The authors build linguistic matrices only for those linguistic variables which yield a linguistic continuum or an ordered linguistic set. Most of the properties enjoyed by real or complex matrices are satisfied by these linguistic matrices. However, we see when we try to define operations on linguistic matrices that they behave in different ways. The possible operations defined on linguistic matrices are only the maximum and minimum. Further, we have different identities for min and max operations on these matrices for the linguistic variable and its associated linguistic words.

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PREFACE

In this book, the authors introduce the linguistic set associated with a linguistic variable and the structure of matrices, which they define as linguistic matrices. The authors build linguistic matrices only for those linguistic variables which yield a linguistic continuum or an ordered linguistic set.

This book is organised into three chapters. The first chapter is introductory, in which we introduce all the basic concepts of linguistic variables and the associated linguistic set to make this book self-contained. Chapter two introduces linguistic matrices and develops basic properties associated with them, like types of matrices, transpose of matrices and diagonal matrices. Most of the properties enjoyed by real or complex matrices are satisfied by these linguistic matrices. Chapter three deals with operations on the linguistic matrices.

However, we see when we try to define operations on linguistic matrices that they behave in very different ways. As the meaning of +; addition, or product × are not defined or cannot be defined on linguistic words, we cannot have the notion of addition or product in the case of linguistic matrices. For instance, the linguistic terms good and very bad cannot be added or multiplied. We see (good + very bad) has no meaning, so on (good × very bad) is not defined. The possible operations that can be defined on linguistic matrices are only the maximum and minimum operations; the maximum is denoted by max, and the minimum operation by min.

Max and min operations can be defined both on the set of linguistic words. Now for the linguistic continuum [worst, best], take good and very bad from the linguistic interval [worst, best], we see min{good, very bad} = very bad and max{ good, very bad} = good. Thus, these operators, min and max, are well defined on the linguistic continuum [worst, best]. Hence, we can define min and max operators on linguistic matrices provided they satisfy compatibility with these operations.

Further, we have different identities for min and max operations on these matrices for the linguistic variable and its associated linguistic words. The empty matrix is one in which every entry is phi (ϕ), the empty symbol.

Results in this direction are developed and discussed. Finally, the notion of linguistic submatrices is discussed and developed. All these concepts are described with examples for the clear understanding of the reader.

Another essential feature of this book is that we suggest many problems at the end of each chapter. As matrix theory happens to be the basis for all mathematical and linguistic models for networks, in AI, in all fuzzy models, these concepts will be a boon to non-mathematicians, for they can quickly work with linguistic matrices in place of usual matrices. This book is an introductory book on linguistic matrices.

We acknowledge Dr K.Kandasamy with gratitude for his continuous support.

W.B.VASANTHA KANDASAMY
K. ILANTHENRAL
FLORENTIN SMARANDACHE

Chapter One

LINGUISTIC VARIABLES, LINGUISTIC TERMS AND THEIR PROPERTIES

In this chapter we proceed on to define the notion of linguistic variables, linguistic terms or linguistic words or linguistic continuum associated with each linguistic variable.

We prove some of these linguistic words / terms form a totally order set, be a finite collection of linguistic terms or be an infinite collection of linguistic terms or be a linguistic continuum. We define on these totally ordered linguistic terms the two binary operation min and max.

At this juncture it is mandatory to recall the notion of linguistic variables was defined in 1975 by Zadeh [25-7]. He defined linguistic variables as a variable whose values are words or sentences in an artificial or in a natural language.

However, we do not use the concept of sentences as we are trying to build a world of linguistic mathematics based on these linguistic terms / words / sets. So throughout this book by w(L), we denote a linguistic set or linguistic words or linguistic terms associated with the linguistic variable L.

As we have different linguistic variables the linguistic word set w(L) will vary.

For instance, if we want to talk about heights of people then the linguistic variable L in this case is the 'height' so the linguistic set / words will be

{tall, tallest, just tall, short, very tall, very very tall, very short, short, just short, very very short, medium, just medium, shortest very very very short, and so on}.

If on the other hand height is to be represented for all humans then this can also be represented as a linguistic continuum and is denoted by the interval S = [shortest, tallest].

This interval [shortest, tallest] corresponds to [0.6 foot, 8 feet], for a human can be shortest at birth say 6 inches to and grow in unusual cases up to 8 feet so numerical [0.6 feet, 8 feet] = [6 inches, 8 feet] is equivalent to the linguistic interval [shortest, tallest].

We see there are infinitely many numbers between 6 and 10; and it is a continuum one for no person after birth suddenly just grows from 1 foot to 1 foot 2" or so on, as time goes the growth take place very slowly from very small fractions of an inch to full inch and so on.

So [0.6 foot, 8 feet] is a section of a continuum on the real scale that is number [0.6, 8] is a section or subinterval on the real line, $(-\infty, \infty)$.

Likewise [shortest, tallest] is a linguistic continuum and shortest may correspond to $\frac{1}{2}$ feet or 6 inches and largest corresponds to the value 8 feet. Thus instead of giving height in the real scale we can also give the height of a person in a linguistic scale which corresponds to the linguistic continuum.

Now it is very natural for a passerby you see or from a group of people you move with; you and your friend discuss about their heights you cannot say he is $6' \ 2''$ or $6'$ or $5' \ 9''$ but instead you may say tall or just tall or short or very tall or medium height and so on. However in this case the linguistic term of assessing his height is more natural though approximate; than saying $6'2''$ or $6'$ or $5'9''$ for none can be true for instance he may exactly when measured be $6' \ 1''$ but we cannot ask a stranger his exact height or so on.

Thus while assessing the height of a person by seeing him it is more natural and appropriate to express it as a linguistic term. This sort of study when introduced in primary and secondary school children it will not only make them understand in a fearless way but they will be forced to appreciate such study which will develop them with strong basics for logic and common intelligence/sense.

Hence the study of linguistic terms and probable mathematics developed based on these happens to be more natural, appropriate and less artificial. So the need for linguistic terms and linguistic variables are a necessity to remove the grip of mathematics phobia in school children.

Another vital reason for making this study is that Artificial Intelligence functions basically on atomic words and atomic sentences so this linguistic language happens to be very useful. It is also pertinent to keep on record all the linguistic variables need not be dependent on time.

Next we consider the linguistic variable age. Age of any human varies linguistically from [youngest, oldest] in this linguistic continuum. The change in age of a person is also only time dependent. Suppose we have set of 10 people and we are interested in finding their age immediately we has to assess that as we cannot ask them and get, it, then in that case we cannot say one person is 60 years 9 months, another one 40 years 2 months another one 4 years 2 months etc., by look we can just say the person is old, older, just old, very old, middle age, just young, youth, just youth, very young and youngest.

These linguistic words or terms on the linguistic continuum can also be referred to as linguistic features or features associated with age. This form of assigning the age is much more natural than putting the age in number when we are not aware of it.

Suppose we want to assess the age we know it lies numerically in the continuum [0, 100] however in linguistic continuum it lies in the range [youngest, oldest] $\cup$ $\{\phi\}$ where ϕ corresponds to the empty linguistic term as that of 0 in case of [0, 100] in numerals. We say this as a linguistic continuum in par with the real continuum [0, 100].

Now if it is the assessment of colour of a person it is not time variant. The colour of a person may be from dark to white, very dark, darkest, just dark, brown, brownish, just brown so on and so forth.

So if a person is given the task of assessing the colour of a person then he may give any of the terms varying from darkest to white however the colours blue, red, orange, green, violet etc; are removed and no ordering is easily possible for the colour may be light brown or dark brown, or just brown or slight brown and so on. A linguistic continuum cannot be formed it is also time independent. Further we cannot give a comparative order in a very precise way. We can only say black, brown , pale white, pale yellow and white.

So the colour red to yellow ordering is not possible or no such ordering can be made. Only sectional ordering can be made from [dark blue to light blue or say up to white] in a little vague way. However the assignment of colour and its linguistic set is not a continuum so it is not in general, possible to give an ordering.

One can order each colour from dark to light or from light to dark other form of ordering is not possible. Further colours cannot be represented numerically unless the fuzzy form of membership is made which is beyond the scope of this book as we not absolutely interested in such study. We proceed on to give an example of the students' performance in the classroom to see the difference in types of linguistic variables.

The performance of 12 students in a class room is given by the following linguistic set / words / terms S associated with the linguistic variable; performance of the students.

S = {very average, average, good, just average, bad, very bad, just bad, very good, very very good}.

Further it is assumed 3 students performance in the class room is good and two students performance is very bad.

Now certain observations about these linguistic set is vital.

First of all we cannot easily give marks because while assessing without any answer paper written, it is not easy to mark them say between 0 to 100, but however a class teacher or a teacher who teaches them can easily say the performance aspect of a student of her class students are like good, average performer, bad, fair (not that good), very good, or very bad and so on.

Now suppose we try to order them; will it be a totally ordered set or a partially ordered set. It is important in order to make this book a self-contained one we just recall the definition of a totally ordered set and a partially ordered set and illustrate them with some examples.

Definition 1.1. *Let S be a set of elements. We say S is totally ordered if for any pair of elements a, b $\in$ S we have a $\leq$ b (or b $\geq$ a). Thus {S, $\leq$} is a totally ordered set if the elements of S can be put in the form $s_1 \leq s_2 \leq s_3 \leq \ldots \leq s_n$, n can be finite or infinite depending on the number of elements in S.*

$$S = \{ s_1, \ldots, s_n \}.$$

We call $s_1 \leq s_2 \leq s_3 \leq \ldots \leq s_n$; to be an increasing chain or an increasing order.

On the other hand if we write S as

$s_n \geq s_{n-1} \geq s_{n-2} \geq \ldots \geq s_3 \geq s_2 \geq s_1$ *; then we call this as decreasing order or a decreasing chain.*

In this book we only use the increasing order chain just like the line of reals or rationals or integers. For more refer [ ].

Example 1.1. Let S = {–28, 2, 4, –1, 0, 7, 8, 90, –42, +1, 3} be a set of positive and negative integers, S is a totally order finite set the increasing chain of S is as follows.

$$-42 \leq -28 \leq -1 \leq 0 \leq 1 \leq 2 \leq 3 \leq 4 \leq 7 \leq 8 \leq 90$$

Order of S is 11 and that of the chain of S is also of length o(S) = 11.

We now proceed onto give another example of a finite totally ordered set.

Example 1.2. Let S = {{a}, {ϕ}, {a, b, c}, {a, b}, {a, b, c, d}, {a, b, c, d, e}, {a, b, c, d, e, f}, {a, b, c, d, e, f, g}, {a, b, c, d, e, f, g, h}} be a set of order 9.

We show S is a totally ordered set the total ordering is done by the 'inclusion' relation $\subseteq$ that is the set containment relation. We may call the inclusion relation as the containment relation.

$$\{\phi\} \subseteq \{a\} \subseteq \{a,b\} \subseteq \{a, b, c\} \subseteq \{a, b, c, d\} \subseteq \{a, b, c, d, e\} \subseteq \{a, b, c, d, e, f\} \subseteq \{a, b, c, d, e, f, g\} \subseteq \{a, b, c, d, e, f, g, h\} \quad \ldots I$$

This is a chain of order 9 and this S is a totally ordered set.

Consider any subset P of S say

P = {{a}, {a, b, c, d}, {a, b, c, d, e}, {a, b, c, d, e, f, g, h}} $\subseteq$ S.

Clearly P is also a totally ordered set under containment relation for

{a} $\subseteq$ {a, b, c, d} $\subseteq$ {a, b, c, d, e} $\subseteq$ {a, b, c, d, e, f, g, h} …II

II is called the subchain of I and the length of the subchain II is of order four.

We have all lengths of subchains of I from 2 to 8, for the length of the chain I is 9.

Example 1.3. Let R be the line of reals or real continuum.

R is a totally ordered set for any two real numbers are comparable.

The chain associated with R is as follows.

$$-\infty < \ldots < -n < \ldots -1 < 0 < 1 < 2 < \ldots < n < \ldots < \infty \quad \ldots I$$

I is the chain of increasing order associated with R.

In fact R is an infinite chain of real numbers as R the reals is of infinite cardinality and it is a totally ordered set.

The chain I contains subchains of both finite and infinite order.

For take P_1 = {–0.0001, –0.004, 81, –9, –0.01, 1, – 1, 0, 0.001, 0.08, 0.9, –8, 19, 8, 9, 98, 18, 1485} $\subseteq$ R.

The cardinality of P_1 is 18 and P_1 is a totally order set and the subchain of I is as follows.

$$-9 \leq -8 \leq -1 \leq -0.01 \leq 4 \leq -0.001 \leq -0.0001 \leq 0 \leq 0.001 \leq 0.08 \leq 0.9 \leq 8 \leq 9 \leq 18 \leq 19 \leq 81 \leq 98 \leq 1485 \quad \ldots \text{II}$$

II is a subchain of I of finite order, that is length of the subchain II is 18.

Consider the subset of R given by $P_2 = \{-\infty, 0\}$ the total ordering of the infinite set P_2 is

$$-\infty < \ldots < -n < \ldots < -5 \leq -4 \leq -3 \leq -2 \leq -1 < 0 \qquad \ldots \text{III}$$

is the subchain of the chain I.

The subchain III is also increasing one but of infinite order.

Thus the infinite chain can have subchains of both finite and infinite order.

Now having seen examples of totally ordered set now we proceed onto define partially ordered set and give examples of them.

Definition 1.2. *Let S be a nonempty set, we say S is a partially ordered set if there exists at least two distinct elements a, b $\in$ S such that a $\leq$ b (or b $\geq$ a).*

It is to be noted that unlike the totally ordered set in which every pair should be comparable in a partially ordered set where at least two distinct elements are comparable; S can be finite set or an infinite set.

We give examples of partially ordered sets.

Example 1.4. Let P(S) be the power set of the set S.

S = {a, b, c, d, e} and P(S) = {{ϕ}, {a}, {b}, {c}, {d}, {e}, {a, b}, {a, c}, {a, d}, {a, e}, {b, c}, {b, d}, {b, e}, {c, d}, {c, e}, {d, e}, {a, b, c}, {a, b, d}, {a, b, e}, {a, c, d}, {a, c, e}, {a, e, d}, {b, c, d}, {b, c, e}, {b, e, d}, {c, e, d}, {a, b, c, d}, {a, b, c, e}, {a, b, d, e}, {a, c, d, e}, {b, c, d, e}, {a, b, c, d, e} = S}

be the power set of S.

Clearly S is only a partially ordered set as we have subsets in P(S) which are not comparable.

For take {a, e} and {b, d} in P(S) we cannot relate them by a containment relation of subsets of S.

Consider {a, b, c}, {c, d, e} $\in$ P(S) we see they cannot be related using any containment relation.

Consider {a, b, c} and {a, b, c, d} $\in$ P(S), we see

{a, b, c} $\subseteq$ {a, b, c, d} so P(S) is a partially ordered set.

However P(S) is not a totally ordered set as we have pairs of distinct elements which do not satisfy containment relation.

We give some subsets of P(S) which forms a totally ordered set.

Consider

B_1 = {{ϕ}, {a, b}, {a, b, c}, {a, b, c, d}, {a, b, c, d, e}} $\subseteq$ P(S).

It is easily verified B_1 is a totally order chain.

The totally ordered chain of B_1 is as follows.

$\{\phi\} \subseteq \{a, b\} \subseteq \{a, b, c\} \subseteq \{a, b, c, d\} \subseteq \{a, b, c, d, e\}$ …I

Clearly the chain I of length 5 and order of B_1 is also 5.

Now consider

$B_2 = \{\{\phi\}, \{a, b\}, \{a, b, c\}, \{b, c, d\}, \{a, b, c, d\}, \{a, b, c, d, e\}\}$ be a subset of P(S).

Clearly B_2 is not a totally ordered set. B_2 is only a partially ordered set as {a, b} is not continued in {b, c, d} and {a, b, c} and {b, c, d} are not comparable we get the following partially ordered figure 1.1.

$$\{\phi\} \subseteq \{a, b\} \subseteq \{a, b, c\} \subseteq \{a, b, c, d\} \subseteq \{a, b, c, d, e\}$$

$$\{b, c, d\} \subseteq \{a, b, c, d\} \subseteq \{a, b, c, d, e\}.$$

Thus we have two branches or two distinct totally ordered chains given by the following.

$\{\phi\} \subseteq \{a, b\} \subseteq \{a, b, c\} \subseteq \{a, b, c, d\} \subseteq \{a, b, c, d, e\}$ …(A)

$\{\phi\} \subseteq \{b, c, d\} \subseteq \{a, b, c, d\} \subseteq \{a, b, c, d, e\}$ … (B)

We see A and B are two distinct chains

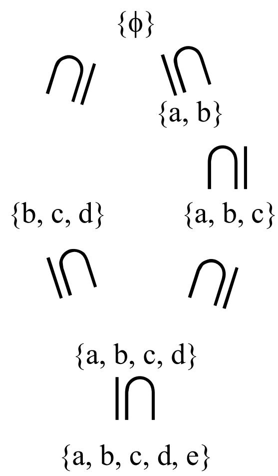

Figure 1.1

We can say the longest totally ordered chain in P(S) is only 6. Some of the totally ordered chains in P(S) is given in the following.

$\{\phi\} \subseteq \{a\} \subseteq \{a,b\} \subseteq \{a, b, c\} \subseteq \{a, b, c, d\} \subseteq \{a,b,c,d,e\}$ …1

$\{\phi\} \subseteq \{a\} \subseteq \{a, c\} \subseteq \{a, b, c\} \subseteq \{a, b, d, c\} \subseteq \{a, b, c, d, e\}$…2

$\{\phi\} \subseteq \{a\} \subseteq \{a, d\} \subseteq \{a, d, c\} \subseteq \{a, b, c, d\} \subseteq \{a, b, c, d, e\}$…3

$\{\phi\} \subseteq \{a\} \subseteq \{a, d\} \subseteq \{a, d, c\} \subseteq \{a, d, c, e\} \subseteq \{a, b, c, d, e\}$…4

$\{\phi\} \subseteq \{a\} \subseteq \{a, e\} \subseteq \{a, e, d\} \subseteq \{a, e, d, c\} \subseteq \{a, b, c, d, e\}$ …5

$\{\phi\} \subseteq \{a\} \subseteq \{a, e\} \subseteq \{a, e, c\} \subseteq \{a, e, d, c\} \subseteq \{a, b, c, d, e\}$…6

and so on.

The task of finding all totally ordered chains of largest order viz 6 is left as an exercise for the reader.

We have seen examples of partially ordered sets.

In fact, the following observations are given without proof which can be taken as a results by the readers for more refer [6,9].

Result 1.1. Every totally order set $\{S, \leq\}$ is also a partially ordered set under '$\leq$'.

Follows from the very definition of a partially ordered sets which demands that there which at least a pair of distinct elements in S which are ordered. Hence the claim.

Result 1.2. A partially ordered set $(S, \leq)$ in general is not a totally ordered set.

Proof. All totally ordered sets, $(S, \leq)$ are partially ordered set $(S, \leq)$ for we can compare any pair of elements in S. However, if $(A, \leq)$ is only a partially ordered set then there exists at least a pair of elements a, b $\in$ A such that a and b are not related. Hence the claim that a partially ordered set is not a totally ordered set.

The classical or very common example of this situation is $\{P(S), \subseteq\}$ the powerset of any non-empty set under the inclusion (or containment) operation, $\subseteq$ as only a partially ordered set and is not a totally ordered set.

Result 1.3 Let S be a non-empty set. P(S) the power set of S. $\{P(S), \subseteq\}$ is only a partially ordered set.

For P(S) has subsets P_1, P_2 such that $P_1 \subseteq P_2$ and also has subset R_1, R_2 such that $R_1 \not\subset R_2$ and $R_2 \not\subset R_1$ for instance take singleton set {a}, {b}, ($a \neq b$ a, $b \in S$) then {a} and {b} are not ordered in any way so {P(S), $\subseteq$} is only a partially ordered set.

We can show several examples of the same.

Result 1.4 Let o(S) = n and P(S) the power set of S. {P(S), $\subseteq$} the partially ordered set P(S) has a totally ordered chain of maximal length of order n + 1.

Proof follows from the simple fact we have a chain

$\{\phi\} \subseteq \{a_1\} \subseteq \{a_1, a_2\} \subseteq \{a, a_2, a_3\} \subseteq \{a_1, a_2, a_3, a_4\} \subseteq$

$\{a_1, a_2, a_3, a_4, a_5\} \subseteq \ldots \subseteq \{a_1, a_2, a_3, a_4, a_5, \ldots, a_n\} = S \qquad \ldots I$

Clearly I is an increasing totally ordered chain and the length of I is n + 1.

Result 1.5 Let {S, $\leq$} be a totally ordered set. Every subset of S is a totally ordered set.

Proof . If there is a subset say; P_1 of S which is not a totally ordered set then we have some elements in P_1 which are not comparable.

But as $P_1 \subseteq S$ and every pair of elements in S is comparable hence the above assumption P_1 is not totally orderable is not possible.

Hence every subset of the set S is again a totally ordered set.

Now we proceed onto obtain results relating with linguistic sets associated with a linguistic variable. If we are studying time dependent linguistic variables certainly they are totally orderable.

Concepts like colour etc, may not be totally orderable however some of them are partially orderable.

However throughout this book we use only those linguistic variables which has a totally ordered linguistic set associated with it.

Only with totally ordered linguistic sets we can perform min and max operations. This assumption is very vital in this book.

We do not deny the existence of linguistic sets which cannot be totally ordered, those linguistic sets which are also time dependent still not totally orderable we avoid them for the present, we are not in a position to use them in this book for we construct linguistic matrix theory in par with classical matrix theory where total or partial order is mandatory for us to perform operations on linguistic matrices only in this book.

Or we feel this is beyond the scope of this book, for we assume every linguistic variables considered in this book yield their respective linguistic set to be a totally ordered set or a linguistic continuum.

We now prove some examples of totally ordered linguistic sets associated with some of the linguistic variables.

Example 1.5. Let S = w(L) be the linguistic set associated with the linguistic variable L age of the person. This linguistic variable is obviously time dependent. In case of age of persons we see the age increases from o to maximum age and each day they become old by a day and so on.

Let

W(L) = {old, just old, middle age, just young, very young, just middle age, young and youngest}

be the linguistic set assigned by an expert for some people under investigation.

It is easily seen that w(L) is a totally ordered set. For the age of the person changes from very young or youngest to very old or oldest as time goes on.

youngest ≤ very young ≤ just young ≤ young ≤ just middle age ≤ middle age ≤ just old ≤ old …I

This is the totally ordered set and is an increasing chain of length 8 which is the same as that of the order of S.

Suppose as discussed earlier, age of a person is discussed from his birth to his very old age then we can use the linguistic continuum

[youngest, very old] ∪ {ϕ}

the linguistic empty term is included as his the age is assessed used linguistic technique this term ϕ denotes the empty linguistic term. This is a continuous one hence it is a totally ordered set.

We get the following increasing totally ordered chain;

ϕ ≤ youngest ≤ very very very young ≤ very very young ≤ very young ≤ just young ≤ young ≤ … ≤ just middle age … ≤ middle age ≤ … ≤ just old ≤ old ≤ very old.

This is an infinite totally ordered linguistic chain which is different from linguistic variable, age of a person.

Now we give yet another real world example of a linguistic variable and its associated linguistic set.

Example 1.6. Let us consider the speed of a vehicle on a busy road which is considered as the linguistic variable.

The expert wishes to work only with a finite number of linguistic terms associated with this linguistic variable.

First of all in the signals the speed of any car will tend to drop to zero, which is denoted by the empty linguistic term ϕ, so its mandatory to add ϕ in our linguistic set. The starting speed at each signal will be very slow or slow and so on for any vehicle on the road.

Let S w(speed of a vehicle in the road)

= {ϕ, very very slow, very slow, slow, medium, just medium, fast, just fast, very fast, very very fast, fastest}.

Now this is a totally ordered set and is totally increasing order chain is as follows.

ϕ ≤ very very slow ≤ very slow ≤ slow ≤ just medium ≤ medium ≤ just fast ≤ fast ≤ very fast ≤ very very fast ≤ fastest …I

Here we wish to state the following facts about the linguistic variable age, is different from the linguistic variable speed of the car; for the linguistic variable age the linguistic terms goes on increasing there is no possibility of decreasing steadily but increases with time; however in case of the linguistic variable speed of the car the starting point is the empty linguistic term ϕ then slow and goes on increasing but just as the time goes on and when the vehicle is near the signal the speed has to reduce and if signal falls the speed reduces to the empty linguistic term ϕ and then starts to increase; further on the speed breakers the speed is again reduced though not to the empty linguistic term ϕ.

Thus one sees in case of speed the terms increases slowly reaching a maximum (that may occur) and slows down depending on the traffic in the road and signals the vehicle has to cross, however this sort of increase or decrease is not possible in case of the linguistic variable age it is ever increasing and after death stops or terminates the chain.

Thus it is important to note the difference between the two linguistic variables.

We give some more examples of changes of a linguistic variable related to time.

Example 1.7. Consider the height of a person from birth to death. The height of a person increases up to some level and then remains a constant.

Thus for the linguistic variable ‘height’ of a person it changes from [shortest, tallest], the shortest being height of the person at birth and the tallest the height of a person which remains a constant after a age.

The growth of height for some may be up to 18 years for some up to the age of 20 years and for some others it may be up to the age of 21 and so on but after age 22 they (humans) do not grow.

This linguistic variable height of a person is distinctly different from the other linguistic variables; age of a person which steadily increases till death of a person or the speed of a car which increases and decreases with increase of time.

A researcher must observe these linguistic variables with intense care for they are not the same for all linguistic variables.

Now we provide yet another example.

Example 1.8. Let us consider the weight of a person. The weight of a person is the linguistic variable. The weight of a person varies from the birth weight increasing with time and also decreases with some health problems after a stage may remain constant for some years increase in the middle age and after that may decrease or reduce with old age and so on.

This linguistic variable is very different from the other 3 linguistic variables, weight, age and height of a person and speed of car.

However just like the speed of the vehicle the weight may not touch the empty linguistic term or the least term.

The researcher or reader is requested to observe all the four linguistic variables.

Now let us consider the linguistic variable "performance" of a worker working in a factory.

How does the performance of a person described as a linguistic set varies from other linguistic variables described so far.

Example 1.9. Let us consider the performance of a specific worker working in a factory.

He may be a very good worker at the time of joining as time goes he may be the best (let us assume) he may become a less performer if he has some problems in the family or with other co-workers or with the employer or if he has some health problems so his performance may fluctuate at this period and after that it may improve or it may deteriorate. Thus the performance of a person may also be changing from time to time. This linguistic variable is different from the other linguistic variable so far discussed. However this linguistic variable is also only time dependent.

Further this variable "performance" also drastically varies from person to person also not like the linguistic variables age or height which increases steadily (in case of height up to some age).

Finally we will briefly discuss about the linguistic variable temperature of water from 0^o to boiling point.

Example 1.10. Let us consider the variable; temperature of water from solid ice state to its boiling point.

The temperature of water is a linguistic variable which is under study, while trying to find the change in temperatures when heated.

The temperature under uniform heat increases its temperature from 0ºC to 100ºC steadily increases and it can be easily mapped on the linguistic scale or interval from

$$[\text{lowest, highest}] \cup \{\phi\}$$

where empty linguistic term word denotes the temperature of water at the 0ºC and lowest just the process of increasing and highest at the 100ºC. This is like the linguistic age variable with uniform temperature; increasing steadily with time, why it is heated uniformly,

However if the same linguistic variable water temperature in a pond or well or sea or lake is studied the condition or the linguistic set associated with it would be entirely different. The trend will be increasing to decreasing, constant for some time decreasing to increases.

Next we discuss the linguistic variable weather report of any place for a week or a month or a day or a year.

We are only measuring the temperature not the humidity or rainfall.

Now for the present the linguistic variable of the weather report with specification to temperature is under study of a particular place in India say, Vellore.

Now the temperature will vary in a day from high or very high in the daytime to low or very low in the night or very early morning of 3 am to 4 am etc. So the linguistic variable of temperature of the day in a specific place will be very varying from high to low.

This is dependent not only in the place we live, it is also dependent on the season the temperature recorded and so on and so forth. This is also different from the other linguistic variables under study. This linguistic variable is different from other linguistic variables though time dependent yet functions very differently.

Further throughout this book we take only those linguistic variables which has its associated linguistic set / term / word to be a totally ordered set for otherwise we will not be in a position to work min or max or both operations on them.

We do not mention those linguistic variables which contribute to linguistic words / terms which are not orderable or partially orderable.

We have given examples of them this never occurs in case of usual number systems or even the memberships which we give using fuzzy membership.

Once we give numbers the situation is very different for it becomes orderable structure.

Say if we have some 9 colours

{black, white, red, brown, blue, green, yellow, orange, pink} we cannot order them however we can order the colour red if bright red to light red among the 'red' property but ordering red with white or blue or green is not possible.

