COMPUTABILITY OF SELF-SIMILAR SETS

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Abstract. We investigate computability of a self-similar set on a Euclidean space. A nonempty compact subset of a Euclidean space is called a self-similar set if it equals to the union of the images of itself by some set of contractions. The main result in this paper is that if all of the contractions are computable, then the self-similar is a recursive compact set. A further result on the case that the self-similar set forms a curve is also discussed.

1. Introduction

The aim of this paper is to find fundamental mathematical tools to investigate self-similar sets from the viewpoint of computability.

First, we should recall what "self-similarity" means. Mandelbrot called a set constructed from some miniatures of the whole a *self-similar set* [5]. A more precise definition was discovered by Hutchinson [3] and generalized by Hata [2]. For any finitely many contractions $T_0, \ldots, T_{m-1} : \mathbb{R}^q \to \mathbb{R}^q$, the set equation

$$X = T_0(X) \cup \cdots \cup T_{m-1}(X)$$

has a unique nonempty compact solution. A nonempty compact set that is the solution of a set equation of this form is called a self-similar set.

Next, we should clarify what "computability" is. Many important studies have been made on real numbers, real function, subsets of the real line, etc from the viewpoint of computability since Rice discovered the real field of all computable real numbers [8]. In this research field, which is often referred to as *classical computable analysis* or simply *computable analysis*, the following *recursiveness* is often used: a nonempty compact subset $K \subset \mathbb{R}^q$ is called *recursive* if $\underline{d}_K : \mathbb{R}^q \to \mathbb{R}$ defined by $\underline{d}_K(x) = \inf_{y \in K} \|x - y\|$ is computable [12] [13].

A question now arises whether the self-similar set is a recursive compact set if all of the contractions are computable transformations. In this paper, we shall answer this question positively. If all of contractions $T_0, \ldots, T_{m-1} : \mathbb{R}^q \to \mathbb{R}^q$ are computable transformations, then the self-similar set with respect to $T_0, \ldots \quad T_{m-1}$ is a recursive compact subset.

Kawamura and Kamo have already shown that the result above holds in the special case when all of the contrancions are affine translations [4]. In other words, some of the results shown in this paper is partly a generalization of those in [4].

In addition, we shall discuss on a further result. Hata [2] proved that if contractions $T_0, \ldots, T_{m-1} : \mathbb{R}^q \to \mathbb{R}^q$ satisfy an additional condition

$$T_1(\mathrm{Fix}(T_0)) = T_0(\mathrm{Fix}(T_{m-1})), \ldots, \text{ and } \quad T_{m-1}(\mathrm{Fix}(T_0)) = T_{m-2}(\mathrm{Fix}(T_{m-1}))$$

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where $\mathrm{Fix}(T)$ denotes the unique fixed point of T, then the self-similar set is (the image of) a curve. We shall show that, under this condition, if all of $T_0, \ldots, T_{m-1}$ are computable transformations, then the curve can be a computable function.

We believe that these results will be the first step for investigation of self-similar sets from the view point of computability.

2. Preliminary

2.1. Self-similarity.

2.1.1. *Self-similar Set.*

Definition 2.1. A nonempty compact subset X of $\mathbb{R}^q$ is *self-similar* with respect to contractions $T_0, \ldots, T_{m-1} : \mathbb{R}^q \to \mathbb{R}^q$ if

$$X = T_0(X) \cup \cdots \cup T_{m-1}(X).$$

We should notice that T_i's in this definition need not be similarity transformations. They may be arbitrary contractions.

We denote the set of all nonempty compact subsets of $\mathbb{R}^q$ by $K(\mathbb{R}^q)$. A complete metric known as the *Hausdorff metric* d_{H} on $K(\mathbb{R}^q)$ is defined by:

$$d_{\mathrm{H}}(X, Y) = \max\{d'(X, Y), d'(Y, X)\}$$

where

$$d'(X, Y) = \sup_{x \in X} \inf_{y \in Y} \|x - y\|.$$

Theorem 2.1 (Hutchinson)**.** *Let $T_0, \ldots, T_{m-1} : \mathbb{R}^q \to \mathbb{R}^q$ be contractions. Then there exists a unique self-similar set X with respect to $T_0, \ldots$ and T_{m-1}. If $\{X_n\}$ is a sequence of nonempty compact sets on $\mathbb{R}^q$ with*

$$X_{n+1} = T_0(X_n) \cup \cdots \cup T_{m-1}(X_n),$$

then X_n converges to the self-similar set X as $n \to \infty$ in the Hausdorff metric.

