (i.e., for any set $X, X \subseteq X$, but $X \notin X$. Or to put more generally, $Z \in W$ is a meaningful expression, though it may be false, provided, for any set, $X: Z \in \wp^{(k)}(X)$ and $W \in \wp^{(k+1)}(X)$, where k is any non-negative integer, and $\wp^{(k)}(X)$ defines the k th-level power-set operation, i.e.: $\wp^{(m)}(X) = \wp(\wp(\dots k \text{ times} \dots(X) \dots))$.)

	<p>sets. Hence 1), 2), 3) say that the empty set and all of X are always open, and finite intersections of open sets are open, while arbitrary unions of open sets are always open. Moreover: 1) Any collection of subsets \mathfrak{S} of X is a <i>basis</i> for X's topology <i>iff</i> for any $U \in \tau_X$, then for any index (discrete or continuous) γ belonging to index-set Γ: if $B_\gamma \in \mathfrak{S}$, then: $\bigcup_{\gamma \in \Delta \subseteq \Gamma} B_\gamma = U \in \tau_X$ (i.e., arbitrary unions of basis elements are open sets.) 2) Any collection of subsets \mathfrak{L} of X is a <i>subbasis</i> if for any $\{S_1, \dots, S_N\} \subseteq \mathfrak{L}$, then $\bigcap_{k=1}^N S_k = B \in \mathfrak{S}$ (I.e. finite intersections of sub-basis elements are basis elements for X's topology.)</p>
III.b) Topological product	<p>For any two topological spaces X, Y, their <i>topological product</i> (denoted $\tau_X \otimes \tau_Y$) is defined by taking, <i>as a sub-basis</i>, the collection: $\{(U, V) \mid U \in \tau_X, V \in \tau_Y\}$. I.e., $\tau_X \times \tau_Y$ is a subbasis for $\tau_X \otimes \tau_Y$. This is immediately apparent since, for U_1 and U_2 open in X, and V_1 and V_2 open in Y: since: $U_1 \times U_2 \cap V_1 \times V_2 = (U_1 \cap V_1) \times (U_2 \cap V_2)$ this indeed forms a basis.</p>
III.c) Continuous mapping	<p>Any mapping from two topological spaces X and Y, preserving openness. I.e. $f: X \rightarrow Y$ is continuous <i>iff</i> for any $U \in \tau_X: f(U) = V \in \tau_Y$</p>
III.d) Homeomorphism (topological space equivalence)	<p>Any <i>continuous bijection</i> h from two topological spaces X and Y. I.e. $h: X \rightarrow Y$ is a homeomorphism <i>iff</i>: a) h is continuous (see III.c), b) h is a bijection (See I.d). Two spaces X and Y are <i>topologically equivalent</i> (i.e., homeomorphic, denoted: $X \cong Y$) <i>iff</i> there exists a homeomorphism connecting them, i.e. $h: X \rightarrow Y$</p>

Table A.1.2

Now the classes of mathematical objects exhibited in Table A.1.2 comprising sets, groups, and topological spaces, all exhibit certain common features:

- The concept of *product* (I.b, II.b, III.b) (or concatenating, in ‘natural manner’ property-preserving structures.) For instance, the Cartesian (I.b) product preserves the ‘set-ness’ property for chains of objects formed from the class of sets, the direct product (II.b) preserves the ‘group-ness’ property under concatenation, etc.
- The concept of ‘morphing’ (I.c, II.c, III.c) from one class of objects to another, in a property-preserving manner. For instance, the continuous map (III.c) respects what makes spaces X and Y ‘topological,’ when morphing from one to another. The homomorphism respects the group properties shared by G and H , when ‘morphing’ from one to another, etc.
- The concept of ‘equivalence in form’ (isomorphism) (I.d, II.d, III.d) defined via conditions placed on ‘how’ one should ‘morph,’ which fundamentally should be in an *invertible* manner. One universally necessary condition for this to hold, is that such a manner is modeled as a bijection. The other necessary conditions of course involve the particular property structure-respecting conditions placed on such morphisms.