But some fuzzy membership property is used these membership values associated with them becomes ordered. However at this juncture we do not want to discuss or debate about its validity in this book.

Repeatedly we keep on record we use in this book only those linguistic variables whose linguistic terms are totally orderable, that is why we can define operations on matrices built over them.

Now we proceed onto describe operations on the linguistic sets associated with a linguistic variable.

Example 1.11. Let L be the linguistic variable associated with the quality of the durability of products made by an industry.

Let

w(L) = {very good, good, fair, bad, just fair, just good, (not upto mark = not bad) not bad, very bad, very very good}.

Now w(L) is totally ordered set it is time independent for it does not vary with time.

Now we have the totally ordered chain associated with w(L) in the following:

very bad ≤ bad ≤ not bad ≤ just fair ≤ fair ≤ just good ≤ good ≤ very good ≤ very very good.

Now we know

$$\min\{x, y\} = \begin{cases} x \text{ if } x \le y \\ y \text{ if } y \le x \end{cases}$$

So min {fair, very very good}

= fair as fair ≤ very very good

min {bad, good}

= bad; from the totally ordered chain as bad ≤ good.

(Recall in a totally ordered system every pair is ordered or 'comparable').

Now we can also define max operator on S. For $x, y \in S$ $(x \neq t)$

$$\max\{x,y\} = \begin{cases} x \text{ if } y \le x \\ y \text{ if } x \le y \end{cases}$$

Further max {fair, very very good}

= very very good as fair ≤ very very good.

max{bad, good} = good as bad ≤ good.

Clearly $\max\{x, y\} \neq \min\{x,y\}$ unless $x = y$.

When $x = y$ we have $\min\{x, x\} = x = \max\{x, x\}$.

$\max\{x, y\} = \min\{x, y\}$ if and only if $x = y$.

Theorem 1.1. *Let S be a linguistic set associated with a linguistic variable. S is a totally ordered set.*

max {x, y} = min{x, y} if and only if x = y; x, y ∈ S.

Proof. To prove max {x,y} = min {x, y} if and only if x = y for x, y ∈ S.

Let max {x, y} = min {x, y} for x, y ∈ S to prove x = y.

max{x, y} = if x y ≤ x …I

min {x, y} = if x x ≤ y …II

From I and II we see x ≤ y and y ≤ x this is possible if and only if x = y. Hence the one way claim.

Suppose x = y (x ≤ y and y ≤ x) to prove

min {x, y} = max {x, y}.

Consider min {x, x} = x and

max {x, x) = x.

So min{x, y} = max {x, y} as x = y. Hence the claim.

We have found min and max operators on a pair of elements from the totally ordered set.

Now we find the min and max operators on a finite sub set of the totally ordered set S.

Consider the subsets

A = {very good, good, not bad, very bad, bad, fair} and

B = {just good, just fair}

be two proper subsets of the linguistic set w(L) given in example related to the variable durability of products made by an industry.

We find out min{A, B} min {A, B}

= min{{very good, good, not bad, very bad, bad, fair}, {just good, just fair}}

= {min{very good, just good}, min{very good, just fair}, min{good, just good}, min {good, just fair}, min{not bad, just good}, min{not bad, just fair}, min{very bad, just good}, min{very bad, just fair}, min{bad, just good}, min{bad, just fair}, min{fair, just good}, min{fair, jut fair}}

= {just good, just fair, not bad, very bad, bad, fair} …I

This is the way min operation is performed on sets.

Now we perform max operation on the subsets A and B of S

max {A, B} =

max{{very good, good, not bad, very bad, bad, fair}, {just good, just fair}}

= {max {very good, just good}, max{very good, just fair}, max{good, just good}, max{good, just fair}, max{not bad, just good}, max{not bad, just fair}, max{bad, just good}, max{bad, just fair}, max{very bad, just good}, max{very bad, just air}, max{fair, just fair}, max{fair, just good}}

= {very good, good, just fair, just good} …II

Clearly I and II are distinct and number of elements in max{A, B} is 4 where as that of min{A, B} is 6.

We just observe $A \cap B = \phi$, that is there are no common elements between A and B.

However min{A, B} $\cap$ max{A, B}

= {just good, just fair, not bad, very bad, bad, fair} $\cap$ {just fair, just good}

= {just fair, just good} that is they have the very subset B to be contained in both min{A, B} and max{A, B}.

Thus B $\subseteq$ min{A, B} and B $\subseteq$ max{A, B} and

min{A, B} $\cap$ max {A, B} = B, however $A \cap B = \phi$.

Now we proceed onto give an example of two linguistic subsets of w(L) for the same linguistic variable "durability of the product" of some factory given in example 1.10.

Let M = {very good, good, bad, fair, not bad} and

N = {bad good, just fair, just good, very bad, very very good} be two subsets of w(L).

Clearly $M \cap N$ = {bad, good} $\neq \phi$.

Now we find out

min{M, N} = min{{very good, good, bad, fair, not bad}, {bad, good, just fair, just good, very bad, very very good}}

= {min{very good, bad}, min{very good, good}, min{very good, just fair}, min{very good, just good}, min{very good, very bad}, min{very good, very very good}, min{good, bad}, min{good, good}, min{good, just fair}, min{good, just good}, min {good, very bad}, min{good, very very good}, min{bad, bad}, min{bad, good}, min{bad, just fair}, min{bad, just good}, min{bad, very bad}, min{bad, very very good}, min{fair, bad}, min{fair, good}, min{fair, just fair}, min{fair, just good}, min{fair, very bad}, min{fair, very very good}, min{not bad, bad}, min{not bad, good}, min{not bad, just fair}, min{not bad, just good}, min{not bad, very bad}, min{not bad, very very good}}

= {bad, good, just fair, just good, very bad, very good, not bad}
…I

Now we find

max{A, B} = max{{very good, good, bad, fair, not bad}}, {bad, good, just fair, just good, very bad, very very good}}

= {max{very good, bad}, max{very good, good}, max{very good, just fair}, max{very good, just good}, max{very good, very bad}, max{very good, very very good}, max{good, bad}, max{good, good}, max{good, just fair}, max{good, just good}, max{good, very bad}, max{good, very very good}, max{bad, bad}, max{bad, good}, max{bad, just fair}, max{bad, just good}, max{bad, very bad}, max{bad, very very good}, max{fair, bad}, max{fair, good}, max{fair, just fair}, max{fair, just good}, max{fair, very bad}, max[fair, very very good}, max{not bad, bad}, max{not bad, good}, max {not bad, just fair}, max{not bad, just good}, max {not bad, very bad}, max{not bad, very very good}}

= {very good, very very good, good, bad, just fair, just good, fair, not bad} …II

I and II are distinctly different i.e. max{M, N} $\neq$ min{M, N}.

Further M $\cap$ N = {good, bad} and

max{M, N} $\cap$ min{M, N} = {bad, good, very good, not bad, fair, just fair, just good}.

Clearly M $\subseteq$ min{M, N} $\cap$ min{M, N} and

N $\not\subset$ max{M, N} $\cap$ min{M, N} as well as

max{M, N} $\cap$ min{M, N} $\subseteq$ N.

Having seen examples of max and min operator on S and subsets of S we now proceed onto give examples of partially ordered linguistic sets.

Example 1.12. Let S = {best, fair, good, very good, bad}} be the linguistic set associated with the linguistic variable performance of a student in a class room.

The power set of S is given by P(S) = {ϕ {best}, {fair}, {good}, {very good}, {bad}, {best, fair}, {best, good}, {best, very good}, {best, bad}, {fair, good}, {fair, very good}, {fair, bad}, {good, very good}, {good, bad}, {very good, bad}, {best, fair, good}, {best fair, very good}, {best, fair, bad}, {best, good, very good}, {best good, bad}, {fair, good, very good}, {fair, good, bad}, {fair, very good, bad}, {good, very good, bad}, {best, bad, very good}, {best, fair, good, very good}, {best, fair, good, bad}, {best, bad, good, very good}, {fair, good, very good, bad}, {best, fair, very good, bad}}.

We see P(S) the linguistic power set of the linguistic set S is only a partially ordered set.

In case of partially ordered set also we define two types of two operations $\cap$, $\cup$ and other is min and max.

In case {P(S), min} is not the same as {P(S), $\cap$}. Similarly {P(S), max} is not same as that of {P(S), $\cup$}.

We prove this by the example.

Consider A = {fair, good, very good, bad} and

B = {best, fair, good, bad} be two linguistic subsets of P(S).

We see A $\cap$ B = {good, bad}, min{A, B}

= min{{fair, good, bad, very good}, {best, fair, good, bad}}

= {min {fair, best}, min{fair, fair} min{fair, good}, min{fair, bad}, min{good, fair}, min{good, fair}, min{good, good}, min{good, bad}, min{bad, best}, min{bad, fair}, min{bad, good}, min{bad, bad}, min{very good, best}, min{very good, fair}, min{very good, good}, min{very good, bad}}

= {fair, bad, good, very good}.

Clearly min{A, B} $\neq$ A $\cap$ B for

{fair, bad, good, very good} $\neq$ {good, bad}.

In fact $A \cap B \subseteq \min\{A, B\}$.

Next we find $A \cup B$ and $\max\{A, B\}$.

$A \cup B$ = {fair, good, very good, bad, best} …II

max{A, B} = max{{fair, good, very good, bad}, {best, fair, good, bad}}

= {max {fair best}, max{fair, fair}, max{fair, good}, max{fair, bad}, max{good, best}, max {good, fair}, max{good, good}, max{good, bad}, max{very good, best}, max{very good, fair}, max{very good, good}, max{very good, bad}, max{bad, best}, max{bad, fair}, max{bad, good}, max{bad, bad}}

= {best, fair, good, very good, bad} …III

$A \cup B$ = {fair, good, very good, bad, best}

= {best, fair, good, very good, bad} = max{A, B}.

In this set both $A \cup B$ and max{A, B} are equal.

We now find for the sets

M = {very good, fair} and

N = {bad, best, good}

be two linguistic subsets of the linguistic power set S.

We see $M \cup N$ = {very good, fair} $\cup$ {bad, best, good}

= {very good, fair, bad, best, good} …(a)

Now we find max{M, N}

= max{{fair, very good}, {bad, best, good}}

= {max{fair, bad}, max{fair, best}, max{fair, good}, max{very good, bad}, max{very good, best}, max{very good, good}}

= {very good, fair, best, good} …(b)

Clearly (a) $\neq$ (b) that is A $\cup$ B $\neq$ max{A, B}.

Thus we see A $\cup$ B = max{A, B} and

A $\cap$ B = min{A, B} in case of the operations on linguistic subsets of a linguistic set or a linguistic power set P(S) are not in general true.

So we see on linguistic subset the two operations $\cup$ and $\cap$ do not coincide so they will not be the same on the linguistic power set P(S) of a linguistic set S. The same for $\cup$ and max.

However on linguistic continuum the two operations coincide when we take singleton set in pairs.

That a $\cap$ b = min{a, b}, a, b $\in$ S(S a totally ordered linguistic set or a linguistic continuum).

Similarly a $\cup$ b = max{a, b} on totally ordered linguistic set or a totally ordered linguistic continuum a, b $\in$ S.

Let us give one more example of this situation.

Example 1.13. Let S = {high, low, lowest, medium, very low, very high, highest, just low, just high, just medium, very very low, very very high} be a linguistic set associated with the linguistic variable temperature of weather in a day.

Let M = {high, low, lowest, medium, very low, very high, highest, just low} and

N = {just high, just medium, very very low, very very high} be two linguistic subset of the linguistic set S.

Clearly $M \cap N = \{\phi\}$ …I

and $M \cup N = S$ …II

Now we find

min{M, N} = min{{high, low, lowest, medium, very low, very high, highest, just low}, {just high, just medium, very very low, very very high}}

= {min{high, just high}, min{high, just medium}, min{high, very very low}, min{high, very very high}, min{low, just high}, min{low, just medium}, min{low, very very low}, min{low, very very high}, min{lowest, just high} min{lowest, just medium), min{lowest, very very low}, min{lowest, very very high}, min{medium, just high}, min{medium, just medium}, min{medium, very very low}, min{medium, very very high}, min{very low, just high}, min{very low, just medium}, min{very low, very very high}, min{very low, very very low}, min{very high, just high}, min{very high, just medium}, min{very high, very very low}, min{very high, very very high}, min{highest, just medium}, min{highest, just high}, min{highest, very very low}, min{highest, very very high} min{just low, just high}, min{just low, just medium}, min{just low, very very low}, min{just low, very very high}}

= {just high, just medium, very very low, high, low, lowest, medium, very very low, very low} …III

Clearly M $\cap$ N $\neq$ min{M, N}.

Now find max{M, N}

= max{high, low, lowest, medium, very low, very high, highest, just low}, {just high, just medium, very very low, very very high}}

= {high, very very high, just high, just medium, medium, very very low, very low, very high, highest, just low} …IV

Clearly II and IV are distinct for S $\neq$ max{M, N}.

Now we find for the same set max{min, M, N}} and min{max {M, N}}.

Equation III gives min{M, N} and IV gives max{M, N}.

Now max{min{M, N}} using III is

max{just high, just medium, very very low, high low, lowest, medium, very very low, very low} = high …V

We find equation IV min{max {M, N}}

= min{high, very very high, just high, just medium, medium, very very low, very low, very high, highest, just low}

= very very low.

Thus max{min{M, N}} ≠ min{max{M, N}}.

Clearly min-max or max min operation can be performed only on proper subsets of the set S.

Now we suggest a few problems for the reader to familiarize the notion of linguistic sets associated with linguistic variables and min and max operations on linguistic sets. The starred problems are difficult to be solved.

SUGGESTED PROBLEMS

1. Give at least four linguistic variables which cannot be ordered (different from colour eyes or ears or skin of humans).

2. Give four examples of linguistic variables which are time dependent and orderable.

3. Does there exist a linguistic variable which is time dependent yet not orderable?

4. Suppose there are 5 persons belonging to five-different parts of the world, their colours are
{white, yellow, dark brown, black, light brown} = C

   i) Can C be totally ordered?

   ii) Can C be partially ordered? Justify your claim.

5. Heights of 7 persons are given by the linguistic set S;
S = {very tall, tall, short, very short, just tall, very very short, medium height}.

i) Is height a time dependent variable? (in general).

ii) Can S be a totally ordered set? (if so order S).

iii) Is S a partially ordered set?

iv) Can we say S is the subset of the linguistic continuum [shortest, tallest] associated with height of a person?

6. Let S = {0.7, 9, 2, 4, –3, 0.09, 4, 1} be the set of real numbers.

   Prove S is a totally ordered set and total order S by '≤' less than or ≥ greater than relation.

7. Let P(S) = {collection of all subsets of the set S = {a, b, c, d}}
   = {{a}, {b}, {c}, {d}, {a, b}, {b, c}, {c, d}, {a, c}, {a, d}, {b, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}, {ϕ}, {a, b, c, d} = S}.

i) Prove S is only a partially ordered set.

ii) Give at least there are 5 subsets (elements) in P(S) which is totally ordered.

iii) Prove S has no collection of sets whose cardinality is greater than 5 can be totally ordered.

iv) Give some examples of subsets of S which cannot be ordered.

8. Let A ={{ϕ}, {1, 2}, {1, 2, 4}, {1, 2, 4, 7}, {1, 2, 4, 7, 9}, {1, 2, 8, 4, 7, 9}, {1, 2, 3, 4, 5, 6, 7, 8, 9}} be a set of subsets.

   i) Is A a totally ordered set?

   ii) What is the length of the chain of A?

   iii) Is A under the operation of ∩ closed? Justify your claim!

   iv) Is A closed under the operation ∪? Justify.

9. Can we say if a set X is a totally ordered set then can we say X is closed under the operation ∪ and not closed under the operation ∩?
   Justify your claim and prove it in a general case!

10. Let S be the linguistic continuum related with the linguistic variable weight of a person [lowest, highest}

    i) Find linguistic subsets of finite order and prove they are totally ordered sets.

    ii) Let M = {low, very low, just low, very very low, medium, just medium, high, very high, just high},

       N = {lowest, very very high, very medium, very very very low, higher, just higher, just lower, lower} be two linguistic subsets of the linguistic continuum S = [lowest, highest].

       a) Calculate the following values.

i) min {M, N},

ii) max {M, N},

iii) $M \cap N$,

iv) $M \cup N$,

v) min{max {M, N}} and

vi) max{min{M, N}}.

Compare the 6 values. How many of them are equal with each other?

b) Find two finite subchains constructed using the chain of S of length 16 and 18.

c) Find the linguistic totally ordered subchains of the linguistic subsets M and N and find their lengths.

11. Let S = {brown, light brown, very light brown, just brown, very very light brown, whitish brown, very very dark brown, very dark brown} be the linguistic set on the linguistic variable shades of the brown colour.

i) Find the linguistic totally ordered increasing chain C of S.

ii) What is the length of the linguistic chain C of S?

iii) Is every linguistic subset of the linguistic set S a totally ordered subchain of C? Justify your claim.

iv) Let A = {brown, light brown, very light brown, just brown, dark brown, just light brown} and B = {just brown, dark brown, very very light brown, whitish brown, very very dark brown} be two linguistic subsets of the linguistic set S.

Find (a) $A \cap B$,

(b) $A \cup B$,

(c) min{A, B},

(d) max{A, B} and compare them.

v) min{max{A, B}} and

vi) max{min{A, B}} and compare them.

12. Let B = {brown, black, white, red, dark red, light red, orange, pink, dark blue, light blue, very light blue, green, light green} be a linguistic set associated with the linguistic variable colour.

i) Is B a totally ordered linguistic set?

ii) Is B a partially ordered linguistic set?

iii) For two linguistic subsets X and Y of (B).

Find the values of

i) $X \cap Y$,

ii) $X \cup Y$,

iii) min{max {X, Y}},

iv) max{min{X, Y}},

v) min{X,Y}

vi) max{X, Y} and compare them.

iv) What is the length of the maximal linguistic chains in P(B)?

v) How many maximal linguistic chains does P(B) contribute?

vi) Find any other special feature enjoyed by the linguistic power set P(B).

13. Give an example of a linguistic variable other than colour for which the linguistic set associated with it is not totally orderable but partially orderable.

14. Let S = {dense, very dense, just dense, scattered, very scattered, just scattered, slightly dense, light barren, slightly scattered, barren} be the linguistic set associated vegetation of a place in India.

    i) Is S a totally ordered set?

    ii) What is the length of the linguistic the chain of S?

15. Let S be a linguistic set associated with the linguistic variable interpersonal relations in a family (between husband and wife); S = {hatred, loving, understanding, fights over frivolous matters, hates each other, neutral, no understanding, …}.

Clearly S cannot be ordered. However it has partial order and P(S) the linguistic power set of S can be ordered.

Find all possible properties associated with S and P(S).

16*. Let S be the linguistic set “indeterminate”

a) Find W(S) the set of linguistic words associated with S.

b) Associate a real world problem which is a linguistic variable L associated with S the indeterminate variable.

Chapter Two

BASIC PROPERTIES OF LINGUISTIC MATRICES

In this chapter we define the notion of linguistic matrices and discuss some the basic properties with examples. The basic notions and notations of linguistic variable L, and its associated linguistic word set w(L) or the linguistic terms w(L) is introduced in chapter I.

Further we have to define on any linguistic term / set we have an ordering as that of the ordering found in the line of integers or rationals or reals.

The linguistic set can be finite or infinite or can also be a continuum. However for any linguistic set we also have with it an associated linguistic variable; the ordering of the linguistic set is dependent on the ling variable. It may be totally orderable or at times unorderable.

Further unlike line of reals or rationals or integers the continuum given by a linguistic variable can be many which solely depends on the linguistic variable L.

All these has been explained earlier but however we recall again and again mainly for the readers to remember them as these concepts of distinctly different from the usual number system.

The matrix is defined as a rectangular array of numbers for instance A is

$$A = \begin{bmatrix} 9 & 3 & 2 & 4 \\ -1 & 0.5 & 5 & 7 \\ 0 & -7 & 0 & -1 \\ 1 & 1 & 0 & 2 \\ 4 & 5 & 6 & 8 \\ 1 & 0 & -1 & 0 \end{bmatrix}.$$

This is a 6×4 rectangular matrix.

Let B be a matrix of number given in the following

$$B = \begin{bmatrix} 6 & 0 & -1 & 2 & 4 \\ 1 & 2 & 0 & 0 & 6 \\ -1 & 3 & -7 & 0.5 & 0 \\ 0 & 1 & 0 & 2 & 0.7 \\ 1 & 0.1 & 6 & 0 & 6 \end{bmatrix}.$$

B is a 5×5 rectangular matrix.

Consider C the matrix of numbers;

$$C = (7, 0, 8, 0.5, 4, 7, 8, 1)$$

C is a 1×8 matrix.

C will be known as a 1×8 row matrix.

Let D be a matrix of number of given by the following

$$D = \begin{bmatrix} 0.8 \\ 0.7 \\ 0.8 \\ 5 \\ 4 \\ 2 \\ 1 \\ 7 \\ 9 \\ 0.6 \\ 1.2 \end{bmatrix}$$

D is known as a 11×1 column matrix.

Thus we have seen 4 types of matrices. We will proceed onto describe zero matrix of all the four types. We first provide an example of a 1×9 zero matrix denoted by

$$(0)_{1\times 9} = (0, 0, 0, 0, 0, 0, 0, 0, 0)$$

We prove and example of a 7 × 4 zero matrix

$$(0)_{7\times 4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(0)_{6\times 6} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is a 6 × 6 zero square matrix,

$$(0)_{8\times 1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{8\times 1} \quad \text{is a zero-column matrix.}$$

Now we define the diagonal matrix of a 7 × 7 square matrix

$$M = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -9 \end{bmatrix}$$

When all the entries of the diagonal are only one and the rest of the elements are zero then we call the square matrix as the identity matrix.

We provide examples of them.

$$I_{4\times4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$I_{7\times7} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$I_{2\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } I_{9\times9} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

are identity matrices of orders 4×4, 7×7, 2×2 and 9×9 respectively.

$$I_{n\times n} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \text{ is a } n \times n \text{ identity matrix.}$$

Now in case of linguistic matrices we proceed onto give examples of them.

Definition 2.1. *Let S be a linguistic set (or (I) a linguistic continuum) and A be a rectangular array of linguistic terms.*

We define A to be linguistic matrix with entries from S (or from the linguistic continuum I).

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

where $a_{ij} \in S$ (or I); $1 \leq i \leq m$ and $1 \leq j \leq n$. We say order of the linguistic matrix A is $m \times n$.

We will provide some examples of them.

Example 2.1. Let S = {very tall, tall, medium height, short, very short, just tall, very very short, just medium height, shortest} be a linguistic finite set associated with the ling variable height of a person.

A = (very short, tall, shortest, short, just tall, tall, short, very tall, just medium)

be a 1×9 linguistic row matrix.

Order of A is 1×9.

$$B = \begin{bmatrix} \text{shortest} \\ \text{short} \\ \text{tall} \\ \text{very tall} \\ \text{medium height} \\ \text{tall} \\ \text{very short} \\ \text{short} \\ \text{tall} \\ \text{shortest} \\ \text{just medium height} \end{bmatrix}_{11\times 1}$$

be a 11 × 1 linguistic column matrix. B is a column linguistic matrix of order 11 × 1.

$$C = \begin{bmatrix} \text{tall} & \text{short} & \text{shortest} & \text{tall} \\ \text{tall} & \text{shortest} & \text{tall} & \text{very short} \\ \text{short} & \text{medium} & \text{short} & \text{short} \\ \text{very short} & \text{tall} & \text{tall} & \text{medium} \\ \text{tall} & \text{just tall} & \text{short} & \text{tall} \\ \text{short} & \text{medium} & \text{very short} & \text{short} \\ \text{shortest} & \text{just tall} & \text{tall} & \text{tall} \\ \text{just tall} & \text{short} & \text{short} & \text{short} \\ \text{medium} & \text{tall} & \text{shortest} & \text{tall} \end{bmatrix}$$

C is 9 × 4 linguistic matrix.

Order of C is 9 × 4.

Now we provide an example of a 5 × 5 square linguistic matrix with entries from S.

$$D' = \begin{bmatrix} \text{very short} & \text{tall} & \text{short} & \text{medium} & \text{just tall} \\ \text{tall} & \text{short} & \text{tall} & \text{very short} & \text{tall} \\ \text{short} & \text{medium} & \text{short} & \text{tall} & \text{tall} \\ \text{medium} & \text{tall} & \text{just tall} & \text{short} & \text{tall} \\ \text{very short} & \text{tall} & \text{short} & \text{shortest} & \text{short} \end{bmatrix}$$

D is a linguistic square matrix of order 5 × 5.

Let E =

$$\begin{bmatrix} \text{short} & \text{tall} & \text{very short} & \text{tall} & \text{short} & \text{tall} & \text{short} \\ \text{tall} & \text{tall} & \text{tall} & \text{tall} & \text{shortest} & \text{short} & \text{short} \\ \text{just tall} & \text{short} & \text{medium} & \text{short} & \text{very short} & \text{tall} & \text{very short} \\ \text{very short} & \text{tall} & \text{very short} & \text{tall} & \text{tall} & \text{very short} & \text{tall} \end{bmatrix}$$

E is a linguistic matrix of order 4 × 7.

Now we proceed onto work with the linguistic matrices with entries from a linguistic continuum.

Example 2.2. Let L_C = [best, worst] be a linguistic continuum associated with the performance of students in a class room.

Clearly the cardinality of L_C is infinite as we have infinite number of linguistic terms in L_C.

Now we give examples of linguistic matrices which are row linguistic matrices, column linguistic matrices, rectangular linguistic matrices and square linguistic matrices.

B = (worst, just good, good, very fair, fair, bad, very bad, best, better, just good)

be a row linguistic matrix of order 1 × 10.

Just as in case of matrices with real numbers we can have entries of a linguistic matrix also to repeat here the term just good repeats twice in the linguistic row matrix B.

It is to be noted that in case there are infinite number of 1×10 linguistic matrices if the linguistic set used is of infinite order or is a linguistic continuum.

If on the other hand the linguistic set is finite the number of linguistic matrices of order 1×10 is only finite in number.

Infact the exact number will be $|S|^{10} = \underbrace{|S| \times |S| \times \ldots \times |S|}_{10\,\text{times}}$.

Now using L_C the linguistic continuum of the linguistic variable; performance of students, we give examples of linguistic column matrices of order 9×1.

$$D = \begin{bmatrix} \text{best} \\ \text{bad} \\ \text{good} \\ \text{fair} \\ \text{very good} \\ \text{bad} \\ \text{just fair} \\ \text{good} \\ \text{bad} \end{bmatrix}_{9\times1}$$

is a linguistic column matrix of order 9×1.

Here the linguistic terms good and bad has repeated twice and thrice respectively.

We see there are infinite number of such 9×1 linguistic column matrices using L_C.

If we use a finite linguistic set S there are

$|S|^9 = |S| \times |S| \times \ldots \times |S|$

number of 1 × 9 linguistic column matrices where |S| denotes the number of elements in S, the linguistic set S.

$$\text{Let } M = \begin{bmatrix} \text{good} & \text{bad} & \text{fair} & \text{best} \\ \text{good} & \text{fair} & \text{bad} & \text{worst} \\ \text{fair} & \text{bad} & \text{very good} & \text{fair} \\ \text{fair} & \text{very bad} & \text{worst} & \text{best} \end{bmatrix}$$

be a linguistic square matrix of order 4 × 4 with entries from L_C.

We see the linguistic term bad has repeated twice, the linguistic terms good, best and worst has repeated twice and the linguistic term, fair has repeated four times.

Number of linguistic square matrices of order 4 × 4 using the linguistic continuum is infinite.

If however instead of L_C we use a finite linguistic set S then the number of 4 × 4 linguistic square matrix is

$$|S|^{16} = \underbrace{|S| \times \ldots \times |S|}_{16\,\text{times}}.$$

Now let N be a 16 × 3 rectangular linguistic matrix with entries from L_C.

Now let N = $\begin{bmatrix} \text{good} & \text{bad} & \text{best} \\ \text{bad} & \text{best} & \text{fair} \\ \text{fair} & \text{best} & \text{bad} \\ \text{good} & \text{good} & \text{good} \\ \text{bad} & \text{bad} & \text{bad} \\ \text{best} & \text{bad} & \text{good} \\ \text{worst} & \text{very good} & \text{fair} \\ \text{very fair} & \text{bad} & \text{bad} \\ \text{good} & \text{good} & \text{goosd} \\ \text{good} & \text{good} & \text{good} \\ \text{best} & \text{bad} & \text{bad} \\ \text{bad} & \text{bad} & \text{bad} \\ \text{good} & \text{best} & \text{best} \\ \text{very fair} & \text{very good} & \text{very bad} \\ \text{just fair} & \text{good} & \text{bad} \\ \text{good} & \text{bad} & \text{good} \end{bmatrix}_{16\times 3}$

We see there are infinite number of 16 × 3 rectangular linguistic matrices.

