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A version of the Jensen–Johnsbråten coding at arbitrary level $n \geq 3$

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Abstract. We generalize, on higher projective levels, a construction of "incompatible" generic Δ^1_3 real singletons given by Jensen and Johnsbråten.

Introduction

In this note, we prove the following theorem:

Theorem 1. *Let* $n \geq 2$. *There is a* CCC *in* L, *the constructible universe, forcing notion* $P = P_n \in \mathrm{L}$ *such that* P*-generic extensions of* L *are of the form* $\mathrm{L}[a]$, *where* $a \subseteq \omega$, $a \notin \mathrm{L}$, $\{a\}$ *is* Π^1_n, *and*

I: *if* $b \in \mathrm{L}[a]$, $b \subseteq \omega$ *is* Σ^1_n *in* $\mathrm{L}[a]$ *then* $b \in \mathrm{L}$ *and* b *is* Σ^1_n *in* L ;

II: *if a transitive model* $\mathscr{M}$ *of* **ZFC** *extends* L *and contains two different* P*-generic sets* a, $a' \subseteq \omega$, *then* $\omega_1^{\mathscr{M}} > \omega_1^{\mathrm{L}}$.

For $n = 2$, this is the result of Jensen and Johnsbråten [4] (then I is a corollary of the Shoenfield absoluteness). In the absense of the "incompatibility" requirement II, the result was proved by Harrington [1] (using a version of the almost disjoint coding of Jensen and Solovay [3]) and, independently, by the author [5,6] (using a version of the Jensen "minimal Δ^1_3" coding [2]). Our proof is a similar modification of the construction in [4].

Recall that the forcing notion used in [4] is the union of a certain increasing ω_1-sequence of its countable initial segments. The construction, reviewed in Section 1, results, roughly speaking, in an ω-long iteration of the forcing by a Souslin tree. It is the specific property of the forcing, discussed in Section 3, that a certain ω-sequence of rationals in the extension can be effectively decoded into a generic sequence of ω_1-branches through the trees. An additional care is taken to guarantee that, if two *different* sequences of rationals can be decoded this way then ω_1^{L} is countable, leading to II of Theorem 1.

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The Jensen–Johnsbråten construction is summarized in Section 2, where we show that, basically, any increasing ω_1-sequence of countable objects called *embrions,* which satisfies certain conditions at each level, produces, in L, a forcing which guarantees the uniqueness of a decodable sequence of rationals in the generic extensions of L, as above. Furthermore, if such a sequence of embrions is Δ_1^{HC} then the unique decodable sequence of rationals in the extension appears to be a Π_2^1 singleton, which is the case considered in [4].

To prove Theorem 1, we employ an increasing sequence of embrions which satisfies an extra requirement (of intersecting certain definable subsets, in the collection of all embrions, Section 4), which lifts the level of definability of the decodable sequence of rationals to be a Π_n^1 singleton, but guarantees I of Theorem 1. In order to obtain the latter property, we show, in Section 5, that the mentioned extra requirement provides an appropriate amount of "symmetry", sufficient to prove that any closed Σ_n^1 formula is decided by the forcing.

Referee's comments[1, 2]

" Your paper requires a committed reader. I think it is largely unavoidable, granted my understanding of the main technical problem. Nevertheless, a larger audience might be attracted if you included an informal discussion of your strategy, perhaps expanding on your remarks at the end of your introduction. Whether to include such a discussion is up to you. For what it worth, here is my understanding of your proof.

(1) Working in L, you define a partial ordering $\mathtt{Emb}$ such that
- $\mathtt{Emb}$ is Δ_1 definable over L_{ω_1} without parameters;
- $\mathtt{Emb}$ is a countably closed tree of height ω_1 ; and
- a cofinal branch through $\mathtt{Emb}$ provides an L_{ω_1}-amenable "JJ-system" $\mathbf{T}$ of length ω_1 .

(2) If $\mathbf{T}$ is a JJ-system of length ω_1 then there exists a partial ordering $\mathbb{P}_{\mathbf{T}}$ such that
- $\mathbb{P}_{\mathbf{T}}$ is Δ_1 definable over $(\mathrm{L}_{\omega_1}, \mathbf{T})$ without parameters;
- $\mathbb{P}_{\mathbf{T}}$ satisfies the ccc; and
- $\mathbb{P}_{\mathbf{T}}$ is countably generated. Indeed, if G is $\mathbb{P}_{\mathbf{T}}$-generic, then there exists a canonical generic real $a \in \mathrm{L}[G]$ coding G in the sense that G is Δ_1 definable from the parameter a over $(\mathrm{L}_{\omega_1}[a], \mathbf{T})$.
- If a and b are independently $\mathbb{P}_{\mathbf{T}}$-generic reals, then $\omega_1^{\mathrm{L}[a,b]} < \omega_1^{\mathrm{L}}$.

[1] The content of this Section is a part of anonymous referee's report on this note, which I received in due course from *AML* editors. It is written in the form of a "letter to the author", and contains insights which did not occur to me in such a perfect form when I wrote this note.

[2] *Addendum from Andreas Blass, editor:* The referee who provided this material was Professor M. Stanley. He has kindly consented to this publication of his comments and to divulging his identity as referee.

(3) $\mathtt{Emb} * \mathbb{P}_{\mathbf{T}}$ is homogeneous.

These elements are used to complete the proof as follows.

(A) Fix $n \geq 1$. If $B * C$ is sufficiently generic for $\mathtt{Emb} * \mathbb{P}_{\mathbf{T}}$ then

$$(\mathrm{L}_{\omega_1}[G], B) \models \varphi \quad \Longleftrightarrow \quad \exists\, (T, t) \in B * G \quad (T, t) \Vdash \varphi\,, \qquad (*)$$

for all sentences φ of the forcing language that are Π_n or Σ_n.

(B) A sufficiently $\mathtt{Emb}$-generic B is definable without parameters over L_{ω_1}. Fix such an B and let $\mathbf{T}$ be the provided JJ-system of length ω_1. Let G be fully $\mathbb{P}_{\mathbf{T}}$ generic over L. Then $(*)$ holds for $B * G$. Let a be the generic real coding G.

(C) Because $\mathtt{Emb} * \mathbb{P}_{\mathbf{T}}$ is homogeneous, it follows from $(*)$ that every real that is Σ_n definable without parameters over $\mathrm{L}_{\omega_1}[a]$ lies in L_{ω_1}. Hence every real that is definable in $\mathrm{L}[a]$ by a Σ^1_{n+1} formula of analysis is constructible and Σ^1_{n+1} definable in L.

(D) Because $\mathbf{T}$ is definable without parameters over L_{ω_1} and $\mathbb{P}_{\mathbf{T}}$ is ccc, “x is $\mathbb{P}_{\mathbf{T}}$-generic” is a property definable without parameters in $\mathrm{L}_{\omega_1}[a]$ and a is the unique real with this property in $\mathrm{L}_{\omega_1}[a]$, since independently generic reals collapse ω_1^{L}.

Obstacle. To get that “x is $\mathbb{P}_{\mathbf{T}}$-generic” is a Π^1_{n+1} property of analysis, an $\mathtt{Emb}$ branch B that is sufficiently generic for (A) must be Δ_n definable in (B). The best estimate (that I know, anyway) is Δ_{n+1}. Your solution is to observe that you only need $(*)$ for φ that are (equivalent to) Π^1_{n+1} sentences of analysis. You show that there is a Δ_n definable $\mathtt{Emb}$ branch that is sufficiently generic to handle the required instances of $(*)$ by carefully restricting the Shoenfield terms that occur in φ. This is the point of the auxiliary forcing relation $t\ \mathtt{forc}_{\mathbf{T}}\ \varphi$. **”**

1. Iterated sequences of Souslin trees

By a *normal tree* we shall understand a tree T, which consists of sequences (so that every $t \in T$ is a function with $\mathtt{dom}\, t \in \mathtt{Ord}$ and the order $<_T$ is the extension order $\subset$) and satisfies conditions 1 – 4:

1. The empty sequence Λ does *not* belong to T.
2. If $t \in T$ and $1 \leq \alpha < \mathtt{dom}\, t$ then $t \restriction \alpha \in T$.