Similar to naïve set theory (NST) Category theory also preserves its form and structure on any level or category ‘type.’ That is to say, any two (or more) categories $C,$

\mathcal{D} can be part of the set of structured objects of a *meta-category* \mathbf{X} whose morphisms (functors) respect the categorical structure of its arguments \mathcal{C}, \mathcal{D} . That is to say:

- **Defn A1.2.** Given two categories $\mathcal{C} = \langle \Omega, \text{MOR}(\Omega), \circ \rangle, \mathcal{D} = \langle \Omega', \text{MOR}(\Omega'), \bullet \rangle$, a *categorical functor* Φ is a morphism in the *meta-category* \mathbf{X} from objects \mathcal{C} to \mathcal{D} assigning each \mathcal{C} -object (in Ω) a \mathcal{D} -object (in Ω') and each \mathcal{C} -morphism (in $\text{MOR}(\Omega)$) a \mathcal{D} -morphism (in $\text{MOR}(\Omega')$) such that:
 - Φ preserves the ‘product’ (compositional) structure of the two categories, i.e., for any $\varphi \in \text{MOR}(\Omega), \psi \in \text{MOR}(\Omega): \Phi(\varphi \circ \psi) = \Phi(\varphi) \bullet \Phi(\psi) \equiv \varphi' \bullet \psi'$ (where φ', ψ' are the **Φ -images** in \mathcal{D} of the functors φ, ψ in \mathcal{C}).
 - Φ preserves identity structure across all categories. That is to say, for any $A \in \Omega, \iota_A \in \text{MOR}(\Omega), \Phi(\iota_A) = \iota_{\Phi(A)} = \iota_{A'}$ where A' is the \mathcal{D} -object (in Ω') assigned by Φ . (I.e., $A' = \Phi(A)$)

Examples of functors include the ‘forgetful functor’ $\mathbf{FOR}: \mathcal{C} \rightarrow \text{SET}$ (where SET is the category of all sets) which has the effect of ‘stripping off’ any extra structure in a mathematical system \mathcal{C} down to its ‘bare-bones’ set-structure only. That is to say, for any \mathcal{C} -object $A \in \Omega, \mathbf{FOR}(A) = S_A$ (where S_A is A ’s underlying set), and for any $\psi \in \text{MOR}(\Omega): \mathbf{FOR}(\psi) = f$ is just the mapping (or functional) property of ψ . Robert Geroch (1985, p. 132, p. 248), for example, builds up the toolchest of the most important mathematical structures applied in physics, via a combination of (partially forgetful⁴²) and (free construction functors.) Part of this toolchest, for example, is suggested in the diagram below. The boxed items represent the categories (of sets, groups, Abelian or commutative groups, etc.), the solid arrows are the (partially) forgetful functors, and the dashed arrows represent the free construction functors.

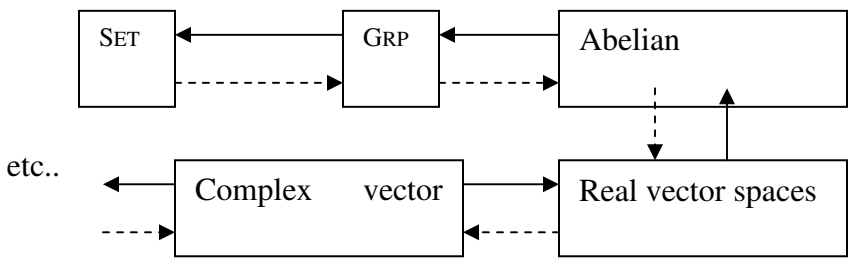


Figure A1.1

⁴² ‘Partially forgetful’ in the sense that the action of such functors does not collapse the structure entirely back to its set-base, just to the ‘nearest’ (simpler) structure.

A.2 Clifford Algebras and Other Algebraic Structures

I proceed here by simply defining the necessary algebraic structures in an increasing hierarchy of complexity:

Defn A2.1: (Group) A group $\langle G, \circ \rangle$ is a set G with a binary operation \circ on G such that:

- a.) \circ is *closed* with respect to G , i.e.: $\forall(x, y) \in G : (x \circ y) \equiv z \in G$ (i.e., \circ is a *mapping into* G or $\circ : G \times G \rightarrow G$, or $\circ(G \times G) \subseteq G$).
- b.) \circ is *associative* with respect to G : $\forall(x, y, z) \in G : (x \circ y) \circ z = x \circ (y \circ z) \equiv x \circ y \circ z$,
- c.) There (uniquely) exists a (left/right) identity element $e \in G : \forall(x \in G) \exists! (e \in G) : x \circ e = x = e \circ x$.
- d.) For every x there exists an *inverse element* of x , i.e.: $\forall(x \in G) \exists(x' \in G) : x \circ x' = e = x' \circ x$.