If we replace the linguistic continuum L_C by a finite linguistic set S then we have

$$|S|^{48} = \underbrace{|S| \times \ldots \times |S|}_{48\,\text{times}}$$

number rectangular linguistic matrices of order 16 × 3.

We cannot provide examples of linguistic diagonal matrices unless we adjoin with the linguistic set S (or with the linguistic continuum L_C) the empty linguistic term ϕ.

This linguistic empty term ϕ of a linguistic set has been introduced in the chapter 1 of this book.

Example 2.3. S be a finite linguistic set associated with the linguistic variable age of a person

S = {old, older, oldest, very young, just old, very old, younger, youngest},

we adjoint the empty linguistic term ϕ with S.

Let $S' = S \cup \{\phi\}$.

The linguistic column matrix C with entries from S′ of order 7 × 1 is as follows.

$$C = \begin{bmatrix} \text{old} \\ \phi \\ \text{young} \\ \phi \\ \text{old} \\ \phi \\ \text{very old} \end{bmatrix}$$

The linguistic empty column matrix of order 9 × 1 is as follows.

$$(\phi)_{9\times1} = \begin{bmatrix} \phi \\ \phi \\ \phi \\ \phi \\ \phi \\ \phi \\ \phi \\ \phi \\ \phi \end{bmatrix}.$$

Now the linguistic row matrix D of order 1×9 with entries from S′ is as follows.

D = (ϕ, young, ϕ, ϕ, old, very young, ϕ, youngest, oldest).

The 1×9 linguistic row empty matrix

$$(\phi)_{1\times9} = (\phi, \phi, \phi, \phi, \phi, \phi, \phi, \phi, \phi).$$

Let E be a 4×9 linguistic row matrix with entries from $S' = S \cup \{\phi\}$ given in the following.

$$E = \begin{bmatrix} \text{young} & \text{old} & \text{youngest} & \phi \\ \phi & \text{oldest} & \phi & \text{old} \\ \text{old} & \phi & \text{young} & \text{very old} \\ \text{just old} & \text{just young} & \phi & \phi \end{bmatrix}$$

$$\begin{matrix} \text{old} & \phi & \text{old} & \phi & \text{oldest} \\ \phi & \text{young} & \phi & \text{old} & \text{oldest} \\ \phi & \text{youngest} & \text{young} & \phi & \phi \\ \text{oldest} & \phi & \phi & \text{oldest} & \phi \end{matrix}\Bigg].$$

Now the 4 × 9 linguistic empty set is as follows.

$$(\phi)_{4\times 9} = \begin{bmatrix} \phi & \phi & \phi & \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi & \phi & \phi & \phi \end{bmatrix}.$$

Finally consider a linguistic square matrix N of order 6 × 6 with entries from $S' = S \cup \{\phi\}$ is given in the following:

$$N = \begin{bmatrix} \text{old} & \phi & \text{young} & \phi & \text{old} & \phi \\ \phi & \text{very young} & \phi & \text{old} & \phi & \text{old} \\ \text{young} & \phi & \text{oldest} & \phi & \text{old} & \text{young} \\ \phi & \text{just old} & \phi & \text{oldest} & \phi & \phi \\ \text{just old} & \phi & \text{old} & \phi & \text{older} & \text{young} \\ \phi & \text{just old} & \text{youngest} & \text{old} & \text{old} & \phi \end{bmatrix}$$

We have only $|S'|^{36} = \underbrace{|S'| \times \ldots \times |S'|}_{36\,\text{times}}$

number linguistic square matrices of order 6 × 6 with entries from S′.

Now the linguistic empty square matrix of order 6 × 6 is as follows.

$$(\phi)_{6\times6} = \begin{bmatrix} \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi \end{bmatrix}.$$

Now we give a few examples of diagonal linguistic square matrices with entries from $S' = S \cup \{\phi\}$.

$$D_{4\times4} = \begin{bmatrix} \text{old} & \phi & \phi & \phi \\ \phi & \text{youngest} & \phi & \phi \\ \phi & \phi & \text{oldest} & \phi \\ \phi & \phi & \phi & \text{just old} \end{bmatrix}$$

$$D_{3\times3} = \begin{bmatrix} \text{very old} & \phi & \phi \\ \phi & \text{very young} & \phi \\ \phi & \phi & \phi \end{bmatrix} \text{ and}$$

$$D_{6\times6} = \begin{bmatrix} \text{old} & \phi & \phi & \phi & \phi & \phi \\ \phi & \text{young} & \phi & \phi & \phi & \phi \\ \phi & \phi & \text{very old} & \phi & \phi & \phi \\ \phi & \phi & \phi & \text{very young} & \phi & \phi \\ \phi & \phi & \phi & \phi & \text{old} & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi \end{bmatrix}$$

$D_{4\times4}$, $D_{3\times3}$ and $D_{6\times6}$ are linguistic diagonal matrices of order 4×4, 3×3 and 6×6 respectively.

However we cannot define the linguistic identity matrix in a unique way as in case of usual matrices. The identity matrix is dependent on two factors

i) The linguistic identity depends on the operation which is defined on that linguistic matrices in contrast with the classical product $\times$ operation in case of usual matrices.

ii) The linguistic identity depends on the linguistic set used to construct the linguistic matrices.

However it is pertinent to recall we have defined two operations on the linguistic set S viz min and max where

$$\min\{x, y\} = x \cap y$$

and $$\max\{x, y\} = x \cup y$$

in chapter 1.

We have not defined these operations on linguistic matrices. We will be defining the notion of min and max on linguistic matrices which can have compatible under min or max.

We can also as in case of usual matrices define the notion of transpose, symmetry etc. in case of linguistic matrices also. We will first illustrate situation by some examples.

Example 2.4. Let S = {heavy, drizzle, very heavy, heaviest, just heavy, just drizzle, very light drizzle, ϕ}

be the linguistic terms associated with the linguistic variable 'rain'. Here ϕ denotes no rain.

Let R = (just drizzle, ϕ, heavy, just heavy, ϕ, heaviest, ϕ, very light drizzle)

be the 1×9 linguistic row matrix with the entries from S.

Now as in case of usual matrices

R^t = (just drizzle, ϕ, heavy, just heavy, ϕ, ϕ, heaviest, ϕ, very light drizzle)^t

$$= \begin{bmatrix} \text{just drizzle} \\ \phi \\ \text{heavy} \\ \text{just heavy} \\ \phi \\ \phi \\ \text{heaviest} \\ \phi \\ \text{very light drizzle} \end{bmatrix}$$

the transpose of a linguistic row matrix of R is a linguistic column matrix.

Thus if R is a linguistic row matrix of order 1×9 then the transpose of R; R^t is a linguistic column matrix of order 9×1.

Now we find out

$$(R^t)^t = \begin{bmatrix} \text{just drizzle} \\ \phi \\ \text{heavy} \\ \text{just heavy} \\ \phi \\ \phi \\ \text{heaviest} \\ \phi \\ \text{very light drizzle} \end{bmatrix}$$

= (just drizzle, ϕ, heavy, just heavy, ϕ, ϕ, heaviest, ϕ, very light) = R.

which is the linguistic row matrix of order 1×9 we started with. Now on the other hand if C is a linguistic column matrix of order 7×1 with entries from S given by

$$C = \begin{bmatrix} \text{heavy} \\ \phi \\ \text{just heavy} \\ \text{very heavy} \\ \phi \\ \text{just drizzle} \\ \phi \end{bmatrix}. \text{ Now } C^t = \begin{bmatrix} \text{heavy} \\ \phi \\ \text{just heavy} \\ \text{very heavy} \\ \phi \\ \text{just drizzle} \\ \phi \end{bmatrix}^t$$

= (heavy, ϕ, just heavy, very heavy, ϕ, just drizzle, ϕ).

Thus transpose of the linguistic column matrix C of order 7×1 is a linguistic row matrix of order 1×7.

Now we find the transpose of transpose of C.

$$(C^t)^t = \left(\begin{bmatrix} \text{heavy} \\ \phi \\ \text{just heavy} \\ \text{very heavy} \\ \phi \\ \text{just drizzle} \\ \phi \end{bmatrix}\right)^t$$

= (heavy, ϕ, just heavy, very heavy, ϕ, just drizzle, ϕ)^t

$$= \begin{bmatrix} \text{heavy} \\ \phi \\ \text{just heavy} \\ \text{very heavy} \\ \phi \\ \text{just drizzle} \\ \phi \end{bmatrix} = C.$$

Thus as in case of usual matrices we see in of linguistic matrices also transpose of the transpose of linguistic matrix M is a linguistic matrix M.

Now we find the linguistic transpose of a square matrix of M order 5 × 5 with entries from S.

M =

$$\begin{bmatrix} \text{heavy} & \text{drizzle} & \phi & \text{very heavy} & \text{drizzle} \\ \phi & \text{just heavy} & \phi & \text{just light drizzle} & \phi \\ \text{heavy} & \phi & \text{just drizzle} & \phi & \text{heavy} \\ \phi & \phi & \phi & \text{heavy} & \phi \\ \text{heavy} & \phi & \text{heavy} & \text{very heavy} & \phi \end{bmatrix}$$

M^t =

$$\begin{bmatrix} \text{heavy} & \text{drizzle} & \phi & \text{very heavy} & \text{drizzle} \\ \phi & \text{just heavy} & \phi & \text{just light drizzle} & \phi \\ \text{heavy} & \phi & \text{just drizzle} & \phi & \text{heavy} \\ \phi & \phi & \phi & \text{heavy} & \phi \\ \text{heavy} & \phi & \text{heavy} & \text{very heavy} & \phi \end{bmatrix}^t$$

$$= \begin{bmatrix} \text{heavy} & \phi & \text{heavy} & \phi & \text{heavy} \\ \text{drizzle} & \text{just heavy} & \phi & \phi & \phi \\ \phi & \phi & \text{just drizzle} & \phi & \text{heavy} \\ \text{very heavy} & \begin{matrix}\text{just light} \\ \text{drizzle}\end{matrix} & \phi & \phi & \text{very heavy} \\ \text{drizzle} & \phi & \text{heavy} & \text{heavy} & \phi \end{bmatrix}$$

≠ M.

Just the transpose of a linguistic square matrix M is again a linguistic square matrix, in general $M \neq M^t$.

We will be characterizing the condition under which a transpose of a linguistic square matrix is equal to linguistic matrix M, i.e. $M = M^t$.

Before we proceed do this about square linguistic matrices we will find the transpose of a linguistic matrix N of order 8×3 with entries from S.

$$N = \begin{bmatrix} \text{heavy} & \phi & \text{just drizzle} \\ \phi & \text{drizzle} & \phi \\ \text{drizzle} & \phi & \text{very light drizzle} \\ \phi & \phi & \phi \\ \text{heavy} & \text{heavy} & \text{heavy} \\ \text{just heavy} & \text{drizzle} & \text{just heavy} \\ \text{very light drizzle} & \phi & \text{heaviest} \\ \text{heaviest} & \phi & \phi \end{bmatrix}.$$

Now we proceed onto find the transpose of N.

$$N^t = \begin{bmatrix} \text{heavy} & \phi & \text{just drizzle} \\ \phi & \text{drizzle} & \phi \\ \text{drizzle} & \phi & \text{very light drizzle} \\ \phi & \phi & \phi \\ \text{heavy} & \text{heavy} & \text{heavy} \\ \text{just heavy} & \text{drizzle} & \text{just heavy} \\ \text{very light drizzle} & \phi & \text{heaviest} \\ \text{heaviest} & \phi & \phi \end{bmatrix}^t$$

$$= \left[\begin{array}{cccc} \text{heavy} & \phi & \text{drizzle} & \phi \\ \phi & \text{drizzle} & \phi & \phi \\ \text{just drizzle} & \phi & \text{very light drizzle} & \phi \end{array}\right.$$

$$\left.\begin{array}{cccc} \text{heavy} & \text{just heavy} & \text{very light drizzle} & \text{heaviest} \\ \text{heavy} & \text{drizzle} & \phi & \phi \\ \text{heavy} & \text{just heavy} & \text{heaviest} & \phi \end{array}\right]$$

We see N is a linguistic matrix of order 8×3 whereas N^t is a linguistic matrix of order 3×8.

It is left for the reader to prove $(N^t)^t = N$.

We just define symmetric linguistic matrix. It is pertinent to keep on record that the notion of symmetry in case of linguistic square matrices it the same as that of usual square matrices.

Definition 2.2. *Let S be linguistic set (or a linguistic continuum). Let*

$$M = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

be a $n \times n$ square linguistic matrix with $a_{ij} \in S$ (or I_C) $1 \leq i,j \leq n$.

We say M is a symmetric linguistic matrix if and only if

$a_{ij} = a_{ji}$ for all $i \neq j$ $1 \leq i, j \leq n$. That is

$$M = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix}$$

or equivalently we can say M is a symmetric linguistic matrix if and only if $M = M^t$.

All square linguistic matrices are not symmetric.

We give examples of symmetric linguistic matrices and non symmetric linguistic square matrices; however we cannot define antisymmetric (or equivalently skew symmetric (or equivalently skew symmetric linguistic matrices as we do not have the concept of negative linguistic terms).

However later on when linguistic logic is developed and when higher level linguistic theory books are defined we will proceed onto develop the negation of the linguistic words which will serve as the negative number.

At this juncture we only talk about linguistic symmetric matrices and not linguistic skew (or anti) symmetric matrices.

Now we provide examples of them.

Example 2.5. S = {low, lowest, just low, high, medium, ϕ, just high, very high, very low} be a linguistic set associated with the linguistic variable temperature of water.

$$\text{Let } A = \begin{bmatrix} \text{low} & \text{lowest} & \text{high} \\ \text{very low} & \phi & \text{high} \\ \text{very high} & \text{just low} & \phi \end{bmatrix}$$

be a 3×3 linguistic square matrix with entries from S.

Clearly A is not a symmetric linguistic matrix as $A \neq A^t$.

$$\text{Consider } H = \begin{bmatrix} \text{very low} & \text{low} & \phi & \text{high} \\ \text{low} & \text{very high} & \text{highest} & \phi \\ \phi & \text{highest} & \phi & \text{lowest} \\ \text{high} & \phi & \text{lowest} & \text{medium} \end{bmatrix}.$$

$$\text{Now } H^t = \begin{bmatrix} \text{very low} & \text{low} & \phi & \text{high} \\ \text{low} & \text{very high} & \text{highest} & \phi \\ \phi & \text{highest} & \phi & \text{lowest} \\ \text{high} & \phi & \text{lowest} & \text{medium} \end{bmatrix} = H.$$

Thus H is a symmetric linguistic matrix as $H = H^t$.

$$K = \begin{bmatrix} \text{low} & \text{high} & \text{low} & \phi & \phi \\ \text{low} & \text{low} & \text{low} & \text{low} & \text{high} \\ \text{high} & \text{highest} & \phi & \text{low} & \phi \\ \phi & \text{high} & \text{low} & \phi & \text{high} \\ \text{low} & \phi & \text{high} & \text{low} & \phi \end{bmatrix}$$

be a 5 × 5 square linguistic matrix with entries from S.

Consider K^t;

$$K^t = \begin{bmatrix} \text{low} & \text{low} & \text{high} & \phi & \text{low} \\ \text{high} & \text{low} & \text{highest} & \text{high} & \phi \\ \text{low} & \text{low} & \phi & \text{low} & \text{high} \\ \phi & \text{low} & \text{low} & \phi & \text{low} \\ \phi & \text{high} & \phi & \text{high} & \phi \end{bmatrix}.$$

Clearly $K \neq K^t$ so K is not a symmetric linguistic square matrix.

Thus in general all square linguistic matrices are not symmetric.

Consider 6 × 6 linguistic diagonal matrix W with entries from S.

$$W = \begin{bmatrix} \text{low} & \phi & \phi & \phi & \phi & \phi \\ \phi & \text{high} & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \text{lowest} & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \text{highest} \end{bmatrix}$$

Now we find W^t;

$$W^t = \begin{bmatrix} \text{low} & \phi & \phi & \phi & \phi & \phi \\ \phi & \text{high} & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \text{lowest} & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \text{highest} \end{bmatrix}$$

Clearly $W = W^t$.

So we can say always a diagonal linguistic matrix is symmetric.

Theorem 2.1. *Let S be a linguistic finite set (or I_C a linguistic continuum). Let D be a $n \times n$ square linguistic diagonal matrix.*

D is a symmetric linguistic matrix.

Proof. To prove D is a symmetric linguistic matrix it is enough if we show $D = D^t$ for any diagonal linguistic matrix D.

$$\text{Let } D = \begin{bmatrix} a_{11} & \phi & \phi & \dots & \phi \\ \phi & a_{22} & \phi & \dots & \phi \\ \phi & \phi & \phi & \dots & \phi \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \phi & \phi & \phi & \dots & a_{nn} \end{bmatrix}$$

be a $n \times n$ linguistic diagonal matrix.

Clearly $D^t = D$ for every $a_{ij} = a_{ji} = \phi$ if $i \neq j$; hence the linguistic diagonal matrix is symmetric.

Next we proceed onto prove that transpose of a transpose of a linguistic matrix M is M.

That $(M^t)^t = M$ and $M^t \neq M$.

Theorem 2.2. *Let S be a linguistic finite set (or I_C the linguistic continuum). Let M be a $s \times n$ ($s \neq n$) linguistic rectangular matrix.*

i) *$M^t \neq M$ (that is transpose of M is not equal to M)*

ii) *$(M^t)^t = M$ (transpose of the transpose of a linguistic matrix M is M).*

Proof. Given M is a $s \times n$ rectangular linguistic matrix ($s \neq n$).

$$M = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & \vdots & \dots & \vdots \\ m_{s1} & m_{s2} & \dots & m_{sn} \end{bmatrix}_{s\times n}$$

where $m_{ij} \in S$ (or I_C); $1 \leq i \leq s$ and $1 \leq j \leq n$.

Now we find $M^t = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & \vdots & \dots & \vdots \\ m_{s1} & m_{s2} & \dots & m_{sn} \end{bmatrix}^t$

$$= \begin{bmatrix} m_{11} & m_{21} & \cdots & m_{s1} \\ m_{12} & m_{22} & \dots & m_{s2} \\ \vdots & \vdots & \dots & \vdots \\ m_{1n} & m_{2n} & \dots & m_{sn} \end{bmatrix}_{s \times n}$$

Clearly M^t is a s × n linguistic matrix (s ≠ n). So M ≠ M^t as both have different orders. Hence (1) is proved.

Now we find $(M^t)^t$.

$$(M^t)^t = \left(\begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & \vdots & \dots & \vdots \\ m_{s1} & m_{s2} & \dots & m_{sn} \end{bmatrix}_{s \times n}^{t} \right)^t = \begin{bmatrix} m_{11} & m_{21} & \cdots & m_{s1} \\ m_{12} & m_{22} & \dots & m_{s2} \\ \vdots & \vdots & \dots & \vdots \\ m_{1n} & m_{2n} & \dots & m_{sn} \end{bmatrix}_{n \times s}^{t}$$

$$= \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & \vdots & \dots & \vdots \\ m_{s1} & m_{s2} & \dots & m_{sn} \end{bmatrix}_{s \times n} = M.$$

Thus $(M^t)^t$ = M, hence (ii) of the theorem is proved.

Now we proceed onto discuss about the notion of submatrix of a linguistic matrix.

We will first define this concept.

Definition 2.3. *Let S be a finite linguistic set (or a linguistic continuum I_C).*

$$Suppose\ A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

be a m × n linguistic matrix with $a_{ij} \in S$ *(or* I_C*);* $1 \leq i \leq m$ *and* $1 \leq j \leq n$.

We call M to be linguistic submatrix of A if we remove some columns or some rows or both some columns and some rows from A.

We will illustrate this situation some examples.

Example 2.6. Let S = {good, very good, average, bad, very bad, lazy, worst, just good, just average, ϕ, best}

be linguistic set of order 11 related with the linguistic variable, performance aspects of a worker in an industry.

$$\text{Let M} = \begin{bmatrix} \text{good} & \phi & \text{average} & \text{best} & \text{worst} \\ \phi & \text{lazy} & \phi & \text{worst} & \phi \\ \text{bad} & \phi & \text{very bad} & \phi & \text{average} \\ \phi & \text{bad} & \text{good} & \text{bad} & \phi \\ \text{bad} & \phi & \phi & \text{good} & \text{bad} \\ \phi & \text{good} & \text{bad} & \phi & \phi \\ \text{worst} & \phi & \phi & \text{bad} & \text{good} \end{bmatrix}_{7\times5}$$

be a 7 × 5 linguistic with entries from S.

Suppose the first row of the linguistic matrix M is removed we get the linguistic submatrix S_1 of M which is as follows.

$$S_1 = \begin{bmatrix} \phi & \text{lazy} & \phi & \text{worst} & \phi \\ \text{bad} & \phi & \text{very bad} & \phi & \text{average} \\ \phi & \text{bad} & \text{good} & \text{bad} & \phi \\ \text{bad} & \phi & \phi & \text{good} & \text{bad} \\ \phi & \text{good} & \text{bad} & \phi & \phi \\ \text{worst} & \phi & \phi & \text{bad} & \text{good} \end{bmatrix}_{6\times5}.$$

Now the 3rd column of M is removed we get the correspond linguistic submatrix S_2 of M of order 6 × 5 which is as follows.

$$S_2 = \begin{bmatrix} \text{good} & \phi & \text{best} & \text{worst} \\ \phi & \text{lazy} & \text{worst} & \phi \\ \text{bad} & \phi & \phi & \text{average} \\ \phi & \text{bad} & \text{bad} & \phi \\ \text{bad} & \phi & \text{good} & \text{bad} \\ \phi & \text{good} & \phi & \phi \\ \text{worst} & \phi & \text{bad} & \text{good} \end{bmatrix}_{7\times4}$$

S_2 is a linguistic submatrix of order 7 × 4.

Let S_3 be the linguistic submatrix obtained from M by removing the 4th row and fourth column.

This is given in the following.

$$S_3 = \begin{bmatrix} \text{good} & \phi & \text{average} & \text{worst} \\ \phi & \text{lazy} & \phi & \phi \\ \text{bad} & \phi & \text{very bad} & \text{average} \\ \text{bad} & \phi & \phi & \text{bad} \\ \phi & \text{good} & \text{bad} & \phi \\ \text{worst} & \phi & \phi & \text{good} \end{bmatrix}_{6\times 4}$$

The order of this linguistic submatrix is 6 × 4.

Let S_4 be linguistic submatrix obtained from M by the first 3 rows and 2nd and 3rd column given in the following.

$$S_4 = \begin{bmatrix} \phi & \text{bad} & \phi \\ \text{bad} & \text{good} & \text{bad} \\ \phi & \phi & \phi \\ \text{worst} & \text{bad} & \text{good} \end{bmatrix}_{4\times 3}$$

The order of the linguistic submatrix S_4 is 4 × 3.

Let S_5 be the linguistic submatrix of M obtained by removing the last 6 rows and last column. The linguistic submatrix S_5 is as follows.

$$S_5 = [\text{good} \quad \phi \quad \text{average} \quad \text{best}]_{1\times 4}$$

We see the linguistic submatrix is a row matrix of order 1 × 4.

Let S_6 be the linguistic submatrix of M obtained by moving the last row and first four columns.

The linguistic submatrix S_6 of M is as follows.

$$S_6 = \begin{bmatrix} \text{worst} \\ \phi \\ \text{average} \\ \phi \\ \text{bad} \\ \phi \end{bmatrix}_{6\times 1}$$

The order of S_6 is 6×1 a column matrix.

Let S_7 be the linguistic submatrix of M got by removing the first three columns and the first and seventh rows of M.

The resulting linguistic submatrix S_7 of M is given in the following.

$$S_7 = \begin{bmatrix} \text{worst} & \phi \\ \phi & \text{average} \\ \text{bad} & \phi \\ \text{good} & \text{bad} \\ \phi & \phi \end{bmatrix}_{5\times 2}$$

is a linguistic submatrix of 5×2.

Let S_8 be the linguistic submatrix of M got by removing the first column and the last four rows. The linguistic submatrix S_8 of M is as follows.

$$S_8 = \begin{bmatrix} \phi & \text{average} & \text{best} & \text{worst} \\ \text{lazy} & \phi & \text{worst} & \phi \\ \phi & \text{very bad} & \phi & \text{average} \end{bmatrix}_{3\times 4}$$

S_8 is linguistic matrix of order 3 × 4.

Let S_9 be the linguistic submatrix of M obtained by moving the last 3 columns and last 5 rows.

The linguistic submatrix S_9 is as follows.

$$S_9 = \begin{bmatrix} \text{good} & \phi \\ \phi & \text{lazy} \end{bmatrix}_{2\times 2}$$

The linguistic submatrix S_9 is of order 2 × 2.

Let S_{10} be the linguistic submatrix of M obtained by removing the first four columns and first 6 rows of M.

S_{10} is given in the following.

$$S_{10} = [\text{good}]$$

From this we make the following observations.

Every single linguistic term / element in the linguistic matrix M is a linguistic submatrix of M.

We have linguistic matrices are all order bound by 7 × 5 to be true of linguistic submatrices.

But all things cannot contribute to linguistic submatrices.

For instance [bad worst] or $\begin{bmatrix} \text{worst} \\ \text{lazy} \end{bmatrix}$ are not linguistic submatrices of M; for they cannot be got by removing some stipulated number of rows and columns of M.

Now finding the number of linguistic submatrices is also a difficult problem. This we have for a linguistic matrix of order 3×4 so that interested students / researchers can find the total number of linguistic submatrices of any $m \times n$ linguistic matrix M.

Example 2.7. Let S be the linguistic set given in example 2.6. Now let.

$$B = \begin{bmatrix} \text{good} & \phi & \text{lazy} & \text{very good} \\ \text{bad} & \text{best} & \text{very bad} & \text{average} \\ \text{just good} & \text{bad} & \text{best} & \text{good} \end{bmatrix}_{3\times 4}$$

be a linguistic matrix of order 3×4 with entries from S.

If we remove the last three columns and last two rows the resulting linguistic submatrix

$S_1 = [\text{good}]_{1\times 1}$ of B is a trivial 1×1 lingusitic submatrix.

Similarly if the first, third and fourth columns and last two rows are removed the resulting 1×1 linguistic submatrix is

$$S_2 = [\phi]_{1\times 1} \text{ of B.}$$

On similar lines if the first, second and fourth columns and the first and 3rd row from B are removed the resulting linguistic submatrix of B is [very bad] is 1×1 trivial linguistic submatrix of B.

Thus we have 12 trivial 1×1 linguistic submatrices of B is given in the following {[good, [ϕ], [lazy], [very good], [bad], [best], [very bad], [average], [just good], [bad], [best] and [good]}

There are 12 trivial 1×1 linguistic submatrices of B.

Suppose we remove the last 2 columns and last two rows of B. Then we get S_3 to be linguistic submatrix of B given by S_3 = [good, ϕ]; S_2 is a linguistic row matrix of order 1×2.

On the other hand if we remove the first two rows and last two columns we get a linguistic submatrix S_4 of B given by

$$S_4 = [\text{just good, bad}]_{1 \times 2}$$

which is again a linguistic row matrix of order 1×2.

Likewise we get the following linguistic submatrices of B of order 1×2 given in the following.

{[good, ϕ], [good, lazy], [good, very good], [ϕ, lazy], [ϕ, very good], [lazy, very good], [bad, best], [bad, very bad], [bad, average], [best, very bad], [best, average], [very bad, average], [just good, bad], [just good, best], [just good, good], [bad, best], [bad, good], [best, good]}.

Thus there are 18 linguistic submatrices of B of order 1×2.

Now suppose we remove the last two rows and the first column, then we get a linguistic submatrix S_5 of B give by

$$S_5 = [\phi, \text{lazy, very good}]_{1\times3}$$

a linguistic row submatrix of order 1×3.

We enumerate all linguistic submatrices of B of order 1×3 in the following.

{[good, ϕ, lazy], [good, lazy, very good], [good, ϕ, very good], [ϕ, lazy, very good], [bad, best, very bad], [bad, best, average], [bad, very bad, average], [best, very bad, average], [just good, bad, best], [just good, bad, good], [just good, best, good], [bad, best, good]}.

We see there 12 1×3 linguistic row submatrices of B,.

Suppose in the matrix B we remove the first two row the resulting linguistic submatrix S_6 of B is given by

S_6 = [just good, bad, best, good]$_{1\times4}$

The order S_6 is 1×4.

We enumerate all the linguistic submatrices of B of order 1×4 in the following.

{[good ϕ lazy very good], [bad, best, very bad, average],

[just good, bad, best, good]}.

There are 3 linguistic submatrices of B of order 1×3.