Let $|t| = \mathtt{dom}\, t$ for any sequence t. It follows from 1, 2 that, for any $\alpha \geq 1$, $T(\alpha) = \{t \in T : |t| = \alpha\}$ is the α-th level of T. (We start counting levels with 1; the missed, for the sake of convenience, level 0 would consist of Λ.) Let $|T|$ be the least ordinal > 0 and $>$ all $|t|$, $t \in T$ (the *height* of T).

For $\alpha < |T|$, let $T \restriction \alpha = \bigcup_{\gamma < \alpha} T(\gamma)$ (the *restriction*).

3. Each non-maximal $t \in T$ has infinitely many immediate successors.
4. Each level $T(\alpha)$ is at most countable.

Let $1 \le \lambda \le \omega_1$. A *normal λ-tree* is a normal tree T satisfying

- $|T| = \lambda$, and, if $t \in T$ and $|t| < \alpha < \lambda$ then t has successors in $T(\alpha)$.

(The only normal 1-tree is the empty tree. Normal 0-trees do not exist.)

Definition 2. Let $2 \le \lambda \le \omega_1$. A system $\mathbf{T} = \langle\langle T_k\rangle_{k\in\omega}, T[\cdot]\rangle$ of normal λ-trees T_k and a map $t \longmapsto T[t]$, satisfying requirements (1)–(8) below, is called a *JJ-system of length* $\lambda = |\mathbf{T}|$. □

(1) $T_k(\alpha) \subseteq (\mathbb{Q}^+)^\alpha$ for all k and $1 \le \alpha < \lambda$.

Fix a recursive partition $\mathbb{Q}^+ = \bigcup_{k,j\in\omega} Q_{kj}$ of positive rationals onto disjoint topologically dense sets $Q_{kj} \subseteq \mathbb{Q}^+$. Put $Q_k = \bigcup_j Q_{kj}$.

(2) $T_k(1) = \{\langle r\rangle : r \in Q_k\}$ and $T_k(\alpha+1) = \{t^\wedge r : t \in T_k(\alpha) \wedge r \in \mathbb{Q}^+\}$.

($t^\wedge r$ is the extension of a sequence t by $r \in \mathbb{Q}^+$ as the rightmost term.) Thus any element $t \in T_k(\alpha)$ is an α-sequence. The trees T_k are pairwise disjoint.

(3) If $t \in T_k(\alpha)$ then $T[t] \subseteq T_{k+1}$ is a normal α-tree; in addition, we have $T_{k+1}{\restriction}\alpha = \bigcup_{t\in T_k(\alpha)} T[t]$.

(4) If $t, t_1 \in T_k$ and $t < t_1$ then $T[t] = T[t_1]{\restriction}|t|$.

(5) If $t \in T_k(\alpha)$, $\alpha+1 < |\mathbf{T}|$, $r \ne r' \in \mathbb{Q}^+$, then $T[t^\wedge r] \cap T[t^\wedge r'] = T[t]$.

We observe that $T[t] = \emptyset$ whenever $t \in T_k(1)$.

Fix once and for all a recursive enumeration $\mathbb{Q}^+ = \{r_m : m \in \omega\}$.

(6) If $t = \langle r, r_m\rangle \in T_k(2)$ then $T[t] = \{\langle r'\rangle : r' \in Q_{km}\}$. If $t \in T_k(\alpha+1)$ and $\alpha + 2 < |\mathbf{T}|$ then $T[t^\wedge r_m] = T[t] \cup \{s^\wedge r : s \in T[t](\alpha) \wedge r \in Q_m\}$ for any m.

Define the *associated map* $\tau = \tau_{\mathbf{T}}$: if $s \in T_{k+1}(\alpha)$, $\alpha+1 < |\mathbf{T}|$, then $\tau(s)$ is the unique, by (4) and (5), $t \in T_k(\alpha+1)$ such that $s \in T[t]$. Thus τ maps any T_{k+1} (except for the top level if $|\mathbf{T}|$ is a successor ordinal) onto non-limit and bigger than 1 levels of T_k, the previous tree. We have

1°. *If $s, s' \in T_{k+1}$, $\tau(s')$ is defined, and $s \subset s'$, then $\tau(s) \subset \tau(s')$. In addition, if $s \in T_{k+1}(\alpha)$ and $\tau(s) \subset t'$, where $t' \in T_k(\beta+1)$, $\alpha < \beta$, then there is $s' \in T_{k+1}(\beta)$ such that $s \subset s'$ and $\tau(s') = t'$.*

(Use (3)–(5).) A few more definitions and requirements.

- Assume that $\{r_\gamma\}_{\gamma<\alpha}$ is a sequence of non-negative rationals. We let $\sum_{\gamma<\alpha} r_\gamma$ to be the supremum of finite partial sums (including the case of $+\infty$). If $s, t \in T_k(\alpha)$ then define $\sum(s,t) = \sum_{\gamma<\alpha} |s_\gamma - t_\gamma|$.
- Say that a normal tree T is *sum-regular* if we have $\sum(s{\restriction}\alpha, t{\restriction}\alpha) < \sum(s,t) < +\infty$ whenever $s, t \in T(\beta)$, $\alpha < \beta < |T|$, and β is limit.

Trees T_k can contain sequences $s \in T_0$ satisfying $\sum s = \infty$, so that some "series" diverge to infinity. However by the next requirement they diverge in "almost parallel" fashion.

(7) The trees T_0 and $T[t]$ for all $t \in T_k$, $k \in \omega$, are sum-regular.

Let τ^m be the superposition: so $\tau^m(t) \in T_{k-m}(\alpha + m)$ if $t \in T_k(\alpha)$.

- Define $\mathbb{P}_{\mathbf{T}} = \bigcup_k T_k$. Order $\mathbb{P}_{\mathbf{T}}$ as follows: $s \le t$ (means: t is stronger) if $s \in T_l$, $t \in T_k$, $l \le k$, $\tau^{k-l}(t)$ is defined, and $s \subseteq \tau^{k-l}(t)$.
- Let $\alpha < \mu < |\mathbf{T}|$, μ being limit. Say: $t \in T_k(\alpha)$, $s \in T_l(\mu)$ are *weakly compatible in* $\mathbb{P}_{\mathbf{T}}$ if either $t \le s$ (then $k \le l$) or $l < k$ and $\tau^{k-l}(t) \subset s$.

Note that $\mathbb{P}_{\mathbf{T}}$ is not a tree. Assuming that $|\mathbf{T}|$ is a limit ordinal, it easily follows from 1° that weak compatibility is equivalent to the true $\le$-compatibility.

- If $\gamma < |\mathbf{T}|$ then we put $\mathbf{T}\restriction\gamma = \langle\langle T_k\restriction\gamma\rangle_{k\in\omega}, T[\cdot]\restriction\gamma\rangle$, where $T[\cdot]\restriction\gamma$ is the restriction of $T[\cdot]$ on the domain $\bigcup_k T_k\restriction\gamma$. (It will be clear that $\mathbf{T}\restriction\gamma$ is a JJ-system of length γ.) We write $\mathbb{P}_{\mathbf{T}}\restriction\gamma$ instead of $\mathbb{P}_{\mathbf{T}\restriction\gamma}$.
- $\mathbf{ZFC}^-$ is the axioms of $\mathbf{ZFC}$ without the power set axiom.
- Assuming that we work in L, let, for any JJ-system $\mathbf{T}$ of countable length, $\vartheta(\mathbf{T})$ denote the least (countable) ordinal ϑ such that L_ϑ models $\mathbf{ZFC}^-$, $\mathbf{T} \in \mathrm{L}_\vartheta$, and both $\mathbb{P}_{\mathbf{T}}$ and $|\mathbf{T}|$ are countable in L_ϑ. Let $\mathscr{M}(\mathbf{T}) = \mathrm{L}_{\vartheta(\mathbf{T})}$.

The final requirement is:

(8) If $\lambda < |\mathbf{T}|$ is a limit ordinal, and $D \in \mathscr{M}(\mathbf{T}\restriction\lambda)$ is a dense subset of $\mathbb{P}_{\mathbf{T}}\restriction\lambda$, then every $t \in \bigcup_k T_k(\lambda)$ is weakly compatible in $\mathbb{P}_{\mathbf{T}}$ with an element of D.