Refer to [3] for the proof.

2.1.2. *Self-similar Curve.* In this paper, we call a continuous function from an interval to $\mathbb{R}^q$ a curve on $\mathbb{R}^q$. We do not call the image of the function a curve. To distinguish a curve and the image of it, we say a curve $f : I \to \mathbb{R}^q$ constructs a set $\gamma \in \mathbb{R}^q$ if $f(I) = \gamma$.

Hata investigated self-similar curves [2]. We summarize here some of the results necessary to proceed out task.

Let $T_0, \ldots, T_{m-1} : \mathbb{R} \to \mathbb{R}$ be contractions. If they satisfy an additional condition

$$T_1(\mathrm{Fix}(T_0)) = T_0(\mathrm{Fix}(T_{m-1})), \ldots, T_{m-1}(\mathrm{Fix}(T_0)) = T_{m-2}(\mathrm{Fix}(T_{m-1})). \tag{1}$$

where $\mathrm{Fix}(T)$ denotes the unique fixed point of T, then the self-similar set is constructed from a curve. Namely, if $T_0, \ldots$ and T_{m-1} satisfy (1), there exists a continuous function $f : [0, 1] \to \mathbb{R}^q$ such that $f([0, 1]) = T_0(f([0, 1])) \cup \cdots \cup T_{m-1}(f([0, 1]))$. Such an f is called a *self-similar curve*.

More precisely, from any given contractions $T_0, \ldots, T_{m-1} : \mathbb{R}^q \to \mathbb{R}^q$ that satisfy (1), we can obtain a self-similar curve through the following proposition.

Proposition 2.1. *Let $T_0, \dots, T_{m-1} : \mathbb{R}^q \to \mathbb{R}^q$ be contractions that satisfy*

$$T_1(\mathrm{Fix}(T_0)) = T_0(\mathrm{Fix}(T_{m-1})), \dots, T_{m-1}(\mathrm{Fix}(T_0)) = T_{m-2}(\mathrm{Fix}(T_{m-1})).$$

Define $f_n : [0,1] \to \mathbb{R}^q$ for $n \in \mathbb{N}$ recursively by

$$f_0(t) = (1-t)\,\mathrm{Fix}(T_0) + t\,\mathrm{Fix}(T_{m-1}),$$

$$f_{n+1}(t) = T_k(f_n(mt-k)) \qquad \text{if } t \in [k/m, (k+1)/m] \text{ and } k \in \{0, \dots, m-1\}.$$

Then the sequence $\{f_n\}$ is uniformly convergent. Using f for the limit, we have

$$f([0,1]) = T_0(f([0,1])) \cup \dots \cup T_{m-1}(f([0,1])).$$

2.2. **Computability in Analysis.**

2.2.1. *Computability of Real Functions.* In this subsection, we will briefly recall the theory of computability on real functions. The definitions here are equivalent to those in [7] although they are expressed differently in detail.

The definitions here are also equivalent to those in another formalization of computability that uses an extended Turing machine named Type 2 machine [9] [10] [12] [13]. The equivalence is not used in this paper. It is however important in application of theoretical computability to computation in the real world.

Definition 2.2. A sequence of rational numbers $\{r_k\}$ is *computable* if there exist recursive functions $s, a, b : \mathbb{N} \to \mathbb{N}$ such that

$$r_k = (-1)^{s(k)} \frac{a(k)}{b(k)+1}.$$

Definition 2.3. A double sequence of reals $\{x_{nk}\}$ *converges to* a sequence of reals $\{x_n\}$ as $k \to \infty$ *effectively* in n and k if there exists a recursive function $a : \mathbb{N}^2 \to \mathbb{N}$ such that for any $n, N \in \mathbb{N}$ and any $k \in \mathbb{N}$ with $k \ge a(n,N)$, $\quad |x_{nk} - x_n| < 2^{-N}$.

