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A nonstandard set theory in the $\in$-language

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Abstract. We demonstrate that a comprehensive nonstandard set theory can be developed in the standard $\in$-language. As an illustration, a nonstandard Law of Large Numbers is obtained.

Introduction

Nonstandard analysis was introduced by A. Robinson in the beginning of the 60s as a concept in foundations of mathematics which allowed to develop such notions as an infinitesimal real or infinitely large natural number adequately and with full mathematical rigor. Those new mathematical objects, called "nonstandard", brought some benefits to several branches of mathematics. However it was soon discovered that this idea naturally led to more and more complicated nonstandard mathematical objects, which could not be effectively tackled in the framework of Robinson's original approach.

Nonstandard set theories represent one of the two known ways of how to develop "nonstandard" mathematics in unified way. (The other setup, called the *model theoretic* version of nonstandard analysis, employs nonstandard extensions of mathematical structures in the "standard" Zermelo – Fraenkel universe, see Lindstrøm [13], which is closer to Robinson's approach.) Any such theory arranges the set universe in such a way that the objects of or-

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dinary mathematics, called “standard”, coexist and interact with “nonstandard” objects (e.g., infinitesimal reals) within the same world of sets.

The best known (and perhaps the only one practically used in non-foundational studies) of those theories is *internal set theory* **IST** of Nelson [14]. Some other nonstandard set theories were invented by Hrbaček [5,6], Kawaï [10,11], and the authors [7,8]. They differ in the manner how the “nonstandard” set universe is arranged, but have a lot in common, in particular, each of them considers the class $\mathbb{S}$ of all *standard sets*, identified with the sets studied by “standard” mathematics, and distinguished by a special unary *standardness predicate* $\mathtt{st}\, x$ (reads: x is standard). In other words, the known nonstandard set theories are theories in the $\mathtt{st}$-$\in$-*language* which contains $\mathtt{st}$ and the membership $\in$ as the only atomic predicates.

The aim of this paper is to present a nonstandard set theory in the $\in$-language, which is yet powerful enough for full scale development of nonstandard analysis. We call it *simplified Hrbaček set theory*, **SHST**.

In principle **SHST** is not the first set theory in the $\in$-language which supports “nonstandard” reasoning. Ballard and Hrbaček [2] demonstrated that **ZFBC**, the Zermelo – Fraenkel – Boffa theory with global Choice [1], also provides such a support. However our approach has the advantage of the existence of elementary extensions which are κ-saturated for every (standard) cardinal κ while elementary extensions defined in [2] are κ-saturated only for a previously fixed cardinal κ. (There is a disadvantage, too: **SHST** does not provide the axioms of Power Set and Choice, while **ZFBC** contains **ZFC**.)

Another (possible) advantage is that the theory **SHST** admits a (Boolean–valued) interpretation in **ZFC** such that the class of all standard sets of the interpretation is isomorphic to the basic **ZFC** universe, a property so far unknown for **ZFBC**. (The consistency proof for **ZFBC** in [1] seems to lead to such an interpretation). It follows that the relations of **SHST** to **ZFC** are somewhat similar to those of complex numbers to the reals: **SHST** sets can be adequately “coded” within the **ZFC** universe.

Our plan will be to start with *Hrbaček set theory* **HST**, a more common nonstandard set theory using $\mathtt{st}$ in the language, and define **SHST** as a “description” of the $\in$-structure of the **HST** universe.

The key observation is that the class $\mathbb{S}$ has an $\in$-definable $\in$-isomorphic copy, the class $\mathbb{V}$ of all well-founded sets.[2] The isomorphism allows us to replace $\mathbb{S}$, as a copy of sets used by “standard” mathematics, by $\mathbb{V}$, and utilize the $\in$-definability of $\mathbb{V}$. The class $\mathbb{I}$ of all internal sets (a saturated elementary

[1] The main feature of **ZFBC** is that the axiom of Regularity is replaced by a strong form of its negation, which implies, in particular, that any extensional binary relation is isomorphic to the membership relation on a transitive set.

[2] This phenomenon was first observed by Hrbaček [6], see also Kawaï [11]. It is an interesting problem to figure out whether $\mathbb{S}$ itself can be $\in$-definable in **HST**.

extension of $\mathbb{S}$ in **HST** and many other nonstandard set theories) is not formally defined in **SHST**, while Saturation, the main nonstandard principle, takes the following form: any transitive $X \in \mathbb{V}$ has an elementary extension $\langle {}^*X\,; \in\rangle$ (with the same relation $\in$!), κ-saturated for any $\mathbb{V}$-cardinal κ.

The theory **SHST**, obtained in this way, provides full support for ordinary "nonstandard" reasoning. Yet the non-uniqueness of saturated elementary extensions is subject to criticism.[3] In order to eliminate this problem, we define an extension **SHST**$^+$ of **SHST**, still a theory in the $\in$-language, strong enough to reasonably define the class $\mathbb{I}$ of internal sets[4], containing saturated elementary extensions of structures in $\mathbb{V}$, which admits a uniquely determined choice of a saturated elementary extension at least for definable structures in $\mathbb{V}$.

This plan was outlined in [8], 1.6; here we present the project in detail.

As an illustration, to show how **SHST** works, we present a contribution to mathematics in hyperfinite domains, namely, a hyperfinite version of the Law of Large Numbers, which is based on "individual" random sequences of hyperfinite length rather than on probability distributions.