2. Construction of JJ-systems

Let us describe how countable JJ-systems extend to longer systems.

- We say that a normal tree T is *sum-dense* if, for all $\beta < \alpha < |T|$, $\varepsilon \in \mathbb{Q}^+$, $t, t' \in T(\beta)$, and $s \in T(\alpha)$, if $t \subset s$ then there exists $s' \in T(\alpha)$ such that $t' \subset s'$ and $\sum(s, s') - \sum(t, t') < \varepsilon$.

Definition 3. An *embrion* is a JJ-system $\mathbf{T}$ of countable length such that T_0 and every $T[t]$ are sum-dense trees. `Emb` is the set of all embrions.

An embrion $\mathbf{T}'$ *extends* $\mathbf{T}$, symbolically $\mathbf{T} \preccurlyeq \mathbf{T}'$, if $\mathbf{T} = \mathbf{T}'\restriction|\mathbf{T}|$. □

Requirements (2) and (6) of Section 1 determine the construction of a unique embrion of length 2. They also show that any embrion of length $\lambda + k$, where λ is a limit ordinal or 0 while $k \ge 2$, admits a unique extension to a embrion of length $\lambda + k + 1$. Clearly the limit of an increasing countable sequence of embrions is an embrion. The following lemma carries out the non-trivial step. It appears to be technically easier to jump from a limit λ immediately to $\lambda + 2$, without a stop at level $\lambda + 1$ – which is reflected in the lemma.

Lemma 4 (assuming $\mathbf{V} = \mathbf{L}$). *Let* $\mathbf{T} = \langle\langle T_k\rangle_{k\in\omega}, T[\cdot]\rangle$ *be an embrion of a limit length* λ. *Then there is an embrion of length* $\lambda + 2$, *extending* $\mathbf{T}$.

Proof. ([4], pp. 283–285.) We have to define the levels $T_k(\lambda)$ and $T_k(\lambda+1)$ and appropriately extend the map $T[\cdot]$. Possible elements of any $T_k(\lambda)$ are branches $b \in (\mathbb{Q}^+)^\lambda$ such that $b \restriction \alpha \in T_k(\alpha)$ for all $\alpha < \lambda$. Let $T[b] = \bigcup_{\alpha<\lambda} T[b \restriction \alpha]$ for any such b. Here the problem is to suitably choose countably many branches b of this kind for any k.

Define $\mathscr{M} = \mathscr{M}(\mathbf{T})$ as in Section 1. Consider the forcing notion $P \in \mathscr{M}$, whose typical element p consists of:

a*. d_p, a finite subset of the set $E = \bigcup_{1\le k<\omega} \omega^k$, such that: if $u \in d_p$ and $1 \le k < |u|$ (the length of u) then $u \restriction k \in d_p$.

b*. For any $u \in d_p$: an element $t_p(u) \in T_{|u|-1}$.

c*. If $u, v \in d_p$, $|u| = |v| = k$, $u \restriction (k-1) = v \restriction (k-1)$ then: $\varepsilon_p(u,v) \in \mathbb{Q}^+$.

It is required that:

d*. If $u \in d_p$ and $v = u^\wedge i \in d_p$ then $t_p(v) \in T[t_p(u)]$.

e*. If $u, v \in d_p$, $|u| = |v| = k$, and $u \restriction (k-1) = v \restriction (k-1) = w$, then $|t_p(u)| = |t_p(v)|$ and $\sum(t_p(u), t_p(u)) < \varepsilon_p(u,v)$.

We set $p \le q$ (that is, q is stronger) iff $d_p \subseteq d_q$, $\varepsilon_p(u,v) = \varepsilon_q(u,v)$ whenever the former is defined, and $t_p(u) \subseteq t_q(u)$ for all $u \in d_p$.

A cumbersome verification in [4], based in particular on the sum-density, essentially shows that any P-generic over $\mathscr{M}$ set $G \subseteq P$ results in a system of λ-branches $b_u = \bigcup_{p\in G,\, u\in d_p} t_p(u) \in (\mathbb{Q}^+)^\lambda$, where $u \in E$, such that

- if $|u| = k$ then $b_u \restriction \alpha \in T_{k-1}$ and $b_{u^\wedge i} \restriction \alpha \in T[b_u \restriction (\alpha+1)]$ for all α, i;
- if $D \in \mathscr{M}$ is a dense subset of $\mathbb{P}_{\mathbf{T}}$ then for any $u \in E$ there is $v \in E$ such that $u \subseteq v$ and $b_v \restriction \alpha \in D$ for some $\alpha < \lambda$;
- define $B_\Lambda = \{b_{\langle i\rangle} : i \in \omega\}$ and $B_{um} = \{b_{u^\wedge j} : j = 2^m(2i+1)-1 \text{ for some } i\}$ for all $u \in E$ and m : then $T_0 \cup B_\Lambda$ and $W_{um} = T[b_u] \cup B_{um}$, for all m and $u \in E$, are sum-dense sum-regular normal $(\lambda+1)$-trees.

Now, to get an embrion of length $\lambda+2$ extending $\mathbf{T}$, we define

$$T_k(\lambda) = \bigcup_{u\in\omega^k,\, m\in\omega} B_{um} \qquad \text{– in particular, } T_0(\lambda) = B_\Lambda\,;$$

$$T_k(\lambda+1) = \{b^\wedge r : b \in T_k(\lambda) \wedge r \in \mathbb{Q}^+\} \quad \text{– according to (2)};$$

and finally $T[b]$ for $b \in T_k(\lambda)$ as above, and $T[b_u{}^\wedge r_m] = T[b_u] \cup B_{um}$ for all $u \in E$ and m. □

3. The structure of generic extensions

Let $\mathbf{T} = \langle \langle T_k \rangle_{k\in\omega}, T[\cdot] \rangle$ be a JJ-system of a limit length $\lambda \le \omega_1$. Put $\tau = \tau_{\mathbf{T}}$. The following is an easy observation.

2°. *Any $\mathbb{P}_{\mathbf{T}}$-generic extension by a generic set $G \subseteq \mathbb{P}_{\mathbf{T}}$ results in a sequence of λ-branches $C_k = \bigcup (G \cap T_k) \in (\mathbb{Q}^+)^\lambda$, such that $C_k \restriction \alpha \in T_k(\alpha)$ for all $1 \le \alpha < \lambda$, and $\tau(C_{k+1} \restriction \alpha) = C_k \restriction (\alpha+1)$ for all $k \in \omega$ and $1 \le \alpha < \lambda$.*

In this case there is a straightforward procedure of "decoding" the branches C_k from the sequence $\langle q_k \rangle_{k\in\omega} \in (\mathbb{Q}^+)^\omega$, where $q_k = C_k(0) \in \mathbb{Q}^+$:

3°. We begin with the values $C_k \restriction 1 = \langle q_k \rangle$, put $C_k \restriction \alpha + 1 = \tau_{\mathbf{T}}(C_{k+1} \restriction \alpha)$ (by induction on α simultaneously for all n), and take unions at all limit steps.

Thus $\langle C_k \rangle_{k\in\omega}$ is constructible from $\langle q_k \rangle_{k\in\omega}$, via procedure 3°, which "converges" in the sense that

4°. *First*, every q_k must be the 1st term of the 2-term sequence $\tau_{\mathbf{T}}(\langle q_{k+1} \rangle)$. *Second*, the unions at limit steps, in the inductive computation of $C_k \restriction \alpha$, must remain in the trees T_k.

The principal idea of [4] is to arrange things so that, in the $\mathbb{P}_{\mathbf{T}}$-generic extension of L, there exists only one sequence $\mathbf{q} = \langle q_k \rangle_{k\in\omega} \in (\mathbb{Q}^+)^\omega$ for which the procedure 3° "converges". Technically, it is realized in such a way that any two different sequences of rationals, for which the procedure 3° "converges", lead to a collapse of ω_1^{L} in the form of an increasing ω_1^{L}-sequence of rationals. Requirement (7) is the main "ingredient" of the argument.