In terms of categories, Defn A2.1 is coextensive with that of a monoid endowed with property A.2.1.d.). A monoid is a category in which all of its left and right identities coincide to one unique element. For example, the integers Z form a monoid under integer multiplication (since, $\forall n \in Z \exists! 1 \in Z$ such that $n \cdot 1 = n = 1 \cdot n$), but not a group, since their multiplicative inverse can violate closure. Whereas, the non-zero rational numbers $Q^* = \{n/m \mid n \neq 0, m \neq 0\}$ form an Abelian (i.e. commutative) group under multiplication.

Defn A2.2: (Subgroups, Normal Subgroups, Simple Groups)

- i.) Let $\langle G, \circ \rangle$ be a group. Then, for any $H \subseteq G$, H is a *subgroup* of G (denoted: $H \trianglelefteq G$) if for any $x, y \in H$, then $x \circ y' \in H$. In other words, H is closed under \circ , $e \in H$, and if $x \in H$ then $x' \in H$. If $H \trianglelefteq G$, and $H \subset G$, then H is a *proper subgroup*, denoted: $H \triangleleft G$. Moreover, if denoted: $\emptyset \subset H$, then H is *non-trivial*.
- ii.) H is a *normal* (or invariant) subgroup of G (denoted: $H \triangleleft G$) if its left and right cosets agree, for any $g \in G$. That is to say, $H \triangleleft G$ iff $\forall g \in G : gH = \{gh \mid h \in H\} = Hg = \{kg \mid k \in H\}$.
- iii.) G is *simple* if G contains no proper, non-trivial, normal subgroups.

Defn A2.3: (Vector Space) A *vector space* is to a structure $\langle V, F, *, \cdot \rangle$ endowed with a (commutative) operation (i.e. $\forall(x, y) \in V : x * y = y * x$, denoted, by convention, by the “+” symbol, though not necessarily to be understood as addition on the real numbers) such that:

- i) $\langle V, * \rangle$ is a commutative (or Abelian) group.

- ii) Given a field⁴³ of scalars F the *scalar multiplication* mapping into $V \cdot : F \times V \rightarrow V$ obeys distributivity (in the following two senses):
- iii) $\forall (\alpha, \beta) \in F \quad \forall \varphi \in V : (\alpha + \beta) \cdot \varphi = (\alpha \cdot \varphi) + (\beta \cdot \varphi)$
- iv) $\forall (\varphi, \phi) \in V \quad \forall \gamma \in F : \gamma \cdot (\varphi + \phi) = (\gamma \cdot \varphi) + (\gamma \cdot \phi)$.

Defn A2.4: (Algebra) An *algebra* A , then, is defined as a *vector space* $\langle V, F, *, \cdot, \bullet \rangle$ endowed with an associative binary mapping \bullet into A (i.e., $\bullet : A \times A \rightarrow A$, such that $\forall (\psi, \varphi, \phi) \in G : (\psi \bullet \varphi) \bullet \phi = \psi \bullet (\varphi \bullet \phi) \equiv \psi \bullet \varphi \bullet \phi$ denoted, by convention, by the “ \times ” symbol, though not necessarily to be understood as ordinary multiplication on the real numbers) This can be re-stated by saying that $\langle A, \bullet \rangle$ forms a *semigroup* (i.e. a set A closed under the binary associative product \bullet), while $\langle A, * \rangle$ forms an Abelian group.

Examples of algebras include the class of *Lie algebras*, i.e. an *algebra* dA whose ‘product’ \bullet is defined by an (associative) Lie product (denoted $[,]$) obeying the Jacobi Identity: $\forall (\zeta, \xi, \zeta) \in dA : [[\zeta, \xi], \zeta] + [[\xi, \zeta], \zeta] + [[\zeta, \zeta], \xi] = 0$. The structure of classes of infinitesimal generators in many applications often form a Lie algebra. Lie algebras, in addition, are often characterized by the behavior of their *structure constants* C . For any elements of a Lie algebra ζ_μ, ξ_ν characterized by their covariant (or contravariant –if placed above) indices (μ, ν) , then a *structure constant* is the indicial function $C(\lambda)^\sigma_{\mu\nu}$ such that, for any $\zeta_\rho \in dA : [\zeta_\mu, \xi_\nu] = \sum_{\sigma=1}^N C^\sigma_{\mu\nu}(\lambda) \zeta_\sigma$, where N is the dimension of dA , and

λ is the Lie Algebra’s *contraction parameter*. A Lie algebra is *stable* whenever:

$\lim_{\lambda \rightarrow \infty, \nu \lambda \rightarrow 0} C(\lambda)^\sigma_{\mu\nu}$ is well-defined for any structure constant $C(\lambda)^\sigma_{\mu\nu}$ and contraction parameter λ .

