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THE EMPTY SET, THE SINGLETON, AND THE ORDERED PAIR

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Dedicated to the memory of Burton S. Dreben

For the modern set theorist the empty set $\emptyset$, the singleton $\{a\}$, and the ordered pair $\langle x, y\rangle$ are at the beginning of the systematic, axiomatic development of set theory, both as a field of mathematics and as a unifying framework for ongoing mathematics. These notions are the simplest building blocks in the abstract, generative conception of sets advanced by the initial axiomatization of Ernst Zermelo [1908a] and are quickly assimilated long before the complexities of Power Set, Replacement, and Choice are broached in the formal elaboration of the 'set of' $\{\}$ operation. So it is surprising that, while these notions are unproblematic today, they were once sources of considerable concern and confusion among leading pioneers of mathematical logic like Frege, Russell, Dedekind, and Peano. In the development of modern mathematical logic out of the turbulence of 19th century logic, the emergence of the empty set, the singleton, and the ordered pair as clear and elementary set-theoretic concepts serves as a motif that reflects and illuminates larger and more significant developments in mathematical logic: the shift from the intensional to the extensional viewpoint, the development of type distinctions, the logical vs. the iterative conception of set, and the emergence of various concepts and principles as distinctively set-theoretic rather than purely logical. Here there is a loose analogy with Tarski's recursive definition of truth for formal languages: The mathematical interest lies mainly in the procedure of recursion and the attendant formal semantics in model theory, whereas the philosophical interest lies mainly in the basis of the recursion, truth and meaning at the level of basic predication. Circling back to the beginning, we shall see how central the empty set, the singleton, and the ordered pair were, after all.

§1. The empty set. A first look at the vicissitudes specific to the empty set, or null class, provides an entrée into our main themes, particularly the

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differing concerns of logical analysis and the emerging set theory. Viewed as part of larger philosophical traditions the null class serves as the extensional focus for age-old issues about Nothing and Negation, and the empty set emerged with the increasing need for objectification and symbolization.

The work of George Boole was a cresting of extensionalism in the 19th century. He introduced "0" without explanation in his *The Mathematical Analysis of Logic* [1847:21], and used it forthwith as an "elective symbol" complementary to his "1" denoting the "'Universe". However, both "0" and "1" were reconstrued variously, as predicates, classes, states of affairs, and as numerical quantities. Significantly, he began his [1847:3] with the assertion that in "Symbolical Algebra ... the validity of the processes of analysis does not depend upon the interpretation of the symbols which are employed, but solely upon the laws of their combination." In his better known *An Investigation into the Laws of Thought* [1854] he had "signs" representing "classes", and incorporating the arithmetical property of 0 that $0 \cdot y = 0$ for every y, assigned [1854:47] to "0" the interpretation "Nothing", the class consisting of no individuals.[1] However shifting his interpretations of "0", it played a natural and crucial role in Boole's "Calculus", which in retrospect exhibited an over-reliance on analogies between logic and arithmetic. It can be justifiably argued that Boole had invented the empty set. Those following in the algebraic tradition, most prominently Charles S. Pierce and Ernst Schröder, adopted and adapted Boole's "0".

Gottlob Frege, the greatest philosopher of logic since Aristotle, aspired to the logical analysis of mathematics rather than the mathematical analysis of logic, and what in effect is the null class played a key role in his analysis of number. Frege in his *Grundlagen* [1884] eschewed the terms "set" ["Menge"] and "class" ["Klasse"], but in any case the extension of the concept "not identical with itself" was key to his definition of zero as a logical object. Schröder, in the first volume [1890] of his major work on the algebra of logic, held a traditional view that a class is merely a collection of objects, without the {} so to speak. In his review [1895] of Schröder's [1890], Frege argued that Schröder cannot both maintain this view of classes and assert that there is a null class, since the null class contains no objects.[2] For Frege, logic enters in giving unity to a class as the extension of a *concept* and thus makes the null class viable.