Definition 2.4. A sequence of reals $\{x_n\}$ is *computable* if there exists a computable double sequence of rational numbers $\{r_{nk}\}$ that converges to $\{x_n\}$ as $k \to \infty$ effectively in n and k.

A real x is called computable if $\{x\}_{k \in \mathbb{N}}$, the sequence all elements of which equal x, is a computable sequence of reals. A sequence of points on a Euclidean space is called computable iff each sequence of its coordinates is computable. A closed rectangle $\prod_{i=1}^q [a_i, b_i]$ is computable iff all of a_i's and b_i's are computable reals.

Definition 2.5. Let $I \subset \mathbb{R}^q$ be a computable closed rectangle. A sequence of functions $\{f_n\}$ with $f_n : I \to \mathbb{R}$ is *computable* if it satisfies the following two conditions.

1. For any computable sequence of points $\{x_k\}$ with $x_k \in I$, the double sequence of reals $\{f_n(x_k)\}$ is computable.
2. There exists a recursive function $a : \mathbb{N}^2 \to \mathbb{N}$ such that for any $n, N \in \mathbb{N}$ and any $x, x' \in I$, if $\|x - x'\| < 2^{-a(n,N)}$, then $|f_n(x) - f_n(x')| < 2^{-N}$ where $\|\text{-}\|$ denotes the Euclidean norm.

Definition 2.6. A sequence of functions $\{f_n\}$ with $f_n : \mathbb{R}^q \to \mathbb{R}$ is *computable* if it satisfies the following two conditions.

1. For any computable sequence of points $\{x_k\}$ on $\mathbb{R}^q$, the double sequence of reals $\{f_n(x_k)\}$ is computable.

2. There exists a recursive function $a : \mathbb{N}^3 \to \mathbb{N}$ such that for any $n, N, M \in \mathbb{N}$ and any $x, x' \in [-M, M]^q$, if $\|x - x'\| < 2^{-a(n,N,M)}$, then $|f_n(x) - f_n(x')| < 2^{-N}$.

We say $\{f_n\}$ is *sequentially computable* if it satisfies the condition 1 and *effectively uniformly continuous* if it satisfies the condition 2.

A function f is called computable if $\{f\}_{k\in\mathbb{N}}$, the sequence all elements of which equal to f, is a computable sequence of functions.

Definition 2.7. Let I be a computable rectangle in $\mathbb{R}^q$. A double sequence of functions $\{f_{nk}\}$ with $f_{nk} : I \to \mathbb{R}$ *converges to* a sequence of functions $\{f_n\}$ with $f_n : I \to \mathbb{R}$ as $k \to \infty$ *uniformly* on I *effectively* in n and k if there exists a recursive function $a : \mathbb{N}^2 \to \mathbb{N}$ such that for any $n, N \in \mathbb{N}$ and any $k \in \mathbb{N}$, if $k \geq a(n, N)$, then for any x in I, $\quad |f_{nk}(x) - f_n(x)| < 2^{-N}$.

We quote the following three facts on computable functions from [7]. Refer to [7] for proofs.

- (Closure under effective convergence) Let $\{x_{nk}\}$ be a computable double sequence of reals and $\{x_n\}$ a sequence of reals such that $x_{nk} \to x_n$ as $k \to \infty$ effectively in n and k. Then $\{x_n\}$ is a computable sequence of reals.
- Let I be a computable rectangle on $\mathbb{R}^q$, $\{f_{nk}\}$ a computable double sequence of functions with $f_{nk} : I \to \mathbb{R}$, and $\{f_n\}$ a sequence of functions with $f_n : I \to \mathbb{R}$ such that $f_{nk}(x) \to f_n(x)$ as $k \to \infty$ effectively in n and uniformly in x. Then $\{f_n\}$ is a computable sequence of functions.
- (Effective version of Maximum Value Theorem) Let I be a computable rectangle in $\mathbb{R}^q$ and $\{f_n\}$ a computable sequence of functions with $f_n : I \to \mathbb{R}$. Then the maximum values $\{\max_{x\in I} f_n(x)\}$ form a computable sequence of real numbers.

