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Elementary Extensions of External Classes in a Nonstandard Universe

Abstract. In continuation of our study of **HST**, *Hrbaček set theory* (a nonstandard set theory which includes, in particular, the **ZFC** Replacement and Separation schemata in the st-$\in$-language, and Saturation for well-orderable families of internal sets), we consider the problem of existence of elementary extensions of inner "external" subclasses of the **HST** universe.

We show that, given a standard cardinal κ, any set $R \subsetneq {}^*\kappa$ generates an "internal" class $\mathbb{S}(R)$ of all sets *standard relatively to* elements of R, and an "external" class $\mathbb{L}[\mathbb{S}(R)]$ of all sets *constructible* (in a sense close to the Gödel constructibility) from sets in $\mathbb{S}(R)$. We prove that under some mild saturation-like requirements for R the class $\mathbb{L}[\mathbb{S}(R)]$ models a certain κ-version of **HST** including the principle of κ^+-saturation; moreover, in this case $\mathbb{L}[\mathbb{S}(R')]$ is an elementary extension of $\mathbb{L}[\mathbb{S}(R)]$ in the st-$\in$-language whenever sets $R \subseteq R'$ satisfy the requirements.

Key words: nonstandard set theory, inner subuniverses, constructibility, iterated elementary extensions.

Introduction

This paper is written in continuation of the series of articles [10]–[13] devoted to set theoretic foundations of nonstandard mathematics in the framework of **HST**, *Hrbaček set theory*, introduced in [11] on the basis of an earlier version of Hrbaček [7].

HST is a theory in the st-$\in$-*language*, containing the membership $\in$ and the standardness st as the basic predicates. The universe $\mathbb{H}$ of all sets (the *universe of discourse*) is arranged by **HST** so that it includes the class $\mathbb{S} = \{x : \mathtt{st}\, x\}$ of all standard sets, together with the bigger class $\mathbb{I} = \{y : \exists^{\mathrm{st}} x\, (y \in x)\}$ of all elements of standard sets, called *internal sets*.

The universe $\mathbb{I}$ is postulated to be an elementary extension of $\mathbb{S}$ in the $\in$-language; in general, $\mathbb{I}$ behaves similarly to the universe of Nelson's *internal set theory* **IST** [16], except for the fact that now there do not exist monstrous sets like an internally finite set containing all standard sets.

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On the other hand, the universe $\mathbb{H}$ of all sets satisfies in **HST** the axioms of Separation and Collection in the $\mathtt{st}$-$\in$-language, which allows us to easily operate with "external sets" (those naturally defined so that the predicate $\mathtt{st}$ occurs in the definitions), which is impossible in theories like **IST** where "external sets" are, generally speaking, not legitimate objects.

In addition, $\mathbb{I}$ is postulated to be κ-saturated, for any well-orderable cardinal κ. Some amount of Choice is provided.[1] Section 1 of the paper contains a brief introduction to **HST**.

The theory **HST** does not include the Power Set axiom, actually incompatible with the unrestricted Saturation property of $\mathbb{I}$. To solve this problem, we introduced in [12] a system of subuniverses $\mathbb{H}_\kappa \subseteq \mathbb{H}$, κ being an infinite standard cardinal, which model $\mathbf{HST}_\kappa$, a version of **HST** with the Saturation axiom suitably restricted by κ (but stronger than simply κ^+-Saturation) and also model the Power Set axiom.

The corresponding internal subclasses $\mathbb{I}_\kappa = \mathbb{H}_\kappa \cap \mathbb{I}$ are elementary submodels of $\mathbb{I}$ and of each other in the $\in$-language. But the classes $\mathbb{H}_\kappa$ are NOT elementary extensions of each other, because they possess essentially different amounts of Saturation, which can be expressed in the $\mathtt{st}$-$\in$-language.

It was mentioned, in the discussion between W. A. J. Luxemburg and the authors in the course of the Oberwolfach meeting (January/February 1994) in nonstandard analysis, that to define representative families of classes of "external" type (those which satisfy the Separation schema in the $\mathtt{st}$-$\in$-language) being elementary extensions of each other would match some known constructions in the model theoretic setting of nonstandard mathematics.

The paper is devoted to this problem.

It turns out that the core of the problem is to define classes which are elementary submodels of some fixed $\mathbb{I}_\kappa$ in the $\mathtt{st}$-$\in$-language, not merely in the $\in$-language. For this purpose, we use classes $\mathbb{S}(R) \subseteq \mathbb{I}_\kappa$ of all sets standard relative to a finite sequence of elements of a given set $R \subseteq {}^*\kappa$. (A generalization of classes introduced by Gordon [6].) These classes are easily provable to be elementary submodels of $\mathbb{I}_\kappa$ and $\mathbb{I}$ in the $\in$-language.

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[1] This differs, in more or less essential details, from other earlier systems of Hrbaček [8] and Kawaï [14], and significantly differs from a more recent development of Ballard and Hrbaček [2] (based on an "anti-well-founded" standard set theory) and Ballard [1], and from the "asterisk" setting of nonstandard analysis, see Chang and Keisler [4], being perhaps closer to a theory introduced by Fletcher [5], at least because the stratified system of subuniverses which Fletcher simply postulates to exist can be defined, even in a more advanced form, as a system of inner classes in **HST**. This also differs from the "extended use of **IST**" by van den Berg [3], essentially a semi-formal treatment of those "external sets" in **IST** which admit a special, although quite broad, kind of $\mathtt{st}$-$\in$-definition.

To expand this property to the $\mathtt{st}$-$\in$-language, we use those sets $R \subseteq {}^*\kappa$ which realize certain types (or, in other words, are in a certain way saturated) — they are called *κ-complete* sets below. Some properties of the classes $\mathbb{I}$ and $\mathbb{I}_\kappa$ described in [10, 12] are employed to prove that a "set-size" collection of types captures the whole proper class of types to be, in principle, realized. (We take advantage of the possibility to convert any $\mathtt{st}$-$\in$-formula to Σ_2^{st} type, that is, to the form $\exists^{\mathrm{st}}\forall^{\mathrm{st}}$ ($\in$-*formula*), equivalent in $\mathbb{I}$, and then to restrict the two leftmost quantifiers by appropriate standard sets.)

It is demonstrated in Section 2 that $\mathbb{S}(R)$ is an elementary submodel of $\mathbb{I}_\kappa$ in the $\mathtt{st}$-$\in$-language, provided the set $R \subseteq {}^*\kappa$ is κ-complete. In particular, $\mathbb{S}(R')$ is an elementary extension of $\mathbb{S}(R)$ in the $\mathtt{st}$-$\in$-language provided the sets $R \subseteq R' \subseteq {}^*\kappa$ are κ-complete.

Section 3 shows how any class of the form $\mathbb{S}(R)$ (where $R \subseteq {}^*\kappa$) can be expanded to an "external" class $\mathbb{L}[\mathbb{S}(R)] \subseteq \mathbb{H}_\kappa$, essentially the class of all sets Gödel-constructible in $\mathbb{H}$, the **HST** universe, from sets $x \in \mathbb{S}(R)$, although in this case the constructibility appears in the form of a rather elementary assembling of sets via definable well-founded trees. (In particular the class $\mathbb{H}_\kappa$ itself turns out to be the same as $\mathbb{L}[\mathbb{I}_\kappa]$.)

Moreover $\mathbb{L}[\mathbb{S}(R)]$ is κ^+-saturated whenever a set $R \subseteq {}^*\kappa$ is κ-complete.

Finally since the construction of $\mathbb{L}[\mathbb{S}(R)]$ is $\mathtt{st}$-$\in$-definable in $\mathbb{S}(R)$, we conclude from the above that $\mathbb{L}[\mathbb{S}(R)]$ is an elementary submodel of $\mathbb{H}_\kappa = \mathbb{L}[\mathbb{I}_\kappa]$ in the $\mathtt{st}$-$\in$-language, hence a model of $\mathbf{HST}_\kappa$, a theory which adequately supports κ^+-saturated nonstandard mathematics. Furthermore $\mathbb{L}[\mathbb{S}(R')]$ is an elementary extension of $\mathbb{L}[\mathbb{S}(R)]$ in the $\mathtt{st}$-$\in$-language provided $R \subseteq R' \subseteq {}^*\kappa$ are κ-complete sets.

Section 4 outlines an application to the problem of getting convenient subuniverses with different "magnitudes" of infinitesimals.

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1. A review of HST

For the convenience of the reader, we begin this section with a brief look at the system of axioms and the set universe of **HST**, then turn to the class of all well-founded sets, obtained from standard sets by a condensation procedure, and finally consider the class of all internal sets. To save space, we shall formulate a number of really simple facts without proof or only with a hint. A complete development is given in our paper [13].

1.1. The axioms

The abbreviations $\exists^{\rm st} x$ and $\forall^{\rm st} x$ (*external quantifiers*) will be used as shorthands for "there exists a standard x" and "for all standard x".

HST deals with three types of sets: standard, internal, and external. *Standard* sets are those x satisfying $\mathtt{st}\, x$. *Internal* sets are those sets x which satisfy $\mathtt{int}\, x$, where $\mathtt{int}\, x$ is the $\mathtt{st}$-$\in$-formula $\exists^{\rm st} y\ (x \in y)$ (saying: x belongs to a standard set). Thus the internal sets are precisely all sets which are elements of standard sets. *External* sets are simply all sets.