1. Hrbaček set theory

The original version of this theory was introduced in Hrbaček [5] under the name: $\mathfrak{N}\mathfrak{S}_1(\mathbf{ZFC})$. An improved version was studied in detail in our papers [7,8]. To make the paper self-contained, we present the list of axioms of **HST**, the *Hrbaček set theory,* with some necessary comments.

Recall that **HST** is a theory in the $\mathtt{st}$-$\in$-language. Thus $\mathtt{st}\,x$ reads: x is standard, while $\mathbb{S} = \{x : \mathtt{st}\,x\}$ is the class of all standard sets.

Elements of standard sets are called *internal sets,* $\mathtt{int}\,x$ is the formula $\exists^{\rm st} y\ (x \in y)$ (of being internal), and $\mathbb{I} = \{x : \mathtt{int}\,x\}$ is the class of all internal sets. The quantifiers $\exists^{\rm st}$ and $\forall^{\rm int}$ below have the obvious meaning.

Axioms for the whole universe. all axioms of **ZFC** with the exception of Regularity, Power Set, and Choice. Note that the schemata of Separation and Replacement apply to formulas in the $\mathtt{st}$-$\in$-language.

Transitivity of $\mathbb{I}$. $\forall^{\rm int} x\, \forall y \in x\, (\mathtt{int}\, y)$ (the internal universe is transitive).

Regularity over $\mathbb{I}$. $\forall X \neq \emptyset\ \exists x \in X\ (x \cap X \subseteq \mathbb{I})$.

ZFC$^{\rm st}$. all statements of the form $\Phi^{\rm st}$ (Φ relativized to $\mathbb{S} = \{x : \mathtt{st}\,x\}$), where Φ is an axiom of **ZFC**.

Transfer. all statements of the form $\Phi^{\rm st} \Longleftrightarrow \Phi^{\rm int}$, where Φ is a closed $\in$-formula with standard parameters.

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[3] See for instance Keisler [12].

[4] As the class of all sets x such that the relation $\in$ is, in a sense, not well-founded over x. This reflects a property of $\mathbb{I}$ in **HST**, discovered in [8].

Standardization. $\forall X \, \exists^{\mathrm{st}} Y \, (X \cap \mathbb{S} = Y \cap \mathbb{S})$. In other words, for any set X there is a standard set Y which contains the same standard elements as X.

These axioms suffice to define the class $\mathbb{V}$ of all well-founded sets[5] and an $\in$-isomorphism $x \mapsto {}^*x : \mathbb{V}$ onto $\mathbb{S}$[6]. It follows that $\mathbb{V}$ interprets **ZFC** and is a transitive class closed under subset formation. Moreover the map $x \mapsto {}^*x$ is an elementary embedding (in the sense of the $\in$-language) of $\mathbb{V}$ into $\mathbb{I}$ by Transfer. (See [8], Sect. 1, for details.)

It is convenient to use $\mathbb{V}$ as the domain for the basic set theoretic notions, such as *ordinals*, *cardinals*, *natural numbers*. Thus a natural number will mean a set $n \in \mathbb{V}$ which is a natural number in the sense of $\mathbb{V}$ (in brief, $\mathbb{V}$-natural number). ω is the set of all natural numbers, as usual.

A *finite set* is a set equinumerous to $n = \{0, 1, ..., n-1\}$, where $n \in \omega$.

A *set of standard size* is a set equinumerous to a set in $\mathbb{V}$. In **HST**, sets of standard size are exactly those which can be well-ordered.

These notions allow us to formulate the last three axioms of **HST**.

Saturation of $\mathbb{I}$. if $\mathcal{X} \subseteq \mathbb{I}$ is a set of standard size such that $\bigcap \mathcal{X}' \neq \emptyset$ for any *nonempty finite* $\mathcal{X}' \subseteq \mathcal{X}$ (the finite intersection property) then $\bigcap \mathcal{X}' \neq \emptyset$.

Standard Size Choice. Choice in the case when the domain of the choice function postulated to exist is a set of standard size (or, which is the same by the above, a well-orderable set).

Dependent Choice. in its usual formulation that allows an ω-sequence of successive picks in the case when the domain of every next pick depends on the results of the previous picks.

Saturation of $\mathbb{I}$ is a source of various nonstandard sets, as usual (see Sect. 5). Standard Size Choice and Dependent Choice partially substitute the absense of full Choice in many important cases. (Full Choice, as well as Power Set and full Regularity would contradict the rest of **HST**.)

The main metamathematical properties of **HST** are summarized in the following theorem proved by Kanovei and Reeken [7,8]. (The equiconsistency part of this theorem is essentially proved in [5].)

Theorem 1. **HST** *and* **ZFC** *are equi-consistent theories. Moreover, there is a Boolean-valued interpretation of* **HST** *in* **ZFC** *the class $\mathbb{S}$ of which is definably $\in$-isomorphic to the basic* **ZFC** *universe.* □

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5 That is, elements of transitive sets X such that $\in \restriction X$ is a well-founded relation.

6 ${}^*x$ is defined to be that unique standard set u whose standard elements are sets of the form ${}^*y$, $y \in x$, and only those sets. The definition utilizes Standardization.

2. Natural nonstandard set theory

The theory **SHST** aims to reflect the $\in$-structure of the **HST** set universe. It follows from the above that **SHST** then must include:

Axioms for the whole universe. all **ZFC** axioms except for Regularity, Power Set, and Choice. (Separation and Replacement in the $\in$-language.)