Defn A2.5: (Clifford Algebra) . A *Clifford Algebra* is a *graded algebra* endowed with the (non-commutative) Clifford product. That is to say:

- i.) For any two elements A, B in a Clifford algebra CL , their Clifford product is defined by: $AB = A \bullet B + A \wedge B$, where $A \bullet B$ is their (commutative and associative) *inner* product, and $A \wedge B$ is their anti-commutative, i.e. $A \wedge B = -B \wedge A$, and associative *exterior* (or Grassmann) product. This naturally makes the Clifford product associative: $A(BC) = (AB)C \equiv ABC$. Less obviously, however, for reasons that will be discussed below, is how the *existence of an*

⁴³ I.e. a an algebraic structure $\langle F, +, \times \rangle$ endowed with two binary operations such that $\langle F, + \rangle$ and $\langle F, \times \rangle$ form commutative groups and $+, \times$ are connected by left (and right, because of commutativity) distributivity, i.e., $\forall (\alpha, \beta, \gamma) \in F : \alpha \times (\beta + \gamma) = (\alpha \times \beta) + (\alpha \times \gamma)$.

inverse A^{-1} for every (nonzero) Clifford element A arises from the Clifford product, i.e.: $A^{-1}A = I = AA^{-1}$, where I is the *unit pseudoscalar* of CL .

- ii.) CL is equipped with an adjoint \uparrow and grade operator $\langle \rangle_r$ (where $\langle \rangle_r$ is defined as isolating the r th grade of a Clifford element A) such that, for any Clifford elements A, B : $\langle AB \rangle_r^\uparrow = (-1)^{C(r,2)} \langle B^\uparrow A^\uparrow \rangle_r$ (where: $C(r,2) = \frac{r!}{2!(r-2)!} = \frac{r(r-1)}{2}$.)

Hence a general Clifford element (or multivector) A of Clifford algebra CL of maximal grade $N = \dim V$ (i.e the dimension of the underlying vector space structure of the Clifford algebra) is expressed by the linear combination:

$$A = \alpha^{(0)}A_0 + \alpha^{(1)}A_1 + \alpha^{(2)}A_2 + \dots + \alpha^{(N)}A_N \quad (\text{A.3.1})$$

where: $\{\alpha^{(k)} \mid 1 \leq k \leq N\}$ are the elements of the scalar field (expansion coefficients) while $\{A_k \mid 1 \leq k \leq N\}$ are the *pure* Clifford elements, i.e. $\langle A_k \rangle_l = A_k$ whenever $k = l$, and $\langle A_k \rangle_l = 0$ otherwise, while for a general multivector (A.3.1), $\langle A \rangle_l = \alpha^{(l)}A_l$, for $1 \leq l \leq N$

Hence, the pure Clifford elements live in their associated closed Clifford subspaces $CL_{(k)}$ of grade k , i.e. $CL = CL_{(0)} \oplus CL_{(1)} \oplus \dots \oplus CL_{(N)}$.

Consider the following example: Let $V = \mathbf{R}^3$, i.e. the underlying vector space for CL is a 3 dimensional Euclidean space $\mathbf{R}^3 = \{\vec{r} = (x,y,z) \mid x \in \mathbf{R}, y \in \mathbf{R}, z \in \mathbf{R}\}$. Then the maximum grade for Clifford Algebra over \mathbf{R}^3 , i.e. $CL(\mathbf{R}^3)$ is $N = \dim \mathbf{R}^3 = 3$. Hence:

$CL(\mathbf{R}^3) = CL_{(0)} \oplus CL_{(1)} \oplus CL_{(2)} \oplus CL_{(3)}$ where: $CL_{(0)}$ (the Clifford subspace of grade 0) is (algebraically) isomorphic to the real numbers \mathbf{R} .⁴⁴ $CL_{(1)}$ (the Clifford subspace of grade 1) is algebraically isomorphic to the Complex numbers \mathbf{C} . $CL_{(2)}$ (the Clifford subspace of grade 2) is algebraically isomorphic the Quaternions \mathbf{H} . $CL_{(3)}$ (the Clifford subspace of grade 3) is algebraically isomorphic to the Octonions \mathbf{O} .