Lemma 5 (proved in [4]). *Let* $\mathbf{T} = \langle \langle T_k \rangle_{k\in\omega}, T[\cdot] \rangle \in \mathrm{L}$ *be a JJ-system of length* ω_1^{L}. *Then* $\mathbb{P}_{\mathbf{T}}$ *is a* CCC *forcing in* L *and each* T_k *is a Souslin tree in* L. *In addition,*

a) *If* $G \subseteq \mathbb{P}_{\mathbf{T}}$ *is* $\mathbb{P}_{\mathbf{T}}$*-generic over* L *and* $\lambda < \omega_1^{\mathrm{L}}$ *is a limit ordinal then* $G \cap (\mathbb{P}_{\mathbf{T}} \restriction \lambda)$ *is* $\mathbb{P}_{\mathbf{T}} \restriction \lambda$*-generic over* $\mathscr{M}(\mathbf{T} \restriction \lambda)$.

b) *In any* $\mathbb{P}_{\mathbf{T}}$*-generic extension of* L, *there is a non-constructible sequence* $\langle q_k \rangle_{k\in\omega} \in (\mathbb{Q}^+)^\omega$ *for which the procedure* 3° *"converges" as in* 4°.

c) *In any extension of* L, *if there are two different sequences* $\langle q_k \rangle_{k\in\omega}$ *for which the procedure* 3° *"converges" as in* 4°, *then* ω_1^{L} *is countable.*

Proof. To see that $\mathbb{P}_{\mathbf{T}}$ is CCC in L, note that, by (8), for every limit $\lambda < \omega_1^{\mathrm{L}}$, any dense subset $D \subseteq \mathbb{P}_{\mathbf{T}} \restriction \lambda$ which belongs to $\mathscr{M}(\mathbf{T} \restriction \lambda)$ remains pre-dense in $\mathbb{P}_{\mathbf{T}} \restriction (\lambda + 1)$, therefore (by 1°) in $\mathbb{P}_{\mathbf{T}}$ as well. It remains to follow usual patterns. This argument also proves a). As for b), set $C_k = \bigcup (G \cap T_k)$ and $q_k = C_k(0)$. The sequence $\langle q_k \rangle_{k\in\omega}$ is not constructible because otherwise the sequence of branches C_k belongs to L, easily leading to contradiction because constructible Souslin trees T_k cannot have cofinal branches in L.

c) Suppose that $\langle q_k \rangle_{k\in\omega}$ and $\langle q'_k \rangle_{k\in\omega}$ are two different sequences of positive rationals for which the procedure 3° "converges", to resp. branches C_k and C'_k in T_k ($k \in \omega$). Now either $C_0 \neq C'_0$ or there is k such that $C_{k+1} \neq C'_{k+1}$ but $C_l = C'_l$ for all $l \leq k$. (Otherwise $q_k = q'_k$ for all k.) In the "either" case C_0 and C'_0 are two different branches in T_0, which implies, by (7), that there exists a strictly increasing ω_1^{L}-sequence of rationals, namely the sequence of sums $\sum(C_0 \restriction \alpha, C'_0 \restriction \alpha)$, $\alpha < \omega_1^{\mathrm{L}}$, hence ω_1^{L} is countable. The "or" case is similar: if $\alpha < \beta < \omega_1^{\mathrm{L}}$ then $C_{k+1} \restriction \alpha$ and $C'_{k+1} \restriction \alpha$ belong to $T[C_k \restriction \beta]$, therefore the sequence of sums $\sum(C_{k+1} \restriction \alpha, C'_{k+1} \restriction \alpha)$ is strictly increasing. □

To present, in brief, the main result of [4], note that, assuming $\mathrm{V} = \mathrm{L}$, there exists, by Lemma 4, an $\preccurlyeq$-increasing Δ_1^{HC} sequence of embrions $\mathbf{T}_\alpha$, $1 \leq \alpha < \omega_1$, each $\mathbf{T}_\alpha$ being of length $\omega\alpha$. Let a JJ-system $\mathbf{T} = \langle\langle T_k\rangle_{k\in\omega}, T[\cdot]\rangle$ of length ω_1 be the "l imit" of such a sequence, so that $\mathbf{T}_\alpha = \mathbf{T}\restriction\omega\alpha$ for all α. Then both the map $T[\cdot]$ and the trees T_k uniformly on k belong to Δ_1^{HC}.

Theorem 6. (proved in [4]). *Every $\mathbb{P}_{\mathbf{T}}$-generic extension of L does not collapse ω_1^{L} and has the form $\mathrm{L}[a]$, where a is a non-constructible Π^1_2 real singleton in* $\mathrm{L}[a]$.

Proof. In view of Lemma 5, it remains to show that 4° can be expressed, in HC, as a Π_1 property of $\langle q_k\rangle_{k\in\omega}$. But this is rather clear: the formula says that *any sequence of some $\alpha < \omega_1^{\mathrm{L}}$ steps in the "procedure" 3° starting from $\langle q_k\rangle_{k\in\omega}$ and satisfying 4° can be extended by one more step so that 4° is not violated*. This is Π_1^{HC} by the choice of $\mathbf{T}$. □

4. Setup for the proof of the main theorem

Theorem 6 is equal to the main theorem (Theorem 1) for $n = 2$. The proof of the general case below follows the scheme of Jensen and Johnsbråten, but contains one more idea: the final JJ-system $\mathbf{T}$ of length ω_1 must be "generic" in the sense that it intersects all dense Δ^1_n subsets in the partially ordered set of all embrions of limit length.

4.1. Auxiliary forcing relation

We argue in L in this Subsection.

- $\mathtt{EmbL}$ is the set of all embrions of limit length.

Let $\mathbf{T} = \langle\langle T_k\rangle_{k\in\omega}, T[\cdot]\rangle \in \mathtt{EmbL}$. Define $\mathscr{M}(\mathbf{T})$ and $\mathbb{P}_{\mathbf{T}} \in \mathscr{M}(\mathbf{T})$ as in Section 1. We employ a special language to carry out the study of analytic phenomena in $\mathbb{P}_{\mathbf{T}}$-generic extensions. Let $\mathscr{L}$ be the language containing variables $i,\ j, ...$ of type 0 (for natural numbers) and $x,\ y, ...$ of type 1 (for subsets of ω), arithmetical predicates for type 0 and the membership predicate $i \in x$.

Define $\mathtt{Trm}(\mathbf{T})$ to be the set of all $\mathbf{T}$-*terms* for subsets of ω, that is, all sets $\tau \subseteq \mathbb{P}_{\mathbf{T}} \times \omega$. Put $\mathtt{Trm}^*(\mathbf{T}) = \mathtt{Trm}(\mathbf{T}) \cap \mathscr{M}(\mathbf{T})$. Let a $\mathbf{T}$-*formula* be a formula φ of $\mathscr{L}$, some (or all) free variables of which, of types 0 and 1, are substituted by resp. natural numbers and elements of $\mathtt{Trm}^*(\mathbf{T})$. In this case, if $G \subseteq \mathbb{P}_{\mathbf{T}}$ then $\varphi[G]$ will denote the formula obtained by substitution, in φ, of each term $\tau \in \mathtt{Trm}^*(\mathbf{T})$ by the set $\tau[G] = \{l \in \omega : \exists\, \mathbf{t} \in G\ (\langle \mathbf{t}, l\rangle \in \tau)\}$. Thus $\varphi[G]$ is a formula of $\mathscr{L}$ containing subsets of ω as parameters.

Let $\mathbf{T}\Sigma^0_\infty$-*formula* be any $\mathbf{T}$-formula which does not contain quantifiers over variables of type 1. Formulas of the form

$$\exists\, x_1\, \forall\, x_2\, \exists\, x_3\ \ldots\ \forall\, (\exists)\, x_m\ \psi \quad \text{and} \quad \forall\, x_1\, \exists\, x_2\, \forall\, x_3\ \ldots\ \exists\, (\forall)\, x_m\ \psi\,,$$

where $\psi \in \mathbf{T}\Sigma^0_\infty$, will be called resp. $\mathbf{T}\Sigma^1_m$-*formulas* and $\mathbf{T}\Pi^1_m$-*formulas*.

We define $\mathbf{t}\ \mathtt{forc}_{\mathbf{T}}\ \varphi$, a relation intended to approximate true forcing. Here it is assumed that $\mathbf{T} \in \mathtt{EmbL}$, $\mathbf{t} \in \mathbb{P}_{\mathbf{T}}$, and φ is a closed $\mathbf{T}$-formula of one of the classes $\mathbf{T}\Sigma^1_m$, $\mathbf{T}\Pi^1_m$. The definition goes on by induction.