This is enough to define the class $\mathbb{V}$ of all well-founded sets and prove that $\mathbb{V}$ is transitive and $(\star)$ any set $x \subseteq \mathbb{V}$ belongs to $\mathbb{V}$. We also add Standard Size Choice, Dependent Choice – as in Sect. 1, and

ZFC$^{\mathrm{wf}}$. all statements of the form Φ^{wf} (Φ relativized to $\mathbb{V} = \{x : \mathtt{wf}\, x\}$), Φ being an axiom of **ZFC**, where $\mathtt{wf}\, x$ says: “x is well-founded”.

In this setup, there is nothing analogous to $\mathbb{I}$, therefore the axioms of Transitivity of $\mathbb{I}$ and Regularity over $\mathbb{I}$ of Sect. 1 are abandoned. The role of Standardization is played by $(\star)$ above.

The following axiom, the last in **SHST**, is the key point in the theory. It simulates both Transfer and Saturation of $\mathbb{I}$. We say that a transitive set ${}^*X$ is *standard size saturated* iff any non-empty set $\mathcal{X} \subseteq {}^*X$ of standard size, satisfying the finite intersection property, has non-empty intersection $\bigcap \mathcal{X}$. (The notions involved in this definition are understood as in Sect. 1.)

Saturated Elementary Extensions. for any transitive set $X \in \mathbb{V}$ there exist a transitive standard size saturated set ${}^*X$ and a map $x \mapsto {}^*x$ from X to ${}^*X$, which is an elementary embedding of $\langle X\,; \in\rangle$ into $\langle {}^*X\,; \in\rangle$.

To see that this holds in **HST** define ${}^*X$ and ${}^*x$ using the $\in$-isomorphism described in footnote 6. (In fact it needs some care to prove that we have an elementary embedding in the model theoretic sense, see [8], Sect. 1.)

Thus, as a subtheory of **HST**, **SHST** shares the content of Theorem 1.

3. Improvement of the theory

It is easy to see that **SHST** supports any sort of ordinary “nonstandard” arguments[7]. However **SHST** is an easy target for the same criticism which the model theoretic version of nonstandard analysis faces: the non-uniqueness of nonstandard versions of usual mathematical structures. (Indeed one can prove in **HST** that, e.g., there are other than the “true” ${}^*\omega$ standard size saturated transitive elementary extensions of ω.)

This problem does not exist in **HST** because the map $x \mapsto {}^*x$ is definable in this theory. However its definition involves the standardness predicate,

7 See Sect. 5 below.

and still it is not known whether there is a definition in the $\in$-language. Even more, there is no clear way how to introduce internal sets in **SHST**. Yet we shall see that a reasonable (although partial) solution exists, and the first step is to $\in$-define internal sets.

3.1. Modeling internal sets

The next lemma (first observed in [8]) shows that the class $\mathbb{I}$ is $\in$-definable in **HST**, leading to a better $\in$-description of **HST** than **SHST** is.

We say that a set x is *quasi-internal* iff there is an ω-sequence $\{x_n\}_{n\in\omega}$ such that $x \in x_{n+1} \in x_n$ for all $n \in \omega$.

Lemma 1. (in **HST**) *A set x belongs to $\mathbb{I}$ iff it is quasi-internal.*

Proof. Suppose that x is internal. Arguing in $\mathbb{I}$, define, by induction on $k \in {}^*\omega$, $y_k = y_{k-1} \cup \{y_{k-1}\}$, starting with $y_0 = x$. Now pick a number $\nu \in {}^*\omega \setminus \omega$ and let $x_n = y_{\nu-n}$ for all $n \in \omega$.

The converse easily follows from Regularity over $\mathbb{I}$.

3.2. The improved theory

Definition 1. *As long as we work in the $\in$-environment, $\mathbb{I}$ is the class of all quasi-internal sets.*[8] □

Now, let **SHST**$^+$ be the theory in the $\in$-language, containing all of **SHST** as above except for Saturated Elementary Extensions (which will be a corollary), together with Transitivity of $\mathbb{I}$, Regularity over $\mathbb{I}$, Saturation of $\mathbb{I}$ – everything just as in Sect. 1, – and the following

"Natural" Transfer. All statements of the form $\Phi^{\text{wf}} \iff \Phi^{\text{int}}$, where Φ is a closed $\in$-formula with parameters in ω.

The last axiom needs some comments as it is not clear from the beginning that $\omega \subseteq \mathbb{I}$. Thus we first accept "Natural" Transfer in the parameter-free version, which is enough to see that $\mathbb{I}$ is a transitive $\in$-model of **ZFC**, so that easily $\omega \subseteq \mathbb{I}$. Now we accept the full "Natural" Transfer.

We hardly can involve more parameters in the formulation of "Natural" Transfer. Indeed, $\mathbb{V} \cap \mathbb{I}$ equals $\mathrm{H}\omega$ (hereditarily finite sets) in **HST** – but parameters in $\mathrm{H}\omega$ are effectively coded by (and easily reducible to) those in ω.

Proposition 1. *Every axiom of* **SHST**$^+$ *is a theorem of* **HST**.

[8] According to Lemma 1, this is compatible with the definition of the class $\mathbb{I}$ in **HST**.