We shall use $\mathbb{S}$, $\mathbb{I}$, $\mathbb{H}$ to denote, in **HST**, the classes of all standard and all internal sets, and the universe of all sets, respectively.

We define ${}^\sigma X = \{x \in X : \mathtt{st}\, x\} = X \cap \mathbb{S}$ for any set X.

The list of axioms of **HST** contains the following three parts.

Axioms for the external universe $\mathbb{H}$. This part includes the **ZFC** *Pair, Union, Extensionality, Infinity* axioms, the schemata of *Separation* (= *Comprehension*), *Collection, Replacement* for all $\mathtt{st}$-$\in$-formulas, and the following axiom of *Weak Regularity*: for any nonempty set X there exists $x \in X$ such that $x \cap X$ contains only internal elements.

This group misses the Power Set, Choice, and Regularity axioms of **ZFC**. Choice (see below) and Regularity are added in weaker forms. This is not a sort of incompleteness of the system; in fact each of the three mentioned axioms contradicts **HST**.

Axioms for standard and internal sets. The first three of four items in this group say that $\mathbb{I}$ is a transitive class in $\mathbb{H}$ and an elementary extension of $\mathbb{S}$ in the $\in$-language; in addition, both $\mathbb{S}$ and $\mathbb{I}$ are **ZFC** universes. ($\Phi^{\rm st}$ and $\Phi^{\rm int}$ denote the relativization of a formula Φ to the classes $\mathbb{S}$ and $\mathbb{I}$ respectively.) The last item, *Standardization*, is of key importance for the development of nonstandard mathematics in **HST**.

1) *Transfer* : $\Phi^{\rm st}(x) \implies \Phi^{\rm int}(x)$,
   where Φ is an $\in$-formula containing only standard sets as parameters.

2) $\Phi^{\rm st}$, where Φ is an arbitrary axiom of **ZFC** in the $\in$-language. Thus *the class* $\mathbb{S}$ *of all standard sets models* **ZFC**.

3) *Transitivity of the internal subuniverse* $\mathbb{I}$: $\forall^{\rm int} x\, \forall y \in x\, (\mathtt{int}\, y)$.

4) *Standardization* : $\forall X\, \exists^{\rm st} Y\, ({}^\sigma X = {}^\sigma Y)$.

Axioms for sets of standard size. A set *of standard size* is any set of the form $\{f(x) : x \in {}^\sigma X\}$, where X is a standard set. The following three axioms refer to the notions of a finite set and a natural number, which will be commented upon below.

Saturation : if $\mathcal{X}$ is a set of standard size such that every $X \in \mathcal{X}$ is internal and the intersection $\bigcap \mathcal{X}'$ is nonempty for any finite nonempty $\mathcal{X}' \subseteq \mathcal{X}$ then $\bigcap \mathcal{X}$ is nonempty.

Standard size Choice : Choice in the case when the domain X of the choice function is a set of standard size.

Dependent Choice : with its ordinary formulation. (Will not be used below.)

Blanket agreement. In the remainder of the paper, the reasoning goes on in **HST** unless explicitly indicated otherwise.

LEMMA 1. *If* $X \subseteq \mathbb{I}$ *then* $X \subseteq S$ *for a standard set* S.

PROOF. Each $x \in X$ is internal, hence belongs to a standard set s. By the Collection axiom of **HST**, there is a set B such that every $x \in X$ belongs to a standard $s \in B$. By Standardization, there exists a standard set A containing the same standard elements as B does. We put $S = \bigcup A$ and use the axiom of transitivity of $\mathbb{I}$ and Transfer. ∎

Remark. It was convenient to adjoin one more axiom, *Extension*, and an axiom schema $\mathbf{BST}^{\text{int}}$ (saying that all the axioms of a certain "internal" nonstandard set theory, **BST**, hold in $\mathbb{I}$) to the list of **HST** axioms in our earlier papers [11, 12]. In fact, those additional axioms can be obtained as formal corollaries of the axioms formulated here (see [13]). We explain in subsection 1.3 below how $\mathbf{BST}^{\text{int}}$ follows from **HST**.

1.2. Condensed universe

Let a *well-founded set* mean: a member of a transitive set X such that the restriction $\in \restriction X$ is a well-founded relation. Let $\mathbb{V}$ denote the class of all well-founded sets in **HST**. Then $\mathbb{V}$ is a transitive subclass of the **HST** universe $\mathbb{H}$ well-founded by the membership $\in$.

We define, by $\in$-induction, a standard set ${}^*x \in \mathbb{S}$ for any set $x \in \mathbb{V}$, so that for any $x \in \mathbb{V}$, ${}^*x$ is the unique (uniqueness – by Transfer, the existence – by the axiom of Standardization) standard set X such that ${}^\sigma X = \{{}^*y : y \in x\}$. One easily shows that the map * is an $\in$-isomorphism of $\mathbb{V}$ onto $\mathbb{S}$, hence by Transfer an elementary embedding of $\mathbb{V}$ into $\mathbb{I}$, the universe of all internal sets. In particular $\mathbb{V}$ models **ZFC**.

Clearly $\mathbb{V}$ is a transitive class, and $y \in \mathbb{V}$ implies $x \in \mathbb{V}$ provided $x \subseteq y$. Therefore many basic set theoretic notions, in particular *ordinals*, *cardinals*, *natural numbers*, *finite sets* have one and the same meaning both in $\mathbb{V}$ and $\mathbb{H}$. Furthermore all ordinals (therefore all cardinals and natural numbers as well) in $\mathbb{H}$ belong to $\mathbb{V}$ as $\mathbb{V}$ comprises all well-founded sets.

Ordinals and cardinals. We let $\mathtt{Ord}$ and $\mathtt{Card}$ denote the classes of all ordinals and cardinals, in the sense of $\mathbb{V}$ or $\mathbb{H}$, which is the same by the above. Then $\mathbb{S}$-*cardinals* (*i. e.* cardinals in $\mathbb{S}$) are sets of the form ${}^*\kappa$ where $\kappa \in \mathtt{Card}$, and only such sets; the same for ordinals.

We observe that the following properties are equivalent in **HST** for any set X: 1) X is a set of standard size, 2) X is well-orderable, 3) X can be put in $1-1$ correspondence with a set in $\mathbb{V}$, 4) X can be put in $1-1$ correspondence with a cardinal $\kappa = \{\xi : \xi < \kappa\} \in \mathtt{Card}$.

In particular, the *cardinality* $\mathtt{card}\, X \in \mathtt{Card}$ is defined, in **HST**, for all sets X of standard size, and only those sets.

Natural numbers. $\mathbb{N}$ will be the set of of all natural numbers, in $\mathbb{V}$ or $\mathbb{H}$. (Equally, $\mathbb{N} = \omega$ is the least limit ordinal.) It does not take much effort to prove that ${}^*n = n$ for any $n \in \mathbb{N}$, therefore $\mathbb{S}$-natural numbers are precisely numbers in $\mathbb{N}$. Furthermore ${}^*\mathbb{N}$ is the set of all $\mathbb{I}$-natural numbers, which we shall call *hypernatural numbers*, following the tradition. Using Standardization and Saturation, one easily proves that $\mathbb{N}$ is a proper initial segment of ${}^*\mathbb{N}$.

Finite sets are those which admit a bijection onto a set of the form $n = \{0, 1, 2, ..., n-1\} \in \mathbb{N} = \omega$. *Hyperfinite* sets are those which admit a bijection onto a set of the form $n = \{0, 1, 2, ..., n-1\} \in {}^*\mathbb{N}$.

One easily shows in **HST** that the collections $X^{<\omega}$ of all finite sequences of elements of X and $\mathcal{P}_{\text{fin}}(X) = \{u \subseteq X : u \text{ is finite}\}$ are SETS for any set X, using Collection, even in the absence of the Power Set axiom.

1.3. Internal sets

HST proves that $\mathbb{I}$, the internal universe, models *bounded set theory* **BST**, a theory[2] in the $\mathtt{st}$-$\in$-language, close to *internal set theory* **IST** of Nelson [16]. **BST** includes all of **ZFC** (in the $\in$-language) together with:

Bounded Idealization **BI** :

$$\forall^{\text{stfin}} A\, \exists x \in X\, \forall a \in A\, \Phi(x, a) \iff \exists x \in X\, \forall^{\text{st}} a\, \Phi(x, a)\,;$$

Standardization **S** : $\forall^{\text{st}} X\, \exists^{\text{st}} Y\, \forall^{\text{st}} x\, [x \in Y \iff x \in X \,\&\, \Phi(x)]\,;$

Transfer **T** : $\exists x\, \Phi(x) \implies \exists^{\text{st}} x\, \Phi(x)\,;$

Boundedness **B** : $\forall x\, \exists^{\text{st}} X\, (x \in X)\,.$

[2] Explicitly introduced in [9], but equal to the "internal part" of a theory in [7].

Here Φ must be an $\in$-formula in **BI** and **T**, and Φ may contain only standard sets as parameters in **T**, but Φ can be any **st**-$\in$-formula in **S** and contain arbitrary parameters in **BI** and **S**. $\forall^{\mathrm{stfin}} A$ stands for: *for all standard finite* A. X is a standard set in **BI**.