Now let S_7 be a linguistic submatrix of B got by removing the last row and the last three columns of B.

$$S_7 = \begin{bmatrix} \text{good} \\ \text{bad} \end{bmatrix}$$

is a linguistic submatrix of B of order 2×1.

We now enumerate all linguistic submatrices of B of order 2 × 1 is the following

$$\{\begin{bmatrix} \text{good} \\ \text{bad} \end{bmatrix}, \begin{bmatrix} \text{good} \\ \text{just good} \end{bmatrix}, \begin{bmatrix} \text{bad} \\ \text{just good} \end{bmatrix}, \begin{bmatrix} \phi \\ \text{best} \end{bmatrix}, \begin{bmatrix} \phi \\ \text{bad} \end{bmatrix}, \begin{bmatrix} \text{best} \\ \text{bad} \end{bmatrix},$$

$$\begin{bmatrix} \text{lazy} \\ \text{very bad} \end{bmatrix}, \begin{bmatrix} \text{lazy} \\ \text{best} \end{bmatrix}, \begin{bmatrix} \text{very bad} \\ \text{best} \end{bmatrix},$$

$$\begin{bmatrix} \text{very good} \\ \text{average} \end{bmatrix}, \begin{bmatrix} \text{very good} \\ \text{good} \end{bmatrix}, \begin{bmatrix} \text{average} \\ \text{good} \end{bmatrix}\}$$

There 12 linguistic column submatrices of order 2 × 1.

Let S_8 be a linguistic submatrix of B got by removing the first 3 colun of B. Now

$$S_8 = \begin{bmatrix} \text{very good} \\ \text{average} \\ \text{good} \end{bmatrix}_{3\times 1}$$

S_8 is a linguistic column submatrix of B of order 3 × 1.

We now enumerate all linguistic submatrices of B of order 3 × 1 in the following.

$$\{\begin{bmatrix} \text{good} \\ \text{bad} \\ \text{just good} \end{bmatrix}, \begin{bmatrix} \phi \\ \text{best} \\ \text{bad} \end{bmatrix}, \begin{bmatrix} \text{lazy} \\ \text{very bad} \\ \text{best} \end{bmatrix}, \begin{bmatrix} \text{very good} \\ \text{average} \\ \text{good} \end{bmatrix}\}$$

there are four (4) linguistic submatrices of B of order 3 × 1.

Let S_9 be a linguistic submatrix of B got by removing the first row and last two columns of B.

We get $$S_9 = \begin{bmatrix} \text{bad} & \text{best} \\ \text{just good} & \text{bad} \end{bmatrix}_{2\times 2}$$

S_9 is a linguistic submatrix of B of order 2 × 2.

We now enumerate all the linguistic submatrices of B of order 2 × 2 in the following.

$$\left\{ \begin{bmatrix} \text{good} & \phi \\ \text{bad} & \text{best} \end{bmatrix}, \begin{bmatrix} \text{good} & \phi \\ \text{just good} & \text{bad} \end{bmatrix}, \begin{bmatrix} \text{bad} & \text{best} \\ \text{just good} & \text{bad} \end{bmatrix}, \right.$$

$$\begin{bmatrix} \text{good} & \text{lazy} \\ \text{bad} & \text{very bad} \end{bmatrix}, \begin{bmatrix} \text{good} & \text{lazy} \\ \text{just good} & \text{best} \end{bmatrix}, \begin{bmatrix} \text{bad} & \text{very bad} \\ \text{just good} & \text{best} \end{bmatrix},$$

$$\begin{bmatrix} \text{good} & \text{very good} \\ \text{bad} & \text{average} \end{bmatrix}, \begin{bmatrix} \text{good} & \text{very good} \\ \text{just good} & \text{good} \end{bmatrix},$$

$$\begin{bmatrix} \text{bad} & \text{average} \\ \text{just good} & \text{good} \end{bmatrix}, \begin{bmatrix} \phi & \text{lazy} \\ \text{best} & \text{very bad} \end{bmatrix}, \begin{bmatrix} \phi & \text{lazy} \\ \text{bad} & \text{best} \end{bmatrix},$$

$$\begin{bmatrix} \text{best} & \text{very bad} \\ \text{bad} & \text{best} \end{bmatrix}, \begin{bmatrix} \phi & \text{very good} \\ \text{very bad} & \text{average} \end{bmatrix}, \begin{bmatrix} \phi & \text{very good} \\ \text{bad} & \text{good} \end{bmatrix},$$

$$\begin{bmatrix} \text{best} & \text{average} \\ \text{bad} & \text{good} \end{bmatrix}, \begin{bmatrix} \text{lazy} & \text{very good} \\ \text{very bad} & \text{average} \end{bmatrix}, \begin{bmatrix} \text{lazy} & \text{very good} \\ \text{best} & \text{good} \end{bmatrix},$$

$$\begin{bmatrix} \text{very bad} & \text{average} \\ \text{best} & \text{good} \end{bmatrix}\}.$$

We see there are 18 linguistic sub matrices of order 2×2.

Let S_{10} be a linguistic submatrix of B of order 3×2 by deleting the last two rows of B.

$$S_{10} = \begin{bmatrix} \text{good} & \phi \\ \text{bad} & \text{best} \\ \text{just good} & \text{bad} \end{bmatrix}_{3\times 2}$$

is a linguistic submatrix of B of order 3×2.

We now enumerate of linguistic submatrices of B of order 3×2 in the following.

$$\{\begin{bmatrix} \text{good} & \phi \\ \text{bad} & \text{best} \\ \text{just good} & \text{bad} \end{bmatrix}, \begin{bmatrix} \text{good} & \text{lazy} \\ \text{bad} & \text{very bad} \\ \text{just good} & \text{best} \end{bmatrix},$$

$$\begin{bmatrix} \text{good} & \text{very good} \\ \text{bad} & \text{average} \\ \text{just good} & \text{good} \end{bmatrix}, \begin{bmatrix} \phi & \text{lazy} \\ \text{best} & \text{very bad} \\ \text{bad} & \text{best} \end{bmatrix},$$

$$\begin{bmatrix} \phi & \text{very good} \\ \text{best} & \text{average} \\ \text{bad} & \text{good} \end{bmatrix}, \begin{bmatrix} \text{lazy} & \text{very good} \\ \text{very bad} & \text{average} \\ \text{best} & \text{good} \end{bmatrix}\}$$

There are 6 linguistic submatrices of B of order 3×2.

Now let S_{11} be a linguistic submatrix of B of order 2×3 got by removing the last row and last column of B.

$$S_{11} = \begin{bmatrix} \text{good} & \phi & \text{lazy} \\ \text{bad} & \text{best} & \text{very bad} \end{bmatrix}_{2\times 3}$$

S_{11} is of order 2×3, we enumerate all linguistic submatrices of B of order 2×3 in the following.

$$\left\{\begin{bmatrix} \text{good} & \phi & \text{lazy} \\ \text{bad} & \text{best} & \text{very bad} \end{bmatrix}, \begin{bmatrix} \text{good} & \phi & \text{lazy} \\ \text{just good} & \text{bad} & \text{best} \end{bmatrix},\right.$$

$$\begin{bmatrix} \text{bad} & \text{best} & \text{very bad} \\ \text{just good} & \text{bad} & \text{best} \end{bmatrix}, \begin{bmatrix} \phi & \text{lazy} & \text{very good} \\ \text{best} & \text{very bad} & \text{average} \end{bmatrix},$$

$$\begin{bmatrix} \phi & \text{lazy} & \text{very good} \\ \text{bad} & \text{best} & \text{good} \end{bmatrix}, \begin{bmatrix} \text{best} & \text{very bad} & \text{average} \\ \text{bad} & \text{best} & \text{good} \end{bmatrix},$$

$$\begin{bmatrix} \text{good} & \text{lazy} & \text{very good} \\ \text{bad} & \text{very bad} & \text{average} \end{bmatrix}, \begin{bmatrix} \text{good} & \text{lazy} & \text{very good} \\ \text{just good} & \text{best} & \text{good} \end{bmatrix},$$

$$\begin{bmatrix} \text{bad} & \text{very bad} & \text{average} \\ \text{just good} & \text{best} & \text{good} \end{bmatrix}, \begin{bmatrix} \text{good} & \phi & \text{very good} \\ \text{bad} & \text{best} & \text{average} \end{bmatrix},$$

$$\left.\begin{bmatrix} \text{good} & \phi & \text{very good} \\ \text{bad} & \text{very bad} & \text{average} \end{bmatrix}, \begin{bmatrix} \text{bad} & \text{best} & \text{average} \\ \text{just good} & \text{best} & \text{good} \end{bmatrix}\right\}.$$

There are 12 linguistic submatrices of B of order 2×3.

Now let S_{12} be a linguistic submatrix of B got by removing the last column.

$$S_{12} = \begin{bmatrix} \text{good} & \phi & \text{lazy} \\ \text{bad} & \text{best} & \text{very bad} \\ \text{just good} & \text{bad} & \text{best} \end{bmatrix}_{3\times3}$$

S_{12} is a linguistic submatrix of B of order 3 × 3.

Now we enumerate all linguistic submatrices of B of order 3 × 3 in the following.

$$\left\{ \begin{bmatrix} \text{good} & \phi & \text{lazy} \\ \text{bad} & \text{best} & \text{very bad} \\ \text{just good} & \text{bad} & \text{best} \end{bmatrix}, \begin{bmatrix} \text{good} & \phi & \text{very good} \\ \text{bad} & \text{best} & \text{average} \\ \text{just good} & \text{bad} & \text{good} \end{bmatrix}, \right.$$

$$\begin{bmatrix} \text{good} & \text{lazy} & \text{very good} \\ \text{bad} & \text{very bad} & \text{average} \\ \text{just good} & \text{best} & \text{good} \end{bmatrix},$$

$$\left. \begin{bmatrix} \phi & \text{lazy} & \text{very good} \\ \text{best} & \text{very bad} & \text{average} \\ \text{bad} & \text{best} & \text{good} \end{bmatrix} \right\}$$

Now we see there are 4 linguistic submatrices of B of order 3 × 3.

Thus we have

12 + 18 + 12 + 3 + 12 + 4 + 18 + 6 + 12 + 4 + 1 = 92

such trivial and non trivial linguistic submatrices of B where order of B is 3 × 4.

We leave it as an exercise for the interested researcher to find a suitable program to find the number of linguistic submatrices of a linguistic matrix.

Next we proceed onto describe the notion of upper triangular and low triangular linguistic matrices. This can be done only in case of square matrices and also the linguistic set (linguistic continuum) under consideration should contain the empty linguistic term ϕ.

We will first define this concept.

Definition 2.4. *Let S be a linguistic set of finite order containing the empty linguistic term ϕ (or the linguistic continuum I_C which contains the linguistic term ϕ).*

A linguistic square matrix is said to be a upper triangular linguistic matrix if all the entries below the main diagonal are empty linguistic term ϕ.

A low triangular linguistic matrix is one in which all the entries above the main diagonal are empty linguistic terms ϕ.

$$\text{That is } U = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \phi & a_{22} & \dots & a_{2n} \\ \phi & \phi & \dots & a_{3n} \\ \vdots & \vdots & \dots & \vdots \\ \phi & \phi & \dots & a_{n-1n} \\ \phi & \phi & \dots & a_{nn} \end{bmatrix}$$

where $a_{ij} \in S$ (or I_C) $1 \leq i, j \leq n$ is the upper triangular linguistic matrix of order $n \times n$.

$$L = \begin{bmatrix} a_{11} & \phi & \ldots & \phi \\ a_{21} & a_{22} & \ldots & \phi \\ a_{31} & a_{32} & \ldots & \phi \\ \vdots & \vdots & \ldots & \phi \\ a_{n-11} & a_{n-12} & \ldots & \phi \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

with $a_{ij} \in S$ *(or* I_C*);* $1 \leq i, j \leq n$ *is the lower triangular linguistic matrix of order* $n \times n$*.*

Now we will illustrate this situation by some examples.

Example 2.8. Let S = {ϕ, fastest, fast, very fast, just fast, slow, very slow, medium speed, just slow, slowest}.

be the linguistic set associated with the linguistic variable, "speed of the car". ϕ the empty linguistic term / word occurs when the car is in the signal (i.e. waiting for the signal so its speed is ϕ).

All other linguistic terms of ϕ are self-explanatory. Consider

$$L = \begin{bmatrix} \text{fast} & \phi & \phi & \phi & \phi & \phi \\ \text{slow} & \text{fastest} & \phi & \phi & \phi & \phi \\ \text{just slow} & \text{fast} & \text{slow} & \phi & \phi & \phi \\ \text{slow} & \text{fast} & \text{fastest} & \text{slow} & \phi & \phi \\ \text{fast} & \phi & \text{slow} & \text{fast} & \text{slow} & \phi \\ \text{very fast} & \text{fast} & \text{slow} & \text{slowest} & \text{just slow} & \text{slow} \end{bmatrix}$$

L is a linguistic lower triangular matrix of order 6 × 6.

Now we give the linguistic upper triangular, matrix of order 4×4 in the following

$$U = \begin{bmatrix} \text{slow} & \text{fast} & \text{fastest} & \text{slow} \\ \phi & \text{very slow} & \text{slow} & \text{fast} \\ \phi & \phi & \text{very fast} & \text{fast} \\ \phi & \phi & \phi & \text{just slow} \end{bmatrix}$$

Now we find transpose of U,

$$U^T = \begin{bmatrix} \text{slow} & \text{fast} & \text{fastest} & \text{slow} \\ \phi & \text{very slow} & \text{slow} & \text{fast} \\ \phi & \phi & \text{very fast} & \text{fast} \\ \phi & \phi & \phi & \text{just slow} \end{bmatrix}$$

$$= \begin{bmatrix} \text{slow} & \phi & \phi & \phi \\ \text{fastest} & \text{very slow} & \phi & \phi \\ \text{fastest} & \text{slow} & \text{very fast} & \phi \\ \text{slow} & \text{fast} & \text{fast} & \text{just fast} \end{bmatrix}_{4\times4}.$$

Clearly U^T, the transpose of the linguistic upper triangular matrix is a linguistic lower triangular matrix.

Now we find L^T of the linguistic low triangular matrix L.

$$L^t = \begin{bmatrix} \text{fast} & \phi & \phi & \phi & \phi & \phi \\ \text{slow} & \text{fastest} & \phi & \phi & \phi & \phi \\ \text{just slow} & \text{fast} & \text{slow} & \phi & \phi & \phi \\ \text{slow} & \text{fast} & \text{fastest} & \text{slow} & \phi & \phi \\ \text{fast} & \phi & \text{slow} & \text{fast} & \text{slow} & \phi \\ \text{very fast} & \text{fast} & \text{slow} & \text{slowest} & \text{just slow} & \text{slow} \end{bmatrix}$$

$$= \begin{bmatrix} \text{fast} & \text{slow} & \text{just slow} & \text{slow} & \text{fast} & \text{very fast} \\ \phi & \text{fastest} & \text{fast} & \text{fast} & \phi & \text{fast} \\ \phi & \phi & \text{slow} & \text{fastest} & \text{slow} & \text{slow} \\ \phi & \phi & \phi & \text{slow} & \text{fast} & \text{slowest} \\ \phi & \phi & \phi & \phi & \text{slow} & \text{just slow} \\ \phi & \phi & \phi & \phi & \phi & \text{slow} \end{bmatrix}$$

Clearly the transpose of a lower triangular linguistic matrix is a upper triangular linguistic matrix.

Now we will show when are two linguistic matrices M and N are equal.

First of all to define or two linguistic matrices M and N to equal we need order of M and N must be the same say $s \times t$.

Suppose

$$M = \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1t} \\ m_{21} & m_{22} & \dots & m_{2t} \\ \vdots & \vdots & \dots & \vdots \\ m_{s1} & m_{s2} & \dots & m_{st} \end{bmatrix}$$

be a linguistic matrix of order s × t and

$$N = \begin{bmatrix} n_{11} & n_{12} & \dots & n_{1t} \\ n_{21} & n_{22} & \dots & n_{2t} \\ \vdots & \vdots & \dots & \vdots \\ n_{s1} & n_{s2} & \dots & n_{st} \end{bmatrix}$$

be another s × t linguistic matrix N = M if and only if $m_{ij} = n_{ij}$ for $1 \leq i \leq s$ and $1 \leq j \leq t$, then we say M and N are equal.

Now we proceed onto suggest some problems to the reader.

By solving these problem one can become more familiar with the concept of linguistic matrices and some of their basic properties.

SUGGESTED PROBLEMS

1. Give a 4 × 5 linguistic matrix using the linguistic set S = {ϕ, good, bad, fair, very bad, best, just fair, just good}

   i) How many linguistic matrices of order 4 × 5 can be got using S?

   ii) How any linguistic matrices of order 5 × 4 can be obtained using S?

   iii) How many linguistic matrices of order 1 × 3 can be got using S?

iv) Find the number linguistic matrices of order 4×4 using S.

2. Given M = $$\begin{bmatrix} \phi & \text{bad} & \text{very bad} & \text{best} \\ \text{fair} & \text{just fair} & \phi & \text{bad} \\ \text{just good} & \phi & \text{good} & \text{best} \\ \text{very bad} & \text{bad} & \phi & \text{bad} \\ \phi & \text{fair} & \phi & \text{good} \\ \text{good} & \phi & \text{bad} & \phi \\ \text{bad} & \text{fair} & \text{fair} & \text{best} \\ \phi & \phi & \text{just good} & \text{fair} \end{bmatrix}_{8\times 4}$$

with entries from S (S given problem 1).

i) Find M^t and prove $M \neq M^t$.

ii) Find $(M^t)^t$ and prove $(M^t)^t = M$.

iii) Find all linguistic submatrices of M.

iv) How many linguistic submatrices of order 3×3 exist in M?

v) Find all linguistic submatrics of M of order 3×2.

vi) Find all linguistic submatrices of M of order 2×3.

vii) Compare the number of linguistic submatrices of order 2×3 and 3×2.

viii) Find all linguistic submatrices of order 4×4 in M.

ix) How many 6×6 upper triangular linguistic matrices using the linguistic set S given problem 1?

3. Find 3 linguistic matrices which are symmetric using the linguistic set S given in problem 1 of orders 10×10, 12×12 and 9×9.

4. Let I_C = [worst, best] be the linguistic continuum related with the linguistic variable performance of a teacher in the class room.

i) Let $C = \begin{bmatrix} \text{worst} \\ \text{bad} \\ \text{very bad} \\ \text{fair} \\ \text{medium} \\ \text{best} \\ \text{just good} \\ \text{good} \\ \text{very good} \\ \text{just good} \\ \text{best} \end{bmatrix}$

be a 11×1 linguistic matrix.

ii) Find C^t.

iii) Find all linguistic submatrices of C.

iv) How many such linguistic submatrices of the linguistic matrix C^t exist?

5. Let I_C = [slowest, fastest] $\cup$ $\{\phi\}$ be the linguistic continuum associated with the linguistic variable speed of the car on the road.

$$\text{Let } C = \begin{bmatrix} \text{slow} & \phi & \text{slowest} & \text{fast} \\ \phi & \text{fast} & \phi & \text{slow} \\ \text{very slow} & \phi & \text{slow} & \phi \\ \phi & \phi & \phi & \phi \\ \text{slow} & \text{slow} & \text{slow} & \text{slowest} \\ \text{fast} & \text{faster} & \text{fast} & \phi \\ \phi & \phi & \text{fast} & \text{just fast} \\ \text{just fast} & \text{just slow} & \phi & \phi \\ \phi & \text{faster} & \phi & \text{fastest} \end{bmatrix}$$

be the linguistic matrix of order 9×4

i) Find all linguistic submatrices of C.

ii) Find the transpose of C.

iii) Find $(C^t)^t$.

6. Obtain any of the striking features of linguistic matrices.

7. Give six distinct linguistic continuum associated with six distinct linguistic variables.

8. Compare the real continuum with the linguistic continuums and show there are infinitely many linguistic continuums.

9. For any given linguistic matrix M of order $n \times m$ ($n \neq m$)

   i) Find the number of linguistic submatrices of M.

   ii) Write a programme to find the number of linguistic submatrices of M.

   iii) What change takes place if M is a square linguistic matrix (i.e, $n = m$)?

10. For the linguistic matrix C given in problem 4 find at least 4 distinct matrices which are not linguistic submatrices of C.

11. Is it possible to find all linguistic matrices of C in problem 4 which are not linguistic submatrices of C.

12. What collection is larger the linguistic submatrices of C or those linguistic matrices built using C which are not linguistic submatrices of C?

13. Can the linguistic row matrix

    D = (good, bad, ϕ, best, fair, ϕ, worst, just good, very good)

    have linguistic matrices which are not linguistic submatrices of D? Justify your claim.

14. Let $F = \begin{bmatrix} \text{best} \\ \text{bad} \\ \phi \\ \text{worst} \\ \text{good} \\ \text{very good} \\ \phi \\ \text{just good} \\ \text{fair} \\ \text{just fair} \\ \phi \end{bmatrix}$

    be a linguistic column matrix of order 11×1.

    Can F have linguistic matrices which are not linguistic submatrices? Justify your claim!

15. Obtain any other special feature associated with linguistic submatrices of a linguistic matrix in general.

16. Suppose $N = (a_1, \ldots, a_n)$ is a linguistic row matrix of order $1 \times n$.

    Prove or disprove N has only $2^n - 1$ number of linguistic submatrices and every linguistic matrix of N is a linguistic submatrix of N.

17. Suppose M be a linguistic column matrix of order $n \times 1$. Prove M has only $2^n - 1$ number of linguistic submatrices and no linguistic matrices of M and every linguistic matrix of M is a linguistic submatrix of M.

18. Let $D = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1p} \\ a_{21} & a_{22} & \ldots & a_{2p} \\ a_{31} & a_{32} & \ldots & a_{3p} \end{bmatrix}$

where $a_{ij} \in S$ (S a linguistic set) $1 \leq i \leq 3$ and $1 \leq j \leq p$ be a linguistic matrix of order $3 \times p$.

i) Prove D has linguistic matrix which are not linguistic submatrices of D.

ii) Find the number of linguistic submatrices of D.

19. Let $E = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ \vdots & \vdots & \vdots \\ a_{r1} & a_{r2} & a_{r3} \end{bmatrix}$

be a linguistic matrix of order $r \times 3$.

i) Prove E has linguistic matrices which are not linguistic submatrices.

ii) Find the total number of linguistic submatrices of E.

iii) Are $S_1 = \begin{bmatrix} a_{11} \\ a_{32} \\ a_{r2} \end{bmatrix}$ and $S_2 = \begin{bmatrix} a_{11} & a_{23} \\ a_{32} & a_{43} \\ a_{63} & a_{71} \\ a_{82} & a_{93} \end{bmatrix}$

linguistic submatrices of E? Justify your claim.

Chapter Three

LINGUISTIC MATRICES AND OPERATIONS ON THEM

In this chapter we for the first time introduce all the basic properties of linguistic matrices in a methodical way. This is mainly carried out with a view to benefit non mathematicians like socio scientist, economists, engineers, medical experts, computer scientist and others can approach this book and construct linguistic mathematical models to analyze any mathematical problems / research.

The general concept of linguistic theory or the basic introduction of linguistic theory can be had from [25].

However, a brief introduction of linguistic theory and their properties are recalled in chapter I of this book to make this book a self-contained one.

Throughout this book S will denote a linguistic set / a continuum and it is always assumed to be kept in an increasing order.

So without loss of generality when we say S is a linguistic set it is assumed always to be ordered and more specifically it is ordered in the increasing order only.

Thus if $S = \{a_1, a_2, \ldots, a_n\}$ then we have

$a_1 < a_2 < \ldots a_n$ where each a_i is a linguistic term.

S can be a collection of linguistic terms associated with height or performance of workers in an industry or teachers' performance in any educational institute or students' performance in studies etc.

We shall first define the following matrices.

Example 3.1. Let $S = \{a_1, a_2, \ldots, a_8, a_9\}$

= {very poor, just poor, poor, very medium, just medium, medium, just good, good, very good}

be the evaluation of an expert in the linguistic expression about the performance of a worker, working in an industry.

Clearly S is a totally ordered linguistic set as

very poor < just poor < poor < very medium < just medium < medium < just good < good < very good.

We see a = (poor, good, very good)

is a 1×3 linguistic row matrix with entries from S or we say order of a is 1×3.

That is $a = (a_3, a_8, a_9)$ = (poor, good, very good).

By varying the linguistic terms from S in the 1×3 linguistic row matrix, we get $9 \times 9 \times 9 = 9^3$ number of such 1×3 linguistic matrices.

$$\text{Let } M = \begin{bmatrix} a_9 \\ a_1 \\ a_2 \\ a_3 \\ a_6 \end{bmatrix}$$

be a linguistic 5 × 1 column matrix where $a_9, a_1, a_2, a_3, a_6 \in S$ or order of M is 5 × 1.

$$\text{Let } N = \left\{ \begin{bmatrix} a_i \\ a_j \\ a_k \\ a_t \\ a_s \end{bmatrix} \middle/ a_i, a_j, a_k, a_t\ a_s \in S; 1 \leq i, j, k, t, s \leq 9 \right\}$$

be the collection of all 5 × 1 linguistic column matrices (of order 5 × 1).

Clearly $|N|$ = order of N = $9 \times 9 \times 9 \times 9 \times 9 = 9^5$.

$$\text{Let } C = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \\ b_9 & b_{10} & b_{11} & b_{12} \\ b_{13} & b_{14} & b_{15} & b_{16} \end{bmatrix}$$

be a 4 × 4 linguistic square matrix with $b_i \in S$; $1 \leq i \leq 16$.

$$\text{If } T = \left\{ \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \\ b_9 & b_{10} & b_{11} & b_{12} \\ b_{13} & b_{14} & b_{15} & b_{16} \end{bmatrix} \middle/ b_i \in S; 1 \leq i \leq 16 \right\}$$

be the collection of all 4 × 4 linguistic square matrices then $o(T) = 9^{16}$.

Let $d = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 & b_{10} & b_{11} & b_{12} \\ b_{13} & b_{14} & b_{15} & b_{16} & b_{17} & b_{18} \\ b_{19} & b_{20} & b_{21} & b_{22} & b_{23} & b_{24} \end{bmatrix}$

be a 4 × 6 linguistic matrix with $b_i \in S$; $1 \leq i \leq 24$.

Suppose

$$W = \left\{ \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 & b_{10} & b_{11} & b_{12} \\ b_{13} & b_{14} & b_{15} & b_{16} & b_{17} & b_{18} \\ b_{19} & b_{20} & b_{21} & b_{22} & b_{23} & b_{24} \end{bmatrix} \middle/ b_i \in S;\ 1 \leq i \leq 24 \right\}$$

be the collection of all 4 × 6 linguistic matrices with entries from S.

Clearly $o(W) = 9^{24}$, we have 9^{24} such 4 × 6 linguistic matrices with entries from S.

In view of these we present now the abstract definition of the linguistic m × n matrices with entries from

$S = \{a_1, a_2, \ldots, a_s\}$, where $2 \leq s < \infty$, a linguistic totally ordered set.

Definition 3.1. *Let $S = \{a_1, a_2, \ldots, a_n\}$ be a linguistic set associated with some problem / concept which is a totally ordered set in the increasing order*

$$A = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ a_{21} & \dots & a_{2m} \\ \vdots & \dots & \vdots \\ a_{p1} & \dots & a_{pm} \end{bmatrix}$$

be rectangular array of linguistic terms with entries from S and $a_{ij} \in S$; $1 \leq i \leq p$ *and* $1 \leq j \leq m$.

We define A as the p × m linguistic matrix having p number of rows and m number of columns. We call p × m as the order of the linguistic matrix A.

If p = 1 then we define as a 1 × m linguistic row matrix. If m = 1 then define A as a p × 1 linguistic column matrix.

If p = m then we define A as a p × p linguistic square matrix. If p ≠ m we define A as a p × m linguistic rectangular matrix.

We will illustrate this situation by some examples.

Example 3.2. Let

S = {lowest, very low, just low, low, moderate, very moderate, just high, high, highest}

be the linguistic terms associated with a industry during a span of 5 years.