Proof. To see that "Natural" Transfer holds in **HST** we first prove in **HST** that ${}^*x = x$ for any $x \in \omega$ by induction on x. Thus $\Phi^{\text{wf}} \iff \Phi^{\text{int}}$ because the map $x \mapsto {}^*x$ is an elementary embedding of $\mathbb{V}$ in $\mathbb{I}$ in **HST**.

Thus **SHST**$^+$ is a subtheory of the $\in$-part of **HST**. It follows that it also satisfies Theorem 1. To see that it extends **SHST** we prove

Lemma 2. (**SHST**$^+$) *We have* Saturated Elementary Extensions, *moreover, the set* ${}^*X$ *can always be chosen in* $\mathbb{I}$.

Proof. Let $X \in \mathbb{V}$ be a transitive set. Using "Natural" Transfer and Saturation of $\mathbb{I}$, we can find a transitive set ${}^*X \in \mathbb{I}$ such that the structures $\langle X\,;\in\rangle$ and $\langle {}^*X\,;\in\rangle$ are elementarily equivalent.

Indeed, let $\mathsf{SAT}_X(\varphi)$ be the usual $\in$-formula of satisfaction, saying that φ is true in X. Here X is a transitive set and φ is an $\in$-formula, with sets in X as parameters, considered as a finite string of (natural numbers used as codes for) logical symbols and sets (elements of X) involved as parameters. Let Φ be the set of all parameter-free $\in$-formulas (viewed in $\mathbb{V}$ as finite strings, see above). Define $\Phi_X = \{\varphi \in \Phi : \mathsf{SAT}_X(\varphi)\}$, the set of all $\in$-formulas true in X. If $\Psi \subseteq \Phi_X$ is finite then the statement $\forall\,\varphi \in \Psi\, \mathsf{SAT}_X(\varphi)$ is true in $\mathbb{V}$. Thus the formula $\exists\, X\ \forall\,\varphi \in \Psi\ \mathsf{SAT}_X(\varphi)$, which we shall denote by $a(\Psi)$, is also true in $\mathbb{V}$. However $a(\Psi)$ is an $\in$-formula which contains only Ψ as parameter. Note that Ψ is a finite set of parameter-free $\in$-formulas; hence a member of Hω (hereditarily finite sets), which can be effectively coded by a natural number. It follows that we can apply "Natural" Transfer, so that $a(\Psi)$ is true in $\mathbb{I}$ – for any finite $\Psi \subseteq \Phi_X$. Applying the **ZFC** Replacement in $\mathbb{I}$, we find a set W such that, for any set Ψ of any kind, if $a(\Psi)$ is true then this is witnessed by a set $X' \in W$. For any formula $\varphi \in \Phi_X$, let us define, in $\mathbb{I}$, $W_\varphi = \{X' \in W : \mathsf{SAT}_{X'}(\varphi)\}$. It follows from the reasoning above that the collection $\mathcal{X} = \{W_\varphi : \varphi \in \Phi_X\}$ has the finite intersection property. By Saturation of $\mathbb{I}$, there is a set ${}^*X \in \bigcap_{\varphi\in\Phi_X} W_\phi$. By the construction, $\langle {}^*X\,;\in\rangle$ and $\langle X\,;\in\rangle$ are elementarily equivalent. (Some extra care should have been taken to provide the transitivity of ${}^*X$, but this is easy.)

Note that $\langle {}^*X\,;\in\rangle$ is standard size saturated by Saturation of $\mathbb{I}$. Hence it remains to define an elementary embedding of $\langle X\,;\in\rangle$ into $\langle {}^*X\,;\in\rangle$.

By the choice of ${}^*X$, for any $\langle x_1,...,x_n\rangle \in X^{<\omega}$ there is $\langle x'_1,...,x'_n\rangle \in {}^*X^{<\omega}$ such that $(*)$ for any $\in$-formula $\varphi(\cdots)$ we have: $\varphi(x_1,...,x_n)$ is true in $\langle X\,;\in\rangle$ iff $\varphi(x'_1,...,x'_n)$ is true in $\langle {}^*X\,;\in\rangle$.[9] By Standard Size Choice there is a $1-1$ length-preserving map $f : X^{<\omega} \to {}^*X^{<\omega}$ such that $(*)$ holds for $\langle x'_1,...,x'_n\rangle = f(\langle x_1,...,x_n\rangle)$ for any $\langle x_1,...,x_n\rangle \in X^{<\omega}$.

Note that it is *not* asserted that $f(\langle x_1,...,x_n\rangle) = \langle f(\langle x_1\rangle),...,f(\langle x_n\rangle)\rangle$.

Let $R_0 = \{f(\langle x\rangle) : x \in X\}$, so that $R_0 \subseteq {}^*X$.

[9] In this proof, *formulas* are understood formally, that is, as certain finite sequences.

For any finite $R \subseteq R_0$ and any finite set F of $\in$-formulas, let $\mathit{\Pi}_{RF}$ be the (internal) set of all internal $1-1$ maps $\pi : {}^*\!X \to {}^*\!X$ such that for any tuple $\langle r_1, ..., r_n \rangle \in R^{<\omega}$ and any $\in$-formula $\varphi(\cdots)$ in F we have

- it is true in $\mathbb{I}$ that $\varphi(r_1, ..., r_n) \iff \varphi(r'_1, ..., r'_n)$, where $\langle r'_1, ..., r'_n \rangle = f(\langle x_1, ..., x_n \rangle)$, and $\langle x_i \rangle = f^{-1}(\langle \pi^{-1}(r_i) \rangle)$ for all i.