Thus **BI** is weaker than the Idealization **I** of **IST** (**I** results by replacing in **BI** the set X by the universe of all sets), but the Boundedness axiom **B** (incompatible with **IST**) is added.

PROPOSITION 2. *The class* $\mathbb{I}$ *of all internal sets in* $\mathbb{H}$ *models* **BST**.

PROOF. Only the Bounded Idealization **BI** needs some care. We prove the direction $\Longrightarrow$ in **BI**. (The other direction is implied by the fact that standard $\mathbb{S}$-finite sets contain only standard elements — an easy consequence of the **HST** Standardization.) Applying the **HST** Collection and Lemma 1 we obtain a standard set A such that it is true in $\mathbb{I}$ for all $x \in X$ that $\forall^{\mathrm{st}} a \in A\, \Phi(x,a) \Longrightarrow \forall^{\mathrm{st}} a\, \Phi(x,a)$. It remains to apply Saturation to the family of sets $X_a = \{x \in X : \Phi^{\mathrm{int}}(x,a)\}$, $a \in {}^{\sigma}\!A$. ■

The proposition allows us to accommodate to **HST** the following two important facts proved earlier for **BST**. Everything asserted to be a theorem of **BST** becomes true in $\mathbb{I}$ in the **HST** setting.

THEOREM 3. (Theorem 1.5 in [10]) *If* $\Phi(x_1,\dots,x_n)$ *is a* **st**-$\in$-*formula then there exists a* $\in$-*formula* $\varphi(a,b,x_1,\dots,x_n)$ *such that* **BST** *proves*

$$\forall x_1 \dots \forall x_1 \, [\, \Phi(x_1,\dots,x_n) \iff \exists^{\mathrm{st}} a \, \forall^{\mathrm{st}} b \, \varphi(a,b,x_1,\dots,x_n) \,] . \qquad ■$$

Formulas of the form $\exists^{\mathrm{st}} a \, \forall^{\mathrm{st}} b$ ($\in$-formula) are called Σ_2^{st} *formulas.* Thus in **HST** any **st**-$\in$-formula is equivalent in $\mathbb{I}$ to a Σ_2^{st} formula.

THEOREM 4. (Lemma 1.1 in [12]) *Let* $\varphi(a,b,x)$ *be a* $\in$-*formula. Then* **BST** *proves that for any* X *there exist standard sets* A *and* B *such that*

$$\forall x \in X \, [\, \exists^{\mathrm{st}} a \, \forall^{\mathrm{st}} b \, \varphi(a,b,x) \iff \exists^{\mathrm{st}} a \in A \, \forall^{\mathrm{st}} b \in B \, \varphi(a,b,x) \,] . \qquad ■$$

2. Partially saturated classes and elementary submodels

It turns out that $\mathbb{I}$ admits various subclasses which are elementary submodels of $\mathbb{I}$ in the $\in$-language and contain all standard sets. We introduced in [12] two types of such subclasses, having the additional advantage of modeling the Power Set axiom in a certain "external envelope" of the subclass.

2.1. Partially saturated classes

In particular we defined in [12] a κ^+-saturated subclass[3] $\mathbb{I}_\kappa \subseteq \mathbb{I}$ for any cardinal κ. We present this construction here referring to [12] for some technical details.

Let $\kappa \in \mathbb{V}$ be a fixed infinite cardinal from now on. We put

$$\begin{aligned} \mathbb{I}_\kappa &= \{x : \exists w \in \mathbb{V}\,(x \in {}^*w \;\&\; \mathtt{card}\, w \le \kappa)\} \\ &= \{x : \exists^{\text{st}} X\,(x \in X \;\&\; \mathtt{card}_{\mathbb{S}} X \le {}^*\kappa \text{ in } \mathbb{S})\} \\ &= \{f(\alpha) : \alpha < {}^*\kappa \;\&\; f \text{ is a standard function } \&\; \mathtt{dom}\, f = {}^*\kappa\}\,. \end{aligned}$$

Note that $\mathbb{S} \subseteq \mathbb{I}_\kappa \subseteq \mathbb{I}$. Obviously $\mathbb{I}_\kappa$ unlike $\mathbb{I}$ is not a transitive class.

THEOREM 5. *$\mathbb{I}_\kappa$ is an elementary submodel of $\mathbb{I}$ in the $\in$-language. Furthermore, $\mathbb{I}_\kappa$ is κ^+-saturated: if a family $\{X_\alpha : \alpha < \kappa\} \subseteq \mathbb{I}_\kappa$ satisfies the finite intersection property (in $\mathbb{H}$) then $\mathbb{I}_\kappa \cap \bigcap_{\alpha<\kappa} X_\alpha \ne \emptyset$.*

PROOF. To see that $\mathbb{I}_\kappa$ is an $\in$-elementary submodel of $\mathbb{I}$, let $\Phi(x)$ be an $\in$-formula containing sets in $\mathbb{I}_\kappa$ as parameters. Since each parameter then necessarily belongs to a standard set of cardinality $\le {}^*\kappa$ in $\mathbb{S}$, we can use the **ZFC** Collection in $\mathbb{S}$ and Transfer to get a standard set X of $\mathbb{S}$-cardinality $\le {}^*\kappa$ such that $\exists\, x\, \Phi(x) \implies \exists\, x \in X\, \Phi(x)$.

Let us prove the saturation assertion. It suffices to obtain a set $W \in \mathbb{V}$ of cardinality $\mathtt{card}\, W \le \kappa$ such that the family of sets $Y_\alpha = X_\alpha \cap {}^*W$ satisfies the finite intersection property. To get such a set W, we observe that for any finite $u \subseteq \kappa$ the intersection $\mathbb{I}_\kappa \cap \bigcap_{\alpha\in u} X_\alpha$ is non-empty because $\mathbb{I}_\kappa$ is an elementary submodel of $\mathbb{I}$. (One easily proves that any finite subset of $\mathbb{I}_\kappa$ is an element of $\mathbb{I}_\kappa$ by induction on the number of elements.) Therefore, using the **HST** standard size Choice axiom, we obtain a function f defined on the set $\mathcal{P}_{\text{fin}}(\kappa) \in \mathbb{V}$ so that, for each finite $u \subseteq \kappa$, the value $f(u) \in \mathbb{V}$ satisfies $\mathtt{card}\, f(u) \le \kappa$ and ${}^*(f(u)) \cap \bigcap_{\alpha\in u} X_\alpha \ne \emptyset$. Note that $f \in \mathbb{V}$ since $\mathbb{V}$ is closed under subsets. It remains to define $W = \bigcup_{u \in \mathcal{P}_{\text{fin}}(\kappa)} f(u)$. ∎

2.2. Elementary submodels

Furthermore, the classes $\mathbb{I}_\kappa$, κ being an infinite cardinal in $\mathbb{V}$, admit even smaller elementary submodels, based on the notion of *relative standardness*, due to Gordon [6] (see Peraire [17] for an axiomatic treatment).

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[3] The class of all internal sets of *order* κ, introduced in [9]. In the particular case $\kappa = \aleph_0$, the definition was given by Luxemburg [15]. The general case was first considered in a unpublished version of Hrbaček [8].

Let x and w be internal sets. The set x is *w-standard*, $w\text{-}\mathtt{st}\,x$ in brief, iff there exists a standard function f such that $f(w)$ is defined and equal to x.[4] Let $\mathbb{S}[w] = \{x : w\text{-}\mathtt{st}\,x\}$; a subclass of $\mathbb{I}$.

It turns out that the *Gordon classes* $\mathbb{S}[w]$ are simply ultrapowers of $\mathbb{V}$. Indeed, let $w \in \mathbb{I}$. By definition there exists a set $W \in \mathbb{V}$ such that $w \in {}^*W$. Consider the set $\mathcal{U} = \mathcal{U}_w = \{U \subseteq W : w \in {}^*U\}$, the *associated ultrafilter*. Then each set $U \subseteq W$ and $\mathcal{U}$ itself belong to $\mathbb{V}$, so in fact $\mathcal{U}$ is an ultrafilter over W in $\mathbb{V}$. One easily proves that $\mathbb{S}[w]$ is $\in$-isomorphic to the *$\mathcal{U}$-ultrapower* of $\mathbb{V}$. (Conversely, given an ultrafilter $\mathcal{U} \in \mathbb{V}$, we obtain, using Saturation, a set $w \in \mathbb{I}$ such that $\mathcal{U} = \mathcal{U}_w$.)

Gordon [6] proved that the classes $\mathbb{S}[w]$ are elementary submodels of $\mathbb{I}$ in the $\in$-language. It is a much more difficult problem to obtain subclasses of some $\mathbb{I}_\kappa$ containing all standard sets and being elementary submodels of $\mathbb{I}_\kappa$ IN THE $\mathtt{st}$-$\in$-LANGUAGE, not merely in the $\in$-language. We shall see that some amount of saturation suffices !

Let us put $\mathbb{S}(R) = \bigcup_{w \in R^{<\omega}} \mathbb{S}[w]$ for any set $R \subseteq \mathbb{I}$.

(We recall that $R^{<\omega}$ is the set of all finite sequences of elements of R. One easily proves that $R^{<\omega} \subseteq \mathbb{I}_\kappa$ provided $R \subseteq \mathbb{I}_\kappa$.)