A = (lowest, low, moderate, high, highest)

is a 1 × 5 linguistic 1 × 5 row matrix with entries from S.

$$\text{Let} \qquad B = \begin{bmatrix} \text{very low} \\ \text{low} \\ \text{low} \\ \text{high} \\ \text{high} \\ \text{highest} \\ \text{mod erate} \\ \text{low} \end{bmatrix}$$

be a 8 × 1 linguistic column matrix with entries from S.

$$\text{Let } C = \begin{bmatrix} \text{high} & \text{low} & \text{moderate} & \text{just low} & \text{lowest} \\ \text{high} & \text{lowest} & \text{moderate} & \text{just low} & \text{highest} \\ \text{high} & \text{low} & \text{very low} & \text{just high} & \text{moderate} \\ \text{high} & \text{high} & \text{moderate} & \text{very low} & \text{very low} \\ \text{low} & \text{high} & \text{lowest} & \text{very low} & \text{very low} \end{bmatrix}$$

be a linguistic 5 × 5 square matrix with entries from S.

$$\text{Let} \qquad D = \begin{bmatrix} \text{lowest} & \text{very ghigh} & \text{highest} \\ \text{low} & \text{high} & \text{very low} \\ \text{high} & \text{lowest} & \text{just high} \\ \text{just high} & \text{low} & \text{low} \\ \text{very low} & \text{low} & \text{high} \\ \text{just high} & \text{high} & \text{lowest} \\ \text{losest} & \text{high} & \text{low} \\ \text{low} & \text{low} & \text{high} \\ \text{moderate} & \text{just low} & \text{low} \\ \text{high} & \text{just high} & \text{highest} \end{bmatrix}$$

be a 10 × 3 rectangular linguistic matrix with entries from S.

We have seen all four types of linguistic matrices.

Example 3.3. Let S = {worst, very very bad, very bad, bad, just fair, fair, very fair, good, best}

be a linguistic set which pertain to the performance of a student in a school.

Let A = (worst, bad, fair, good, best)

be a 1×5 linguistic matrix.

Let $B = \begin{bmatrix} \text{bad} \\ \text{bad} \\ \text{worst} \\ \text{good} \\ \text{good} \\ \text{fair} \\ \text{fair} \\ \text{bad} \end{bmatrix}$ be a 8×1 column linguistic matrix.

For the 6×6 linguare matrix we have the following.

$$C = \begin{bmatrix} \text{fair} & \text{very bad} & \text{good} & \text{fair} & \text{best} & \text{very bad} \\ \text{bad} & \text{very fair} & \text{bad} & \text{just fair} & \text{best} & \text{just fair} \\ \text{good} & \text{just fair} & \text{good} & \text{bad} & \text{bad} & \text{very fair} \\ \text{good} & \text{very bad} & \text{bad} & \text{bad} & \text{good} & \text{very very bad} \\ \text{bad} & \text{very bad} & \text{good} & \text{good} & \text{best} & \text{good} \\ \text{fair} & \text{good} & \text{very bad} & \text{best} & \text{fair} & \text{best} \end{bmatrix}$$

be the linguistic square matrix of order 6×6.

The following gives the rectangular linguistic matrix

$$D = \begin{bmatrix} \text{fair} & \text{fair} & \text{very bad} & \text{good} & \text{worst} & \text{very bad} & \text{good} & \text{fair} & \text{best} \\ \text{good} & \text{bad} & \text{very fair} & \text{bad} & \text{very very bad} & \text{worst} & \text{bad} & \text{bad} & \text{good} \\ \text{bad} & \text{best} & \text{just fair} & \text{best} & \text{very fair} & \text{bad} & \text{fair} & \text{good} & \text{best} \\ \text{best} & \text{good} & \text{very bad} & \text{fair} & \text{best} & \text{good} & \text{fair} & \text{worst} & \text{bad} \end{bmatrix}$$

be a 4 × 9 rectangular linguistic matrix with entries from S.

Now having seen examples of linguistic matrix we now proceed onto describe a few of its properties.

Now we first describe the notion of transpose of any linguistic matrix. It is important to record that the transpose of a linguistic matrix is also the same as that of a classical matrix. However to make this book a complete one on linguistic matrices we fist exhibit them by some examples.

Example 3.4. Let A = (worst, bad, fair, good, best) be the 1 × 5 linguistic row matrix described in example 3.3. We denote the transpose of a linguistic matrix A as A^t.

$$A^t = \begin{bmatrix} \text{worst} \\ \text{bad} \\ \text{fair} \\ \text{good} \\ \text{best} \end{bmatrix}$$

is the linguistic transpose of the 1×5 linguistic row matrix.

Clearly A^t is a 5×1 linguistic column matrix.

Consider the 8×1 linguistic column matrix B given in example 3.3. We find the transpose of B;

B^t = (bad, bad, worst, good, good, fair, fair, bad) is the 1×8 linguistic row matrix.

So we see as in the case of classical matrices the transpose of a linguistic row matrix is a linguistic column matrix and the transpose of a column linguistic matrix is a linguistic row matrix.

$$\text{Thus } (A^t)^t = A \text{ and } (B^t)^t = B.$$

For now

$(A^t)^t$ = (worst, bad, fair, good, best) = A which is the linguistic 1×5 row matrix.

$$\text{Now } (B^t)^t = \begin{bmatrix} \text{bad} \\ \text{bad} \\ \text{worst} \\ \text{good} \\ \text{good} \\ \text{fair} \\ \text{fair} \\ \text{bad} \end{bmatrix} = B \text{ is linguistic column } 8 \times 1 \text{ matrix.}$$

Now we find the transpose of the 6×6 square linguistic matrix given in example 3.3.

$C^t =$

$$\begin{bmatrix} \text{fair} & \text{bad} & \text{good} & \text{good} & \text{bad} & \text{fair} \\ \text{very bad} & \text{very fair} & \text{just fair} & \text{very bad} & \text{very bad} & \text{good} \\ \text{good} & \text{bad} & \text{good} & \text{bad} & \text{good} & \text{very bad} \\ \text{fair} & \text{just fair} & \text{bad} & \text{bad} & \text{good} & \text{very bad} \\ \text{best} & \text{best} & \text{bad} & \text{good} & \text{best} & \text{fair} \\ \text{very bad} & \text{just fair} & \text{very fair} & \begin{matrix}\text{very} \\ \text{very bad}\end{matrix} & \text{good} & \text{best} \end{bmatrix}$$

is again a 6×6 square linguistic matrix, but C^t is different from C.

Now we find the transpose of D, the 4×9 linguistic matrix from example 3.3.

$$D^t = \begin{bmatrix} \text{fair} & \text{good} & \text{bad} & \text{best} \\ \text{fair} & \text{bad} & \text{very fair} & \text{bad} \\ \text{very bad} & \text{very fair} & \text{just fair} & \text{very bad} \\ \text{good} & \text{bad} & \text{best} & \text{fair} \\ \text{worst} & \text{very very bad} & \text{very fair} & \text{best} \\ \text{very bad} & \text{worst} & \text{bad} & \text{good} \\ \text{good} & \text{bad} & \text{fair} & \text{fair} \\ \text{fair} & \text{bad} & \text{good} & \text{worst} \\ \text{best} & \text{good} & \text{best} & \text{bad} \end{bmatrix}$$

is the 9×4 linguistic matrix.

Clearly $D \neq D^t$.

It is interesting to note that one can have only in case of linguistic square matrices under special conditions we may have the matrix and its transpose are identical.

We will first illustrate this situation by an example and then give a definition.

Example 3.5. Let s be as an example 3.3. Consider a linguistic 5 × 5 square matrix A given by

$$A = \begin{bmatrix} \text{best} & \text{good} & \text{fair} & \text{bad} & \text{worst} \\ \text{good} & \text{fair} & \text{bad} & \text{worst} & \text{good} \\ \text{fair} & \text{bad} & \text{very fair} & \text{good} & \text{best} \\ \text{bad} & \text{worst} & \text{good} & \text{good} & \text{bad} \\ \text{worst} & \text{good} & \text{best} & \text{bad} & \text{best} \end{bmatrix}$$

now the find the A^t of A.

$$A^t = \begin{bmatrix} \text{best} & \text{good} & \text{fair} & \text{bad} & \text{worst} \\ \text{good} & \text{fair} & \text{bad} & \text{worst} & \text{good} \\ \text{fair} & \text{bad} & \text{very fair} & \text{good} & \text{best} \\ \text{bad} & \text{worst} & \text{good} & \text{good} & \text{bad} \\ \text{worst} & \text{good} & \text{best} & \text{bad} & \text{best} \end{bmatrix}.$$

We see $A^t = A$; this is the case when both are equal.

We make the following observations.

Only in case of square linguistic matrices (as in classical case) we can get the main diagonal concept.

In case of A the main diagonal elements are denoted by (best, fair, very fair, good, best).

For thus we need to have the diagonal linguistic matrix.

Now in our linguistic sets we do not have the concept of zero. So we in order to define a non-mathematical term we denote in any linguistic matrix A by 'ϕ' the linguistic term is empty. Then comes the problem will S contain ϕ. To this end to adjoin with S the empty symbol S.

So $S' = \{\phi, S\}$ and ϕ will be leading least element in any increasing order set (that is an ordered set).

We can define only in case of taking terms from S' form a linguistic diagonal matrix which will always be a square matrix.

$$A = \begin{bmatrix} \text{good} & \phi & \phi & \phi \\ \phi & \text{bad} & \phi & \phi \\ \phi & \phi & \text{fair} & \phi \\ \phi & \phi & \phi & \text{best} \end{bmatrix}$$

(this S' is taken from Example 3.3).

We call A the diagonal 4×4 linguistic matrix.

We see our linguistic matrices have ϕ as an element; it is not mandatory that ϕ must be present in every linguistic matrix. Only in case of diagonal linguistic matrices we have every element in it is ϕ the diagonal terms can be empty or may not be empty.

We have the empty row linguistic matrix

$$(\phi) = (\phi, \phi, \phi, \phi, \phi, \phi, \phi, \phi)$$

which is a 1×8 row empty linguistic matrix.

$$\text{Let } (\phi) = \begin{bmatrix} \phi \\ \phi \\ \phi \\ \phi \\ \phi \\ \phi \\ \phi \\ \phi \\ \phi \end{bmatrix}$$

be a 9 × 1 linguistic empty column matrix.

$$(\phi) = \begin{bmatrix} \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi \end{bmatrix}$$

is a 6 × 6 linguistic empty matrix.

$$(\phi) = \begin{bmatrix} \phi & \phi & \phi & \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi & \phi & \phi & \phi \end{bmatrix}$$

is a 4 × 9 empty linguistic matrix.

We will be using these concepts in the next section of this chapter.

Thus using $S' = S \cup \{\phi\}$ where S is taken from the example 3.3. We give some linguistic matrices with entries from S′.

Let A = (good, ϕ, bad, just fair, fair, ϕ, best)

is a 1 ×7 row linguistic matrix with entries from S′.

Consider the 9 × 1 linguistic column matrix with entries from S′.

$$B = \begin{bmatrix} \text{bad} \\ \text{good} \\ \phi \\ \text{fair} \\ \phi \\ \text{best} \\ \phi \\ \text{best} \\ \text{bad} \end{bmatrix}$$

is a 9 × 1 linguistic column matrix with entries from S′.

$$\text{Let } C = \begin{bmatrix} \text{good} & \text{bad} & \phi & \text{fair} & \text{very fair} \\ \text{bad} & \text{best} & \text{fair} & \text{bad} & \text{bad} \\ \phi & \text{very fair} & \text{bad} & \text{best} & \text{best} \\ \text{good} & \text{fair} & \text{fair} & \text{best} & \text{bad} \\ \text{best} & \text{fair} & \text{bad} & \text{very fair} & \text{best} \end{bmatrix}$$

is 5 × 5 linguistic square matrix with entries from S′.

$$\text{Likewise D} = \begin{bmatrix} \phi & \text{bad} & \text{best} & \text{good} \\ \text{good} & \phi & \text{bad} & \text{fair} \\ \text{very fair} & \text{bad} & \phi & \text{best} \\ \phi & \phi & \text{bad} & \text{fair} \\ \text{just fair} & \phi & \phi & \text{bad} \\ \text{bad} & \text{best} & \text{bad} & \phi \\ \phi & \text{best} & \phi & \text{bad} \end{bmatrix}$$

is a 8 × 4 linguistic rectangular matrix with entries from S′.

We now proceed onto define the notion of symmetric linguistic matrix.

Definition 3.2. *Let S′ = {S ∪ ϕ} be the linguistic set with the empty linguistic term.*

Let A be any n × n linguistic square matrix with entries from S′. We say A is a symmetric linguistic square matrix if and only if $A = A^t$.

That is if and only if $a_{ij} = a_{ji}$ for $i \neq j$, $1 \leq i, j \leq n$ ($a_{ij} \in S'$)

$$\text{where } A = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}.$$

Clearly the diagonal linguistic matrix is symmetric.

We have already given examples of them.

We will now provide more examples of them.

Example 3.6 Let S′ = {S ∪ ϕ} where S is taken from example 3.3.

$$A = \begin{bmatrix} \text{good} & \phi & \text{bad} & \text{fair} & \text{best} \\ \phi & \text{best} & \phi & \text{bad} & \phi \\ \text{bad} & \phi & \text{bad} & \phi & \text{very fair} \\ \text{fair} & \text{bad} & \phi & \text{just fair} & \text{bad} \\ \text{best} & \phi & \text{very fair} & \text{bad} & \phi \end{bmatrix}$$

be a 5 × 5 linguistic square matrix. Clearly $A = A^t$ so A is a symmetric linguistic 5 × 5 matrix.

We see $a_{ij} = a_{ji}$ for all $i \neq j$; $1 \leq i, j \leq 5$.

$$\text{Consider } B = \begin{bmatrix} \text{bad} & \phi & \phi \\ \phi & \text{best} & \phi \\ \phi & \phi & \text{just fair} \end{bmatrix}$$

B is a linguistic diagonal matrix of order 3 × 3.

Trivially B is a symmetric linguistic matrix as

$$a_{ij} = a_{ji} = \phi.\ i \neq j;\ 1 \leq i, j \leq 3.$$

It is pertinent to keep on record that we will not be in a position to define skew symmetric matrix as we do not have the notion of negative terms in linguistic sets.

Next we proceed onto define some operations on these linguistic matrices.

ϕ will be the least element we know S′ is a totally ordered set (order is defined as an increasing order) with ϕ as the least element.

We first define max operation on these linguistic matrices.

To have a better understanding of this concept we first define max operation on the set S′ non abstractly by taking an example.

Example 3.7. Let S′ = {ϕ, lowest, lower, very low, just low, low, moderate, very moderate, very high, high, higher, highest} be the linguistic set; say associated with the linguistic variable weather report.

S′ is ordered in the increasing order as follows:

ϕ < lowest < lower < very low < just low < low < moderate < very moderate < high < very high < higher < highest.

If we define max operator on S′ we see

max{low, very high} = very high as low < very high.

Clearly max{low, very high}

= max {very high, low} = very high.

So we can define max operator on S′ as

max{x, y} = y if and only if x ≤ y (x ≠ y).

Using this definition of max operator we can define max{A, B} where A and B are two 1×6 linguistic row matrices given as

A = (ϕ, fair, bad, good, ϕ, best) and

B = (fair, ϕ, ϕ, bad, good, bad) the elements of them are taken from example 3.7, which are 1×6 row linguistic matrices

max (A, B)

= max{(ϕ, fair, bad, good, ϕ, best), (fair, ϕ, ϕ, bad, good bad)}

= (max {ϕ, fair}, max{fair, ϕ}, max{bad, ϕ}, max{good, bad}, max{ϕ, good}, max{best, bad})

= (fair, far, bad, good, good, best)

is again a linguistic 1×6 row matrix with entries from S′.

We show how max operation is performed on 8×1 column linguistic matrices given by M and N where

$$M = \begin{bmatrix} \phi \\ \text{bad} \\ \text{good} \\ \text{just fair} \\ \text{best} \\ \phi \\ \text{very fair} \\ \phi \end{bmatrix} \text{ and } N = \begin{bmatrix} \text{best} \\ \text{bad} \\ \phi \\ \text{good} \\ \text{best} \\ \phi \\ \text{fair} \\ \text{just fair} \end{bmatrix} \text{ respectively.}$$

$$\max\{M, W\} = \max\{\begin{bmatrix} \phi \\ \text{bad} \\ \text{good} \\ \text{just fair} \\ \text{best} \\ \phi \\ \text{very fair} \\ \phi \end{bmatrix}, \begin{bmatrix} \text{best} \\ \text{bad} \\ \phi \\ \text{good} \\ \text{best} \\ \phi \\ \text{fair} \\ \text{just fair} \end{bmatrix}\}$$

$$= \begin{bmatrix} \max\{\phi, \text{best}\} \\ \max\{\text{bad}, \text{bad}\} \\ \max\{\text{good}, \phi\} \\ \max\{\text{just fair}, \text{good}\} \\ \max\{\text{best}, \text{best}\} \\ \max\{\phi, \phi\} \\ \max\{\text{very fair}, \text{fair}\} \\ \max\{\phi, \text{just fair}\} \end{bmatrix} = \begin{bmatrix} \text{best} \\ \text{bad} \\ \text{good} \\ \text{good} \\ \text{best} \\ \phi \\ \text{very fair} \\ \text{just fair} \end{bmatrix}$$

is again a 8 × 1 linguistic column matrices.

Let A and B two 4 × 4 square linguistic matrices with entries from $S' = S \cup \{0\}$ of problem 3.3 given in the following.

$$A = \begin{bmatrix} \phi & \text{good} & \text{bad} & \phi \\ \text{fari} & \phi & \phi & \text{good} \\ \phi & \text{very fair} & \phi & \text{best} \\ \text{bad} & \phi & \text{good} & \phi \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \text{good} & \text{bad} & \text{best} & \text{fair} \\ \phi & \phi & \phi & \phi \\ \text{fair} & \text{just fair} & \text{good} & \text{bad} \\ \phi & \text{bad} & \phi & \text{good} \end{bmatrix}.$$

We find

$$\max\{A, B\} = \max\left\{ \begin{bmatrix} \phi & \text{good} & \text{bad} & \phi \\ \text{fari} & \phi & \phi & \text{good} \\ \phi & \text{very fair} & \phi & \text{best} \\ \text{bad} & \phi & \text{good} & \phi \end{bmatrix}, \begin{bmatrix} \text{good} & \text{bad} & \text{best} & \text{fair} \\ \phi & \phi & \phi & \phi \\ \text{fair} & \text{just fair} & \text{good} & \text{bad} \\ \phi & \text{bad} & \phi & \text{good} \end{bmatrix} \right\}$$

$$= \begin{bmatrix} \max\{\phi, \text{good}\} & \max\{\text{good}, \text{bad}\} & \max\{\text{bad}, \text{best}\} & \max\{\phi, \text{fair}\} \\ \max\{\text{fair}, \phi\} & \max\{\phi, \phi\} & \max\{\phi, \phi\} & \max\{\text{good}, \phi\} \\ \max\{\phi, \text{fair}\} & \max\{\text{very fair}, \text{just fair}\} & \max\{\phi, \text{good}\} & \max\{\text{best}, \text{bad}\} \\ \max\{\text{bad}, \phi\} & \max\{\phi, \text{bad}\} & \max\{\text{good}, \phi\} & \max\{\phi, \text{good}\} \end{bmatrix}$$

$$= \begin{bmatrix} \text{good} & \text{good} & \text{best} & \text{fair} \\ \text{fair} & \phi & \phi & \text{good} \\ \text{fair} & \text{very fair} & \text{good} & \text{best} \\ \text{bad} & \text{bad} & \text{good} & \text{good} \end{bmatrix}$$

is again a 4 × 4 linguistic square matrix.

Finally we give the max operation on two rectangular 4 × 6 linguistic matrices X and Y using entries from S′ of problem 3.3.

$$\text{Let } X = \begin{bmatrix} \phi & \text{fair} & \text{good} & \text{bad} & \text{just fair} & \phi \\ \text{fair} & \text{bad} & \phi & \phi & \text{best} & \text{bad} \\ \text{bad} & \text{good} & \text{bad} & \text{fair} & \phi & \phi \\ \text{best} & \phi & \phi & \phi & \text{good} & \text{bad} \end{bmatrix}$$

and

$$Y = \begin{bmatrix} \text{best} & \phi & \text{fair} & \text{good} & \phi & \phi \\ \phi & \text{bad} & \phi & \phi & \text{bad} & \text{best} \\ \text{just fair} & \phi & \text{bad} & \text{bad} & \phi & \phi \\ \phi & \text{good} & \text{fair} & \phi & \text{bad} & \text{bad} \end{bmatrix}$$

be the 4 × 6 linguistic rectangular matrices with entries from S′ = {S ∪ ϕ} where S is taken from 3.3.

Now we perform the max operation on X and Y;

$$\max\{X, Y\} = \max\{\begin{bmatrix} \phi & \text{fair} & \text{good} & \text{bad} & \text{just fair} & \phi \\ \text{fair} & \text{bad} & \phi & \phi & \text{best} & \text{bad} \\ \text{bad} & \text{good} & \text{bad} & \text{fair} & \phi & \phi \\ \text{best} & \phi & \phi & \phi & \text{good} & \text{bad} \end{bmatrix}$$

,

$$\begin{bmatrix} \text{best} & \phi & \text{fair} & \text{good} & \phi & \phi \\ \phi & \text{bad} & \phi & \phi & \text{bad} & \text{best} \\ \text{just fair} & \phi & \text{bad} & \text{bad} & \phi & \phi \\ \phi & \text{good} & \text{fair} & \phi & \text{bad} & \text{bad} \end{bmatrix}\}$$

$$= \begin{bmatrix} \max\{\phi, \text{best}\} & \max\{\text{fair}, \phi\} & \max\{\text{good}, \text{fair}\} \\ \max\{\text{fair}, \phi\} & \max\{\text{bad}, \text{bad}\} & \max\{\phi, \phi\} \\ \max\{\text{bad}, \text{just fair}\} & \max\{\text{good}, \phi\} & \max\{\text{bad}, \text{bad}\} \\ \max\{\text{best}, \phi\} & \max\{\phi, \text{good}\} & \max\{\phi, \text{fair}\} \end{bmatrix}$$

$$\begin{bmatrix} \max\{\text{just fair}, \phi\} & \max\{\phi, \phi\} \\ \max\{\text{best}, \text{bad}\} & \max\{\text{bad}, \text{best}\} \\ \max\{\phi, \phi\} & \max\{\phi, \phi\} \\ \max\{\text{good}, \text{bad}\} & \max\{\text{bad}, \text{bad}\} \end{bmatrix}$$

$$= \begin{bmatrix} \text{best} & \text{fair} & \text{good} & \text{good} & \text{just fair} & \phi \\ \text{fair} & \text{bad} & \phi & \phi & \text{best} & \text{best} \\ \text{just fair} & \text{good} & \text{bad} & \text{fair} & \phi & \phi \\ \text{best} & \text{good} & \text{fair} & \phi & \text{good} & \text{bad} \end{bmatrix};$$

clearly $\max\{X, Y\}$ is again a 4 × 6 linguistic rectangular matrix with elements from $S' = S \cup \{\phi\}$, S given in Example 3.3.

Now we make the formal definition of the max operator on linguistic matrices of same order.

Definition 3.3. *Let S′ be a linguistic set with empty set. Suppose A and B are two linguistic matrices of same order p × q, where A = (a_{ij}) and B = (b_{ij}); a_{ij}, b_{ij} ∈ S′ with $1 \leq i \leq p$ and $1 \leq j \leq q$. We define maximum of A and B as max{A,B} = max{(a_{ij}), (b_{ij})} for all i, j with $1 \leq i \leq p$ and $1 \leq j \leq q$.*

We have already given several examples of this. In the first place we could define the operation on these linguistic matrices as only when they are of same order; otherwise it would not be possible to define max operation on them. However it is pertinent to keep on record we can define max operation on two linguistic matrices A and B of different orders provided we have the number of columns in A and the number of rows in B are the same.

We make a little deviation for this, which we will first illustrate by some examples.

Example 3.8. Let S′ = S∪{ϕ} where S is taken as in example 3.3. We define instead of max define max{max} operator on

A and B where A = (ϕ, bad fair, good, just fair} and

$$B = \begin{bmatrix} \text{best} \\ \text{bad} \\ \phi \\ \text{fair} \\ \text{good} \end{bmatrix}$$

be two linguistic matrices with entries from S′ under the new composite operation max{max}; we see order of A is 1 × 5 and order of B is 5 × 1 so we have compatibility of the max{max} of A and B.

$$\max \{\max \{A, B\}\}$$

$$= \max\{\max (\phi, \text{bad, fair, good, just fair}\} \begin{bmatrix} \text{best} \\ \text{bad} \\ \phi \\ \text{fair} \\ \text{good} \end{bmatrix} \}$$

$$= \max\{\{\max\{\phi, \text{best}\}, \max\{\text{bad}, \text{bad}\}, \max\{\text{fair}, \phi\}, \max\{\text{good, fair}\}, \max\{\text{just fair, good}\}\}$$

= max{best, bad, fair, good, good}

= {best} is a 1 × 1 trivial linguistic matrix.

Now is the max{max{B, A}} defined}, it is defined as number of columns in B is 1 and the number of rows in A is 1.

Now we find

$$\max\{\begin{bmatrix} \text{best} \\ \text{bad} \\ \phi \\ \text{fair} \\ \text{good} \end{bmatrix}, (\phi, \text{bad, fair, good, just fair})\}$$

$$= \left[\begin{matrix} \max\{\text{best}, \phi\} & \max\{\text{best}, \text{bad}\} & \max\{\text{best}, \text{fair}\} \\ \max\{\text{bad}, \phi\} & \max\{\text{bad}, \text{bad}\} & \max\{\text{bad}, \text{fair}\} \\ \max\{\phi, \phi\} & \max\{\phi, \text{bad}\} & \max\{\phi, \text{fair}\} \\ \max\{\text{fair}, \phi\} & \max\{\text{fair}, \text{bad}\} & \max\{\text{fair}, \text{fair}\} \\ \max\{\text{good}, \phi\} & \max\{\text{good}, \text{bad}\} & \max\{\text{good}, \text{fair}\} \end{matrix}\right.$$

$$\left.\begin{matrix} \max\{\text{best}, \text{good}\} & \max\{\text{best}, \text{just}\} \\ \max\{\text{bad}, \text{good}\} & \max\{\text{bad}, \text{just fair}\} \\ \max\{\phi, \text{good}\} & \max\{\text{good}, \text{just fair}\} \\ \max\{\text{fair}, \text{good}\} & \max\{\text{good}, \text{just fair}\} \\ \max\{\text{good}, \text{good}\} & \max\{\text{good}, \text{just fair}\} \end{matrix}\right.$$

$$= \begin{bmatrix} \text{best} & \text{best} & \text{best} & \text{best} & \text{best} \\ \text{bad} & \text{bad} & \text{fair} & \text{good} & \text{just fair} \\ \phi & \text{bad} & \text{fair} & \text{good} & \text{good} \\ \text{fair} & \text{fair} & \text{fair} & \text{good} & \text{good} \\ \text{good} & \text{good} & \text{good} & \text{good} & \text{good} \end{bmatrix}.$$

It is to be recorded this is one way using only max operator.

We will give some more examples of using max operator on linguistic matrices.

$$\text{Let } A = \begin{bmatrix} \text{fair} & \phi & \text{good} & \phi \\ \phi & \text{bad} & \phi & \text{fair} \\ \text{best} & \phi & \text{fair} & \phi \\ \text{worst} & \text{fair} & \phi & \text{good} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \text{fair} & \text{good} \\ \text{bad} & \phi \\ \phi & \text{best} \\ \text{fair} & \phi \end{bmatrix}$$ be any two linguistic matrices.

Clearly number of columns of A is 4 and the number of rows of B is four so maxmax {A, B} is well defined.

We find max{max(A, B)}

$$= \max\{\max\{ \begin{bmatrix} \text{fair} & \phi & \text{good} & \phi \\ \phi & \text{bad} & \phi & \text{fair} \\ \text{best} & \phi & \text{fair} & \phi \\ \text{worst} & \text{fair} & \phi & \text{good} \end{bmatrix}, \begin{bmatrix} \text{fair} & \text{good} \\ \text{bad} & \phi \\ \phi & \text{best} \\ \text{fair} & \phi \end{bmatrix} \}$$

$$= \left[\begin{array}{l} \max\{\max\{\text{fair}, \text{fair}\}, \max\{\phi, \text{bad}\}, \max\{\text{good}, \phi\} \max\{\phi, \text{fair}\}\} \\ \max\{\text{mad}\{\phi, \text{fair}\}, \max\{\text{bad}, \text{bad}\}, \max\{\phi, \phi\}, \max\{\text{fair}, \text{fair}\}\} \\ \{\max\{\max\{\text{best}, \text{fair}\}, \max\{\phi, \text{bad}\}, \max\{\text{fair}, \phi\}, \max\{\phi, \text{fair}\}\} \\ \max\{\max\{\text{worst}, \text{fair}\}, \text{mad}\{\text{fair}, \text{bad}\}, \max\{\phi, \phi\}, \\ \qquad\qquad\qquad\qquad \max\{\text{good}, \text{fair}\} \end{array} \right.$$

$$\left. \begin{array}{l} \max\{\max\{\text{fair}, \text{good},\}, \max\{\phi, \phi\}, \max\{\text{good}, \text{best}, \max\{\phi, \phi\}\} \\ \max\{\max\{, \text{good}\} \max\{\text{bad}, \phi\}, \max\{\phi, \text{best}\}, \max\{\text{fair}, \phi\} \\ \max\{\max\{\text{best}, \text{good}\}, \max\{\phi, \phi\}, \max\{\{\text{fair}, \text{best}\} \max\{\phi, \phi\}, \\ \max\{\max\{\text{worst}, \text{good}\}, \max\{\text{fair}, \phi\}, \max\{\phi, \text{best}\} \\ \qquad\qquad\qquad\qquad \max\{\text{good}, \phi\}\} \end{array} \right]$$

$$= \begin{bmatrix} \max\{\text{fair}, \text{bad}, \text{good}, \text{fair}\} & \max\{\text{good}, \phi, \text{best}, \phi\} \\ \max\{\text{fair}, \text{bad}, \phi, \text{fair}\} & \max\{\text{good}, \text{bad}, \text{best}, \text{fair}\} \\ \max\{\text{best}, \text{fair}, \text{bad}, \text{fair}\} & \max\{\text{best}, \phi, \text{best}, \phi\} \\ \max\{\text{fair}, \text{fair}, \phi, \text{good}\} & \max\{\text{good}, \text{fair}, \text{best}, \text{good}\} \end{bmatrix}$$

$$= \begin{bmatrix} \text{good} & \text{best} \\ \text{fair} & \text{best} \\ \text{best} & \text{best} \\ \text{good} & \text{best} \end{bmatrix}$$

is a different linguistic matrix which is neither A nor B.