It quickly follows from the choice of ${}^*\!X$ and f that the family of sets $\mathit{\Pi}_{RF}$ is a standard size family of non-empty sets satisfying the finite intersection property. (Basically $\mathit{\Pi}_{R_1F_1} \cap \mathit{\Pi}_{R_2F_2} \supseteq \mathit{\Pi}_{R_1 \cup R_2\,,\, F_1 \cup F_2}$.) Therefore there is an internal $1-1$ map $\pi : {}^*\!X \to {}^*\!X$ which belongs to every relevant set $\mathit{\Pi}_{RF}$, so that • holds for all $\langle r_1, ..., r_n \rangle \in {R_0}^{<\omega}$ and all $\in$-formulas φ.

One easily sees that $p(x) = \pi(r)$, where $\langle r \rangle = f(\langle x \rangle))$, is an elementary embedding of $\langle X\,; \in \rangle$ into $\langle {}^*\!X\,; \in \rangle$. $\square$

4. Uniqueness of the asterisks

Let us return to the problem of uniqueness of sets ${}^*\!X$. Suppose that a transitive set $X \in \mathbb{V}$ is "mathematically unique" in the sense that there is an $\in$-formula $\varphi(\cdot)$ so that X is the only transitive set satisfying $\varphi(X)$ in $\mathbb{V}$. (For instance this includes the sets like ω and $V_{\omega+\omega}$.) It follows from "Natural" Transfer that there is a unique set ${}^*\!X \in \mathbb{I}$ satisfying $\varphi({}^*\!X)$ in $\mathbb{I}$. Using "Natural" Transfer again, we conclude that then the structures $\langle X\,; \in \rangle$ and $\langle {}^*\!X\,; \in \rangle$ are elementarily equivalent. Now the proof of Lemma 2 yields an elementary embedding of X into ${}^*\!X$.

Note that this construction results in a really unique standard size saturated elementary extension ${}^*\!X$ of X!

Of course, the merits of this result are restricted by the assumption that X is definable by an $\in$-formula in $\mathbb{V}$. To show the difficulties connected with this problem, let us demonstrate that, even in this case, the elementary embedding itself may be not unique.

Let $P = \mathcal{P}(\omega) = \{x : x \subseteq \omega\}$, so that $P \in \mathbb{V}$ is $\in$-definable in $\mathbb{V}$.

Lemma 3. **(HST)** *There exist two different elementary embeddings of the structure* $\langle P\,; \in \rangle$ *into* $\langle {}^*\!P\,; \in \rangle$.[10]

Proof. As $\mathtt{card}\, P = 2^{\aleph_0} > \aleph_0$ in $\mathbb{V}$, there is a set $y \in P$ which is not definable in P by a parameter-free $\in$-formula.[11] This means that for any $\in$-formula $\varphi(\cdot)$ there is a set $y' \in P$, $y' \neq y$, such that $\varphi(y) \iff \varphi(y')$ in $\langle P\,; \in \rangle$.

[10] Here and in the proof below ${}^*\!P$ etc. is understood in the sense of the embedding $x \mapsto {}^*x$ defined in footnote 6.

[11] See footnote 9.

By Transfer, this is true in $\langle {}^*P\,;\in\rangle$, too, in the sense that for any $\varphi(\cdot)$ there is a set $y' \in {}^*P$, $y' \neq {}^*y$, such that $\varphi({}^*y) \Longleftrightarrow \varphi(y')$ in $\langle {}^*P\,;\in\rangle$. Therefore, by Saturation of $\mathbb{I}$, there is one particular $y' \in {}^*P$, $y' \neq {}^*y$, such that the equivalence $\varphi({}^*y) \Longleftrightarrow \varphi(y')$ holds in $\langle {}^*P\,;\in\rangle$ for any parameter-free $\in$-formula $\varphi(\cdot)$. Now, using the same argument as in the proof of Lemma 2, we can define an elementary embedding $p : P \to {}^*P$ such that $p(y) = y'$; hence p is different from the restriction $(x \mapsto {}^*x) \restriction P$ just because $y' \neq {}^*y$.

In fact, another kind of uniqueness is consistent with **SHST**$^+$. We gave in [9] the following formulation of the *isomorphism property*, first considered by Henson in the study of nonstandard models in the **ZFC** universe:

Isomorphism Property, IP**.** any two internally presented elementarily equivalent structures of a first-order language containing (standard size)–many symbols are isomorphic.

(A structure is *internally presented* iff its universe and all of its relations and functions are internal.) We proved in [9] that IP is consistent with **HST**. Therefore IP is consistent with **SHST**$^+$ as well, by the above. But in **SHST**$^+$ plus IP all internally presented elementary extensions of any fixed structure $\langle X\,;\in\rangle$, $X \in \mathbb{V}$, are isomorphic (although not necessarily via an internal isomorphism).

5. Towards applications

This section is written to explain how the theories **SHST** and **SHST**$^+$ can be used to formalize "nonstandard" reasoning.