In particular, by definition $\mathbb{I}_\kappa = \mathbb{S}({}^*\kappa)$ for any cardinal $\kappa \in \mathbb{V}$.

DEFINITION 6. Let $\kappa \in \mathbb{V}$ be an infinite cardinal, $\lambda = 2^\kappa$ in $\mathbb{V}$. A set $R \subseteq {}^*\kappa$ is *κ-complete* if for each family $\{X_\alpha : \alpha < \lambda\} \subseteq \mathbb{S}(R)$ of sets $X_\alpha \subseteq {}^*\kappa$, satisfying the finite intersection property (in $\mathbb{H}$), $R \cap \bigcap_{\alpha<\lambda} X_\alpha \neq \emptyset$.

Notice that the κ-completeness of $R \subseteq {}^*\kappa$ implies $R = \mathbb{S}(R) \cap {}^*\kappa$.

Thus the κ-completeness is a special type of $(2^\kappa)^+$-saturation of $\mathbb{S}(R)$: only families of subsets of ${}^*\kappa$ are relevant. The next theorem proves that this suffices for ${}^*\kappa$-saturation for sets not necessarily restricted by ${}^*\kappa$. What is even more important, this also solves the question of elementary submodels in the $\mathtt{st}$-$\in$-language. It is not clear whether a smaller amount of Saturation in Definition 6 would be sufficient.

THEOREM 7. *Let $\kappa \in \mathbb{V}$ be an infinite cardinal. Suppose that $R \subseteq {}^*\kappa$ is a κ-complete set. Then $\mathbb{S}(R)$ is an elementary submodel of $\mathbb{I}_\kappa$ in the $\mathtt{st}$-$\in$-language. In addition $\mathbb{S}(R)$ is κ^+-saturated* (as in Theorem 5).

PROOF. The proof of the property of being an elementary submodel proceeds by induction on the complexity of the formulas involved. Since clearly $\mathbb{S}(R) \subseteq \mathbb{I}_\kappa$, and on the other hand $\mathbb{I}_\kappa$ is definable in $\mathbb{I}$ by a $\mathtt{st}$-$\in$-formula

[4] Another definition of relative standardness also proposed in [6] defines $w\text{-}\mathtt{st}\,x$ iff $x \in f(w)$ where f is a standard function such that $f(w)$ is a hyperfinite set.

with ${}^*\kappa \in \mathbb{S}$ as the only parameter[5], we have to prove the following: given a st-$\in$-formula $\Phi(x)$ with parameters in $\mathbb{S}(R)$, if (1) there exists $x \in \mathbb{I}_\kappa$ satisfying $\Phi(x)$ in $\mathbb{I}$ then (2) such a set x can be found in $\mathbb{S}(R)$.

First of all, we can suppose that $\Phi(\cdot)$ contains only standard sets and some $w_0 \in R^n$, $n \in \mathbb{N}$, as parameters, so Φ will be written as $\Phi(x, w_0)$.

We can further assume that $\Phi(x, \cdot)$ explicitly says that x is an $\mathbb{I}$-ordinal and $x < {}^*\kappa$, simply because $\mathbb{S}(R) \subseteq \mathbb{I}_\kappa = \mathbb{S}({}^*\kappa)$.

It can be also assumed by Proposition 2 and Theorem 3 that $\Phi(x, \cdot)$ is a $\Sigma_2^{\rm st}$ formula with standard parameters. Since $\mathbb{S} \subseteq \mathbb{S}(R)$, the leftmost quantifier $\exists^{\rm st}$ can be eliminated, so that we finally can assume that $\Phi(x, \cdot)$ has the form $\forall^{\rm st} b\, \varphi(b, x, \cdot)$, where φ is an $\in$-formula with standard parameters.

The problem takes the form: if (1) there exists an $\mathbb{I}$-ordinal $\xi < {}^*\kappa$ such that $\forall^{\rm st} b\, \varphi(b, \xi, w_0)$ in $\mathbb{I}$ then (2) such an ordinal ξ exists in R. To show this we restrict the variable b by a standard set of cardinality $\leq {}^*\lambda$ in $\mathbb{S}$.

As $w_0 \in R^n \subseteq \mathbb{I}_\kappa$, there exists a set $W \in \mathbb{V}$ such that $\operatorname{card} W \leq \kappa$ and $w_0 \in {}^*W$. Let, in $\mathbb{I}$, $\Xi_b = \{\langle \xi, w\rangle \in {}^*\kappa \times {}^*W : \neg\, \varphi(b, \xi, w)\}$ for all internal b, so that $\Xi_b \in \mathbb{I}$ for each $b \in \mathbb{I}$ because φ is an $\in$-formula. Applying in $\mathbb{I}$ (which is an $\in$-model of **ZFC**) the **ZFC** Collection and Choice, we get a set B of cardinality $\leq {}^*\lambda$ in $\mathbb{I}$ such that $\forall b\, \exists b' \in B\, (\Xi_b = \Xi_{b'})$.

Such a set B can be chosen in $\mathbb{S}$ by Transfer. Then, by Transfer again, we have $\forall^{\rm st} b\, \exists^{\rm st} b' \in B\, (\Xi_b = \Xi_{b'})$. This implies, in $\mathbb{I}$,

$$\forall \xi < {}^*\kappa\ \forall w \in {}^*W\ [\exists^{\rm st} b \neg\, \varphi(b, \xi, w) \implies \exists^{\rm st} b \in B \neg\, \varphi(b, \xi, w)].$$

We observe that, since B is a standard set satisfying $\operatorname{card} B \leq {}^*\lambda$ in $\mathbb{S}$, there exists a surjection h mapping $\lambda = 2^\kappa$ (a cardinal in $\mathbb{V}$) onto the set ${}^\sigma B$ of all standard $b \in B$. Now the last displayed formula takes the form:

$$\forall \xi < {}^*\kappa\ \forall w \in {}^*W\ [\forall^{\rm st} b\, \varphi(b, \xi, w) \iff \forall \nu < \lambda\, \varphi(h(\nu), \xi, w)]. \qquad (*)$$

Let us define $X_\nu = \{\xi < {}^*\kappa : \varphi(h(\nu), \xi, w_0)\}$ for every $\nu < \lambda$. Hypothesis (1) implies $\bigcap_{\nu<\lambda} X_\nu \neq \emptyset$. Then there exists $\xi \in \mathbb{S}(R) \cap \bigcap_{\nu<\lambda} X_\nu$ (because R is complete). But this implies $\forall^{\rm st} b\, \varphi(b, \xi, w_0)$ by $(*)$, as required.

The proof that the class $\mathbb{S}(R)$ is κ^+-saturated is entirely similar to the proof of the saturation statement in Theorem 5. ■

2.3. The existence of complete sets

Thus the question of the existence of classes being st-$\in$-elementary submodels of $\mathbb{I}_\kappa$ (for an infinite cardinal κ in $\mathbb{V}$) is reduced to the construction of

[5] It is pointed out by the referee that the "occasional" definability of $\mathbb{I}_\kappa$ in $\mathbb{I}$ can be avoided and the proof of the theorem can be carried out completely within $\mathbb{I}_\kappa$ where appropriate versions of theorems 3 and 4 hold.

complete (as in Definition 6) sets $R \subseteq {}^*\kappa$. We give the construction in the particular but quite comprehensive case that R is a set of standard size.

THEOREM 8. *Let $\kappa \in \mathbb{V}$ be an infinite cardinal. Suppose that $R_0 \subseteq {}^*\kappa$ is a set of standard size. Then there exists a κ-complete set $R \subseteq {}^*\kappa$ of standard size satisfying $R_0 \subseteq R$.*

PROOF. Let $\lambda = 2^\kappa$ in $\mathbb{V}$. Suppose that $Q \subseteq R \subseteq {}^*\kappa$. We say that R *completes* Q iff, for every set of the form $\{X_\nu : \nu < \lambda\} \subseteq \mathbb{S}(Q)$ such that $X_\nu \subseteq {}^*\kappa$ for all ν and the intersection $X = \bigcap_{\nu<\lambda} X_\nu$ is non-empty, the intersection $X \cap R$ is non-empty as well.

CLAIM 9. *Let $Q \subseteq {}^*\kappa$ be a set of standard size. Suppose that $\Omega > \lambda$ is an infinite cardinal in $\mathbb{V}$ such that $\mathtt{card}\, Q \le \Omega$ and $\Omega^\lambda = \Omega$. Then there exists a set $R \subseteq {}^*\kappa$ of $\mathtt{card}\, R \le \Omega$ which completes Q.*

PROOF. To obtain a required set R, let us first of all enumerate all the relevant λ-sequences of sets $X \in \mathbb{S}(Q)$, $X \subseteq {}^*\kappa$, by ordinals $\delta < \Omega$.

We present Q as $Q = \{\rho_\gamma : \gamma < \Omega\}$.