2.2.2. *Recursiveness of compact subsets of Euclidean Spaces.* Defining computability of a subset of Euclidean spaces is not a straightforward task. For $A \subset \mathbb{R}^q$, the characteristic function $\chi_A : \mathbb{R}^q \to \mathbb{R}$ defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise} \end{cases}$$

is useless to investigate computability of A since χ_A is computable iff $A = \emptyset$ or $A = \mathbb{R}^q$.

An alternative for the characteristic function is $\underline{d}_A : \mathbb{R}^q \to \mathbb{R}$ defined by

$$\underline{d}_A(x) = \inf_{y\in A} \|x - y\|.$$

Recursiveness defined as follows is often used in computable analysis.

Definition 2.8 (Weihrauch)**.** A nonempty compact subset $K \subset \mathbb{R}^q$ is *recursive* if $\underline{d}_K$ is computable.

We should notice that the definition of a recursive compact subset is position-dependent. In other words, a compact subset may not be recursive even if it is congruent to a recursive compact subset. For example, let $x_0 \in \mathbb{R}^q$ be a non-computable point. The compact subset $\{x_0\}$ is not recursive although it is congruent to a recursive compact subset $\{0\}$.

2.2.3. *Computability of curves.* Defining computability of a curve is a straightforward task. A curve on a Euclidean space is (or is considered to be) a continuous function from an interval to a Euclidean space. Thus we can define computability of a curve by using computability of a function.

Definition 2.9. A *computable curve* is a computable function $f : [0, 1] \to \mathbb{R}^q$.

Restriction of the domain to $[0, 1]$ causes no loss of generality. Let $f : [a, b] \to \mathbb{R}^q$ be a continuous function from an interval $[a, b]$ with computable endpoints. Computability of f is equivalent to that of $f' : [0, 1] \to \mathbb{R}^q$ defined by $f'(t) = f(a + (b - a)t)$.

We should notice that the definition of a computable curve is also position-dependent.

Theorem 2.2. *If $f : [0, 1] \to \mathbb{R}^q$ is a computable function, then $f([0, 1])$ is a recursive compact set.*

Proof. Apply the effective version of Maximum Value Theorem to

$$\underline{d}_{f([0,1])}(x) = \min_{t\in[0,1]} \|x - f(t)\|.$$

□

In other words, computability of a curve as a function is a stronger condition than recursiveness of a curve as a compact set.

3. Recursiveness of a Self-similar Set

In this section, we investigate computability of a self-similar set. Recursiveness of a self-similar set with respect to a set of contractions is of course dependent on computability of the contractions. We will claim that a self-similar set with respect to computable contractions is a recursive compact set.

First, we will show three lemmas.

Lemma 3.1. *Let $T_0, \dots, T_{m-1} : \mathbb{R}^q \to \mathbb{R}^q$ be computable contractions and $a \in \mathbb{R}^q$ a computable point. Define a sequence of nonempty compact sets $\{X_n\}$ on $\mathbb{R}^q$ recursively by*

$$X_0 = \{a\},$$
$$X_{n+1} = T_0(X_n) \cup \dots \cup T_{m-1}(X_n).$$

Then $\{\underline{d}_{X_n}\}$ is a computable sequence of functions.

Proof. For $i \in \{0, \dots, m^n - 1\}$, we abbreviate $T_{i_{n-1}} \circ \dots \circ T_{i_0}$ to T_i^n if $i_0, \dots, i_{n-1} \in \{0, \dots, m - 1\}$ and $i = i_0 m^0 + \dots + i_{n-1} m^{n-1}$.

By induction on n, it is straightforward to show that each X_n is well-defined and each $\underline{d}_{X_n}$ satisfies

$$\underline{d}_{X_n}(x) = \min_{i\in\{0,\dots,m^n-1\}} \|x - T_i^n(a)\|.$$

Clearly, $\{\underline{d}_{X_n}\}$ is a computable sequence of functions. □

Lemma 3.2. *Let $T_0, \dots, T_{m-1} : \mathbb{R}^q \to \mathbb{R}^q$ be computable contractions and $\{X_n\}$ a sequence of nonempty compact sets on $\mathbb{R}^q$ such that*

$$X_{n+1} = T_0(X_n) \cup \dots \cup T_{m-1}(X_n).$$

Then $\{X_n\}$ converges to some nonempty compact set X effectively in n in the Hausdorff metric as $n \to \infty$.