We give yet another example of the max{max{P, Q}} where P and Q, 2×3 and 3×2 linguistic matrices respectively.

$$P = \begin{bmatrix} \text{fair} & \phi & \text{good} \\ \text{bad} & \text{best} & \phi \end{bmatrix} \text{ and } Q = \begin{bmatrix} \text{good} & \phi \\ \text{fair} & \text{best} \\ \phi & \text{bad} \end{bmatrix}$$

with entries from the linguistic set $S' = S \cup \{\phi\}$ where S is the linguistic set defined in example 3.3.

We fine max max{Q,P}; max{max{Q,P}}

$$= \max\{\max \begin{bmatrix} \text{good} & \phi \\ \text{fair} & \text{best} \\ \phi & \text{bad} \end{bmatrix}, \begin{bmatrix} \text{fair} & \phi & \text{good} \\ \text{bad} & \text{best} & \phi \end{bmatrix})\}$$

$$= \begin{bmatrix} \max\{\max\{\text{good}, \text{fair}\}, \max\{\phi, \text{bad}\}\} \\ \max\{\text{mad}\{\text{fair}, \text{fair}\}, \max\{\text{best}, \text{bad}\}\} \\ \max\{\max\{\phi, \text{fair}\}, \max\{\text{bad}, \text{bad}\} \end{bmatrix}$$

$$\begin{matrix} \max\{\max\{\text{good},\phi\},\max\{\phi,\text{best}\}, \\ \max\{\max\{\text{good},\text{good}\},\max\{\phi,\phi\} \\ \max\{\max\{\text{fair},\phi\},\max\{\text{best},\text{best}\}, \\ \max\{\max\{\text{fair},\text{good}\}\max\{\text{best},\phi\} \\ \max\{\max\{\phi,\phi\},\max\{\text{bad},\text{best}\} \\ \max\{\max\{\phi,\text{good}\}\max\{\text{bad},\phi\}\} \end{matrix}\Bigg]$$

$$= \begin{bmatrix} \max\{\text{good},\text{bad}\},\max\{\text{good},\text{best}\},\max\{\text{good},\phi\} \\ \max\{\text{fair},\text{best}\},\max\{\text{fair},\text{best}\},\max\{\text{good},\text{best}\} \\ \max\{\text{fair},\text{bad}\},\max\{\phi,\text{best}\}\max[\text{good},\text{bad}\} \end{bmatrix}$$

$$= \begin{bmatrix} \text{good} & \text{best} & \text{good} \\ \text{best} & \text{best} & \text{best} \\ \text{fair} & \text{best} & \text{good} \end{bmatrix} \qquad \ldots \text{I}$$

Clearly $\max\{\max\{Q,P\}\}$ is a 3×3 linguistic matrix given by I. Consider $\max\{\max\{P,Q\}\}$

$$= \max\{\max\begin{bmatrix} \text{fair} & \phi & \text{good} \\ \text{bad} & \text{best} & \phi \end{bmatrix},$$

$$\begin{bmatrix} \text{good} & \phi \\ \text{fair} & \text{best} \\ \phi & \text{bad} \end{bmatrix}\}\}$$

$$= \Bigg[\begin{matrix} \max\{\max\{\text{fair},\text{good}\},\max\{\phi,\text{fair}\},\max\{\text{good},\phi\} \\ \max\{\max\{\text{bad},\text{good}\}\max\{\text{best},\text{fair}\},\max\{\phi,\phi\}\} \end{matrix}$$

$$\begin{matrix} \max\{\max\{\text{fair},\phi\},\max\{\phi,\text{best}\},\max\{\text{good},\text{bad}\}\} \\ \max\{\max\{\text{bad},\phi\},\max\{\text{best},\text{best}\},\max\{\phi,\text{bad}\}\} \end{matrix}\Bigg]$$

$$= \begin{bmatrix} \max\{\text{good}, \text{fair}, \text{good}\} & \max\{\text{fair}, \text{best}, \text{good}\} \\ \max\{\text{good}, \text{bestm}, \phi\} & \max\{\text{bad}, \text{best}, \text{bad}\} \end{bmatrix}$$

$$= \begin{bmatrix} \text{good} & \text{best} \\ \text{best} & \text{best} \end{bmatrix} \qquad \ldots \text{II}$$

$\max\{\max\{P,Q\}\}$ gives a 2×2 linguistic matrix.

We make the following observations:

i) Both $\max\{\max\{P,Q\}\}$ and $\max\{\max\{Q, P\}\}$ are defined.

ii) $\max\{\max\{P,Q\}\} \neq \{\max\{\max\{Q, P\}\}$.

It important to note if P and Q are of different order

$$\max\{\max\{P, Q\}\} \neq \max\{\max\{Q,P\}\}.$$

iii) In this case both $\max\{\max\{P, Q\}\}$ and $\max\{\max \{Q,P\}\}$ are square linguistic matrices of different order.

This occurs only when both $\max\{\max\{P,Q\}\}$ and $\max\{\max\{Q,P\}\}$ are defined.

If one of them is defined then it will only be a linguistic square matrix.

Let M and N be two linguistic matrices of order 5×2 and 2×3 respectively where

$$M = \begin{bmatrix} \text{good} & \phi \\ \text{fair} & \text{bad} \\ \phi & \phi \\ \text{best} & \text{bad} \\ \phi & \text{good} \end{bmatrix}$$

and

$$N = \begin{bmatrix} \text{good} & \phi & \text{bad} \\ \text{best} & \text{bad} & \phi \end{bmatrix}$$

where M and N take their entries from S′ = S ∪ {ϕ} where S is defined as in example 3.3.

Only max{max{M, N}} is defined, however max{max{N, M}} is undefined.

We find max{max{M,N}}; max{max{M,N}

$$= \begin{bmatrix} \max\{\max\{\text{good}, \text{good}\}, \max\{\phi, \text{best}\}\} \\ \max\{\max\{\text{fair}, \text{good}\}, \max\{\text{bad}, \text{best}\}\} \\ \max\{\max\{\phi, \text{good}\}, \max\{\phi, \text{best}\}\} \\ \max\{\max\{\text{best}, \text{good}\}, \max\{\text{bad}, \text{best}\}\} \\ \max\{\max\{\phi, \text{good}\}, \max\{\text{good}, \text{best}\}\} \end{bmatrix}$$

$$\left.\begin{matrix} \max\{\max\{\text{good}, \phi\}, \max\{\phi, \text{bad}\}\} \\ \max\{\max\{\text{good}, \text{bad}\}, \max\{\phi, \phi\}\} \\ \max\{\max\{\text{fair}, \phi\}, \max\{\text{bad}, \text{bad}\}\} \\ \max\{\max\{\text{fair}, \text{bad}\}, \max\{\text{bad}, \phi\}\} \\ \max\{\max\{\phi, \phi\}, \max\{\phi, \text{bad}\}\} \\ \max\{\max\{\phi, \text{bad}\}, \max\{\phi, \phi\} \\ \max\{\max\{\text{best}, \phi\}, \max\{\text{bad}, \text{bad}\}\} \\ \max\{\max\{\text{bad}, \text{best}\} \max\{\text{bad}, \phi\} \\ \max \max\{\phi, \phi\}, \max\{\text{good}, \text{bad}\}\} \\ \max\{\max\{\phi, \text{bad}\}, \max\{\text{good}, \phi\}\} \end{matrix}\right]$$

$$= \begin{bmatrix} \max\{\text{good}, \text{best}\}, \max\{\text{good}, \text{bad}\} \max[\text{good}, \phi\} \\ \max\{\text{good}, \text{best}\}, \max\{\text{fair}, \text{bad}\}, \max\{\text{fair}, \text{bad}\} \\ \max\{\text{good}, \text{best}\}, \max\{\phi, \text{bad}\}, \max\{\text{bad}, \phi\} \\ \max\{\text{good}, \text{best}\}, \max\{\text{best}, \text{bad}\} \max\{\text{bad}, \text{bad}\} \\ \max\{\text{good}, \text{best}\}, \max\{\phi, \text{good}\}, \max\{\text{bad}, \text{good}\} \end{bmatrix}$$

$$= \begin{bmatrix} \text{best} & \text{good} & \text{good} \\ \text{best} & \text{fair} & \text{fair} \\ \text{best} & \text{bad} & \text{bad} \\ \text{best} & \text{best} & \text{bad} \\ \text{best} & \text{good} & \text{good} \end{bmatrix}$$

is a 5×3 linguistic matrix which is neither the order of M nor the order of N.

Next we proceed onto define max{max {A, A^t}} and max{max{A^t, A}} where A is a linguistic matrix.

We will illustrate this by an example.

Example 3.9. Let S′ = S ∪ {ϕ} where S is taken as in example 3.3.

$$\text{Let } A = \begin{bmatrix} \text{good} & \phi \\ \phi & \text{bad} \\ \text{good} & \text{fair} \\ \text{best} & \text{good} \\ \text{fair} & \phi \\ \phi & \text{best} \\ \text{bad} & \text{best} \end{bmatrix}$$

be a 7 × 2 linguistic matrix with entries from S′,

Now we get the transpose of A;

$$A^t = \begin{bmatrix} \text{good} & \phi & \text{good} & \text{best} & \text{fair} & \phi & \text{bad} \\ \phi & \text{bad} & \text{fair} & \text{good} & \phi & \text{best} & \text{best} \end{bmatrix};$$

clearly A^t is a 2 × 7 linguistic matrix.

$$\text{We find } \max\{\max\{A, A^t\}\} = \max\{\max\{\begin{bmatrix} \text{good} & \phi \\ \phi & \text{bad} \\ \text{good} & \text{fair} \\ \text{best} & \text{good} \\ \text{fair} & \phi \\ \phi & \text{best} \\ \text{bad} & \text{best} \end{bmatrix},$$

$$\begin{bmatrix} \text{good} & \phi & \text{good} & \text{best} & \text{fair} & \phi & \text{bad} \\ \phi & \text{bad} & \text{fair} & \text{good} & \phi & \text{best} & \text{best} \end{bmatrix}\}\}$$

$$= \begin{bmatrix} \text{good} & \text{good} & \text{good} & \text{best} & \text{good} & \text{best} & \text{best} \\ \text{good} & \text{bad} & \text{good} & \text{best} & \text{fair} & \text{best} & \text{best} \\ \text{good} & \text{good} & \text{good} & \text{best} & \text{good} & \text{best} & \text{best} \\ \text{best} & \text{best} & \text{best} & \text{best} & \text{best} & \text{best} & \text{best} \\ \text{good} & \text{fair} & \text{good} & \text{best} & \text{fair} & \text{best} & \text{best} \\ \text{best} & \text{best} & \text{best} & \text{best} & \text{best} & \text{best} & \text{best} \\ \text{best} & \text{best} & \text{best} & \text{best} & \text{best} & \text{best} & \text{best} \end{bmatrix}$$

is a 7×7 linguistic square matrix which is symmetric.

Now consider the operation max(max $\{A^t, A\}$) of A^t with A;

max(max$\{A^t$, A$\}$) =

$$\max\{\max\{\begin{bmatrix} \text{good} & \phi & \text{good} & \text{best} & \text{fair} & \phi & \text{bad} \\ \phi & \text{bad} & \text{fair} & \text{good} & \phi & \text{best} & \text{best} \end{bmatrix},$$

$$\begin{bmatrix} \text{good} & \phi \\ \phi & \text{bad} \\ \text{good} & \text{fair} \\ \text{best} & \text{good} \\ \text{fair} & \phi \\ \phi & \text{best} \\ \text{bad} & \text{best} \end{bmatrix}\} =$$

$$\left[\begin{matrix} \max\{\max\{\text{good}, \text{good}\}, \max\{\phi, \phi\}, \max\{\text{good}, \text{good}\}, \\ \max\{\text{best}, \text{best}\}, \max\{\text{fair}, \text{fair}\}, \max\{\phi, \phi\}, \max\{\text{bad}, \text{bad}\}\} \\ \max\{\max\{\phi, \text{good}\}, \max\{\text{bad}, \phi\}, \max\{\text{fair}, \text{good}\}, \\ \max\{\text{good}, \text{best}\}, \max\{\phi, \text{fair}\}, \max\{\text{best}, \phi\}, \max\{\text{best}, \text{bad}\}\} \end{matrix}\right.$$

$$\left.\begin{matrix} \max\{\max\{\text{good}, \phi\}, \max\{\phi, \text{bad}\}, \max\{\text{good}, \text{fair}\} \\ \max\{\text{best}, \text{good}\}, \max\{\text{fair}, \phi\}, \max\{\phi, \text{best}\} \max\{\text{bad}, \text{best}\}\} \\ \max\{\max\{\phi, \phi\}, \max\{\text{bad}, \text{bad}\}, \max\{\text{fair}, \text{fair}\}, \\ \max\{\text{good}, \text{good}\}, \max\{\phi, \phi\}, \max\{\text{best}, \text{best}\}, \\ \max\{\text{best}, \text{best}\}\} \end{matrix}\right]$$

$$= \left[\begin{matrix} \max\{\text{good}, \phi, \text{good}, \text{best}, \text{fair}, \phi, \text{bad}\} \\ \max\{\text{good}, \text{bad}, \text{good}, \text{best}, \text{fair}, \text{best}, \text{best}\} \end{matrix}\right.$$

$$\left.\begin{matrix} \max\{\text{good}, \text{bad}, \text{good}, \text{best}, \text{fair}, \text{best}, \text{best}\} \\ \max\{\phi, \text{bad}, \text{fair}, \text{good}, \phi, \text{best}, \text{best}\} \end{matrix}\right]$$

$$= \begin{bmatrix} \text{best} & \text{best} \\ \text{best} & \text{best} \end{bmatrix}$$

is a 2×2 linguistic matrix.

We make the following observation.

The order of the linguistic rectangular matrix A is 7×2 and its transpose A^t is of order 2×7. The $\max\{\max\{A, A^t\}\}$ is a 7×7 square linguistic symmetric matrix whereas $\max\{\max\{A^t, A\}\}$ is a 2×2 square linguistic matrix which is also symmetric.

Next for the same set $S' = S \cup \{\phi\}$; S given in example 3.3.

We find the $\max\{\max\{A, A^t\}\}$ and max)max$\{A^t, A\}\}$ where A is a 5×1 column linguistic matrix given in the following:

$$A = \begin{bmatrix} \text{good} \\ \text{bad} \\ \phi \\ \text{fair} \\ \text{best} \end{bmatrix}$$

is a 5 × 1 linguistic column matrix and its transpose

A^t = (good, bad, ϕ, fair, best) is a 1 × 5 row linguistic matrix.

We find

$$\max\{\max\{A, A^t\}\}$$

$$= \max\{\max\{\begin{bmatrix} \text{good} \\ \text{bad} \\ \phi \\ \text{fair} \\ \text{best} \end{bmatrix}, (\text{good, bad, } \phi\text{, fair, best})\}$$

$$= \begin{bmatrix} \text{good} & \text{good} & \text{good} & \text{good} & \text{best} \\ \text{good} & \text{bad} & \text{bad} & \text{fair} & \text{best} \\ \text{good} & \text{bad} & \phi & \text{fair} & \text{best} \\ \text{good} & \text{fair} & \text{fair} & \text{fair} & \text{best} \\ \text{best} & \text{best} & \text{best} & \text{best} & \text{best} \end{bmatrix}$$

is a linguistic 5 × 5 square matrix which is symmetric.

Now we find max{max {A^t, A}}

$$= \max\{\max\{(\text{good, bad, } \phi, \text{ fair, best}), \begin{bmatrix} \text{good} \\ \text{bad} \\ \phi \\ \text{fair} \\ \text{best} \end{bmatrix}\}\}$$

$$= \max\{\text{good, bad, } \phi, \text{ fair, best}\}$$

$$= (\text{best})$$

is a 1 × 1 linguistic trivial matrix.

In view of this example we have the following.

i) Since the order of A is 5 × 1 we have order of A^t to be 1 × 5 so max {max{A, A^t}} is a 5 × 5 a square linguistic symmetric matrix.

ii) A is of order 5 × 1 and that of A^t is 1 × 5.

iii) max{max{A^t, A}} is a 1 × 1 trivial linguistic matrix which is nothing but the maximum of all the terms using in the column linguistic matrix A.

Now consider A to be a 6 × 6 square linguistic matrix with entries from S′ = S ∪{ϕ} where S is the linguistic set taken from example 3.3.

$$A = \begin{bmatrix} \phi & \text{bad} & \text{fair} & \phi & \text{best} & \phi \\ \text{bad} & \phi & \text{best} & \text{bad} & \phi & \phi \\ \phi & \text{good} & \phi & \phi & \text{fair} & \text{bad} \\ \text{bad} & \phi & \text{good} & \text{fair} & \phi & \text{good} \\ \phi & \text{good} & \phi & \text{bad} & \text{fair} & \phi \\ \text{best} & \phi & \text{fair} & \phi & \text{bad} & \phi \end{bmatrix}$$

The transpose of A is

$$A^t = \begin{bmatrix} \phi & \text{bad} & \phi & \text{bad} & \phi & \text{best} \\ \text{bad} & \phi & \text{good} & \phi & \text{good} & \phi \\ \text{fair} & \text{best} & \phi & \text{good} & \phi & \text{fair} \\ \phi & \text{bad} & \phi & \text{fair} & \text{bad} & \phi \\ \text{best} & \phi & \text{fair} & \phi & \text{fair} & \text{bad} \\ \phi & \phi & \text{bad} & \text{good} & \phi & \phi \end{bmatrix}$$

A^t is also a 6×6 linguistic matrix.

Our job to find max{max{A, A^t}} and max{max{A^t, A}} and compare them.

Consider max{max{ $\begin{bmatrix} \phi & \text{bad} & \text{fair} & \phi & \text{best} & \phi \\ \text{bad} & \phi & \text{best} & \text{bad} & \phi & \phi \\ \phi & \text{good} & \phi & \phi & \text{fair} & \text{bad} \\ \text{bad} & \phi & \text{good} & \text{fair} & \phi & \text{good} \\ \phi & \text{good} & \phi & \text{bad} & \text{fair} & \phi \\ \text{best} & \phi & \text{fair} & \phi & \text{bad} & \phi \end{bmatrix}$,

$$\begin{bmatrix} \phi & \text{bad} & \phi & \text{bad} & \phi & \text{best} \\ \text{bad} & \phi & \text{good} & \phi & \text{good} & \phi \\ \text{fair} & \text{best} & \phi & \text{good} & \phi & \text{fair} \\ \phi & \text{bad} & \phi & \text{fair} & \text{bad} & \phi \\ \text{best} & \phi & \text{fair} & \phi & \text{fair} & \text{bad} \\ \phi & \phi & \text{bad} & \text{good} & \phi & \phi \end{bmatrix}\}\}$$

$$= \begin{bmatrix} \text{best} & \text{best} & \text{best} & \text{best} & \text{best} & \text{best} \\ \text{best} & \text{best} & \text{best} & \text{best} & \text{best} & \text{best} \\ \text{best} & \text{best} & \text{good} & \text{good} & \text{good} & \text{best} \\ \text{best} & \text{best} & \text{good} & \text{good} & \text{good} & \text{best} \\ \text{best} & \text{best} & \text{good} & \text{good} & \text{good} & \text{best} \\ \text{best} & \text{best} & \text{best} & \text{best} & \text{best} & \text{best} \end{bmatrix}.$$

We see $\max\{\max\{A, A^t\}\}$ is a 6×6 linguistic symmetric square matrix.

Now we find out $\max\{\max\{A^t, A\}\}$ and compare it with $\max\{\max\{A, A^t\}\}$.

$\max\{\max\{A^t, A\}\}$

$$= \max\{\max\{\begin{bmatrix} \phi & \text{bad} & \phi & \text{bad} & \phi & \text{best} \\ \text{bad} & \phi & \text{good} & \phi & \text{good} & \phi \\ \text{fair} & \text{best} & \phi & \text{good} & \phi & \text{fair} \\ \phi & \text{bad} & \phi & \text{fair} & \text{bad} & \phi \\ \text{best} & \phi & \text{fair} & \phi & \text{fair} & \text{bad} \\ \phi & \phi & \text{bad} & \text{good} & \phi & \phi \end{bmatrix},$$

$$\left[\begin{array}{cccccc} \phi & \text{bad} & \text{fair} & \phi & \text{best} & \phi \\ \text{bad} & \phi & \text{best} & \text{bad} & \phi & \phi \\ \phi & \text{good} & \phi & \phi & \text{fair} & \text{bad} \\ \text{bad} & \phi & \text{good} & \text{fair} & \phi & \text{good} \\ \phi & \text{good} & \phi & \text{bad} & \text{fair} & \phi \\ \text{best} & \phi & \text{fair} & \phi & \text{bad} & \phi \end{array}\right]\}\} =$$

$$\left[\begin{array}{cccccc} \text{best} & \text{best} & \text{best} & \text{best} & \text{best} & \text{best} \\ \text{best} & \text{good} & \text{best} & \text{good} & \text{best} & \text{good} \\ \text{best} & \text{best} & \text{best} & \text{best} & \text{best} & \text{best} \\ \text{best} & \text{good} & \text{best} & \text{fair} & \text{best} & \text{best} \\ \text{best} & \text{best} & \text{best} & \text{best} & \text{best} & \text{best} \\ \text{best} & \text{good} & \text{best} & \text{good} & \text{best} & \text{good} \end{array}\right].$$

Clearly $\max\{\max\{A^t, A\}\} \neq \max\{\max\{A, A^t\}\}$. However both $\max\{\max\{A, A^t\}\}$ and $\max\{\max\{A^t, A\}\}$ are 6×6 linguistic symmetric square matrices with entries from $S' = S \cup \{\phi\}$ where S is as given in example 3.3.

In view of this we prove the following theorem.

Theorem 3.1. *Let S' be a linguistic set together with the empty linguistic term ϕ.*

Let $A = (a_{ij})$ be a $m \times n$ linguistic matrix with entries from S' that is $a_{ij} \in S'$; $1 \leq i \leq m$ and $1 \leq j \leq n$.

$\max\{\max\{A, A^t\}\}$ and $\max\{\max\{A^t, A\}\}$ are square linguistic matrices of order $m \times m$ and $n \times n$ respectively.

In case $m = n$ we see $\max\{\max A^t, A\}\}$ and

$\max\{\max A, A^t\}\}$ are $n \times n$ square symmetric linguistic matrices which are in general distinct.

If $m = 1$ (or $n = 1$) we get $\max\{\max\{A, A^t\}\}$ be a trivial linguistic matrix.

If $n = 1$ then $\max\{\max, \{A^t, A\}\}$ is a trivial linguistic matrix.

Proof. If A is a $m \times m$ linguistic matrix A^t is a $n \times m$ matrix then it follows from the simple fact when we take the max max operator then we get $m \times m$ and $n \times n$ square matrices only.

Further as the column and rows are exchanged the max operator is performed and as the max operator is commutative we get $\max\{a_{ij}, a_{ji}\}$ in each case leading to a symmetry of the linguistic square matrix.

Finally we see in case $m \neq n$,

$$\max\{\max\{A^t, A\}\} \neq \max\{\max, \{A, A^t\}\}$$

with very order to be different.

However in case of $m = n$ that is A a square that

$$\max\{\max\{A, A^t\}\} \text{ and } \max\{\max\{A^t, A\}\}$$

are square linguistic symmetric matrices of same order however in general they are distinct.

Now we can say if A itself is symmetric linguistic matrix then we will have $\max\{\max\{A^t, A\}\} = \max\{\max\{A, A^t\}\}$ as $A^t = A$.

Further if a researcher is interested in constructing such symmetric linguistic matrices one can easily do so by considering a linguistic matrix A of desired order and finding max{max{A, A^t}} or max{max{A^t, A}}.

In the following section we study min{min{A, B^t}}, min{A, B} and min{min{B, A}} for any linguistic matrices A and B.

In the following we assume S′ is a linguistic set with the linguistic empty word ϕ included in it.

That if S′ is a totally ordered set order and ϕ the empty linguistic term is the least linguistic term.

We by order of the linguistic matrix A mean the number of rows and number of columns A has and is denoted by $m \times n$ if there are m rows and n columns.

So if S′ is a linguistic set ordered in the increasing order say

$\phi < a_1 < a_2 \ldots < a_n$ where $a_i \in S'$; $1 \leq i \leq n$, then

$$\min(a_i, a_j) \begin{cases} = a_i \text{ if } a_i < a_j \\ = a_j \text{ if } a_j < a_i \\ = a_i \text{ if } a_i = a_j \end{cases}$$

for all $1 \leq i, j \leq n$ and $\min\{\phi, a_i\} = \phi$ for all $a_i \in S$ $1 \leq i \leq n$.

With this in mind we first provide some examples of min operation on linguistic matrices of same order.

Example 3.10. Let

$$A = \begin{bmatrix} \text{good} & \phi & \text{fair} \\ \phi & \text{bad} & \text{bad} \\ \text{best} & \phi & \text{best} \\ \text{fair} & \text{fair} & \phi \\ \phi & \phi & \text{good} \end{bmatrix}$$

and

$$B = \begin{bmatrix} \text{fair} & \text{fair} & \text{bad} \\ \text{good} & \text{very bad} & \phi \\ \text{fair} & \phi & \text{good} \\ \text{bad} & \text{good} & \text{bad} \\ \text{good} & \phi & \text{bad} \end{bmatrix}$$

be two linguistic 5×3 matrices with entries by $S' = S \cup \{\phi\}$ where S is a linguistic set provided in example 3.3 of this chapter.

We find min{A, B}; min{A, B}

$$= \min\{ \begin{bmatrix} \text{good} & \phi & \text{fair} \\ \phi & \text{bad} & \text{bad} \\ \text{best} & \phi & \text{best} \\ \text{fair} & \text{fair} & \phi \\ \phi & \phi & \text{good} \end{bmatrix}, \begin{bmatrix} \text{fair} & \text{fair} & \text{bad} \\ \text{good} & \text{very bad} & \phi \\ \text{fair} & \phi & \text{good} \\ \text{bad} & \text{good} & \text{bad} \\ \text{good} & \phi & \text{bad} \end{bmatrix}$$

$$= \begin{bmatrix} \min\{\text{good}, \text{fair}\} & \min\{\phi, \text{fair}\} & \min\{\text{fair}, \text{bad}\} \\ \min\{\phi, \text{good}\} & \min\{\text{bad}, \text{very bad}\} & \min\{\text{bad}, \phi\} \\ \min\{\text{best}, \text{fair}\} & \min\{\phi, \phi\} & \min\{\text{best}, \text{good}\} \\ \min\{\text{fair}, \text{bad}\} & \min\{\text{fair}, \text{good}\} & \min\{\phi, \text{bad}\} \\ \min\{\phi, \text{good}\} & \min\{\phi, \phi\} & \min\{\text{good}, \text{bad}\} \end{bmatrix}$$

$$= \begin{bmatrix} \text{fair} & \phi & \text{fair} \\ \phi & \text{very bad} & \phi \\ \text{fair} & \phi & \text{good} \\ \text{bad} & \text{fair} & \phi \\ \phi & \phi & \text{bad} \end{bmatrix}.$$

is again a 5 × 3 linguistic matrix with entries from S′ = S ∪{ϕ} where S is defined as in example 3.3.

Consider $M = \begin{bmatrix} \text{good} \\ \text{bad} \\ \phi \\ \text{fair} \\ \text{best} \\ \phi \\ \text{just fair} \\ \text{very bad} \end{bmatrix}$ and $N = \begin{bmatrix} \text{best} \\ \text{fair} \\ \phi \\ \text{bad} \\ \text{bad} \\ \text{bad} \\ \text{fair} \\ \text{bad} \end{bmatrix}$

be any two 8 × 1 linguistic matrices with entries from

S′ = S ∪ {ϕ} where S is as in example 3.3.