Let us consider, in $\mathbb{V}$, a pair of functions, $h : \lambda \times \kappa^{<\omega} \longrightarrow \mathcal{P}(\kappa)$ and $g : \lambda \longrightarrow \Omega^{<\omega}$. Let $\nu < \lambda$. Then $g(\nu) = \langle \gamma_1, ..., \gamma_n \rangle \in \Omega^{<\omega}$. We define $w_\nu = \langle \rho_{\gamma_1}, ..., \rho_{\gamma_n} \rangle \in Q^{<\omega}$, so that $w_\nu \in ({}^*\kappa)^{<\omega}$. We further put $h_\nu(w) = h(\nu, w)$, for any $w \in \kappa^{<\omega}$, so $h_\nu \in \mathbb{V}$ maps $\kappa^{<\omega}$ into $\mathcal{P}(\kappa)$. We finally put $X_\nu^{gh} = ({}^*(h_\nu))(w_\nu)$, hence $\{X_\nu^{gh} : \nu < \lambda\}$ is a set of subsets of ${}^*\kappa$, each X_ν^{gh} being a member of $\mathbb{S}(Q)$ as all functions ${}^*(h_\nu)$ are standard.

One easily proves that for *every* set of the form $\{X_\nu : \nu < \lambda\} \subseteq \mathbb{S}(Q)$ such that $X_\nu \subseteq {}^*\kappa$ for all ν, there exists a pair of functions g, $h \in \mathbb{V}$ of this type satisfying $X_\nu = X_\nu^{gh}$ for all $\nu < \lambda$.

We observe that, since $\Omega^\lambda = \Omega$, the set of all pairs of functions h, g of the mentioned type has cardinality Ω in $\mathbb{V}$. Now the standard size Choice axiom of **HST** allows to pick up a set $Q' = \{\sigma_\delta : \delta < \Omega\} \subseteq {}^*\kappa$, so that, for any such a pair of functions g, h, there exists an index $\delta < \Omega$ satisfying the following: if the intersection $\bigcap_{\nu<\lambda} X_\nu^{gh}$ is non-empty then $\sigma_\delta \in \bigcap_{\nu<\lambda} X_\nu^{gh}$. It remains to define $R = Q \cup Q'$. ∎

We come back to the proof of the theorem. In principle, had we a sufficient amount of choice, the proof could be accomplished by setting $R = \bigcup_{\alpha<\lambda^+} R_\alpha$, where λ^+ is the next cardinal (in $\mathbb{V}$), $R_\alpha = \bigcup_{\beta<\alpha} R_\beta$ at limit steps α, and $R_{\alpha+1}$ completing R_α for all α. Apparently **HST** does not directly provide such kind of constructions involving uncountable Dependent Choice. However, an indirect reasoning leads to the goal.

To begin with, let us consider $U = {}^*(\kappa^\Omega)$, the (standard) set of all internal functions $u : {}^*\Omega \longrightarrow {}^*\kappa$. For every $u \in U$, we put

$$[u] = \{u({}^*\gamma) : \gamma < \Omega\} = \{u(\gamma) : \gamma < {}^*\Omega \ \& \ \mathtt{st}\,\gamma\}.$$

Thus $[u]$ is a subset of ${}^*\kappa$ of cardinality $\leq \Omega$. Conversely, it can be easily proved using the **HST** Saturation axiom that each subset of ${}^*\kappa$ of cardinality $\leq \Omega$ has the form $[u]$ for some $u \in U$. In particular, there exists $u^0 \in U$ such that $[u^0] = R_0$, the given set.

For $u,\ v \in U$, we write $u \prec v$ iff $[v]$ completes $[u]$ (see above).

Let $\vartheta = \lambda^+$ in $\mathbb{V}$. We observe that $\vartheta \leq \Omega$ (because $\Omega^\lambda = \Omega$ in $\mathbb{V}$). Therefore, to prove Theorem 8 it would be sufficient to define a $\prec$-increasing sequence $\langle u_\alpha : \alpha < \vartheta\rangle$ of $u_\alpha \in U$ such that $u_0 = u^0$.

Starting the construction of such a sequence, we note that the relation $u \prec v$ on the standard set P can be expressed in $\mathbb{I}$ by a $\mathtt{st}$-$\in$-formula with standard parameters. It follows from Theorem 3 and Proposition 2 that $u \prec v$ can be expressed by a Σ_2^{st} formula, *i.e.* a formula of the form $\exists^{\mathrm{st}} a\, \forall^{\mathrm{st}} b\, \varphi(a,b,u,v)$, where φ is an $\in$-formula with standard parameters. By Theorem 4 the quantifiers $\exists^{\mathrm{st}} a$ and $\forall^{\mathrm{st}} b$ can be restricted by standard sets, say ${}^*A$ and ${}^*B$. This presents the relation $\prec$ in the form

$$u \prec v \quad \text{iff} \quad \exists\, a \in A\, \forall\, b \in B\; Q_{ab}(u,v),$$

where $A,\ B \in \mathbb{V}$ while $Q_{ab} \subseteq U \times U$ is an internal set for all $a \in A$ and $b \in B$. Let us say that $a \in A$ *witnesses* $u \prec v$ iff $\forall\, b \in B\, Q_{ab}(u,v)$.

For any $\alpha \leq \vartheta$, we let $\mathcal{A}_\alpha$ be the set of all functions $\mathbf{a} : \alpha \times \alpha \longrightarrow A$ such that there exists a function $\mathbf{u} : \alpha \longrightarrow U$ satisfying $\mathbf{u}(0) = u^0$ and the requirement that $\mathbf{a}(\beta,\gamma) \in A$ witnesses $\mathbf{u}(\beta) \prec \mathbf{u}(\gamma)$ whenever $\beta < \gamma < \alpha$. Then each function $\mathbf{a} \in \bigcup_{\alpha \leq \vartheta} \mathcal{A}_\alpha$, every set $\mathcal{A}_\alpha$, and even the sequence $\langle \mathcal{A}_\alpha : \alpha \leq \vartheta\rangle$ belong to $\mathbb{V}$ since this subuniverse is closed under subsets.

Claim 10. *If $\alpha < \vartheta$ then every function $\mathbf{a} \in \mathcal{A}_\alpha$ can be extended by some $\mathbf{a}' \in \mathcal{A}_{\alpha+1}$.*

Proof. By definition there is a function $\mathbf{u} : \alpha \longrightarrow U$ such that $\mathbf{u}(0) = u^0$ and $\mathbf{a}(\beta,\gamma)$ witnesses $\mathbf{u}(\beta) \prec \mathbf{u}(\gamma)$ whenever $\beta < \gamma < \alpha$. There is $u \in U$ satisfying $\mathbf{u}(\beta) \prec u$ for all $\beta < \alpha$. (Indeed if $\alpha = \beta + 1$ then this follows from Claim 9. If α is a limit ordinal then we have $u \in U$ such that $[u] = \bigcup_{\beta<\alpha}[\mathbf{u}(\beta)]$.) Using standard size Choice we get a function $f : \alpha \longrightarrow A$ such that $f(\beta)$ witnesses $\mathbf{u}(\beta) \prec u$ for each $\beta < \alpha$. Now expand $\mathbf{a}$ to the required $\mathbf{a}' \in \mathcal{A}_{\alpha+1}$ by $\mathbf{a}'(\beta,\alpha) = f(\beta)$ for all $\beta < \alpha$. ∎

CLAIM 11. *If* $\alpha \le \vartheta$ *is a limit ordinal and a function* $\mathbf{a} : \alpha \times \alpha \longrightarrow A$ *satisfies* $\mathbf{a} \restriction (\beta \times \beta) \in \mathcal{A}_\beta$ *for all* $\beta < \alpha$ *then* $\mathbf{a} \in \mathcal{A}_\alpha$.

PROOF. Suppose that $\beta < \gamma < \alpha$ and $b \in B$. We let $H_{b\beta\gamma}$ be the set of all internal functions $\eta : {}^*\alpha \longrightarrow U$ which satisfy $Q_{\mathbf{a}(\beta,\gamma)\,b}(\eta({}^*\beta), \eta({}^*\gamma))$ and $\eta({}^*0) = u^0$. The sets $H_{b\beta\gamma}$ are internal because so are all Q_{ab}.

We assert that the intersection $H_\beta = \bigcap_{b \in B;\ \gamma < \gamma' < \beta} H_{b\gamma\gamma'}$ is non-empty for any $\beta < \alpha$. Indeed, since $\mathbf{a} \restriction (\beta \times \beta) \in \mathcal{A}_\beta$, there exists a function $\mathbf{u} : \beta \longrightarrow U$ such that $\mathbf{a}(\gamma, \gamma')$ witnesses $\mathbf{u}(\gamma) \prec \mathbf{u}(\gamma')$ whenever $\gamma < \gamma' < \beta$. The **HST** Saturation axiom gives an internal function η defined on ${}^*\alpha$ and satisfying $\eta({}^*\gamma) = \mathbf{u}(\gamma)$ for all $\gamma < \alpha$. Then $\eta \in H_\beta$.

Then the total intersection $H = \bigcap_{b \in B;\ \beta < \gamma < \alpha} H_{b\beta\gamma}$ is non-empty by Saturation. Let $\eta \in H$. By definition, we have $Q_{\mathbf{a}(\beta,\gamma)\,b}(\eta({}^*\beta), \eta({}^*\gamma))$ whenever $\beta < \gamma < \alpha$ and $b \in B$. Now, to see that $\mathbf{a} \in \mathcal{A}_\alpha$, let us define $\mathbf{u}(\beta) = \eta({}^*\beta)$ for all $\beta < \alpha$. ■

To complete the proof of Theorem 8, we note that claims 10 and 11 imply $\mathcal{A}_\vartheta \ne \emptyset$. Taking any $\mathbf{a} \in \mathcal{A}_\vartheta$, we obtain, by definition, a $\prec$-increasing sequence of terms $u_\alpha = \mathbf{u}(\alpha) \in U$ such that $u_0 = u^0$, as required. ■

3. External closure of an internal class

A general method of extending a class $I \subseteq \mathbb{I}$ to a subclass of $\mathbb{H}$ which models an appropriate part of **HST**, in particular, models the schemata of Separation and Collection in the $\mathtt{st}$-$\in$-language, was introduced in [12]. For the benefit of the reader we start with a review of relevant notation.