Proof. The existence of X is guaranteed by Theorem 2.1.

Since $T_0, \dots$ and T_{m-1} are contractions, there exists a computable real α with $0 < \alpha < 1$ such that for any $i \in \{0, \dots, m-1\}$ and any $x, y \in \mathbb{R}^q$,

$$\|T_i(x) - T_i(y)\| \leq \alpha \|x - y\|.$$

Analogously to Hutchinson's proof of Theorem 2.1, this implies:

$$d_{\mathrm{H}}(X_n, X) \leq \frac{\alpha^n}{1-\alpha} d_{\mathrm{H}}(X_0, X_1).$$

The right-hand side forms a computable sequence of reals since

$$d_{\mathrm{H}}(X_0, X_1) = \max_{i \in \{0, \dots, m-1\}} \|a - T_i(a)\|.$$

Therefore, $d_{\mathrm{H}}(X_n, X) \to 0$ effectively in n as $n \to \infty$. □

Lemma 3.3. *In the notation of Lemma 3.2, $\underline{d}_{X_n}(x)$ converges to $\underline{d}_X(x)$ uniformly in x and effectively in n as $n \to \infty$.*

Proof. For an arbitrary point x in $\mathbb{R}^q$, we will show that

$$|\underline{d}_{X_n}(x) - \underline{d}_X(x)| \leq d_{\mathrm{H}}(X_n, X).$$

We obtain that for any $x \in \mathbb{R}^q$,

$$\begin{aligned} \underline{d}_X(x) - \underline{d}_{X_n}(x) &= \sup_{y \in X_n} \inf_{z \in X} (\|x - z\| - \|x - y\|) \\ &\leq \sup_{y \in X_n} \inf_{z \in X} \|y - z\|. \end{aligned}$$

By exchanging X_n and X, we also obtain that

$$\underline{d}_{X_n}(x) - \underline{d}_X(x) \leq \sup_{z \in X} \inf_{y \in X_n} \|z - y\|.$$

Thus

$$|\underline{d}_{X_n}(x) - \underline{d}_X(x)| \leq d_{\mathrm{H}}(X_n, X).$$

Now an application of Lemma 3.2 yields that $\underline{d}_{X_n}(x)$ converges to $\underline{d}_X(x)$ uniformly in x and effectively in n as $n \to \infty$. □

As an immediate consequence of Theorem 2.1 and Lemmas 3.1 and 3.3, we obtain the following according to Definition 2.8.

Theorem 3.1. *Let $T_0, \dots, T_{m-1} : \mathbb{R}^q \to \mathbb{R}^q$ be computable contractions. Then the unique nonempty compact set X such that*

$$X = T_0(X) \cup \cdots \cup T_{m-1}(X)$$

is a recursive compact set.

Example 3.1. A Cantor ternary set which is a self-similar set on $\mathbb{R}$ with respect to contractions T_0 and T_1 defined by

$$T_0(x) = \frac{1}{3}x, \qquad T_1(x) = \frac{1}{3}x + \frac{2}{3},$$

is a recursive compact set.

Example 3.2. A Sierpiński gasket which is a self-similar set on $\mathbb{R}^2$ with respect to contractions T_0, T_1 and T_2 defined by

$$T_0(x) = \frac{1}{2}x, \qquad T_1(x) = \frac{1}{2}x + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \qquad T_2(x) = \frac{1}{2}x + \begin{pmatrix} 1/4 \\ \sqrt{3}/4 \end{pmatrix},$$

is a recursive compact set.

4. Computability of a Self-similar Curve

As shown in Proposition 2.1, contractions $T_0, \dots, T_{m-1} : \mathbb{R}^q \to \mathbb{R}^q$ that satisfy

$$T_1(\mathrm{Fix}(T_0)) = T_0(\mathrm{Fix}(T_{m-1})), \dots, T_{m-1}(\mathrm{Fix}(T_0)) = T_{m-2}(\mathrm{Fix}(T_{m-1}))$$

generate a curve $f : [0,1] \to \mathbb{R}^q$. In this case, computability of f is also of our interest.