We now find

$$\min\{M, N\} = \min\{\begin{bmatrix} \text{good} \\ \text{bad} \\ \phi \\ \text{fair} \\ \text{best} \\ \phi \\ \text{just fair} \\ \text{very bad} \end{bmatrix}, \begin{bmatrix} \text{best} \\ \text{fair} \\ \phi \\ \text{bad} \\ \text{bad} \\ \text{bad} \\ \text{fair} \\ \text{bad} \end{bmatrix}\}$$

$$= \begin{bmatrix} \min\{\text{good}, \text{best}\} \\ \min\{\text{fair}, \text{bad}\} \\ \min\{\phi, \phi\} \\ \min\{\text{fair}, \text{bad}\} \\ \min\{\text{best}, \text{bad}\} \\ \min\{\text{bad}, \phi\} \\ \min\{\text{fair}, \text{just fair}\} \\ \min\{\text{bad}, \text{very bad}\} \end{bmatrix} = \begin{bmatrix} \text{good} \\ \text{bad} \\ \phi \\ \text{bad} \\ \phi \\ \text{just fair} \\ \text{very bad} \end{bmatrix}$$

is again a 8 × 1 linguistic matrix with entries from S′.

The following observations is mandatory;

min{A, B+ = min{B, A} that is the min operation is commutative which operation on linguistic matrices.

$$\text{Let P} = \begin{bmatrix} \text{good} & \text{bad} & \phi & \text{fair} & \text{best} \\ \phi & \text{fair} & \text{good} & \phi & \phi \\ \text{best} & \text{bad} & \phi & \text{very bad} & \phi \\ \phi & \phi & \text{just fair} & \phi & \text{good} \\ \text{bad} & \text{good} & \phi & \text{bad} & \text{bad} \end{bmatrix}$$

$$\text{and Q} = \begin{bmatrix} \phi & \text{fair} & \phi & \text{bad} & \text{very bad} \\ \text{bad} & \phi & \text{bad} & \phi & \text{bad} \\ \phi & \text{good} & \text{best} & \text{bad} & \text{fair} \\ \text{fair} & \phi & \text{good} & \phi & \text{bad} \\ \text{good} & \text{good} & \phi & \text{best} & \text{very bad} \end{bmatrix}$$

be any two 5 × 5 square linguistic matrices with entries from S′ = S ∪ {ϕ} S given in example 3.3.

We find $\min\{P, Q\}$; $\left\{\begin{bmatrix} \text{good} & \text{bad} & \phi & \text{fair} & \text{best} \\ \phi & \text{fair} & \text{good} & \phi & \phi \\ \text{best} & \text{bad} & \phi & \text{very bad} & \phi \\ \phi & \phi & \text{just fair} & \phi & \text{good} \\ \text{bad} & \text{good} & \phi & \text{bad} & \text{bad} \end{bmatrix}\right.$

$$\left.\begin{bmatrix} \phi & \text{fair} & \phi & \text{bad} & \text{very bad} \\ \text{bad} & \phi & \text{bad} & \phi & \text{bad} \\ \phi & \text{good} & \text{best} & \text{bad} & \text{fair} \\ \text{fair} & \phi & \text{good} & \phi & \text{bad} \\ \text{good} & \text{good} & \phi & \text{best} & \text{very bad} \end{bmatrix}\right\}$$

$$= [\begin{bmatrix} \min\{\text{good}, \phi\} & \min\{\text{bad}, \text{fair}\} \\ \min\{\phi, \text{bad}\} & \min\{\text{fair}, \phi\} \\ \min\{\phi, \text{best}\} & \min\{\text{bad}, \text{good}\} \\ \min\{\phi, \text{fair}\} & \min\{\phi, \phi\} \\ \min\{\text{bad}, \text{good}\} & \min\{\text{good}, \text{good}\} \end{bmatrix}.$$

$$\begin{matrix} \min\{\phi, \phi\} \min\{\text{fair}, \text{bad}\} & \min\{\text{best}, \text{very bad}\} \\ \min\{\text{good}, \text{bad}\} \min\{\phi, \phi\} & \min\{\phi, \text{bad}\} \\ \min\{\phi, \text{best}\} \min\{\text{very bad}, \text{bad}\} & \min\{\phi, \text{fair}\} \\ \min(\text{just fair}, \text{good}\} \min\{\phi, \phi\} & \min\{\text{good}, \text{bad}\} \\ \min\{\phi, \phi\} \min\{\text{bad}, \text{best}\} & \min\{\text{bad}, \text{very bad}\} \end{matrix}\Bigg]$$

$$= \begin{bmatrix} \phi & \text{bad} & \phi & \text{bad} & \text{very bad} \\ \phi & \phi & \text{bad} & \phi & \phi \\ \phi & \text{bad} & \phi & \text{very bad} & \phi \\ \phi & \phi & \text{just fair} & \phi & \text{bad} \\ \text{bad} & \text{good} & \phi & \text{bad} & \text{very bad} \end{bmatrix}$$

is clearly a 5 × 5 linguistic square matrices with entries from $S' = S \cup \{\phi\}$.

$$\text{Let } A = \begin{bmatrix} \text{good} & \phi & \text{best} & \text{just fair} \\ \phi & \text{good} & \phi & \text{very bad} \\ \text{bad} & \text{fair} & \phi & \text{worst} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \phi & \text{bad} & \text{good} & \text{bad} \\ \phi & \text{best} & \phi & \text{bad} \\ \text{worst} & \text{bad} & \phi & \text{good} \end{bmatrix}$$

be any two 3 × 4 linguistic rectangular matrices with entries from $S' = S \cup \{\phi\}$ where S is a linguistic set given in example 3.3.

$$\min\{\begin{bmatrix} \text{good} & \phi & \text{best} & \text{just fair} \\ \phi & \text{good} & \phi & \text{very bad} \\ \text{bad} & \text{fair} & \phi & \text{worst} \end{bmatrix},$$

$$\begin{bmatrix} \phi & \text{bad} & \text{good} & \text{bad} \\ \phi & \text{best} & \phi & \text{bad} \\ \text{worst} & \text{bad} & \phi & \text{good} \end{bmatrix}\}$$

$$= \begin{bmatrix} \min\{\text{good},\phi\} & \min\{\phi,\text{bad}\} & \min\{\text{best},\text{good}\} \\ \min\{\phi,\phi\} & \min\{\text{good},\text{best}\} & \min\{\phi,\phi\} \\ \min\{\text{bad},\text{worst}\} & \text{miin}\{\text{fair},\text{bad}\} & \min\{\phi,\phi\} \end{bmatrix}$$

$$\begin{matrix} \min\{\text{just fair},\text{bad}\} \\ \min\{\text{very bad},\text{bad}\} \\ \min[\text{worst},\text{good}\} \end{matrix}\Bigg]$$

$$= \begin{bmatrix} \phi & \phi & \text{good} & \text{bad} \\ \phi & \text{good} & \phi & \text{very bad} \\ \text{worst} & \text{bad} & \phi & \text{worst} \end{bmatrix}$$

is a 3×4 linguistic rectangular matrix with entries from $S' = S \cup \{\phi\}$ where S is given as in example 3.3.

Now having seen examples of min operator on linguistic matrices of same order we now proceed onto define the min operator on two linguistic matrices of same order.

Definition 3.4. *Let A and B be any two linguistic matrices of order $m \times n$ with entries from the set $S' = S \cup \{\phi\}$ where S' is a totally ordered set with increasing order.*

If A (a_{ij}) and B (b_{ij}) with $a_{ij}, b_{ij} \in S'$; $1 \leq i \leq m$ and $1 \leq j \leq n$; then $\min\{A, B\} = \min\{(a_{ij}), (b_{ij})\} = (c_{ij})$ $1 \leq i \leq m$; $1 \leq j \leq n$ is again a linguistic matrix of order $m \times n$.

Here

$$\min\{a_{ij}, b_{ij}\} \quad \begin{cases} = a_{ij} = c_{ij} \text{ if } a_{ij} \leq b_{ij} \\ = b_{ij} = c_{ij} \text{ if } b_{ij} \leq a_{ij} \\ 1 \leq i \leq m \text{ and } 1 \leq j \leq n \end{cases}$$

Then $\min\{A, B\} = (c_{ij})$.

Now having seen the min operator we proceed onto describe min min operator on linguistic matrices by some examples.

Example 3.11. Let S′ = S ∪ {ϕ} be the linguistic set with linguistic empty term ϕ.

We take two linguistic matrices A and B of order 1 × 6 and 6 × 3 respectively described below.

A = (good, ϕ, fair, bad, just fair, best)

and

$$B = \begin{bmatrix} \text{good} & \phi & \text{fair} \\ \text{bad} & \text{fair} & \phi \\ \phi & \text{bad} & \text{good} \\ \text{just fair} & \text{bad} & \phi \\ \text{fair} & \text{good} & \text{bad} \\ \text{best} & \text{fair} & \phi \end{bmatrix}$$

be linguistic matrices of order 1 × 6 and 6 × 3 respectively.

The entries of A and B are taken from S′ = S ∪{ϕ} where S is the linguistic set given in example 3.3.

min{min{A, B}}

$$= \min\{\min\{(\text{good}, \phi, \text{fair}, \text{bad}, \text{just fair}, \text{best}),$$

$$\begin{bmatrix} \text{good} & \phi & \text{fair} \\ \text{bad} & \text{fair} & \phi \\ \phi & \text{bad} & \text{good} \\ \text{just fair} & \text{bad} & \phi \\ \text{fair} & \text{good} & \text{bad} \\ \text{best} & \text{fair} & \phi \end{bmatrix}\}\}$$

$$= \begin{bmatrix} \min\{\min\{\text{good}, \text{good}\} \min\{\phi, \text{bad}\}, \min\{\text{fair}, \phi\} \\ \min\{\text{bad}, \text{just fair}\}, \min\{\text{just fair}, \text{fair}\} \min\{\text{best}, \text{best}\}\} \\ \min\{\min\{\text{goo}, \text{d}\phi\}, \min\{\phi, \text{fair}\}, \min\{\text{fair}, \text{bad}\} \\ \min\{\text{bad}, \text{bad}\}, \min\{\text{just fair}, \text{good}\} \min\{\text{best}, \text{fair}\}\} \\ \min\{\min\{\text{good}, \text{fair}\}, \min\{\phi, \phi\}, \min(\text{fair}, \text{good}\}, \min\{\text{bad}, \phi\} \\ \min\{\text{just fair}, \text{bad}\}, \min\{\text{best}, \phi\}\} \end{bmatrix}$$

= [(min good, ϕ, ϕ, bad, just fair, best), min{ϕ, ϕ, bad, bad, just fair, fair), min{fair, ϕ, fair, ϕ, bad, ϕ}]

= (ϕ, ϕ, ϕ) is a 1×3 linguistic empty row matrix. Only min{min{A,B}} is defined however min{min{B, A}} is not defined.

Consider

$$A = \begin{bmatrix} \text{bad} & \phi & \text{bad} & \text{fair} \\ \text{good} & \text{fair} & \phi & \text{just fair} \\ \phi & \text{best} & \phi & \text{bad} \end{bmatrix}$$

and

$$B = \begin{bmatrix} \phi & \text{fair} & \phi \\ \text{fair} & \text{bad} & \text{good} \\ \text{very bad} & \text{bad} & \text{bad} \\ \text{bad} & \text{fair} & \phi \end{bmatrix}$$

two linguistic matrices with entries from $S' = S \cup \{\phi\}$ where S is given in example 3.3.

We find min{min{A, B}} and min{min{B, A}}.

$$\min\{\min\{A,B\}\} = \min\{\min \begin{bmatrix} \text{bad} & \phi & \text{bad} & \text{fair} \\ \text{good} & \text{fair} & \phi & \text{just fair} \\ \phi & \text{best} & \phi & \text{bad} \end{bmatrix},$$

$$\begin{bmatrix} \phi & \text{fair} & \phi \\ \text{fair} & \text{bad} & \text{good} \\ \text{very bad} & \text{bad} & \text{bad} \\ \text{bad} & \text{fair} & \phi \end{bmatrix}\}\}$$

$$= \left[\begin{array}{c} \min\{\min\{\text{bad},\phi\}\min\{\phi,\text{fair}\} \\ \min\{\text{bad},\text{very bad}\}\min\{\text{fair},\text{bad}\}\} \\ \min\{\min\{\text{good},\phi\},\min\{\text{fair},\text{fair}\}, \\ \min\{\phi,\text{very bad}\},\min\{\text{just fair},\text{bad}\}\} \\ \min\{\min\{\phi,\phi\},\min\{\text{best},\text{fair}\} \\ \min\{\phi,\text{bad}\}\min\{\text{bad},\text{bad}\}\} \end{array}\right.$$

$$\begin{array}{c} \min\{\min\{\text{bad},\text{just fair}\},\min\{\phi,\text{fair}\},\min\{\text{bad},\text{bad}\}, \\ \min\{\text{fair},\text{fair}\}\} \\ \min\{\min\{\text{good},\text{just fair}\},\min\{\text{fair},\text{bad}\},\min\{\phi,\text{bad}\} \\ \min(\text{just fair},\text{fair}\}\} \\ \min\{\min\{\phi,\text{just fair}\},\min\{\text{best},\text{bad}\}\min\{\phi,\text{bad}\}, \\ \min\{\text{bad},\text{fair}\}\} \end{array}$$

$$\left.\begin{array}{c} \min\{\min\{\text{bad},\phi\},\min\{\phi,\text{good}\},\min\{\text{bad},\text{bad}\} \\ \min\{\text{fair},\phi\}\} \\ \min\{\min\}\{\text{good},\phi\},\min\{\text{fair},\text{good}\},\min\{\phi,\text{bad}\}, \\ \min\{\text{just fair},\phi\}\} \\ \min\{\min\{\phi,\phi\},\min\{\text{best},\text{good}\},\min\{\phi,\text{bad}\}, \\ \min\{\text{bad},\phi\} \end{array}\right]$$

$$= \begin{bmatrix} \min\{\phi, \phi, \text{bad}, \text{very bad}\} & \min\{\text{bad}, \phi, \text{bad}, \text{fair}\} & \min\{\phi, \phi, \text{bad}, \phi\} \\ \min\{\phi, \text{fair}, \phi, \text{bad}\} & \min\{\text{just fair}, \text{bad}, \phi, \text{just fair}\} & \min\{\phi, \text{good}, \phi, \phi\} \\ \min[\phi, \text{fair}, \phi, \text{bad}\} & \min\{\phi, \text{bad}, \phi, \text{bad}\} & \min\{\phi, \text{good}, \phi, \phi\} \end{bmatrix}$$

$$= \begin{bmatrix} \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \end{bmatrix}$$

= (ϕ) is a 3 × 3 empty linguistic term matrix.

Trivially symmetric.

Now we find min{min{B, A}}.

On similar lines of calculation we can get

min{min{B, A}}.

$$\min(\min\{B, A\}\} = \begin{bmatrix} \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \text{bad} \\ \phi & \phi & \phi & \text{very bad} \\ \phi & \phi & \phi & \phi \end{bmatrix}$$

which is a 4 × 4 linguistic matrix.

Now we show min{min{A, A^t}} and min{min{A^t, A}} are distinct for any linguistic matrix A and both are defined.

In case A is a square linguistic matrix we see A both min{min{A, A^t}} and min{min{A^t, A}} are in general two distinct symmetric square linguistic matrices.

$$\min\{\min\{A, A^t\}\} = \min\{\min \begin{bmatrix} \text{bad} & \phi & \text{bad} & \text{fair} \\ \text{good} & \text{fair} & \phi & \text{just fair} \\ \phi & \text{best} & \phi & \text{bad} \end{bmatrix},$$

$$\begin{bmatrix} \text{bad} & \text{good} & \phi \\ \phi & \text{fair} & \text{best} \\ \text{bad} & \phi & \phi \\ \text{fair} & \text{just fair} & \text{bad} \end{bmatrix}\}$$

$$= \begin{bmatrix} \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \end{bmatrix} = (\phi)$$

is a 3 × 3 linguistic trivially symmetric matrix.

We now find min{max{A^t, A}

$$= \min\{\min \begin{bmatrix} \text{bad} & \text{good} & \phi \\ \phi & \text{fair} & \text{best} \\ \text{bad} & \phi & \phi \\ \text{fair} & \text{just fair} & \text{bad} \end{bmatrix},$$

$$\begin{bmatrix} \text{bad} & \phi & \text{bad} & \text{fair} \\ \text{good} & \text{fair} & \phi & \text{just fair} \\ \phi & \text{best} & \phi & \text{bad} \end{bmatrix}\}\}$$

$$= \begin{bmatrix} \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \text{bad} \end{bmatrix}$$

is clearly 4 × 4 linguistic square matrix which is symmetric.

Further $\min\{\min\{A, A^t\}\} \neq \min\{\min\{A^t, A\}\}$.

Now consider the linguistic matrix $A^t = \begin{bmatrix} \text{fair} \\ \text{good} \\ \text{bad} \\ \text{just fair} \\ \text{best} \\ \text{good} \\ \text{bad} \\ \text{fair} \\ \text{very bad} \\ \text{good} \end{bmatrix}$

where A is a 1 × 10 linguistic matrix.

$$\min\{\min\{A^t, A\}\} = \min\{\min\{\begin{bmatrix} \text{fair} \\ \text{good} \\ \text{bad} \\ \text{just fair} \\ \text{best} \\ \text{good} \\ \text{bad} \\ \text{fair} \\ \text{very bad} \\ \text{good} \end{bmatrix},$$

[fair, good, bad, just fair, best, good, bad, fair, very bad, good]}}

= (very bad); is a 1 × 1 trivial linguistic matrix

min{min{A, A^t) = min{min{(fair, good, bad, just fair, best,

$$\text{good, bad, fair, very bad, good)}, \begin{bmatrix} \text{fair} \\ \text{good} \\ \text{bad} \\ \text{just fair} \\ \text{best} \\ \text{good} \\ \text{bad} \\ \text{fair} \\ \text{very bad} \\ \text{good} \end{bmatrix}\}\}$$

$$= \begin{bmatrix} \text{very bad} & \text{very bad} & \dots & \text{very bad} \\ \text{very bad} & \text{very bad} & \dots & \text{very bad} \\ \text{very bad} & \text{very bad} & \dots & \text{very bad} \\ \vdots & \vdots & \dots & \vdots \\ \text{very bad} & \text{very bad} & \dots & \text{very bad} \\ \text{very bad} & \text{very bad} & \dots & \text{very bad} \end{bmatrix}_{10\times10}$$

is a 10 × 10 linguistic matrix.

It is trivially symmetric.

Finally we find min{min{A, A^t}} and min{min{A^t, A}} where A is a 4 × 4 linguistic square matrix from S′ = S∪{ϕ} where the linguistic set S is taken from the example 3.3.

$$A = \begin{bmatrix} \text{best} & \text{good} & \text{fair} & \text{bad} \\ \text{bad} & \text{bad} & \text{bad} & \text{best} \\ \text{fair} & \text{best} & \text{bad} & \text{best} \\ \text{good} & \text{best} & \text{good} & \text{fair} \end{bmatrix} \text{ and}$$

$$A^t = \begin{bmatrix} \text{best} & \text{bad} & \text{fair} & \text{good} \\ \text{good} & \text{bad} & \text{best} & \text{best} \\ \text{fair} & \text{bad} & \text{bad} & \text{good} \\ \text{bad} & \text{best} & \text{best} & \text{fair} \end{bmatrix}$$

be the given linguistic square matrix and its transpose respectively.

min{min{A, A^t}}

$$= \min\{\min \begin{bmatrix} \text{best} & \text{good} & \text{fair} & \text{bad} \\ \text{bad} & \text{bad} & \text{bad} & \text{best} \\ \text{fair} & \text{best} & \text{bad} & \text{best} \\ \text{good} & \text{best} & \text{good} & \text{fair} \end{bmatrix},$$

$$\begin{bmatrix} \text{best} & \text{bad} & \text{fair} & \text{good} \\ \text{good} & \text{bad} & \text{best} & \text{best} \\ \text{fair} & \text{bad} & \text{bad} & \text{good} \\ \text{bad} & \text{best} & \text{best} & \text{fair} \end{bmatrix}\}\}$$

$$= \begin{bmatrix} \text{bad} & \text{bad} & \text{bad} & \text{bad} \\ \text{bad} & \text{bad} & \text{bad} & \text{bad} \\ \text{bad} & \text{bad} & \text{bad} & \text{bad} \\ \text{bad} & \text{bad} & \text{bad} & \text{bad} \end{bmatrix}.$$

Clearly min{min{A, A^t}} is a 4 × 4 linguistic symmetric matrix.

Now we find

$$\min\{\min\{A^t, A\}\} = \min\{\min \begin{bmatrix} \text{best} & \text{bad} & \text{fair} & \text{good} \\ \text{good} & \text{bad} & \text{best} & \text{best} \\ \text{fair} & \text{bad} & \text{bad} & \text{good} \\ \text{bad} & \text{best} & \text{best} & \text{fair} \end{bmatrix}, \begin{bmatrix} \text{best} & \text{good} & \text{fair} & \text{bad} \\ \text{bad} & \text{bad} & \text{bad} & \text{best} \\ \text{fair} & \text{best} & \text{bad} & \text{best} \\ \text{good} & \text{best} & \text{good} & \text{fair} \end{bmatrix}\}\}$$

$$= \begin{bmatrix} \text{bad} & \text{bad} & \text{bad} & \text{bad} \\ \text{bad} & \text{bad} & \text{bad} & \text{bad} \\ \text{bad} & \text{bad} & \text{bad} & \text{bad} \\ \text{bad} & \text{bad} & \text{bad} & \text{bad} \end{bmatrix}.$$

min{min{A^t, A}} is a 4 × 4 linguistic symmetric matrix.

In fact trivially symmetric.

Further min{min{A, A^t}} ≠ min{min{A^t, A}} in general.

However both yield a symmetric linguistic matrix of same order as that of A.

In view of all these we have the following result.

Theorem 3.2. *Let S' be a linguistic set together with linguistic empty term associated with a linguistic variable.*

Suppose $A = (a_{ij})$ be a $m \times n$ linguistic matrix with entries $a_{ij} \in S'$ ($1 \leq i \leq m$ and $1 \leq j \leq n$). Then $min(min(A, A^t))$ and $min(min(A^t, A))$ are linguistic $m \times m$ and $n \times n$ square symmetric matrices respectively.

- *i)* *If $m = 1$ or $n = 1$ then it yield a trivial single element linguistic matrix.*
- *ii)* *If $m \neq n$ then we get two symmetric linguistic matrices of different orders.*
- *iii)* *If $m = n \neq 1$ we get two distinct symmetric linguistic matrices of same order; in fact as that of A; that is $m \times m$.*

Proof. Given A is a linguistic matrix of order $m \times m$. If $m = 1$ or $n = 1$ then A is a $1 \times n$ row linguistic matrix or A is a $m \times 1$ column linguistic matrix respectively.

In both case $\min\{\min\{A, A^t\}\}$ when $m = 1$ and $\min\{\min\{A, A^t\}\}$ when $n = 1$ are trivial one element linguistic matrices. Thus proof (i) is complete.

Proof of (ii) If $m \neq n$ we see $\min\{\min\{A, A^t\}\}$ is a $m \times n$ square linguistic symmetric matrix; proof as in case of $\max\{\max\{A, A^t\}\}$ refer theorem 3.1.

Similarly $\min\{\min\{A^t, A\}\}$ is a $n \times n$ square linguistic symmetric matrix (proof as in case of max max operator).

Proof of (iii) If $m = n \neq 1$ then A is a square linguistic matrix and both min{min{A, A^t}} and min{A, A^t}} are of order $m \times m$ they are symmetric in nature as proved in theorem 3.1. For max. max.

This method can be used to get many or desired order symmetric linguistic matrices.

Next we proceed onto describe the two operations max{min{A, B}} and min{max {A, B}} for compatible linguistic matrices A and B.

In this section we proceed onto describe min max and max min operators on the linguistic matrices A and B which are such that the number of columns of A is equal to the number of rows of B or the number of columns of B equal to the number of row of A.

We will first illustrate the operation min max by some examples.

Example 3.12. Let $S' = S \cup \{\phi\}$ where S is the linguistic set defined in example 3.3.

Let A = (very bad, fair, good, best, bad) and

$$B = \begin{bmatrix} \text{bad} \\ \text{good} \\ \text{best} \\ \text{bad} \\ \text{fair} \end{bmatrix}$$

be two linguistic matrices of order 1×5 and 5×1 respectively.

Clearly both min max{A, B} and min{max{B, A}} are defined.

We first find min{max{A, B}}

$$= \min\{\max\{(\text{very bad, fair, good, best, bad}), \begin{bmatrix} \text{bad} \\ \text{good} \\ \text{best} \\ \text{bad} \\ \text{fair} \end{bmatrix}\}\}$$

= min{max{very bad, bad}, max{fair, good}, max{good, best}, max{best, bad}, max{bad, fair}}

= min {bad, good, best, best, fair}

= (bad) is a singleton linguistic set of order 1 × 1.

Now we find max{min{A, B}}

$$= \max\ \{\min\{(\text{very bad, fair, good, best, bad}) \begin{bmatrix} \text{bad} \\ \text{good} \\ \text{best} \\ \text{bad} \\ \text{fair} \end{bmatrix}\}\}$$

= max{min {very bad, bad}, min{fair, good}, min{good, best}, min{best, bad}, min{bad, fair}}

= max{very bad, fair, good, bad, bad}

= {good}.

This is a 1 × 1 matrix.

Clearly max[min{A, B}} ≠ min{max {A, B}}.

Now we proceed onto work with upper triangular linguistic matrices and lower triangular linguistic matrices in the following.

Example 3.13. Let S = {very old, old, oldest, just old, young, very young, ϕ, just young, youngest} be a linguistic set associated with the linguistic variable age of a person.

Let

$$A = \begin{bmatrix} \text{old} & \phi & \phi & \phi & \phi & \phi \\ \phi & \text{young} & \phi & \phi & \phi & \phi \\ \text{very young} & \text{old} & \text{young} & \phi & \phi & \phi \\ \text{old} & \text{young} & \text{old} & \text{young} & \phi & \phi \\ \phi & \phi & \text{old} & \text{old} & \text{young} & \phi \\ \text{very old} & \text{oldest} & \text{young} & \text{young} & \text{old} & \text{old} \end{bmatrix}$$

be the lower triangular linguistic matrix.

min{max{A, A}} =

$$\min\{\max\{= \begin{bmatrix} \text{old} & \phi & \phi & \phi & \phi & \phi \\ \phi & \text{young} & \phi & \phi & \phi & \phi \\ \begin{matrix}\text{very}\\ \text{young}\end{matrix} & \text{old} & \text{young} & \phi & \phi & \phi \\ \text{old} & \text{young} & \text{old} & \text{young} & \phi & \phi \\ \phi & \phi & \text{old} & \text{old} & \text{young} & \phi \\ \text{very old} & \text{oldest} & \text{young} & \text{young} & \text{old} & \text{old} \end{bmatrix},$$

$$\left.\left.\begin{bmatrix} \text{old} & \phi & \phi & \phi & \phi & \phi \\ \phi & \text{young} & \phi & \phi & \phi & \phi \\ \text{very young} & \text{old} & \text{young} & \phi & \phi & \phi \\ \text{old} & \text{young} & \text{old} & \text{young} & \phi & \phi \\ \phi & \phi & \text{old} & \text{old} & \text{young} & \phi \\ \text{very old} & \text{oldest} & \text{young} & \text{young} & \text{old} & \text{old} \end{bmatrix}\right\}\right\}$$

$$= \begin{bmatrix} \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi \end{bmatrix}.$$

We see the resultant is not a lower triangular linguistic matrix under min{max {A, B}}.

Now we find in this case of A min{max{A, A}} = {(ϕ)} the empty linguistic term matrix.