3.1. Elementary external sets

It turns out that the internal subuniverse $\mathbb{I}$ "codes" information about quite a big class of external sets. Of course $\mathbb{I}$ does not contain non-internal sets. In particular sets like the set $\mathbb{N}$ of all standard natural numbers do not belong to $\mathbb{I}$. But actually **HST** allows to implicitly incorporate some external sets in $\mathbb{I}$.

Let an *elementary external set* mean an arbitrary (not necessarily internal) $\mathtt{st}$-$\in$-definable subset of an internal set. This looks like an unsound, "metamathematical" definition, but fortunately sets of this type admit a uniform description given by:

$\mathcal{C}_p = \bigcup_{a \in {}^\sigma A} \bigcap_{b \in {}^\sigma B} \eta(a, b)$, where $p = \langle A, B, \eta \rangle \in \mathbb{I}$, A and B are standard sets, η being an internal function defined on $A \times B$.

(And, we recall, ${}^\sigma S = S \cap \mathbb{S}$.)
If $p \in \mathbb{I}$ is not of the mentioned form then we set $\mathcal{C}_p = \emptyset$.

PROPOSITION 12. (see [11]) *The class* $\mathbb{E} = \{\mathcal{C}_p : p \in \mathbb{I}\}$ *contains all internal sets and satisfies Separation for all* $\mathtt{st}$-$\in$*-formulas.* ∎

Thus $\mathbb{E}$ is quite a comprehensive class, containing all internal sets and also those external sets which are $\mathtt{st}$-$\in$-definable in $\mathbb{I}$; it is closed under the Separation schema. On the other hand, one can determine in $\mathbb{I}$, by appropriate $\mathtt{st}$-$\in$-formulas, the truth of the elementary predicates $\mathcal{C}_p \in \mathcal{C}_q$, $\mathcal{C}_p = \mathcal{C}_q$, $\mathtt{st}\,\mathcal{C}_p$, so $\mathbb{E}$ has a definable model in the smaller universe $\mathbb{I}$.

Take notice that each set $X \in \mathbb{E}$ satisfies $X \subseteq \mathbb{I}$ as internal sets contain only internal elements.

3.2. Assembling sets from internal sets

Let $\mathtt{Seq}$ denote the class of all finite sequences of internal sets. (Then $\mathtt{Seq} \subseteq \mathbb{I}$.) For $t \in \mathtt{Seq}$ and $a \in \mathbb{I}$, let $t^\wedge a$ be the sequence in $\mathtt{Seq}$ obtained by adjoining a as the rightmost additional term to t .

A *tree* is a non-empty (possibly non-internal) set $T \subseteq \mathtt{Seq}$ such that, whenever $t', t \in \mathtt{Seq}$ satisfy $t' \subset t$, $t \in T$ implies $t' \in T$. Thus every tree T contains Λ, the *empty sequence*, and satisfies $T \subseteq \mathbb{I}$. A tree T is *well-founded* (*wf tree*, in brief) iff every non-empty set $T' \subseteq T$ contains a $\subseteq$-maximal in T' element.

By $\mathtt{Max}\,T$ we denote the set of all elements $t \in T$, $\subseteq$*-maximal* in T .

Let T be a wf tree. Associate a set $F(t)$ with each $t \in \mathtt{Max}\,T$. If $t \notin \mathtt{Max}\,T$ then define $F(t)$ as the collection of all already defined sets of the form $F(t^\wedge a)$. The following definition realizes this idea.

DEFINITION 13. A *wf pair* is any pair $\langle T, F\rangle$ such that T is a wf tree and F is a function, $F : \mathtt{Max}\,T \longrightarrow \mathbb{I}$. In this case, the family of sets $F_T(t)$ $(t \in T)$ is defined, by induction on the "rank" of t in T, as follows:

1) if $t \in \mathtt{Max}\,T$ then $F_T(t) = F(t)$;
2) if $t \in T \setminus \mathtt{Max}\,T$ then $F_T(t) = \{F_T(t^\wedge a) : t^\wedge a \in T\}$.

We finally put $F[T] = F_T(\Lambda)$. ∎

Let *e.g.* $T = \{\Lambda, \langle a\rangle\}$ and $F(a) = x \in \mathbb{I}$. Then $F[T] = F_T(\Lambda) = \{x\}$.

DEFINITION 14. $\boldsymbol{\pi}$ is the class of all wf pairs $\langle T, F\rangle$ s. t. T, $F \in \mathbb{E}$.[6]

[6] This class of wf pairs was introduced in [12], as $\mathcal{H}$. Note that T, $F \in \mathbb{E}$ is not sufficient for $\langle T, F\rangle \in \mathbb{E}$ because $\mathbb{E}$ contains only subsets of $\mathbb{I}$.

3.3. Sets constructible from internal sets

In principle the equality $\mathbb{L}[\mathbb{I}] = \{F[T] : \langle T, F\rangle \in \boldsymbol{\pi}\}$ does not look like the definition of any kind of constructibility. But it turns out that the class $\mathbb{L}[\mathbb{I}]$ defined this way is the least subclass of $\mathbb{H}$ which includes $\mathbb{I}$ and satisfies **HST** (in particular the Separation and Collection schemata in the $\mathtt{st}$-$\in$-language), which is precisely what in general the class $\mathbb{L}[\mathbb{I}]$ should be (see our paper [13]). Furthermore this gives the same result as the ordinary definition of constructible sets suitably accommodated to the **HST** environment, but with much less effort in this particular case.

However we are rather interested in a proper definition of classes of the form $\mathbb{L}[\mathbb{S}(R)]$, R being a subset of $\mathbb{I}$. (We shall consider only standard size sets R of $\mathbb{I}$-ordinals as in subsections 2.2 and 2.3.)

The plan will be as follows. We first recall the definition of $\mathbb{L}[\mathbb{I}_\kappa]$, a class which models a suitable κ-version of **HST**, from [12]. Then, using the fact that $\mathbb{L}[\mathbb{I}_\kappa]$ is somehow $\mathtt{st}$-$\in$-expressible in $\mathbb{I}_\kappa$ (although $\mathbb{L}[\mathbb{I}_\kappa] \not\subseteq \mathbb{I}_\kappa$) and, on the other hand, that classes of the form $\mathbb{S}(R)$, where R is κ-complete, are elementary submodels of $\mathbb{L}[\mathbb{I}_\kappa]$ in the $\mathtt{st}$-$\in$-language, we simply restrict the definition of $\mathbb{L}[\mathbb{I}_\kappa]$ to $\mathbb{S}(R)$, getting the required class $\mathbb{L}[\mathbb{S}(R)]$.

Let κ be a fixed infinite cardinal in $\mathbb{V}$, the desired amount of saturation in the universes we are going to define.

DEFINITION 15. $\mathbb{E}_\kappa = \{\mathcal{C}_p : p \in \mathbb{I}_\kappa\}$.

One might define $\mathbb{L}[\mathbb{I}_\kappa]$ as the collection of all sets of the form $F[T]$ where $\langle T, F\rangle$ is a wf pair in $\boldsymbol{\pi}$ such that both T and F belong to $\mathbb{E}_\kappa$. However this would not be a good definition because such a class is not extensional: it contains different sets having the same elements in $\mathbb{L}[\mathbb{I}_\kappa]$. This obstacle led us to a more sophisticated definition in [12].

DEFINITION 16. $\boldsymbol{\pi}_\kappa$ is the collection of all wf pairs $\langle T, F\rangle \in \boldsymbol{\pi}$ such that both T and F belong to $\mathbb{E}_\kappa$, $T \subseteq \mathbb{I}_\kappa$, $F : \mathtt{Max}\, T \longrightarrow \mathbb{I}_\kappa$, and T does not contain elements $t \in T$ such that there exists a set $I \in \mathbb{I}_\kappa$ satisfying $I \cap \mathbb{I}_\kappa = F_T(t) \neq I$.[7]

We set $\mathbb{L}[\mathbb{I}_\kappa] = \{F[T] : \langle T, F\rangle \in \boldsymbol{\pi}_\kappa\}$ (denoted by $\mathbb{H}_\kappa$ in [12]).

The restriction imposed on elements of the tree T in the definition is not harmful: the removed sets can be suitably replaced by internal sets.

[7] Elements t of this type are called *κ-illegal* in [12]. Alternatively one could get the same result by the following change in the definition of $F[T]$. For any $t \notin \mathtt{Max}\, T$, if t is not illegal then still $F_T(t) = \{F_T(t^\wedge a) : t^\wedge a \in T\}$. Otherwise $F_T(t) = I$, where $I \in \mathbb{I}_\kappa$ witnesses that t is illegal. With this change, one eliminates the notion of illegality.