To understand *why* the Clifford algebra over \mathbf{R}^3 would invariably involve closed subspaces with elements related to the unit imaginary $i = \sqrt{-1}$ (and some of its derivative notions thereon, in the case of the Quaternions and Octonions) entails a closer study of the nature of the Clifford product. Defn. A.2.4 i) deliberately leaves the Grassman product under-specified. I now fill in the details here. First, it is important to note that \wedge is a *grade-raising* operation: for any pure Clifford element A_k (where $k < N = \dim V$) and

⁴⁴ Since the real numbers are a *field*, they're obviously describable as an algebra, in which their underlying 'vector space' structure is identical to their field of scalars. In other words, scalar multiplication is the same as the 'vector' product \bullet .

B_1 , then $\langle A_k B_1 \rangle = k + 1$. It is for this reason that pure Clifford elements of grade k are often called *multivectors*. Conversely, the inner product \bullet is a *grade-lowering* operation: for any pure Clifford element A_k (where $k < N = \dim V$) and B_1 , then $\langle A_k \bullet B_1 \rangle = k - 1$. (Hence the inner product is often referred to as a *contraction*).

The reason for the grade-raising, anti-commutative nature of the Grassman product is historically attributed to Grassman's geometric notions of (directed) line segments, (rays) areas, volumes, hypervolumes, etc. For example, in the case of two vectors \vec{A}, \vec{B} , their associated directed area segments $\vec{A} \wedge \vec{B}, \vec{B} \wedge \vec{A}$ are illustrated below:



Fig. A.2.1

The notion of directed area, volume, hypervolume segments indeed survives, to a certain limited sense, in the vector-algebraic notion of ‘cross-product.’ For example, the magnitude of the cross-product $\vec{A} \times \vec{B}$ is precisely the area of the parallelogram spanned by \vec{A}, \vec{B} as depicted in Fig. A.2.1. The difference, however, lies in the fixity of grade in the case of $\vec{A} \times \vec{B}$, in the sense that the anti-commutativity is geometrically attributed to the directionality of the *vector* $\vec{A} \times \vec{B}$ (of positive sign in the case of right-handed coordinate system) perpendicular to the plane spanned by \vec{A}, \vec{B} . This limits the notion of the vector cross-product, as it can only be defined for spaces of maximum dimensionality 3.⁴⁵ On the other hand, the Grassmann product of multivectors interpreted as directed areas, volumes, and hypervolumes is unrestricted by the dimensionality of the vector space.

The connection with the algebraic behavior of $i = \sqrt{-1}$ lies in the inherently anti-commutative aspect (i.e. the Grassmann component) of the Clifford product, as discussed above. To see this, consider the even simpler case of $V = \mathbf{R}^2$ (as discussed, for example,

⁴⁵ “[T]he *vector algebra* of Gibbs... was effectively the end of the search for a unifying mathematical language and the beginning of a proliferation of novel algebraic systems, created as and when they were needed; for example, spinor algebra, matrix and tensor algebra, differential forms, etc.” (Lansby, et. al. (2000), 21)