A. If $\varphi \in \mathbf{T}\Sigma^0_\infty \cup \mathbf{T}\Sigma^1_1 \cup \mathbf{T}\Pi^1_1$ then $\mathbf{t}\ \mathtt{forc}_{\mathbf{T}}\ \varphi$ iff ($\mathbf{T}$, $\mathbf{t}$, φ are as above and) $\mathbf{t} \Vdash_{\mathbf{T}} \varphi$, where $\Vdash_{\mathbf{T}}$ is the ordinary forcing in the sense of $\mathscr{M}(\mathbf{T})$ as the initial model and $\mathbb{P}_{\mathbf{T}}$ as the notion of forcing.

B. Let $m \ge 1$, $\varphi(x) \in \mathbf{T}\Pi^1_m$. Define $\mathbf{t}\ \mathtt{forc}_{\mathbf{T}}\ \exists x\, \varphi(x)$, iff there is a term $\tau \in \mathtt{Trm}^*(\mathbf{T})$ such that $\mathbf{t}\ \mathtt{forc}_{\mathbf{T}}\ \varphi(\tau)$.

C. Let $m \ge 2$, φ be a closed $\mathbf{T}\Pi^1_m$ formula. Put $\mathbf{t}\ \mathtt{forc}_{\mathbf{T}}\ \varphi$ iff $\neg\, \mathbf{s}\ \mathtt{forc}_{\mathbf{S}}\ \varphi^-$ for any embrion $\mathbf{S} \in \mathtt{EmbL}$ which extends $\mathbf{T}$ and any $\mathbf{s} \in \mathbb{P}_{\mathbf{S}}$, $\mathbf{s} \ge \mathbf{t}$, where φ^- is the result of the transformation of $\neg\, \varphi$ to $\mathbf{T}\Sigma^1_m$.

The following statement is true for the usual forcing, hence true for the relation $\mathtt{forc}$ restricted on formulas φ in $\mathbf{T}\Sigma^0_\infty \cup \mathbf{T}\Sigma^1_1 \cup \mathbf{T}\Pi^1_1$, while the extension on more complicated formulas is easily carried out by induction.

5°. $\mathbf{t}\ \mathtt{forc}_{\mathbf{T}}\ \varphi$ *and* $\mathbf{t}\ \mathtt{forc}_{\mathbf{T}}\ \varphi^-$ *are incompatible.*

Lemma 7. *If* $\mathbf{t}\ \mathtt{forc}_{\mathbf{T}}\ \varphi$ *and an embrion* $\mathbf{S} \in \mathtt{EmbL}$ *extends* $\mathbf{T}$, $\mathbf{s} \in \mathbb{P}_{\mathbf{S}}$, $\mathbf{s} \ge \mathbf{t}$, *then* $\mathbf{s}\ \mathtt{forc}_{\mathbf{S}}\ \varphi$.

Proof. The induction step is trivial, so we concentrate on the case when φ belongs to $\mathbf{T}\Sigma^0_\infty \cup \mathbf{T}\Sigma^1_1 \cup \mathbf{T}\Pi^1_1$. The key observation is that, by (8), any set $D \in \mathscr{M}(\mathbf{T})$, which is a dense subset of $\mathbb{P}_{\mathbf{T}}$, remains pre-dense in $\mathbb{P}_{\mathbf{S}}$. It follows that, given a $\mathbb{P}_{\mathbf{S}}$-generic over $\mathscr{M}(\mathbf{S})$ set $G \subseteq \mathbb{P}_{\mathbf{S}}$, the restriction $G' = G \cap \mathbb{P}_{\mathbf{T}}$ is $\mathbb{P}_{\mathbf{T}}$-generic over $\mathscr{M}(\mathbf{T})$. It is also clear that $\varphi[G]$ coincides with $\varphi[G']$. It remains to apply usual forcing arguments, together with the fact that sentences of classes Σ^1_1 and Π^1_1 are absolute for transitive models of $\mathbf{ZFC}^-$, to show that $\mathbf{t} \Vdash_{\mathbf{T}} \varphi$ iff $\mathbf{t} \Vdash_{\mathbf{S}} \varphi$. $\square$

Consider the *complexity* of the relation $\mathtt{forc}$.

Let $\varphi(x_1, \dots, x_m, l_1, \dots, l_\mu)$ be a parameter–free formula of $\mathscr{L}$. Put

$$\begin{aligned}\mathtt{Forc}(\varphi) = \{\langle \mathbf{T}, \mathbf{t}, \tau_1, \dots, \tau_m, l_1, \dots, l_\mu\rangle : \mathbf{T} \in \mathtt{EmbL} \wedge \tau_1, \dots, \tau_m \in \mathtt{Trm}^*(\mathbf{T}) \\ \wedge\, \mathbf{t} \in \mathbb{P}_{\mathbf{T}} \wedge l_1, \dots, l_\mu \in \omega \wedge \mathbf{t}\ \mathtt{forc}_{\mathbf{T}}\ \varphi(\tau_1, \dots, \tau_m, l_1, \dots, l_\mu)\}\,.\end{aligned}$$

Theorem 8. *If* φ *is a formula of one of the classes* Σ^0_∞, Σ^1_1, Π^1_1, Σ^1_2, *then* $\mathtt{Forc}(\varphi) \in \Delta^{\mathrm{HC}}_1$. *If* $m \ge 2$ *and* $\varphi \in \Pi^1_m$ *or* Σ^1_{m+1} *then* $\mathtt{Forc}(\varphi) \in \Pi^{\mathrm{HC}}_{m-1}$.

Proof. The base part follows from the uniform $\Delta^{\mathrm{HC}}_1(\mathbf{T})$ definability of the usual forcing $\Vdash_{\mathbf{T}}$ in the model $\mathscr{M}(\mathbf{T})$. The induction step is clear. $\square$

4.2. Forcing to prove the main theorem

Let us fix a natural number $n \ge 3$ for which we prove Theorem 1.

Arguing in L, we easily define, using Theorem 8, an $\preccurlyeq$-increasing Δ^{HC}_n sequence of embrions $\mathbf{T}_\alpha$, $1 \le \alpha < \omega_1$, each $\mathbf{T}_\alpha$ of length $\omega\alpha$, satisfying

6°. If $\alpha < \omega_1$, $\mathbf{t} \in \mathbb{P}_{\mathbf{T}_\alpha}$, and φ is a closed $\mathbf{T}_\alpha\Sigma^1_n$ formula, then there exist $\alpha \le \beta < \omega_1$ and a condition $\mathbf{t}' \in \mathbb{P}_{\mathbf{T}_\beta}$, $\mathbf{t}' \ge \mathbf{t}$, such that $\mathbf{t}'\ \mathtt{forc}_{\mathbf{T}_\beta}\ \varphi$ or $\mathbf{t}'\ \mathtt{forc}_{\mathbf{T}_\beta}\ \varphi^-$.

Let a JJ-system $\mathbf{T} = \langle\langle T_k\rangle_{k\in\omega}, T[\cdot]\rangle$ of length ω_1 be the "limit" of this sequence, so that $\mathbf{T}_\alpha = \mathbf{T}\restriction\omega\alpha$ for all α. Then both the map τ and the trees T_k uniformly on k belong to Δ^{HC}_n (in L).

Let $\Vdash$ be the ordinary forcing in the sense of L as the initial model and $\mathbb{P}_{\mathbf{T}}$ as the notion of forcing. Define $\mathtt{Trm}(\mathbf{T})$ to be the set of all countable sets $\tau \subseteq \mathbb{P}_{\mathbf{T}} \times \omega$, so that $\mathtt{Trm}(\mathbf{T}) = \bigcup_{\alpha<\omega_1} \mathtt{Trm}(\mathbf{T}_\alpha)$. Define $\mathbf{T}\Sigma^1_m$ and $\mathbf{T}\Pi^1_m$ as in Subsection 4.1. Let, finally, $\mathbf{t}\ \mathtt{forc}_{\mathbf{T}}\ \varphi$ mean that $\mathbf{t}\ \mathtt{forc}_{\mathbf{T}_\alpha}\ \varphi$ for some ordinal $\alpha < \omega_1$. The following lemma ties $\Vdash$ and $\mathtt{forc}_{\mathbf{T}}$.