To match the universe $\mathbb{L}[\mathbb{I}_\kappa]$, we let $\mathbf{HST}_\kappa$ be the theory containing:

(1) All of **HST**, with the following reservations: first, $\mathtt{card}\,\mathfrak{X} \le \kappa$ in Saturation, and second, the domain of the choice function claimed to exist in the standard size Choice axiom is a set of cardinality $\le \kappa$;

(2) The Power Set axiom.

Of course (1) is weaker than **HST**, but the difference is relevant only to the amount of Saturation and standard size Choice. As soon as we fix a particular application, where all the cardinals involved are naturally bounded by a certain cardinal, the possibilities (1) offers are practically equal to those of **HST**; but we now have the Power Set axiom !

The following theorem is proved in [12] (Theorem 3.10).

THEOREM 17. *$\mathbb{L}[\mathbb{I}_\kappa]$ models $\mathbf{HST}_\kappa$. In addition $\mathbb{I} \cap \mathbb{L}[\mathbb{I}_\kappa] = \mathbb{I}_\kappa$ and each set $X \subseteq \mathbb{L}[\mathbb{I}_\kappa]$ of cardinality $\mathtt{card}\,X \le \kappa$ belongs to $\mathbb{L}[\mathbb{I}_\kappa]$.* ■

It follows that $\mathbb{I}_\kappa$ is the class of all formally internal sets in $\mathbb{L}[\mathbb{I}_\kappa]$ (those satisfying the formula $\mathtt{int}\,x$, *i. e.* $\exists^{\mathrm{st}} X\,(x \in X)$).

3.4. Definability of the construction

We now prove that the construction of $\mathbb{L}[\mathbb{I}_\kappa]$ can be expressed in $\mathbb{I}_\kappa$ by suitable $\mathtt{st}$-$\in$-formulas. The principal step is to get definability in $\mathbb{E}$, the class of all elementary external sets; the result then will be strengthened to definability in $\mathbb{I}$ and finally to $\mathbb{I}_\kappa$.

LEMMA 18. *The class $\boldsymbol{\pi} \subseteq \mathbb{E} \times \mathbb{E}$ and the relations $F[T] = F'[T']$ and $F[T] \in F'[T']$ (in four arguments) and $\mathtt{st}\,F[T]$ (in two arguments) are definable in $\mathbb{E}$ by parameter-free $\mathtt{st}$-$\in$-formulas. The class $\boldsymbol{\pi}_\kappa$ is definable in $\mathbb{E}$ by a $\mathtt{st}$-$\in$-formula which contains only ${}^*\kappa \in \mathbb{S}$ as a parameter.*

PROOF. Consider the class $\boldsymbol{\pi}$ of all wf pairs $\langle T, F\rangle$. It would be sufficient to prove that the notion of being a well-founded tree is absolute for $\mathbb{E}$. In other words we have to prove that if a tree $T \in \mathbb{E}$, $T \subseteq \mathsf{Seq}$ is well-founded in $\mathbb{E}$ then it is well-founded in $\mathbb{H}$, the universe of all sets.

We observe that, since $\mathbb{E}$ satisfies Separation (in the $\mathtt{st}$-$\in$-language) and contains all internal sets, the well-foundedness of T in $\mathbb{E}$ implies the existence of a rank function in $\mathbb{E}$, mapping T into $\mathbb{S}$-ordinals. But such a function witnesses that T is well-founded in the universe as well.

As for the equality $F[T] = F'[T']$, it actually means the existence of a "computation" of truth values of equalities of the form $F_T(t) = F'_{T'}(t')$

(where $t \in T$ and $t' \in T'$) by induction on the ranks of t in T and t' in T', which gives `true` for the final equality $F_T(\Lambda) = F'_{T'}(\Lambda)$. Assuming that T, T', F, F' belong to $\mathbb{E}$, the unique existing computation belongs to $\mathbb{E}$ because this class satisfies Separation.

The other relations, $\in$ and $\mathtt{st}$, can be reduced to the equality. ■

COROLLARY 19. *There exist* $\mathtt{st}$-$\in$-*formulas* $\mathtt{P}(\cdot,\cdot,\cdot)$, $\mathtt{ST}(\cdot,\cdot)$, $\mathtt{EQ}(\cdot,\cdot,\cdot,\cdot)$, *and* $\mathtt{IN}(\cdot,\cdot,\cdot,\cdot)$ *such that for any infinite cardinal* $\kappa \in \mathbb{V}$ *and all sets* p, p', q, $q' \in \mathbb{I}_\kappa$ *we have*

$$\begin{array}{rlll} \mathtt{P}({}^*\kappa, p, q) & \textit{is true in } \mathbb{I}_\kappa & \textit{iff} & \langle \mathcal{C}_q, \mathcal{C}_p \rangle \in \boldsymbol{\pi}_\kappa \quad ; \\ \mathtt{ST}(p,q) & \textit{is true in } \mathbb{I}_\kappa & \textit{iff} & \mathtt{st}\, \mathcal{C}_p[\mathcal{C}_q] \quad ; \\ \mathtt{IN}(p,q,p',q') & \textit{is true in } \mathbb{I}_\kappa & \textit{iff} & \mathcal{C}_p[\mathcal{C}_q] \in \mathcal{C}_{p'}[\mathcal{C}_{q'}] \quad ; \\ \mathtt{EQ}(p,q,p',q') & \textit{is true in } \mathbb{I}_\kappa & \textit{iff} & \mathcal{C}_p[\mathcal{C}_q] = \mathcal{C}_{p'}[\mathcal{C}_{q'}] \quad . \end{array}$$

PROOF. Consider the first item; the other three of them are treated similarly. It follows from Lemma 18 that there exists a $\mathtt{st}$-$\in$-formula $\mathtt{P}'(\cdot,\cdot,\cdot)$ such that, for all T, $F \in \mathbb{E}$, $\langle T, F \rangle \in \boldsymbol{\pi}_\kappa$ iff $\mathtt{P}'({}^*\kappa, T, F)$ is true in $\mathbb{E}$.

By the definition of $\mathbb{E}$ there exists another $\mathtt{st}$-$\in$-formula $\mathtt{P}''(\cdot,\cdot,\cdot)$ such that $\langle \mathcal{C}_q, \mathcal{C}_p \rangle \in \boldsymbol{\pi}_\kappa$ iff $\mathtt{P}''({}^*\kappa, p, q)$ is true in $\mathbb{I}$.

Let $\mathtt{P}(\cdot,\cdot,\cdot)$ be a Σ_2^{st} formula equivalent in $\mathbb{I}$ to $\mathtt{P}''(\cdot,\cdot,\cdot)$ (we refer to Theorem 3). Notice that $\mathtt{P}$, as any other Σ_2^{st} formula, is absolute for $\mathbb{I}_\kappa$ because this class is an elementary submodel of $\mathbb{I}$ in the $\in$-language and contains all standard sets. ■

3.5. Constructibility over internal subuniverses

Suppose now that $\kappa \in \mathbb{V}$ is an infinite cardinal, as above, and $R \subseteq {}^*\kappa$, so that $\mathbb{S} \subseteq \mathbb{S}(R) \subseteq \mathbb{I}_\kappa = \mathbb{S}({}^*\kappa) \subseteq \mathbb{I}$. The following definition introduces "lightface" counterparts of the notions above.

DEFINITION 20. Let $R \subseteq {}^*\kappa$. $\pi_\kappa(R)$ is the collection of all wf pairs $\langle T, F \rangle \in \boldsymbol{\pi}_\kappa$ such that $T = \mathcal{C}_p$ and $F = \mathcal{C}_q$ for some p, $q \in \mathbb{S}(R)$.

We define $\mathbb{L}[\mathbb{S}(R)] = \{F[T] : \langle T, F \rangle \in \pi_\kappa(R)\}$.

In particular $\pi_\kappa({}^*\kappa) = \boldsymbol{\pi}_\kappa$ and $\mathbb{L}[\mathbb{S}({}^*\kappa)] = \mathbb{L}[\mathbb{I}_\kappa]$.

This is also a kind constructibility: the definition can be converted to a sort of ordinary definition of the class of all sets constructible (in the **HST** universe $\mathbb{H}$) in the usual sense from sets in $\mathbb{S}(R)$, although with some care as $\mathbb{S}(R)$ is a non-transitive proper class.

THEOREM 21. *Suppose that* $R \subseteq {}^*\kappa$ *is a* κ*-complete set. Then* $\mathbb{L}[\mathbb{S}(R)]$ *is an elementary submodel of* $\mathbb{L}[\mathbb{I}_\kappa]$ *in the* $\mathtt{st}$-$\in$*-language. In particular* $\mathbb{L}[\mathbb{S}(R)]$ *models* $\mathbf{HST}_\kappa$ (by Theorem 17). *In addition,*

(i) $\mathbb{L}[\mathbb{S}(R)] \cap \mathbb{I} = \mathbb{S}(R)$ *and* $\mathbb{L}[\mathbb{S}(R)] \cap {}^*\kappa = R$;

(ii) *any set* $X \subseteq \mathbb{L}[\mathbb{S}(R)]$ *of cardinality* $\mathtt{card}\, X \le \kappa$ *belongs to* $\mathbb{L}[\mathbb{S}(R)]$;

(iii) *the class* $\mathbb{S}(R)$ *is* κ^+*-saturated in* $\mathbb{L}[\mathbb{S}(R)]$: *if* $\{X_\alpha : \alpha < \kappa\}$ *is a family of internal sets* $X_\alpha \in \mathbb{S}(R)$ *satisfying the finite intersection property then the intersection* $\mathbb{L}[\mathbb{S}(R)] \cap \bigcap_{\alpha<\kappa} X_\alpha$ *is nonempty.*

PROOF. Recall that $\mathbb{S}(R)$ is a $\mathtt{st}$-$\in$-elementary submodel of $\mathbb{I}_\kappa = \mathbb{S}({}^*\kappa)$ by Theorem 7. Therefore $\mathbb{L}[\mathbb{S}(R)]$ is actually an elementary submodel of $\mathbb{L}[\mathbb{I}_\kappa]$ in the $\mathtt{st}$-$\in$-language, by Corollary 19.