Suppose we consider two linguistic matrices A and B where A is a upper triangular linguistic matrix of order 4 × 4 and that of B is a linguistic order 4 × 4 given in the following:

$$A = \begin{bmatrix} \text{old} & \text{just old} & \text{young} & \text{old} \\ \phi & \text{just young} & \text{old} & \text{old} \\ \phi & \phi & \text{young} & \text{old} \\ \phi & \phi & \phi & \text{youngest} \end{bmatrix}$$

and

$$B = \begin{bmatrix} \text{young} & \phi & \phi & \phi \\ \text{old} & \text{young} & \phi & \phi \\ \text{just old} & \text{youngest} & \text{just old} & \phi \\ \text{old} & \text{young} & \text{young} & \text{old} \end{bmatrix}.$$

We find min{max{A, B}}

$$= \min\{\max\{\begin{bmatrix} \text{old} & \text{just old} & \text{young} & \text{old} \\ \phi & \text{just young} & \text{old} & \text{old} \\ \phi & \phi & \text{young} & \text{old} \\ \phi & \phi & \phi & \text{youngest} \end{bmatrix},$$

$$\begin{bmatrix} \text{young} & \phi & \phi & \phi \\ \text{old} & \text{young} & \phi & \phi \\ \text{just old} & \text{youngest} & \text{just old} & \phi \\ \text{old} & \text{young} & \text{young} & \text{old} \end{bmatrix}\}\}$$

$$= \begin{bmatrix} \text{old} & \text{young} & \text{just old} & \text{young} \\ \text{young} & \phi & \phi & \phi \\ \text{young} & \phi & \phi & \phi \\ \text{young} & \phi & \phi & \phi \end{bmatrix}.$$

This is not a upper triangular linguistic matrix or a low triangular linguistic matrix.

Next we find max{min{A, B}

$$\max\{\min\{ = \begin{bmatrix} \text{old} & \text{just old} & \text{young} & \text{old} \\ \phi & \text{just young} & \text{old} & \text{old} \\ \phi & \phi & \text{young} & \text{old} \\ \phi & \phi & \phi & \text{youngest} \end{bmatrix},$$

$$\begin{bmatrix} \text{young} & \phi & \phi & \phi \\ \text{old} & \text{young} & \phi & \phi \\ \text{just old} & \text{youngest} & \text{young} & \phi \\ \text{old} & \text{young} & \text{young} & \text{old} \end{bmatrix}\}\}$$

$$= \begin{bmatrix} \text{just old} & \text{young} & \text{young} & \text{old} \\ \text{old} & \text{young} & \text{young} & \text{old} \\ \text{old} & \text{young} & \text{young} & \text{old} \\ \text{youngest} & \text{youngest} & \text{youngest} & \text{youngest} \end{bmatrix}$$

$$\max\{\min\{A, B\}\} \neq \min\{\max\{A, B\}\}.$$

It is important to note that none of them are upper triangular or lower triangular linguistic matrices.

We now determine min{max{B, A}}

$$= \min\{\max\{\begin{bmatrix} \text{young} & \phi & \phi & \phi \\ \text{old} & \text{young} & \phi & \phi \\ \text{just old} & \text{youngest} & \text{just old} & \phi \\ \text{old} & \text{young} & \text{young} & \text{old} \end{bmatrix},$$

$$\begin{bmatrix} \text{old} & \text{just old} & \text{young} & \text{old} \\ \phi & \text{just young} & \text{old} & \text{old} \\ \phi & \phi & \text{young} & \text{old} \\ \phi & \phi & \phi & \text{youngest} \end{bmatrix}\}\}$$

$$= \begin{bmatrix} \phi & \phi & \phi & \text{youngest} \\ \phi & \phi & \phi & \text{youngest} \\ \phi & \phi & \phi & \text{youngest} \\ \text{young} & \text{young} & \text{young} & \text{old} \end{bmatrix}$$

Now we find out max{min{B, A}}

$$= \max\{\min\{\begin{bmatrix} \text{young} & \phi & \phi & \phi \\ \text{old} & \text{young} & \phi & \phi \\ \text{just old} & \text{youngest} & \text{just old} & \phi \\ \text{old} & \text{young} & \text{young} & \text{old} \end{bmatrix},$$

$$\begin{bmatrix} \text{old} & \text{just old} & \text{young} & \text{old} \\ \phi & \text{just young} & \text{old} & \text{old} \\ \phi & \phi & \text{young} & \text{old} \\ \phi & \phi & \phi & \text{youngest} \end{bmatrix}\}\}$$

$$= \begin{bmatrix} \text{young} & \text{young} & \text{young} & \text{young} \\ \text{old} & \text{just old} & \text{young} & \text{old} \\ \text{just old} & \text{just old} & \text{young} & \text{just old} \\ \text{old} & \text{just old} & \text{young} & \text{old} \end{bmatrix}.$$

Both min{max{B, A}} $\neq$ max{min{B, A}}.

Now we find min{max{A, A^t}}

$$= \min\{\max\{\begin{bmatrix} \text{old} & \text{just old} & \text{young} & \text{old} \\ \phi & \text{just young} & \text{old} & \text{old} \\ \phi & \phi & \text{young} & \text{old} \\ \phi & \phi & \phi & \text{youngest} \end{bmatrix},$$

$$\begin{bmatrix} \text{old} & \phi & \phi & \phi \\ \text{just old} & \text{just young} & \phi & \phi \\ \text{young} & \text{old} & \text{young} & \phi \\ \text{old} & \text{old} & \text{old} & \text{youngest} \end{bmatrix}\}\}$$

$$= \begin{bmatrix} \text{young} & \text{just young} & \text{young} & \text{young} \\ \text{just old} & \phi & \phi & \phi \\ \text{young} & \phi & \phi & \phi \\ \text{old} & \phi & \phi & \phi \end{bmatrix}$$

We find min{max{A^t, A}}

$$= \begin{bmatrix} \phi & \phi & \phi & \text{youngest} \\ \phi & \phi & \phi & \text{youngest} \\ \phi & \phi & \phi & \text{youngest} \\ \text{youngest} & \text{youngest} & \text{youngest} & \text{youngest} \end{bmatrix}$$

We see min {max{A^t, A} $\neq$ min{max{A, A^t}.

Now we proceed onto propose some problems so that by solving them one becomes familiar with the new concept linguistic matrices and their properties.

SUGESSTED PROBLEMS

1. Let S be a linguistic set associated with the linguistic variable speed of a car running in the road given by S = {ϕ, fastest, fast, just fast, slow, very slow, very fast, just slow, slowest}.

   Let B = (slow, ϕ, fast, fastest, just slow, fast, fastest) be a 1 × 7 row linguistic matrix.

   i) Show the number of linguistic row matrices of order 1 × 7 with entries from S is finite a number and find the number of them.

ii) Find B^t.

iii) Determine min$\{B, B^t\}$ and max$\{B, B^t\}$.

iv) Find max$\{B^t, B\}$ and min$\{B^t, B\}$.

v) Is the min-max and max-min operation defined on $\{B, B^t\}$ and $\{B^t, B\}$? Justify your claim.

vi) What is the identity linguistic row matrix of order 1×7 using S under min operation.

vii) Is the identity matrix of this linguistic row matrix of order 1×7 has the same identity linguistic matrix under max operation also?

2. Let S = [lowest, highest] be a linguistic continuum associated with the linguistic variable temperature of water (from 0 to 100° when starts to boil).

i) Let $A = \begin{bmatrix} \text{low} \\ \text{very low} \\ \text{lowest} \\ \text{high} \\ \text{very high} \\ \text{just low} \\ \text{highest} \end{bmatrix}_{7\times 1}$

be a linguistic column matrix of order 7×1. Find min(max$\{A, A^t\}\}$.

ii) Let B = $\begin{bmatrix} \text{low} & \text{high} & \text{highest} & \text{low} \\ \text{highest} & \text{just low} & \text{low} & \text{lowest} \\ \text{hgih} & \text{high} & \text{high} & \text{low} \\ \text{highest} & \text{high} & \text{low} & \text{lowest} \end{bmatrix}$

Find a) min{max{A, A}},

b) max{min{A, A}},

c) max{min{A, A^t}},

d) min{max{A^t, A}},

e) max{min{A^t, A}} and

f) min{max{A^t, A}}.

Are all the 6 result distinct? Which of them are equal?

iv) Let M =

$$\begin{bmatrix} \text{low} & \text{high} & \text{low} & \text{lowest} \\ \text{just high} & \text{low} & \text{very low} & \text{high} \\ \text{very high} & \text{high} & \text{just high} & \text{very low} \\ \text{lowest} & \text{high} & \text{high} & \text{low} \\ \text{low} & \text{low} & \text{lowest} & \text{high} \\ \text{high} & \text{high} & \text{high} & \text{low} \\ \text{low} & \text{very low} & \text{just low} & \text{low} \\ \text{high} & \text{very high} & \text{just high} & \text{low} \end{bmatrix}_{8\times4}$$

be a linguistic matrix of 8 × 4 order.

a) Find min{max{M, M^t}},

b) Find max$\{$min(M, $M^t\}\}$,

c) Find min max$\{\{M^t, M\}\}$ and

d) min max$\{\{M^t, M\}\}$.

Prove all the four are distinct.

Do they contribute to symmetric linguistic matrices?

v) Let L = $\begin{bmatrix} \text{very high} & \text{low} & \text{high} & \text{higest} \\ \text{low} & \text{high} & \text{lowest} & \text{low} \\ \text{high} & \text{low} & \text{high} & \text{low} \\ \text{highest} & \text{high} & \text{low} & \text{very low} \end{bmatrix}$

be a linguistic matrix.

Find i) max$\{L, L^t\}$,

ii) min$\{L, L^t\}$,

iii) max$\{L^t, L\}$,

iv) min$\{L^t, L\}$,

v) max$\{$min$\{L, L^t\}\}$,

vi) min$\{$max$\{L, L^t\}\}$,

vii) max$\{$min$\{L^t, L\}\}$ and

viii) min$\{$max$\{L^t, L\}\}$.

v) Find the identity linguistic matrix of order 4×1 under min operation and max operation.

Are they different or the same? Justify your claim.

3. Let S be linguistic finite set given in problem 1, associated with the linguistic variable speed of the car.

$$\text{Let } M = \begin{bmatrix} \text{slow} & \phi & \text{fast} \\ \phi & \text{just fast} & \phi \\ \text{slowest} & \phi & \text{very fast} \\ \phi & \text{very slow} & \phi \\ \text{slow} & \phi & \text{fast} \\ \text{fast} & \text{fast} & \text{slow} \\ \text{just fast} & \phi & \phi \\ \phi & \phi & \text{just slow} \\ \phi & \text{fastest} & \phi \end{bmatrix}_{9\times 3}$$

be a linguistic matrix of order 9 × 3.

a) Find all linguistic submatrices of M.

b) How many of these linguistic submatrices are square matrices?

c) Find M^t of M.

d) Prove $(M^t)^t = M$.

e) Find min{max{M, M^t}} and min{max{M^t, M}}.

f) Find max{min{M, M^t}} and max{min{M^t, M}}.

4. Let S be as in problem 1, the linguistic set associated with the linguistic variable speed of the car.

$$\text{Let } D = \begin{bmatrix} \text{fast} & \text{slow} & \text{just fast} & \text{very fast} \\ \phi & \text{fast} & \text{slowest} & \text{fastest} \\ \phi & \phi & \text{slow} & \text{fast} \\ \phi & \phi & \phi & \text{just slow} \end{bmatrix}$$

be the upper triangular linguistic matrix of order 4×4

a) Find D^t, prove $(D^t)^t = D$.

b) Suppose X = (fast ϕ slow, slowest) is a row linguistic matrix of order 1×4.

c) Find max{X, D}.

d) Find min{X, D}.

e) Calculate max{X, D^t}and min{X, D^t}.

f) Find min{D, X^t} and max{D, X^t}

5. Let A =

$$\begin{bmatrix} \text{fast} & \phi & \text{fast} & \text{just slow} & \text{very fast} & \text{slow} & \phi & \phi \\ \phi & \text{slowest} & \phi & \phi & \text{very slow} & \text{fast} & \text{slow} & \phi \\ \text{slow} & \text{slow} & \phi & \text{just fast} & \phi & \text{fast} & \text{slow} & \phi \end{bmatrix}_{3\times 8}$$

be a linguistic matrix of order 3×8 with entries from the linguistic set S given in problem 1 related with the linguistic variable speed of the car.

i) Find A^t, prove $(A^t)^t = A$.

ii) If X = (slow, fast, ϕ)$_{1\times3}$ a 1×3 linguistic row matrix find max{X, A} and min{X, A}.

iii) Find min{A^t, X^t} and max{A^t, X^t}.

iv) Let Y = (fast, slow, just slow, very slow, very fast, just fast, ϕ, slowest)$_{1\times8}$

be a linguistic row matrix of order 1×8

a) Find max {Y, A^t}.

b) Find min{Y, A^t}.

c) Calculate min{A, Y^t} and max{A, Y^t}.

6. Obtain any other special feature associated with linguistic square matrices.

7. Using the linguistic set S given in problem 1.

a) Find linguistic diagonal matrices of order 4×4, 7×7, 8×8 and 9×9.

b) Find linguistic symmetric matrices of order 3×3, 6×6, 10×10 and 8×8.

8. What are the advantages or drawbacks of using min-min and max-max operators instead of min-max or max-min operators?

9. Let B = {good, bad, ϕ, worst, very bad, fair, just good, very good, very fair, just bad, very very bad, very very very good, best}

be a linguistic set associated with the performance of a student in a classroom.

i) Let M =

$$\begin{bmatrix} \text{good} & \text{good} & \text{bad} & \phi \\ \text{fair} & \text{very good} & \phi & \text{bad} \\ \phi & \phi & \text{very bad} & \text{fair} \\ \text{just good} & \text{very bad} & \text{just bad} & \text{best} \\ \text{worst} & \phi & \phi & \phi \\ \text{good} & \text{bad} & \text{best} & \text{fair} \\ \text{fair} & \phi & \phi & \text{best} \\ \text{good} & \text{best} & \text{bad} & \text{best} \\ \phi & \text{very bad} & \text{very very bad} & \phi \end{bmatrix}_{9\times 4}$$

linguistic matrix of order 9×4 with entries from B.

a) Find M^t and prove $(M^t)^t = M$.

b) Find all linguistic submatrices of M.

c) Find all linguistic column sub matrices of M.

d) $A = \begin{bmatrix} \text{fair} \\ \text{best} \\ \text{very very bad} \end{bmatrix}$,

Is A a linguistic submatrix of M? Justify your claim.

e) Is A^t a linguistic submatrix of M? Justify your claim.

f) Find min$\{M, M^t\}$ and max$\{M, M^t\}$.

g) Find min$\{$min$\{M, M^t\}\}$ and max$\{$max$\{M, M^t\}\}$.

h) Find min$\{$max$\{M, M^t\}\}$ and max min $\{M, M^t\}\}$.

i) Find min$\{$max$\{M^t, M\}\}$ and max min $\{M^t, M\}\}$.

j) Calculate min$\{$min $\{M^t, M\}\}$ and

max max $\{M^t, M\}\}$.

k) Determine min$\{M^t, M\}$ and max$\{M, M^t\}$.

l) Does anyone the operations mentioned in the problem (a) to (k) yield a linguistic symmetric matrix? Justify your claim.

10. Let T =

$$\begin{bmatrix} \text{good} & \text{bad} & \text{best} & \phi & \text{worst} & \text{fair} \\ \phi & \text{best} & \phi & \text{good} & \phi & \text{just bad} \\ \text{very good} & \phi & \text{good} & \text{bad} & \text{best} & \phi \\ \text{very bad} & \text{bad} & \text{bad} & \text{best} & \phi & \text{bad} \\ \text{just bad} & \text{just good} & \phi & \text{good} & \text{bad} & \phi \\ \phi & \phi & \text{bad} & \text{bad} & \text{good} & \text{bad} \\ \text{bad} & \text{bad} & \phi & \text{bad} & \text{good} & \phi \end{bmatrix}_{7\times 6}$$

be a linguistic matrix of order 7×6. With entries from the linguistic set B.

i) Show the transpose of all linguistic row submatrices is a linguistic column submatrix of T.

ii) Prove in general the transpose of a linguistic column submatrix of T is not a linguistic row submatrix of T.

iii) Prove T is not a symmetric linguistic matrix.

iv) $$J = \begin{bmatrix} \text{good} & \text{bad} \\ \phi & \text{best} \\ \text{very good} & \phi \\ \text{very bad} & \text{bad} \\ \text{jst bad} & \text{just good} \\ \phi & \phi \\ \text{bad} & \text{bad} \end{bmatrix}$$

be a linguistic submatrix of T. Is J^t a linguistic submatrix of T? Justify your claim.

v) Prove or disprove the transpose of every linguistic square submatrix of T is a linguistic square submatrix of T.

vi) Will the claim in (v) be true for all linguistic square submatrices of W and W any linguistic matrix?

vii) Characterize all those linguistic submatrices of T for which their transpose is a linguistic submatrix.

viii) C can these claims in (i) to (vii) be true for any linguistic matrix?

11. Find any special feature associated with linguistic symmetric matrices.

12. Let

$$V = \begin{bmatrix} \text{good} & \text{bad} & \phi & \text{good} & \text{fair} \\ \text{fair} & \phi & \text{bad} & \phi & \phi \\ \text{very fair} & \text{fair} & \phi & \text{very bad} & \phi \\ \text{very good} & \phi & \text{best} & \phi & \text{very bad} \\ \phi & \text{best} & \phi & \text{very fair} & \phi \\ \text{best} & \text{bad} & \text{very bad} & \phi & \text{good} \end{bmatrix}$$

be a linguistic square matrix of order 6 × 6.

i) Will min min {A, A^t} and min min {A^t, A} be only linguistic empty term square matrix of order 6 × 6.

ii) What can we say about max{max{A, A^t}} and max{max{A^t, A}}?

iii) Clearly $P = \begin{bmatrix} \text{good} \\ \text{fair} \\ \text{very fair} \\ \text{very good} \\ \phi \\ \text{best} \end{bmatrix}$

is a linguistic submatrix of V will P^t be a linguistic submatrix of V?

iv) Let W = (good, bad, ϕ, good, fair) be the linguistic submatrix of V.

Can W be a linguistic submatrix of V? Justify your claim

v) $$A = \begin{bmatrix} \text{bad} & \phi \\ \phi & \text{bad} \end{bmatrix}$$

is a linguistic submatrix of V. It A^t a linguistic submatrix of V? Justify your claim.

vi) $$B = \begin{bmatrix} \text{very bad} & \phi \\ \phi & \text{very bad} \end{bmatrix}$$

and $$C = \begin{bmatrix} \text{very fair} & \phi \\ \phi & \text{good} \end{bmatrix}$$

are linguistic submatrices of V. Prove both B^t and C^t are linguistic submatrices of V.

vii) What is specialty about A, B and C to be linguistic submatrices of V whose transpose are also linguistic submatrices of V?

viii) T = [bad, ϕ] is a linguistic submatrix of V prove T^t is also a linguistic submatrix of V.

ix) $L = \begin{bmatrix} \phi \\ \text{very bad} \\ \phi \end{bmatrix}$ is linguistic submatrix of V prove L^t also a linguistic submatrix of V.

x) Find all linguistic submatrices of V which are such that their transpose is also a linguistic submatrices.

xi) $\begin{bmatrix} \text{fair} \\ \phi \end{bmatrix}$ is a linguistic submatrix prove [fair, ϕ] is also a linguistic submatrix of V.

xii) Let X = (very good, ϕ, best) be the linguistic submatrix of V. Is X^t a linguistic submatrix of V?

13. Does there exist a linguistic matrix Y such that for every linguistic submatrix of Y its transpose is again a linguistic submatrix?

14. Let E = $\begin{bmatrix} \text{good} & \text{bad} & \text{best} \\ \text{bad} & \text{good} & \text{fair} \\ \text{best} & \text{fair} & \text{good} \end{bmatrix}$ be a linguistic square matrix which is symmetric.

    Prove every linguistic submatrix of E is such that its transpose is a linguistic submatrix of E.

15. Let $F = \begin{bmatrix} \text{good} & \text{bad} & \text{best} \\ \text{bad} & \text{just fair} & \text{fair} \\ \text{best} & \text{fair} & \text{just good} \end{bmatrix}$

    be a linguistic symmetric square matrix.

Is the transpose of every linguistic submatrix again a linguistic submatrix of F?

16. Prove all symmetric n × n linguistic matrices M are scuh that every linguistic submatrices of M is such that their transpose are also a linguistic submatrices.

    (Exploit the fact transpose of a symmetric linguistic matrix M is itself, that is $M = M^t$).

17. Let S be a symmetric linguistic matrix of order 7 with entries from the linguistic set B give n in problem 9.

$$S = \begin{bmatrix} \text{good} & \text{bad} & \phi & \text{best} & \phi & \text{very bad} & \text{good} \\ \text{bad} & \phi & \text{best} & \phi & \text{fair} & \phi & \text{bad} \\ \phi & \text{best} & \phi & \text{very bad} & \phi & \text{good} & \phi \\ \text{best} & \phi & \text{very bad} & \phi & \text{very fair} & \phi & \text{just fair} \\ \phi & \text{fair} & \phi & \text{very fair} & \text{good} & \text{fair} & \phi \\ \text{very bad} & \phi & \text{good} & \phi & \text{fair} & \text{best} & \text{good} \\ \text{good} & \text{bad} & \phi & \text{just fair} & \phi & \text{good} & \text{bad} \end{bmatrix}.$$

i) Find all linguistic submatrices of S and prove the transpose of these linguistic submatrices are again linguistic submatrices of S.

ii) Find the transpose of the following linguistic submatrices of S.

a) $$\begin{bmatrix} \text{good} & \text{bad} & \phi \\ \text{bad} & \phi & \text{best} \\ \phi & \text{best} & \phi \\ \text{best} & \phi & \text{very bad} \\ \phi & \text{fair} & \phi \\ \text{very bad} & \phi & \text{good} \\ \text{good} & \text{bad} & \phi \end{bmatrix} = P$$

b) $$W = \begin{bmatrix} \phi & \text{best} & \phi & \text{fair} & \phi & \text{bad} \\ \text{best} & \phi & \text{very bad} & \phi & \text{good} & \phi \\ \phi & \text{very bad} & \phi & \text{very fair} & \phi & \text{just fair} \\ \text{fair} & \phi & \text{very fair} & \text{good} & \text{fair} & \phi \end{bmatrix}$$

c) $$Q = \begin{bmatrix} \text{good} & \text{bad} & \phi & \text{best} & \phi \\ \text{bad} & \phi & \text{best} & \phi & \text{fair} \\ \phi & \text{best} & \phi & \text{very bad} & \phi \\ \text{best} & \phi & \text{very bad} & \phi & \text{very fair} \\ \phi & \text{fair} & \phi & \text{very fair} & \text{good} \end{bmatrix}$$

d) $R = \begin{bmatrix} \text{best} \\ \phi \\ \text{very bad} \\ \phi \\ \text{very fair} \\ \phi \\ \text{just fair} \end{bmatrix}$

e) N = (very bad, ϕ, good, ϕ, fair, best, good_

f) $L = \begin{bmatrix} \text{very bad} & \phi & \text{very fair} \\ \phi & \text{very fair} & \text{good} \end{bmatrix}$

18. Let M =

$$\begin{bmatrix} \text{good} & \phi & \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \text{bad} & \phi & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \begin{matrix}\text{very}\\\text{good}\end{matrix} & \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \begin{matrix}\text{just}\\\text{fair}\end{matrix} & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \begin{matrix}\text{just}\\\text{good}\end{matrix} & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \text{best} & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi & \text{bad} & \phi \\ \phi & \phi & \phi & \phi & \phi & \phi & \phi & \text{worst} \end{bmatrix}$$

be a linguistic diagonal matrix of order 8 × 8 with entries from the linguistic set B given in problem 9.

i) Let $N = \begin{bmatrix} \phi & \phi & \text{very good} & \phi \\ \phi & \phi & \phi & \text{just fair} \\ \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \end{bmatrix}$

be a linguistic submatrix of M. Prove N^t is also a linguistic submatrix of M.

ii) Let $P = \begin{bmatrix} \text{just fair} & \phi & \phi & \phi \\ \phi & \text{just good} & \phi & \phi \\ \phi & \phi & \text{best} & \phi \\ \phi & \phi & \phi & \text{bad} \end{bmatrix}$

be a linguistic submatrix of M, show $P = P^t$.

19. Can there be any other linguistic matrix other than the linguistic symmetric matrix (which includes, diagonal, linguistic matrices) which are such that the transpose of every linguistic submatrix is again a linguistic submatrix?

20. Prove in case of the row linguistic matrix R none of its linguistic submatrices whose transpose is a linguistic submatrix of R (of course we do not take trivial 1×1 linguistic matrices into consideration).

21. Study (20) by replacing linguistic row matrix by the linguistic column matrix C.

FURTHER READING

1. Broumi S, Ye J, Smarandache F. An extended TOPSIS method for multiple attribute decision making based on interval neutrosophic uncertain linguistic variables. Infinite Study; 2015 Jul 1.

2. Dat LQ, Thong NT, Ali M, Smarandache F, Abdel-Basset M, Long HV. Linguistic approaches to interval complex neutrosophic sets in decision making. IEEE access. 2019 Mar 4;7:38902-17.

3. Fan C, Feng S, Hu K. Linguistic neutrosophic numbers einstein operator and its application in decision making. Mathematics. 2019 Apr 28;7(5):389.

4. Fan C, Ye J, Hu K, Fan E. Bonferroni mean operators of linguistic neutrosophic numbers and their multiple attribute group decision-making methods. Information. 2017 Sep 1;8(3):107.

5. Fang Z, Ye J. Multiple attribute group decision-making method based on linguistic neutrosophic numbers. Symmetry. 2017 Jul 7;9(7):111.

6. Herstein., I.N., Topics in Algebra, Wiley Eastern Limited, (1975).

7. Kamacı H. Linguistic single-valued neutrosophic soft sets with applications in game theory. International Journal of Intelligent Systems. 2021 Aug;36(8):3917-60.

8. Kandasamy WV, Ilanthenral K, Smarandache F. Algebraic Structures on MOD Planes. Infinite Study; 2015.

9. Lang, S., Algebra, Addison Wesley, (1967).

10. Li YY, Zhang H, Wang JQ. Linguistic neutrosophic sets and their application in multicriteria decision-making problems. International Journal for Uncertainty Quantification. 2017;7(2).

11. Meyer A, Zimmermann HJ. Applications of fuzzy technology in business intelligence. International Journal of Computers Communications & Control. 2011 Sep 1;6(3):428-41.

12. Peng X, Dai J, Smarandache F. Research on the assessment of project-driven immersion teaching in extreme programming with neutrosophic linguistic information. International Journal of Machine Learning and Cybernetics. 2022 Oct 10:1-6.

13. Peng, X., Dai, J. & Smarandache, F. Research on the assessment of project-driven immersion teaching in extreme programming with neutrosophic linguistic information. Int. J. Mach. Learn. & Cyber. (2022). https://doi.org/10.1007/s13042-022-01669-6.

14. Smarandache F, Broumi S, Singh PK, Liu CF, Rao VV, Yang HL, Patrascu I, Elhassouny A. Introduction to neutrosophy and neutrosophic environment. InNeutrosophic Set in Medical Image Analysis 2019 Jan 1 (pp. 3-29). Academic Press.

15. Smarandache F. Linguistic paradoxes and tautologies. Collect. Pap. Vol. III. 2000;3:122.

16. Smarandache, Florentin, Neutrosophic Logic—Generalization of the Intuitionistic Fuzzy Logic, Special Session on Intuitionistic Fuzzy Sets and Related Concepts, International EUSFLAT Conference, Zittau, Germany, 10-12 September 2003.

17. Tian ZP, Wang J, Zhang HY, Chen XH, Wang JQ. Simplified neutrosophic linguistic normalized weighted Bonferroni mean operator and its application to multi-criteria decision-making problems. Filomat. 2016 Jan 1;30(12):3339-60.

18. Vasantha Kandasamy WB, Smarandache F, Amal K. Fuzzy Linguistic Topological Spaces. arXiv e-prints. 2012 Mar:arXiv-1203.

19. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., MOD planes, EuropaNova, (2015).

20. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., MOD Functions, EuropaNova, (2015).

21. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., Multidimensional MOD planes, EuropaNova, (2015).

22. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., Natural Neutrosophic numbers and MOD Neutrosophic numbers, EuropaNova, (2015).

23. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., Linguistic functions, Editorial Global knowledge, (2022).

24. Ye J. Some aggregation operators of interval neutrosophic linguistic numbers for multiple attribute decision making. Journal of Intelligent & Fuzzy Systems. 2014 Jan 1;27(5):2231-41.

25. Zadeh LA. The concept of a linguistic variable and its application to approximate reasoning—I. Information sciences. 1975 Jan 1;8(3):199-249.

26. Zadeh LA. The concept of a linguistic variable and its application to approximate reasoning—II. Information sciences. 1975 Jan 1;8(4):301-57.

27. Zadeh LA. The concept of a linguistic variable and its application to approximate reasoning-III. Information sciences. 1975 Jan 1;9(1):43-80.

28. Zimmermann HJ. Description and optimization of fuzzy systems. International journal of general System. 1975 Jan 1;2(1):209-15.

29. Zimmermann HJ. Fuzzy set theory. Wiley interdisciplinary reviews: computational statistics. 2010 May;2(3):317-32.

30. Zimmermann HJ. Fuzzy set theory—and its applications. Springer Science & Business Media; 2011 Jun 27.

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