First of all recall that **SHST** is a theory in the $\in$-language, containing all of **ZFC** – except for Regularity, Power Set, and Choice, – with some additional axioms which involve the class $\mathbb{V} = \{x : \mathtt{wf}\, x\}$ of all well-founded sets, namely, Standard Size Choice and Dependent Choice as in Sect. 1 and **ZFC**$^{\mathrm{wf}}$ as in Sect. 2, and finally Saturated Elementary Extensions (Sect. 2) as the source of nonstandard sets.

SHST$^+$ extends **SHST** (minus the axiom of Saturated Elementary Extensions which becomes a corollary) by some axioms, related to the class $\mathbb{I}$ of all quasi-internal sets (see Definition 1) viewed as a substitute for internal sets. Those additional axioms include Transitivity of $\mathbb{I}$, Regularity over $\mathbb{I}$, Saturation of $\mathbb{I}$ – as in **HST**, and "Natural" Transfer which says that $\mathbb{V}$ and $\mathbb{I}$ are elementarily equivalent w. r. t. formulas with natural numbers as parameters. (See Sect. 3.)

Let us outline the development of nonstandard analysis in **SHST**. First of all, we shall, informally, consider $\mathbb{V}$ as the "standard" mathematical universe. Since **SHST** satisfies Theorem 1 (being a subtheory of **HST**), the **SHST** universe can be viewed as an auxiliary (although well defined by means of $\mathbb{V}$) extension of the "true" set universe $\mathbb{V}$ – in the same manner as $\mathbb{C}$ (complex numbers) is an extension of $\mathbb{R}$. Thus **SHST** is not a merely syntactical tool: there is a sound interpretation.

It is known that the set $X = V_{\omega+\omega}$, defined in $\mathbb{V}$, is an adequate domain for a large part of mathematics. In particular the sets $\mathbb{N} = \omega$ (the natural numbers) and $\mathbb{R}$ (the reals) belong to X.

Applying Saturated Elementary Extensions, we obtain a transitive standard size saturated set ${}^*X = {}^*V_{\omega+\omega}$ and an elementary embedding $x \mapsto {}^*x$ of $\langle V_{\omega+\omega}\,; \in\rangle$ into $\langle {}^*V_{\omega+\omega}\,; \in\rangle$. (If we work in **SHST**$^+$ rather than in **SHST**, it can be specified that ${}^*V_{\omega+\omega} \in \mathbb{I}$, see Lemma 2. In this case, ${}^*V_{\omega+\omega}$ is just the $\mathbb{I}$-analog of $X = V_{\omega+\omega}$: formally, ${}^*V_{\omega+\omega} = V_{{}^*\omega+{}^*\omega}$ in $\mathbb{I}$.)

One easily proves that if $n \in \mathbb{N}$ then ${}^*n = n \in {}^*\mathbb{N}$ (for instance by induction on n); hence $\mathbb{N}$ is an initial segment of ${}^*\mathbb{N}$. To see that $\mathbb{N}$ is a *proper* initial segment of ${}^*\mathbb{N}$, use the saturation property of X with respect to the family of sets $S_n = \{k \in {}^*\mathbb{N} : k > n\}$, $n \in \mathbb{N}$. Note that elements of ${}^*\mathbb{N}$ are just $\mathbb{I}$-natural numbers, which could also be called *hypernatural numbers*. Numbers in ${}^*\mathbb{N} \setminus \mathbb{N}$ can be called *infinitely large*.

As for the reals, we have both $\mathbb{R} \in V_{\omega+\omega}$ and $\mathbb{R} \subseteq V_{\omega+\omega}$; hence ${}^*\mathbb{R} \in {}^*V_{\omega+\omega}$ and ${}^*x \in {}^*\mathbb{R}$ have been defined for all $x \in \mathbb{R}$. Elements of ${}^*\mathbb{R}$, i.e. $\mathbb{I}$-reals, can be called *hyperreals*. One now defines, in the ordinary manner, the notions of being an *infinitesimal, infinitely large, limited* hyperreal, and the relation $\approx$ of being *infinitely close*. Let us prove

Lemma 4. *If $x \in {}^*\mathbb{R}$ is limited then there is $z \in \mathbb{R}$ such that $x \approx {}^*z$.*

Proof. The sets $A = \{y \in \mathbb{R} : {}^*y \le x\}$ and $B = \{y \in \mathbb{R} : {}^*y > x\}$ are non-empty because x is limited. Moreover they belong to $\mathbb{V}$ because this class is closed under the subset formation. (See Sect. 2.) Therefore, arguing in $\mathbb{V}$, we obtain a real z which is either the greatest in A or the least in B. One easily sees that z is as required. $\square$

This outline demonstrates that typical "nonstandard" reasoning is fully supported in **SHST**. (As for more advanced examples, like the Loeb measure or descriptive set theory in hyperfinite domains, see Kanovei and Reeken [8], 2.2 and 2.3. We demonstrated there that the usual way of reasoning goes through. The absence of Power Set in the **SHST** universe in principle causes some technical problems, which however can be eliminated.)

6. Randomness in hyperfinite domain

Following the setup of the previous section, we shall call sets in ${}^*V_{\omega+\omega}$ *internal* and sets of the form ${}^*x$, $x \in V_{\omega+\omega}$, *standard*. Thus every standard set is internal, and the embedding $x \mapsto {}^*x$ is a $1-1$ map of $V_{\omega+\omega}$ onto the set of all standard elements of ${}^*V_{\omega+\omega}$.