(i) Suppose that $x \in \mathbb{L}[\mathbb{S}(R)] \cap \mathbb{I}$. Then $x \in \mathbb{I}_\kappa$ as $\mathbb{L}[\mathbb{S}(R)] \subseteq \mathbb{L}[\mathbb{I}_\kappa]$ while $\mathbb{L}[\mathbb{I}_\kappa] \cap \mathbb{I} = \mathbb{I}_\kappa$ by Theorem 17. By definition there exist $p,\, q \in \mathbb{S}(R)$ such that $x = F[T]$ where $T = \mathcal{C}_p$, $F = \mathcal{C}_q$, and $\langle T, F\rangle \in \boldsymbol{\pi}_\kappa$. We can easily define sets λ and ϕ_x for any internal x in an absolute way so that $\mathcal{C}_\lambda = \{\Lambda\}$ and $\mathcal{C}_{\phi_x} = \{\langle \Lambda, x\rangle\}$ for all x; then clearly $\langle \mathcal{C}_\lambda, \mathcal{C}_{\phi_x}\rangle \in \boldsymbol{\pi}_\kappa$ whenever $x \in \mathbb{I}_\kappa$, and $\mathcal{C}_{\phi_x}[\mathcal{C}_\lambda] = x$. We conclude that the statement "*there exists* x *satisfying* $\mathtt{EQ}(p, q, \lambda, \phi_x)$ " is true in $\mathbb{I}_\kappa$ by the choice of the formula $\mathtt{EQ}$ (see Corollary 19). Then it is true in $\mathbb{S}(R)$ as well because $\mathbb{S}(R)$ is a $\mathtt{st}$-$\in$-elementary submodel of $\mathbb{I}_\kappa$ by Theorem 7. This yields a set $x' \in \mathbb{S}(R)$ such that $x' = F[T]$; in other words, $x \in \mathbb{S}(R)$.

(ii) Let $X \subseteq \mathbb{L}[\mathbb{S}(R)]$ be a set of cardinality $\mathtt{card}\, X \le \kappa$. By the **HST** standard size Choice axiom we have $X = \{x_\alpha : \alpha < \kappa\}$ where $x_\alpha = F_\alpha[T_\alpha]$ and $\langle T_\alpha, F_\alpha\rangle \in \pi_\kappa(R)$ for all $\alpha < \kappa$. Using the axiom of Standardization and Theorem 7 (the fact that $\mathbb{S}(R)$ is κ^+-saturated) we easily present the sets T_α and F_α in the form $T_\alpha = \mathcal{C}_{t({}^*\alpha)}$ and $F_\alpha = \mathcal{C}_{f({}^*\alpha)}$ where $t,\, f \in \mathbb{S}(R)$ are functions defined on ${}^*\kappa$.

It is clear that $X = F[T]$, where the wf pair $\langle T, F\rangle$ is defined by $T = \{\Lambda\} \cup \{({}^*\alpha)^\wedge t : \alpha < \kappa \;\&\; t \in T_\alpha\}$ and $F(({}^*\alpha)^\wedge t) = F_\alpha(t)$ for all $\alpha < \kappa$ and $t \in \mathtt{Max}\, T_\alpha$. On the other hand, one obtains in $\mathbb{S}(R)$ (using the functions $t,\, f$) sets $p,\, q \in \mathbb{S}(R)$ such that $T = \mathcal{C}_p$ and $F = \mathcal{C}_q$. Thus if $\langle T, F\rangle$ belongs to $\boldsymbol{\pi}_\kappa$ then $X \in \mathbb{L}[\mathbb{S}(R)]$ by definition.

It remains to consider the case when $\langle T, F\rangle \notin \boldsymbol{\pi}_\kappa$. As $\langle T, F\rangle$ is composed from pairs in $\boldsymbol{\pi}_\kappa$, this can happen only in the case when Λ is κ-illegal (see footnote 7), *i. e.* there exists a set $I \in \mathbb{I}_\kappa$ satisfying $I \cap \mathbb{I}_\kappa = F[T] = X \ne I$. Let us check that this is impossible. Indeed, by Lemma 2.3 in [12], if $I \in \mathbb{I}_\kappa$ and the intersection $I \cap \mathbb{I}_\kappa$ is a set of standard size (as our set X is) then actually $I \subseteq \mathbb{I}_\kappa$, which is a contradiction.

(iii) Each X_α belongs to $\mathbb{S}(R)$ by (i). Now use Theorem 7. ■

4. An application

It was demonstrated by Gordon [6] that an adequate nonstandard treatment of some patterns of standard reasoning in analysis (especially those involving real functions of several variables) needs infinitesimals and infinitely large numbers of different magnitude. More exactly, given an infinitesimal $\alpha > 0$, we have to involve a "much smaller" infinitesimal $\beta > 0$. [8] This is of course equivalent to the following: given an infinitely large integer n, define a "much bigger" number $m > n$.

Classes of the form $\mathbb{L}[\mathbb{S}(R)]$ provide a good environment for such kind of reasoning in **HST**.

Let $\kappa \in \mathbb{V}$ be an infinite cardinal, the amount of saturation we look for.

Assume we consider at the moment a class $\mathbb{L}[\mathbb{S}(R)]$ generated by a κ-complete set $R \subseteq {}^*\kappa$ which is a set of standard size. It easily follows from the **HST** Saturation axiom and Theorem 21(i) that there exists a hypernatural number $m \in {}^*\mathbb{N}$ bigger than all numbers $n \in {}^*\mathbb{N}$ in $\mathbb{L}[\mathbb{S}(R)]$. We are interested to adjoin such a number to $\mathbb{L}[\mathbb{S}(R)]$.

It follows from Theorem 8 that there exists a κ-complete set $R' \subseteq {}^*\kappa$ containing m and all elements of R.

Now the class $\mathbb{L}[\mathbb{S}(R')]$ is an elementary extension of $\mathbb{L}[\mathbb{S}(R)]$ in the st-$\in$-language, containing m, by Theorem 21. Moreover $\mathbb{L}[\mathbb{S}(R')]$ (as well as $\mathbb{L}[\mathbb{S}(R)]$) is a κ^+-saturated universe and a model of $\mathbf{HST}_\kappa$.

We recall that $\mathbf{HST}_\kappa$ is a κ-version of **HST** including (unlike **HST**) the Power Set axiom. Such a theory allows to freely develop κ^+-saturated nonstandard analysis.

5. A problem

Let $R \subseteq {}^*\kappa$. The class $\mathbb{S}(R)$ admits another type of "external envelope". Indeed let $\pi'(R)$ denote the class of all wf pairs $\langle T, F\rangle \in \boldsymbol{\pi}$ such that both T and F are subsets of $\mathbb{S}(R)$. The class $\mathbb{A}(R)$ of all sets of the form $F[T]$ where $\langle T, F\rangle \in \pi'(R)$ may then be viewed as the collection of all sets one can assemble in $\mathbb{H}$ from sets in $\mathbb{S}(R)$.

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[8] Consider for instance a double limit $\lim_{x\to 0}\lim_{y\to 0} F(x,y)$ where F is a standard function. In the spirit of nonstandard analysis one should pick up an infinitesimal α and consider the limit $\lim_{y\to 0} F(\alpha, y)$ of $F(\alpha, y)$ as a function of y. It is inappropriate to replace y by an arbitrary infinitesimal β as $F(\alpha, y)$ as a function of y is a nonstandard function. The point is that β, the second infinitesimal, must be "much smaller" than α, to handle the case in proper way. Taking β so that $|\beta|$ smaller than any α-standard positive infinitesimal, one gets an adequate treatment of the case as it is shown by Gordon [6].

Classes $\mathbb{A}(R)$ give a little bit more than those of the form $\mathbb{L}[\mathbb{S}(R)]$: in particular they also model $\mathbf{HST}_\kappa$; but in addition they model *the full Choice* (in the form that *every set can be well-ordered*; in fact it is true in $\mathbb{A}(R)$ that all sets are sets of standard size), only provided R is a set of standard size. (We refer to [12], Section 4, where a similar type of universes, denoted there by $\mathbb{H}'_\kappa$, is considered.)

The problem is to find reasonable requirements for sets of standard size $R \subseteq R' \subseteq {}^*\kappa$ which would guarantee that $\mathbb{A}(R)$ is an elementary submodel of $\mathbb{A}(R')$ (at least in the $\in$-language). The method applied for the classes $\mathbb{L}[\mathbb{S}(R)]$ does not work as the construction of $\mathbb{A}(R)$ now cannot be expressed in $\mathbb{S}(R)$.

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