Lemma 9. *Let $1 \le m \le n$. Assume that φ is a closed $\mathbf{T}\Sigma^1_m$ or $\mathbf{T}\Pi^1_m$ formula, and $\mathbf{t} \in \mathbb{P}_{\mathbf{T}}$. Then $\mathbf{t} \Vdash \varphi$ and $\mathbf{t}\ \mathtt{forc}_{\mathbf{T}}\ \varphi^-$ are incompatible.*

Proof. We argue by induction on m. Assume that φ belongs to $\mathbf{T}\Sigma^0_\infty \cup \mathbf{T}\Sigma^1_1 \cup \mathbf{T}\Pi^1_1$. Let α be any ordinal such that $\mathbf{t} \in \mathbb{P}_{\mathbf{T}_\alpha}$ and φ is a formula in $\mathbf{T}_\alpha\Sigma^1_m$ or $\mathbf{T}_\alpha\Pi^1_m$. By definition $\mathbf{t}\ \mathtt{forc}_{\mathbf{T}_\alpha}\ \varphi$ means that $\mathbf{t} \Vdash_\alpha \varphi$, where $\Vdash_\alpha$ is the ordinary forcing in the sense of $\mathscr{M}(\mathbf{T}_\alpha)$ as the initial model and $\mathbb{P}_{\mathbf{T}_\alpha} = \mathbb{P}_{\mathbf{T}}\restriction\omega\alpha$ as the notion of forcing. On the other hand, by a) of Lemma 5, if $G \subseteq \mathbb{P}_{\mathbf{T}}$ is $\mathbb{P}_{\mathbf{T}}$-generic over L then $G_\alpha = G \cap \mathbb{P}_{\mathbf{T}_\alpha}$ is $\mathbb{P}_{\mathbf{T}_\alpha}$-generic over $\mathscr{M}(\mathbf{T}_\alpha)$. Finally, by the choice of α, the formulas $\varphi[G]$ and $\varphi[G_\alpha]$ coincide. It follows, by the usual forcing technique and the absoluteness argument applied in the proof of Lemma 7., that $\mathbf{t} \Vdash \varphi$ iff $\mathbf{t} \Vdash_\alpha \varphi$. In other words, $\Vdash$ and $\mathtt{forc}_{\mathbf{T}}$ coincide for formulas in $\mathbf{T}\Sigma^0_\infty \cup \mathbf{T}\Sigma^1_1 \cup \mathbf{T}\Pi^1_1$, as required.

Now we carry out the step. Prove the result for a $\mathbf{T}\Sigma^1_{m+1}$ formula φ of the form $\exists\, x\ \psi(x)$, assuming that $m < n$. Suppose, on the contrary, that $\mathbf{t} \Vdash \varphi$ and $\mathbf{t}\ \mathtt{forc}_{\mathbf{T}}\ \varphi^-$. As $\mathbb{P}_{\mathbf{T}}$ is CCC by Lemma 5, there is a term $\tau \in \mathtt{Trm}(\mathbf{T})$ such that $\mathbf{t} \Vdash \psi(\tau)$. By 6°, there is a condition $\mathbf{t}' \in \mathbb{P}_{\mathbf{T}}$, $\mathbf{t}' \ge \mathbf{t}$, such that $\mathbf{t}'\ \mathtt{forc}_{\mathbf{T}}\ \psi(\tau)$ or $\mathbf{t}'\ \mathtt{forc}_{\mathbf{T}}\ \psi(\tau)^-$. Clearly we have the latter: otherwise this would contradict the assumption $\mathbf{t}\ \mathtt{forc}_{\mathbf{T}}\ \varphi^-$ by Lemma 7. and 5°. But this contradicts $\mathbf{t} \Vdash \psi(\tau)$ by the induction hypothesis.

Prove the result for a $\mathbf{T}\Pi^1_{m+1}$ formula φ of the form $\forall\, x\ \psi(x)$. Suppose, on the contrary, that $\mathbf{t} \Vdash \varphi$ and $\mathbf{t}\ \mathtt{forc}_{\mathbf{T}}\ \varphi^-$. The latter, by definition, implies $\mathbf{t}\ \mathtt{forc}_{\mathbf{T}}\ \psi(\tau)^-$, for a term $\tau \in \mathtt{Trm}(\mathbf{T})$. On the other hand, the former implies $\mathbf{t} \Vdash \psi(\tau)$, which is a contradiction by the induction hypothesis. □

Corollary 10. *Let φ be a closed parameter-free Σ^1_n or Π^1_n formula. If $G \subseteq \mathbb{P}_{\mathbf{T}}$ is $\mathbb{P}_{\mathbf{T}}$-generic over* L *then φ is true in* $\mathrm{L}[G]$ *iff there is $\mathbf{t} \in G$ such that $t\,\mathtt{forc}_{\mathbf{T}}\,\varphi$.*
(Apply 6° and Lemma 9..) □

4.3. The proof of the main theorem

Let us show that the forcing $\mathbb{P}_{\mathbf{T}}$ suffices for Theorem 1.

Everything here, except I of Theorem 1, is just the same as for Theorem 6. Thus we can concentrate on requirement I of Theorem 1. The next theorem (proved below) is the key part of the proof.

Theorem 11. *Let $m \ge 1$. Assume that φ is a parameter-free Σ^1_m formula, $\mathbf{T}$, $\mathbf{T}' \in \mathtt{EmbL}$, and $\mathbf{t} \in \mathbb{P}_{\mathbf{T}}$, $\mathbf{t}' \in \mathbb{P}_{\mathbf{T}'}$. Then $\mathbf{t}\ \mathtt{forc}_{\mathbf{T}}\ \varphi$ is inconsistent with $\mathbf{t}'\ \mathtt{forc}_{\mathbf{T}'}\ \varphi^-$.*

Assuming this theorem, we can improve the end of Corollary 10. as follows, so that G does not occur any more:

- *... iff there are an embrion* $\mathbf{S} \in \mathtt{EmbL}$ *and* $\mathbf{s} \in \mathbb{P}_{\mathbf{S}}$ *such that* $\mathbf{s}\ \mathtt{forc}_{\mathbf{S}}\ \varphi$.

It immediately follows that, in a $\mathbb{P}_{\mathbf{T}}$-generic extension of L, every Σ^1_n subset of ω belongs to L – actually, is Σ^1_n in L by Theorem 8.

$\square$ (*Theorem 1*)

5. Proof of the homogeneity theorem

In this Section, devoted to the proof of Theorem 11, we argue only in L. The proof is based on transformations of $\mathtt{Emb}$. Let $\mathbf{T} = \langle\langle T_k\rangle_{k\in\omega}, T[\cdot]\rangle$ and $\mathbf{T}' = \langle\langle T'_k\rangle_{k\in\omega}, T'[\cdot]\rangle$ be two embrions, of equal length $\eta < \omega_1$.

- An *isomorphism of* $\mathbf{T}$ *onto* $\mathbf{T}'$ is a $\subset$-isomorphism $h : \mathbb{P}_{\mathbf{T}} \xrightarrow{\text{onto}} \mathbb{P}_{\mathbf{T}'}$ which maps T_0 onto T'_0 and $T[s]$ onto $T'[h(s)]$ for any $s \in \mathbb{P}_{\mathbf{T}}$.

As $T_k \cap T_l = T'_k \cap T'_l = \emptyset$ provided $k \neq l$ (see Section 1), there is no need to split h in a sequence of separate maps $h_k : T_k \xrightarrow{\text{onto}} T'_k$.

Let $\mathtt{Isom}(\mathbf{T}, \mathbf{T}')$ denote the set of all isomorphisms $\mathbf{T}$ onto $\mathbf{T}'$.

5.1. *Existence of isomorphisms*

Let $\mathbf{T} = \langle\langle T_k\rangle_{k\in\omega}, T[\cdot]\rangle$ be an embrion of length $\lambda + 1$, where $\lambda < \omega_1$ is a limit ordinal. For any $b \in T_k(\lambda)$ define $T_+[b] = T[b] \cup U[b]$, where $U[b]$ is the set of all $s \in T_{k+1}(\lambda)$ such that $s \restriction \alpha \in T[b]$ for all $\alpha < \lambda$.