Let $w,\ y$ be internal sets. We defined in [8] y to be w-*standard* iff there is a standard function f such that $w \in \mathtt{dom}\, f$ and $y = f(w)$. If w is standard then "w-standard" and "standard" is one and the same.

We also defined a real x to be

– w-*infinitesimal* iff $|x| < \varepsilon$ for some w-standard infinitesimal ε;
– w-*infinitely large* iff $|x| > c$ for some w-standard infinitely large c.

This definition makes sense iff there really exist w-standard infinitesimals and infinitely large numbers. In particular it does not make sense (and will not be used) in the case when w is standard.

Suppose that w is any internal set while $\mathcal{X}$ is a strictly hyperfinite (i.e. hyperfinite but not finite) internal set, the *hyperfinite domain*. Let μ be a *hyperprobability measure* on $\mathcal{X}$, i.e. a hyperfinitely additive internal measure on $\mathcal{X}$ such that $\mu(\mathcal{X}) = 1$.

We defined in [8] an element $x \in \mathcal{X}$ to be w-*random in* $\mathcal{X}$ *w. r. t.* μ iff x does not belong to any $\langle w, \mathcal{X}\rangle$-standard set $X \subseteq \mathcal{X}$ such that $\mu(X)$ is $\mathcal{X}$-infinitesimal. (Here w may be standard and may be nonstandard.) Thus nonstandard analysis allows to consistently consider the notion of an *individual* random element rather than a random variable as a function defined on a probability space. The following lemma shows that, in agreement with the intuition, non-random elements form a rare family.

Lemma 5. *If $w,\ \mathcal{X},\ \mu$ are as above then the set $\complement R$ of all elements $x \in \mathcal{X}$, non-w-random in $\mathcal{X}$ w. r. t. μ, can be covered by an internal set $Z \subseteq \mathcal{X}$ such that $\mu(Z)$ is infinitesimal.*

It is not asserted here that $\mu(Z)$ is, e.g., w-infinitesimal. We cannot demand that $\mu(\complement R)$ is infinitesimal because μ is defined only for internal subsets of $\mathcal{X}$ while R is, generally speaking, external.

Proof. First of all, by Saturation, there is an infinitesimal $\delta > 0$ such that $\delta > \varepsilon$ for any $\mathcal{X}$-infinitesimal ε. Then any $x \in \complement R$ is covered by a $\langle w, \mathcal{X}\rangle$-standard set X such that $\mu(X) < \delta$.

Let W be a standard set, containing both w and $\mathcal{X}$, and such that any internal subset of $\mathcal{X}$ also belongs to W. The set F of all standard functions $f : W^2 \to W$ such that $f(w, \mathcal{X})$ is a subset of $\mathcal{X}$ satisfying $\mu(X) < \delta$ – is a set of standard size. Thus there is an internal set F' containing $< 1/\sqrt{\delta}$ elements, such that $F \subseteq F'$. For any $f \in F'$ define X_f to be $f(w, \mathcal{X})$ iff f is a function, $X = f(w, \mathcal{X})$ is defined and is a subset of $\mathcal{X}$, and $\mu(X) < \delta$, while $X_f = \emptyset$ otherwise.

Then $\complement R \subseteq Z = \bigcup_{f \in F'} X_f$ and $\mu(Z) \le \delta/\sqrt{\delta} = \sqrt{\delta}$ is infinitesimal. □

Let w be any internal set.

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are two strictly hyperfinite sets, while μ and ν are hyperprobability measures on resp. $\mathcal{X}$ and $\mathcal{Y}$.

Lemma 6. (Fubini) *Assume that ν is $\langle w, \mathcal{X}, \mathcal{Y}\rangle$-standard. Let $\langle x, y\rangle \in \mathcal{X} \times \mathcal{Y}$ be w-random in $\mathcal{X} \times \mathcal{Y}$ w. r. t. $\mu \times \nu$. Then x is w-random in $\mathcal{X}$ w. r. t. μ while y is $\langle w, x\rangle$-random in $\mathcal{Y}$ w. r. t. ν.*

Suppose that, in addition, $\mathcal{Y}$ is $\mathcal{X}$-standard and $\mathcal{X}$ is $\mathcal{Y}$-standard. Let $x \in \mathcal{X}$ be w-random in $\mathcal{X}$ w. r. t. μ while $y \in \mathcal{Y}$ be $\langle w, x\rangle$-random in $\mathcal{Y}$ w. r. t. ν. Then $\langle x, y\rangle$ is w-random in $\mathcal{X} \times \mathcal{Y}$ w. r. t. $\mu \times \nu$.

Proof. Let $X \subseteq \mathcal{X}$ be a $\langle w, \mathcal{X}\rangle$-standard set of measure $\mu(X) < \varepsilon$, where ε is $\mathcal{X}$-infinitesimal. Assume on the contrary that $x \in X$. Then $\langle x, y\rangle \in P$, where $P = X \times \mathcal{Y}$ is $\langle w, \mathcal{X}, \mathcal{Y}\rangle$-standard and satisfies $(\mu \times \nu)(P) < \varepsilon$, which is a contradiction. (Note, in passing by, that to be $\langle w, \mathcal{X}, \mathcal{Y}\rangle$-standard and to be $\langle w, \mathcal{X} \times \mathcal{Y}\rangle$-standard is one and the same.)