in Lasenby, et. al. (2000), 26-29). Then; $N = \dim \mathbf{R}^2 = 2$. Moreover, $\mathbf{R}^2 = \langle (\hat{e}_1, \hat{e}_2) \rangle$, where $\langle \dots \rangle$ denotes the *span* and (\hat{e}_1, \hat{e}_2) are the ordered pair of orthonormal vectors (parallel, for example, to the x and y axes.) Hence: $\hat{e}_1^2 = \hat{e}_2^2 = 1$, and $\hat{e}_1 \bullet \hat{e}_2 = \hat{e}_2 \bullet \hat{e}_1 = 0$. So: $\hat{e}_1 \hat{e}_2 = \hat{e}_2 \bullet \hat{e}_1 + \hat{e}_1 \wedge \hat{e}_2 = \hat{e}_1 \wedge \hat{e}_2 = -\hat{e}_2 \wedge \hat{e}_1 = -\hat{e}_2 \hat{e}_1$. Hence: $(\hat{e}_1 \hat{e}_2)^2 = (\hat{e}_1 \hat{e}_2)(\hat{e}_1 \hat{e}_2) = \hat{e}_1 (\hat{e}_2 \hat{e}_1) \hat{e}_2 = -\hat{e}_1 (\hat{e}_1 \hat{e}_2) \hat{e}_2 = -(\hat{e}_1 \hat{e}_1)(\hat{e}_2 \hat{e}_2) = -(\hat{e}_1^2)(\hat{e}_2^2) = -1$ (using the anti-commutativity and associativity of the Clifford product.) Hence, the multivector $\hat{e}_1 \hat{e}_2$ is algebraically isomorphic to $i = \sqrt{-1}$. Moreover, $(\hat{e}_1 \hat{e}_2) \hat{e}_1 = -\hat{e}_2$ and $(\hat{e}_1 \hat{e}_2) \hat{e}_2 = \hat{e}_1$, by the same simple algebraic maneuvering. Geometrically, then, the multivector $\hat{e}_1 \hat{e}_2$ when multiplying on the left has the effect of a *clockwise* $\pi/2$ -rotation. Represented then in the matrix algebra $M_2(\mathbf{R})$ (the algebra of real-valued 2x2 matrices):

$$\hat{e}_1 \hat{e}_2 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{where: } \hat{e}_1 \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \hat{e}_2 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Moreover, for $CL(\mathbf{R}^2)$ the multivector $\hat{e}_1 \hat{e}_2$ is the *unit pseudoscalar*, i.e. the element of maximal grade. In general, for any Clifford Algebra $CL(V)$, where $\dim V = N$, and $V = \langle (\gamma_1, \gamma_2, \dots, \gamma_N) \rangle$, where the basis elements aren't necessarily orthonormal, the unit pseudoscalar I of $CL(V)$ is: $I = \gamma_1 \gamma_2 \dots \gamma_N$. In general, for grade k (where $1 \leq k \leq N$) the closed subspaces $CL_{(k)}$ of grade k in $CL(V) = CL_{(0)} \oplus CL_{(1)} \oplus \dots \oplus CL_{(N)}$ have dimensionality $C(N, k) = \frac{N!}{k!(N-k)!}$, i.e. are spanned by $C(N, k) = \frac{N!}{k!(N-k)!}$ multivectors of degree k . Hence the total number of Clifford basis elements generated by the Clifford product acting on the basis elements of the underlying vector space is: $2^N = \sum_{k=0}^N C(N, k)$.

The unit pseudoscalar is therefore the (one) multivector (only one there are $C(N, N) = 1$ of them, modulo sign or order of multiplication) spanning the closed Clifford subspace of maximal grade N .

For example, in the case of $CL(\mathbf{R}^3) = CL_{(0)} \oplus CL_{(1)} \oplus CL_{(2)} \oplus CL_{(3)}$, where:
 $\mathbf{R}^3 = \langle (\hat{e}_1, \hat{e}_2, \hat{e}_3) \rangle$:
 $CL_{(0)} = \langle 1 \rangle \cong \mathbf{R}$, $CL_{(1)} = \langle (e_1, e_2, e_3) \rangle$, $CL_{(2)} = \langle (e_{12}, e_{13}, e_{23}) \rangle$, $CL_{(3)} = \langle I \rangle = \langle e_{123} \rangle$

(where the abbreviation $e_{i\dots k} = \hat{e}_i \dots \hat{e}_k$ is adopted). As demonstrated in the case of $CL(\mathbf{R}^2)$ the multivector, the unit pseudoscalar I should *not* be interpreted as a multiplicative identity, i.e. it is certainly *not* the case that for any $A \in CL(V)$, $AI = A = IA$. Rather, the unit pseudoscalar is adopted to define an element of dual grade A^* : for any pure Clifford element A_k (where $0 \leq k < N$): the grade of AI (or A^*) is $N - k$, and vice versa. Thus an inverse element A^{-1} can in principle be constructed, for every nonzero $A \in CL(V)$. So the linear equation $AX = B$ has the formal solution $X = A^{-1}B$ in $CL(V)$. “Much of the power of geometric (Clifford) algebra lies in this property of invertibility.” (Lasenby, et. al. (2000), 25)

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