We have already claimed that computability of f is a stronger condition than recursiveness of $f([0,1])$. We have also obtained that if T_0, …, and T_{m-1} are computable, then $f([0,1])$ is a recursive compact set. The next question is the computability of f.

We will first prove two lemmas.

Lemma 4.1. *Let $T_0, \dots, T_{m-1} : \mathbb{R}^q \to \mathbb{R}^q$ be contractions that satisfy*

$$T_1(\mathrm{Fix}(T_0)) = T_0(\mathrm{Fix}(T_{m-1})), \dots, \text{ and } \quad T_{m-1}(\mathrm{Fix}(T_0)) = T_{m-2}(\mathrm{Fix}(T_{m-1})).$$

Define $f_n : [0,1] \to \mathbb{R}^q$ recursively by

$$f_0(t) = (1-t)\,\mathrm{Fix}(T_0) + t\,\mathrm{Fix}(T_{m-1}),$$
$$f_{n+1}(t) = T_i(f_n(mt - i)) \qquad \text{if } t \in [i/m, (i+1)/m] \text{ with } i \in \{0, \dots, m-1\}.$$

If all of T_i's are computable, then $\{f_n\}$ forms a computable sequence of functions.

Proof. We use the abbreviation T_i^n in Lemma 3.1.

It is straightforward to show, by induction on n, that each f_n is well-defined and satisfies, for any $i \in \{0, \dots, m^n - 1\}$,

$$f_n(t) = T_i^n(f_0(m^n t - i)) \qquad \text{if } t \in [i/m^n, (i+1)/m^n]. \tag{2}$$

Clearly, $\{f_n\}$ is effectively uniformly continuous. We however confront with a difficulty here in showing sequential computability of $\{f_n\}$. Calculation from a real t of the integer i in (2) is not effective. This makes it impossible to choose T_i^n effectively for calculation of $f_n(t)$.

Take any computable sequence $\{t_k\}$ with $t_k \in [0,1]$. There exists a double sequence of rational numbers $\{\tau_{kl}\}$ such that $\tau_{kl} \to t_k$ effectively in k and l as $l \to \infty$. To overcome the difficulty and to show that $\{f_n(t_k)\}$ is a computable double sequence of points, we shall investigate the triple sequence of the points $\{f_n(\tau_{kl})\}$.

We have, for any $i \in \{0, \dots, m^n - 1\}$,

$$f_n(\tau_{kl}) = T_i^n(f_0(m^n \tau_{kl} - i)) \qquad \text{if } \tau_{kl} \in [i/m^n, (i+1)/m^n]. \tag{3}$$

In this case, calculation of i from n, k and l in (3) *is* effective since the relation $\le$ in $\mathbb{Q}$ is effective. More precisely, we can construct a recursive function that computes i from n, k and l by using a recursive function that corresponds to the relation $\le$ in $\mathbb{Q}$. Hence $\{f_n(\tau_{kl})\}$ is a computable triple sequence of points.

Now we are ready to show computability of $\{f_n(t_k)\}$. Since $\tau_{kl} \to t_k$ effectively in k and l as $l \to \infty$ and $\{f_n\}$ is effectively uniformly continuous, we obtain that

$f_n(\tau_{kl}) \to f_n(t_k)$ effectively in n, k and l as $l \to \infty$. We conclude that $\{f_n(t_k)\}$ is a computable double sequence of points since it is a list of a computable and effectively convergent triple sequence of points. □

Lemma 4.2. *Let $T_0, \ldots, T_{m-1} : \mathbb{R}^q \to \mathbb{R}^q$ be contractions that satisfy*

$$T_1(\mathrm{Fix}(T_0)) = T_0(\mathrm{Fix}(T_{m-1})), \ldots, \text{ and } \quad T_{m-1}(\mathrm{Fix}(T_0)) = T_{m-2}(\mathrm{Fix}(T_{m-1})),$$

and $\{f_n\}$ curves with $f_n : [0,1] \to \mathbb{R}^q$ such that

$$f_{n+1}(t) = T_i(f_n(mt - i)) \quad \text{if } t \in [i/m, (i+1)/m] \text{ with } i \in \{0, \ldots, m-1\}.$$

If all of T_i's are computable, $\{f_n\}$ is effectively uniformly convergent as $n \to \infty$.