- Say that $\mathbf{T}$ is *top-correct* if, whenever $k \in \omega$, $T_+[b]$ is a normal $(\lambda + 1)$-tree for any $b \in T_k(\lambda)$, and $T_{k+1}(\lambda) = \bigcup_{b\in T_k(\lambda)} T_+[b]$.

In particular, if $\mathbf{T}'$ is an embrion of length at least $\lambda + 2$, then $\mathbf{T} = \mathbf{T}' \restriction (\lambda + 1)$ is easily top-correct.

Lemma 12. *Let* $\mathbf{T} = \langle\langle T_k\rangle_{k\in\omega}, T[\cdot]\rangle$, $\mathbf{T}' = \langle\langle T'_k\rangle_{k\in\omega}, T'[\cdot]\rangle$ *be top-correct embrions of length* $\lambda + 1$, λ *being a limit ordinal,* $\mathscr{M}(\mathbf{T}') \subseteq \mathscr{M}(\mathbf{T})$, $\mathbf{t} \in T_j(\lambda)$, *and* $\mathbf{t}' \in T'_j(\lambda)$. *Then there is an isomorphism* $h \in \mathtt{Isom}(\mathbf{T}, \mathbf{T}') \cap \mathscr{M}(\mathbf{T})$ *such that* $h\mathbf{t} = \mathbf{t}'$.

Proof. The proof is based on the following statement:

7°. *Let* T *and* T' *be countable normal* $(\lambda + 1)$*-trees,* $t \in T(\lambda)$, *and* $t' \in T'(\lambda)$. *Then there is a* $\subset$*-isomorphism* $h : T \xrightarrow{\text{onto}} T'$ *with* $h(t) = t'$.

To prove this, we first define, using a kind of back-and-forth argument, a map $h : T(\lambda) \xrightarrow{\text{onto}} T'(\lambda)$ such that $h(t) = t'$ and, for all $s_1, s_2 \in T(\lambda)$, the maximal $\alpha < \lambda$ such that $s_1 \restriction \alpha = s_2 \restriction \alpha$ is equal to the maximal $\alpha' < \lambda$ such that $h(s_1) \restriction \alpha' = h(s_2) \restriction \alpha'$. Now pull h down: define, for $u \in T(\alpha)$, $\alpha < \lambda$, $h(u) = h(s) \restriction \alpha$, where $s \in T(\lambda)$ is any satisfying $u = s \restriction \alpha$.

Note that, by the top-correctness, for any $s \in T_{k+1}(\lambda)$ there is a unique $b \in T_k(\lambda)$ such that $s \in T_+[b]$, and the same for $s' \in T'_{k+1}(\lambda)$. Applying 7° consecutively for T_0, then for each $T_+[b]$, where $b \in T_0(\lambda)$, then for each $T_+[b]$, where $b \in T_1(\lambda)$, *etc.*, we get a $\subset$-isomorphism $h : \mathbb{P}_{\mathbf{T}} \xrightarrow{\text{onto}} \mathbb{P}_{\mathbf{T}'}$, mapping T_0 onto T'_0 and any $T_+[b]$ onto $T'_+[h(b)]$, with $h(\mathbf{t}) = \mathbf{t}'$.

The construction of h can be maintained in $\mathscr{M}(\mathbf{T})$ because both $\mathbf{T}$ and $\mathbf{T}'$ belong to and are countable in $\mathscr{M}(\mathbf{T})$. □

5.2. Extensions of isomorphisms on longer embrions

Let us assume the following in this Subsection:

8°. $\mathbf{T} = \langle\langle T_k\rangle_{k\in\omega}, T[\cdot]\rangle$ and $\mathbf{T}' = \langle\langle T'_k\rangle_{k\in\omega}, T'[\cdot]\rangle$ are top-correct embrions of one and the same length $\lambda + 1$, $\lambda < \omega_1$ being limit, $\mathscr{M}(\mathbf{T}') \subseteq \mathscr{M}(\mathbf{T})$, and $h \in \mathscr{M}(\mathbf{T}) \cap \mathtt{Isom}(\mathbf{T}, \mathbf{T}')$.

In this case, the action of h can be correctly extended on any embrion $\mathbf{S} = \langle\langle S_k\rangle_{k\in\omega}, S[\cdot]\rangle \in \mathtt{Emb}$ which extends $\mathbf{T}$. Indeed assume that $s \in S_k(\gamma)$. If $\gamma \le \lambda$ then $s \in T_k(\gamma)$, and we put $h^+(s) = h_k(s)$. If $\lambda < \gamma$ then define $s' = h^+(s) \in (\mathbb{Q}^+)^\gamma$ so that $s'\restriction\lambda = h(s\restriction\lambda)$ while $s'(\alpha) = s(\alpha)$ for all $\alpha \ge \lambda$. Let $S'_k = \{h^+(s) : s \in S_k\}$, for each k. To define the associated map $S'[\cdot]$, assume that $s' = h^+(s) \in S'_k$, so that $s \in S_k$. Put $S'[s'] = \{h^+(t) : t \in S[s]\}$. This ends the definition of $\mathbf{S}' = \langle\langle S'_k\rangle_{k\in\omega}, S'[\cdot]\rangle$. We shall write $\mathbf{S}' = h\mathbf{S}$.

Lemma 13. *In this case,* $\mathbf{S}' = h\mathbf{S}$ *is an embrion extending* $\mathbf{T}'$, $\mathscr{M}(\mathbf{S}') = \mathscr{M}(\mathbf{S})$, *and* $h^+ \in \mathtt{Isom}(\mathbf{S}, \mathbf{S}') \cap \mathscr{M}(\mathbf{S})$.

Proof. It suffices to check only (8), (7), and the sum-density for $\mathbf{S}'$ above λ; the rest of requirements is quite obvious.

Prove (8). Let $\eta < |\mathbf{S}| = |\mathbf{S}'|$ be a limit ordinal, and $D' \in \mathscr{M}(\mathbf{S}'\restriction\eta)$ be a dense subset of $\mathbb{P}_{\mathbf{S}'}\restriction\eta$. Prove that any $s' = h^+(s) \in \bigcup_{k\in\omega} S'_k(\eta)$ is compatible with an element of D'. The case $\eta \le \lambda$ is clear: apply (8) for $\mathbf{T}'$. Assume that $\lambda < \eta < |\mathbf{S}'|$. Then $\mathscr{M}(\mathbf{S}'\restriction\eta) \subseteq \mathscr{M}(\mathbf{S}\restriction\eta)$ because $h \in \mathscr{M}(\mathbf{T})$. It follows that the set $D = \{t \in \mathbb{P}_{\mathbf{S}}\restriction\eta : h^+(t) \in D'\}$ belongs to $\mathscr{M}(\mathbf{S}\restriction\eta)$. Moreover, D is a dense subset of $\mathbb{P}_{\mathbf{S}}\restriction\eta$. Therefore s is compatible with an element of D. Then s' is compatible with an element of D', as required.

Prove (7). Suppose that W' is S'_0 or $S'[s']$ for some $s' = h^+(s) \in S'_k$, $\alpha < \eta < |\mathbf{S}'|$, η is limit, and $s'_1 = h(s_1)$, $s'_2 = h(s_2)$ belong to $W'(\eta)$. (Then s_1 and s_2 belong to the set W which is equal to resp. S_0 or $S[s]$.) We have to prove that $\sum(s'_1\restriction\alpha, s'_2\restriction\alpha) < \sum(s'_1, s'_2) < +\infty$. Assume $\lambda < \eta$ (the nontrivial case). To prove the right inequality note that

$$\textstyle\sum(s'_1, s'_2) = \sum(s'_1\restriction\lambda, s'_2\restriction\lambda) + \sum_{\lambda\le\gamma<\lambda}|s_1(\gamma) - s_2(\gamma)|\,,$$

so the result follows from the fact that $\mathbf{S}$ and $\mathbf{T}'$ are embrions. The left inequality is demonstrated similarly.