Let $Y \subseteq \mathcal{Y}$ be a $\langle w, x, \mathcal{Y}\rangle$-standard set of measure $\nu(Y) < \varepsilon$, where ε is $\mathcal{Y}$-infinitesimal. Suppose on the contrary that $y \in Y$. By definition we have $Y = f(w, x, \mathcal{Y})$, where f is a standard function. Let P be the set of all pairs $\langle x', y'\rangle \in \mathcal{X} \times \mathcal{Y}$ such that $Y_{x'} = f(w, x', \mathcal{Y})$ is a subset of $\mathcal{Y}$ satisfying the inequality $\nu(Y_{x'}) < \varepsilon$, and $y' \in Y_{x'}$. Note that P is $\langle w, \mathcal{X}, \mathcal{Y}\rangle$-standard by the assumptions above, and $(\mu \times \nu)(P) \le \varepsilon$. On the other hand, $\langle x, y\rangle \in P$ by definition, which is a contradiction.

To prove the converse, consider a $\langle w, \mathcal{X}, \mathcal{Y}\rangle$-standard set $P \subseteq \mathcal{X} \times \mathcal{Y}$ of measure $(\mu \times \nu)(P) < \varepsilon$, where ε is a $\langle \mathcal{X}, \mathcal{Y}\rangle$-infinitesimal; hence $\mathcal{X}$-infinitesimal by the assumption. Put $P_{x'} = \{y \in \mathcal{Y} : \langle x', y\rangle \in P\}$ for any $x' \in \mathcal{X}$. The set $X = \{x' \in \mathcal{X} : \nu(P_{x'}) \ge \sqrt{\varepsilon}\}$ is $\langle w, \mathcal{X}\rangle$-standard by the assumption, and $\mu(X) \le \sqrt{\varepsilon}$ because $(\mu \times \nu)(P) < \varepsilon$. Therefore $x \notin X$ by the randomness of x. Thus the $\langle w, \mathcal{X}, x\rangle$-standard (therefore $\langle w, \mathcal{Y}, x\rangle$-standard) set $Y = P_x$ satisfies $\nu(Y) < \sqrt{\varepsilon}$. However $y \in Y$, which contradicts the randomness of y. □

7. Law of Large Numbers

In classical probability, this is a common name for several important theorems saying that, under some conditions, the arithmetic mean $\frac{\xi_1 + \ldots + \xi_n}{n}$ of jointly independent random variables ξ_i tends to the arithmetic mean of their expectations $\frac{m_1 + \ldots + m_n}{n}$ as $n \to \infty$. (See [15], Sect. 2.)

Our aim will be to obtain a hyperfinite version.

Let $\mathcal{X}$ be a hyperfinite set and μ a hyperprobability measure on $\mathcal{X}$, as above. Assume, in addition, that $\mathcal{X} \subseteq {}^*\mathbb{R}$. We define

- $\mathsf{E}\mu = \sum_{x \in \mathcal{X}} x\, \mu(\{x\})$, the *expectation* of μ;
- $\mathtt{Var}\,\mu = \sum_{x \in \mathcal{X}} (x - \mathsf{E}\mu)^2\, \mu(\{x\})$, the *variance* of μ.

Note that the expectation and variance are functions of the measure (= the probability distribution) rather than of random elements as we defined them.

Suppose that $H \in {}^*\mathbb{N} \setminus \mathbb{N}$, and for any $n = 1, 2, \ldots H$, we have a hyperfinite set $\mathcal{X}_n \subseteq {}^*\mathbb{R}$ and a hyperprobability measure μ_n on $\mathcal{X}_n$, so that the maps $n \mapsto \mathcal{X}_n$ and $n \mapsto \mu_n$ are internal. Let $m_n = \mathsf{E}\mu_n$ and $v_n = \mathtt{Var}\,\mu_n$ for all n. Define $\mathcal{X} = \prod_{n=1}^{H} \mathcal{X}_n$ and let $\mu = \prod_{n=1}^{H} \mu_n$ be the product hyperprobability measure on $\mathcal{X}$.

Theorem 2. (Hyperfinite Law of Large Numbers)
Assume that $v = H^{-1} \sum_{n=1}^{H} v_n$ *is a limited number. Suppose that the measure* μ *is* $\mathcal{X}$*-standard. Then, for any sequence* $x = \{x_n\}_{n=1}^{H}$*, random* (i.e. 0-random) *in* $\mathcal{X}$ *w. r. t.* μ*, the difference*

$$\Delta(x) = \frac{x_1 + \ldots + x_H}{H} - \frac{m_1 + \ldots + m_H}{H}$$

is infinitesimal.

Proof. By Kolmogorov's inequality (see, e.g., [15], Theorem 12.2), applied in the internal universe ${}^*V_{\omega+\omega}$, for any $s > 0$ we have

$$\mu(\{y \in \mathcal{X} : \Delta(y) \geq s\}) \leq \frac{v}{Hs^2} .$$

By the assumption, vs^{-2} is a limited number whenever $s > 0$ is standard. Thus the set $X_s = \{y \in \mathcal{X} : \Delta(y) \geq s\}$ has an H-infinitesimal measure $\mu(X_s)$ whenever $s > 0$ is standard. On the other hand, if s is standard then X_s is $\langle \mathcal{X}, \mu \rangle$-standard; hence $\mathcal{X}$-standard because μ is $\mathcal{X}$-standard. It follows, by definition, that $x \notin X_s$ for any standard $s > 0$, as required. □

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