Proof. Let α be a computable real with $0 < \alpha < 1$ such that

$$\forall i \, \forall x, y \, \|T_i(x) - T_i(y)\| \le \|x - y\|.$$

We will show, by induction on n, that for any $n \in \mathbb{N}$,

$$\sup_{t \in [0,1]} \|f_n(t) - f_{n+1}(t))\| \le \alpha^n \sup_{t \in [0,1]} \|f_0(t) - f_1(t))\|.$$

The induction base is clear. The remaining is the induction step.

Suppose

$$\sup_{t \in [0,1]} \|f_n(t) - f_{n+1}(t))\| \le \alpha^n \sup_{t \in [0,1]} \|f_0(t) - f_1(t))\|$$

and evaluate $\|f_{n+1}(t) - f_{n+2}(t)\|$. If $t \in [i/m, (i+1)/m]$, then

$$\begin{aligned} \|f_{n+1}(t) - f_{n+2}(t)\| &= \|T_i(f_n(mt - i)) - T_i(f_{n+1}(mt - i))\| \\ &\le \alpha \|f_n(mt - i) - f_{n+1}(mt - i)\| \\ &\le \alpha \sup_{t \in [0,1]} \|f_n(t) - f_{n+1}(t)\|. \end{aligned}$$

Thus

$$\sup_{t \in [0,1]} \|f_{n+1}(t) - f_{n+2}(t))\| \le \alpha^{n+1} \sup_{t \in [0,1]} \|f_0(t) - f_1(t))\|.$$

We have computed the induction step.

This implies that $\{f_n\}$ converges uniformly to a continuous function as $n \to \infty$. Using f for the limit, we have

$$\begin{aligned} \|f_n(t) - f(t)\| &\le \sum_{k=n}^{\infty} \|f_k(t) - f_{k+1}(t)\| \\ &\le \frac{\alpha^n}{1 - \alpha} \sup_{t \in [0,1]} \|f_1(t) - f_0(t)\|. \end{aligned}$$

Choosing a computable real such that $\beta \ge \sup_{t \in [0,1]} \|f_1(t) - f_0(t)\|$, we have

$$\sup_{t \in [0,1]} \|f_n(t) - f(t)\| \le \frac{\alpha^n \beta}{1 - \alpha}.$$

Therefore, $\{f_n\}$ is effectively uniformly convergent as $n \to \infty$. □

As an immediate consequence of Lemmas 4.1 and 4.2, we obtain the following.

Theorem 4.1. *Let* $T_0, \dots, T_{m-1} : \mathbb{R}^q \to \mathbb{R}^q$ *be computable contractions that satisfy*

$$T_1(\mathrm{Fix}(T_0)) = T_0(\mathrm{Fix}(T_{m-1})), \dots, \text{ and } \quad T_{m-1}(\mathrm{Fix}(T_0)) = T_{m-2}(\mathrm{Fix}(T_{m-1})).$$

Then there exist a computable curve $f : [0,1] \to \mathbb{R}^q$ *such that*

$$f([0,1]) = T_0(f([0,1])) \cup \cdots \cup T_{m-1}(f([0,1])).$$

Example 4.1. A Koch curve which is a self-similar set on $\mathbb{R}^2$ with respect to contractions T_0 and T_1 defined by

$$T_0(x) = \begin{pmatrix} 1/2 & 1/(2\sqrt{3}) \\ 1/(2\sqrt{3}) & -1/2 \end{pmatrix} x,$$
$$T_1(x) = \begin{pmatrix} 1/2 & -1/(2\sqrt{3}) \\ -1/(2\sqrt{3}) & -1/2 \end{pmatrix} x + \begin{pmatrix} 1/2 \\ 1/(2\sqrt{3}) \end{pmatrix},$$

is a computable curve.

5. Conclusion

A finite set of contractions on a Euclidean space construct a self-similar set. If all of the contractions are computable, then the self-similar set with respect to them is a recursive compact set. If the contractions additionally satisfy Hata's curve condition, then the self-similar curve with respect to the contractions is a computable curve.

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