Finally prove the sum-density. Suppose that W' is S'_0 or $S'[s']$ for some $s' = h(s) \in S'_k$, $\varepsilon \in \mathbb{Q}^+$, $\beta \le \lambda < \alpha \le |W'|$ (the nontrivial case), and

$t'_1 = h(t_1)$, $t'_2 = h(t_2)$ belong to $W'(\beta)$, and $s'_1 = h(s_1) \in W'(\alpha)$, $t'_1 \subset s'_1$. We have to find $s'_2 \in W'(\alpha)$ such that $t'_2 \subset s'_2$ and $\sum(s'_1, s'_2) - \sum(t'_1, t'_2) < \varepsilon$.

Let $u'_1 = s'_1 \restriction \lambda$, so $u'_1 = h(u_1) \in W'(\lambda)$, where $u_1 \in W(\lambda)$ and $W = S_0$ or $W = S[s]$. As $\mathbf{T}' \in \mathtt{Emb}$, there is $u'_2 = h(u_2) \in W'(\lambda)$ ($u_2 \in W(\lambda)$) such that $t'_2 \subset u'_2$ and $\sum(u'_1, u'_2) - \sum(t'_1, t'_2) < \varepsilon/2$. As $\mathbf{S} \in \mathtt{Emb}$, there is $s_2 \in W(\alpha)$ with $u_2 \subset s_2$ and $\sum(s_1, s_2) - \sum(u_1, u_2) < \varepsilon/2$. Now $s'_2 = h(s_2)$ is as required since $\sum(s'_1, s'_2) - \sum(u'_1, u'_2) = \sum(s_1, s_2) - \sum(u_1, u_2)$. □

Let us assume the following stronger version of 8°:

9°. $\mathbf{T}$, $\mathbf{T}'$, λ, and h are as above in 8° and $\mathscr{M}(\mathbf{T}') = \mathscr{M}(\mathbf{T})$.

Now consider any embrion $\mathbf{S} \in \mathtt{EmbL}$ extending $\mathbf{T}$. Define $\mathbf{S}' = h\mathbf{S}$ (also an embrion by Lemma 13., clearly satisfying $\mathscr{M}(\mathbf{S}') = \mathscr{M}(\mathbf{S})$ by 9°), and let $h^+ \in \mathtt{Isom}(\mathbf{S}, \mathbf{S}') \cap \mathscr{M}(\mathbf{S})$ be defined as above. If $\tau \in \mathtt{Trm}(\mathbf{S})$ then $h\tau = \{\langle h^+(t), l\rangle : \langle t, l\rangle \in \tau\}$ belongs to $\mathtt{Trm}(\mathbf{S}')$. Moreover, $h\tau \in \mathtt{Trm}^*(\mathbf{S}')$ whenever $\tau \in \mathtt{Trm}^*(\mathbf{S})$, and further, if (assuming 9°) Φ is an $\mathbf{S}$-formula then the formula $h\Phi$, obtained by changing every term $\tau \in \mathtt{Trm}^*(\mathbf{S})$ in Φ by $h\tau$, is an $\mathbf{S}'$-formula.

Note finally that $h^{-1} \in \mathtt{Isom}(\mathbf{T}', \mathbf{T})$, and the consecutive action of h and h^{-1} on conditions, terms, and formulas, is identity.

Lemma 14. *Assume that* $\mathbf{t} \in \mathbb{P}_{\mathbf{S}}$ *and* Φ *is a* $\mathbf{S}$*-formula. Then* $\mathbf{t}\ \mathtt{forc}_{\mathbf{S}}\ \Phi$ *iff* $h\mathbf{t}\ \mathtt{forc}_{\mathbf{S}'}\ h\Phi$.

Proof. We argue by induction on the complexity of Φ.

Let Φ be a formula in $\mathbf{S}\Sigma^0_\infty \cup \mathbf{S}\Sigma^1_1 \cup \mathbf{S}\Pi^1_1$ (case A in Subsection 4.1). Then h defines, in $\mathscr{M}(\mathbf{S}) = \mathscr{M}(\mathbf{S}')$, an order isomorphism $\mathbb{P}_{\mathbf{S}}$ onto $\mathbb{P}_{\mathbf{S}'}$, such that $\varphi[G]$ is equal to $(h\varphi)[h''G]$ for any set $G \subseteq \mathbb{P}_{\mathbf{S}}$ and any $\mathbf{S}$-formula φ. This implies the result by the ordinary forcing theorems. ($h''G$ is the h-image of G.)

The induction steps B and C in Section 4.1 do not cause any problem. (However Lemma 13. and 9° participate in the consideration of step C.) □

5.3. *Proof of Theorem 11*

Let us suppose, to the contrary, that $\mathbf{t}_0\ \mathtt{forc}_{\mathbf{T}_0}\ \varphi$ and $\mathbf{t}'_0\ \mathtt{forc}_{\mathbf{T}'_0}\ \varphi^-$. We may assume that $\mathbf{T}_0$ and $\mathbf{T}'_0$ are embrions of one and the same limit length $\lambda < \omega_1$. By Lemma 4, there are embrions $\mathbf{S}$ and $\mathbf{S}''$, of length $\lambda + \omega$, extending resp. $\mathbf{T}_0$ and $\mathbf{S}_0$. Then $\mathbf{T} = \mathbf{S} \restriction (\lambda + 1)$ and $\mathbf{T}' = \mathbf{S}'' \restriction (\lambda + 1)$ are top-correct embrions of length $\lambda + 1$, still extending resp. $\mathbf{T}_0$ and $\mathbf{S}_0$.

We can assume that $\mathscr{M}(\mathbf{T}) = \mathscr{M}(\mathbf{T}')$.

(Indeed, suppose that, say, $\vartheta(\mathbf{T}') < \vartheta(\mathbf{T})$. Let $\eta = \lambda + \vartheta(\mathbf{T})$, a limit ordinal. Let $\mathbf{T}_1$ be an embrion of length $\eta + \omega$, extending $\mathbf{T}$. Choose, by Lemma 12., $h \in \mathscr{M}(\mathbf{T}) \cap \mathtt{Isom}(\mathbf{T}, \mathbf{T}')$. Define $\mathbf{T}'_1 = h\mathbf{T}_1$: an embrion of length $\eta + \omega$ by Lemma 13.. Note that $\mathbf{T}_2 = \mathbf{T}_1 \restriction (\eta + 1)$ and $\mathbf{T}'_2 = \mathbf{T}'_1 \restriction (\eta + 1)$ are top-correct embrions of length $\eta + 1$, extending resp. $\mathbf{T}$ and $\mathbf{T}'$. Finally, as η is long enough, we have $h \in \mathscr{M}(\mathbf{T}'_2)$, which easily implies $\mathscr{M}(\mathbf{T}_2) = \mathscr{M}(\mathbf{T}'_2)$. Now we can take $\mathbf{T}_2$ and $\mathbf{T}'_2$ instead of $\mathbf{T}$, $\mathbf{T}'$.)

Fix conditions $\mathbf{t} \in \mathbb{P}_{\mathbf{T}}$ and $\mathbf{t}' \in \mathbb{P}_{\mathbf{T}'}$ such that $\mathbf{t}_0 \le \mathbf{t}$, $\mathbf{t}'_0 \le \mathbf{t}'$, and $\mathbf{t} \in T_j(\lambda)$, $\mathbf{t}' \in T'_j(\lambda)$ for one and the same j. Choose, by Lemma 12., $h \in \mathscr{M}(\mathbf{T}) \cap \mathtt{Isom}(\mathbf{T}, \mathbf{T}')$ such that $h(\mathbf{t}) = \mathbf{t}'$. Then $\mathbf{S}' = h\mathbf{S}$ is an embrion of length $\lambda + \omega$ extending $\mathbf{T}'$ by Lemma 13., and $\mathbf{t}' \in \mathbb{P}_{\mathbf{S}'}$. Then, by Lemma 7., $\mathbf{t}\ \mathtt{forc}_{\mathbf{S}}\ \varphi$ – thus $\mathbf{t}'\ \mathtt{forc}_{\mathbf{S}'}\ h\varphi$ by Lemma 14., – and $\mathbf{t}'\ \mathtt{forc}_{\mathbf{S}'}\ \varphi^-$. However $h\varphi$ is φ because φ does not contain terms, which is a contradiction by 5°.

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