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[1]Earlier Boole had written [1854:28]: "By a class is usually meant a collection of individuals, to each of which a particular name or description may be applied; but in this work the meaning of the term will be extended so as to include the case in which but a single individual exists, answering to the required name or description, as well as the cases denoted by the terms 'nothing' and 'universe,' which as 'classes' should be understood to comprise respectively 'no beings,' 'all beings.' " Note how Boole had already entertained the singleton.

[2]Edmund Husserl in his review [1891] of Schröder's [1890] also criticized Schröder along similar lines. Frege had already lodged his criticism in the *Grundgesetze* [1893:2–3] and written that in Husserl's review "the problems are not solved."

Giuseppe Peano [1889], the first to axiomatize mathematics in a symbolic language, used "Λ" to denote both the falsity of propositions (Part II of Logical Notations) and the null class (Part IV). He later [1897] provided a definition of the null class as the intersection of all classes, making it more explicit that there is exactly one null class. More importantly, [1897] had the first occurrence of "∃", used there to indicate that a class is not equal to the null class. Frege had taken the existential quantifier to be derivative from the universal quantifier in the not-for-all-not formulation; in Peano's development, the first indication of the existential quantifier is intimately tied to the null class. Thus, in the logical tradition the null class played a focal and pregnant role.

It is among the set theorists that the null class, *qua* empty set, emerged to the fore as an elementary concept and a basic building block. Georg Cantor himself did not dwell on the empty set. Early on in his study of "pointsets" ["Punktmengen"] of real numbers Cantor did write [1880:355] that "the identity of two pointsets P and Q will be expressed by the formula $P \equiv Q$"; defined disjoint sets as "*lacking intersection*"; and then wrote [1880:356] "for the absence of points ... we choose the letter O; $P \equiv O$ indicates that the set P contains *no single point*." So, "$\equiv O$" is arguably more like a predication for being empty at this stage.

Richard Dedekind in his groundbreaking essay on arithmetic *Was sind und was sollen die Zahlen?* [1888:(2)] deliberately excluded the empty set [Nullsystem] "for certain reasons", though he saw its possible usefulness in other contexts.[3] Indeed, the empty set, or null class, is not necessary in his analysis: Whereas Frege the logician worked to get at what numbers *are* through definitions based on equi-numerosity, Dedekind the mathematician worked to define the number *sequence* structurally, up to isomorphism or "inscrutability of reference." Whereas zero was crucial to Frege's logical development, the particular "base element" for Dedekind was immaterial and he denoted it by the symbol "1" before proceeding to define the numbers "by abstraction."

Ernst Zermelo [1908a], the first to provide a full-fledged axiomatization of set theory, wrote in his Axiom II: "There exists a (improper [uneigentliche]) set, the *null set* [*Nullmenge*] 0, that contains no element at all." Something of intension remained in the "(improper [uneigentliche])", though he did point out that because of his Axiom I, the Axiom of Extensionality, there is a single empty set.

Finally, Felix Hausdorff, the first developer of the transfinite after Cantor and the first to take the sort of extensional, set-theoretic approach to mathematics that would dominate in the years to come, unequivocally opted for the empty set [Nullmenge] in his classic *Grundzüge der Mengenlehre* dedicated to Cantor. However, a hint of predication remained when he wrote

[3]But see footnote 10.

[1914:3]: "... the equation $A = 0$ means that the set A has no element, vanishes [verschwindet], is empty." In a footnote Hausdorff did reject the formulation that the set A does not exist, insisting that it exists with no elements in it. The use to which Hausdorff put "0" in the *Grundzüge* is much as "$\emptyset$" is used in modern mathematics, particularly to indicate the extension of the conjunction of mutually exclusive properties.

The set theorists, unencumbered by philosophical motivations or traditions, attributed little significance to the empty set beyond its usefulness. Interestingly, there would be latter-day attempts in philosophical circles to nullify the null set.[4] Be that as it may, just as zero became basic to numerical notation as a place holder, so also did $\emptyset$ to the algebra of sets and classes to indicate the empty extension.[5]

§2. Inclusion vs. membership. In 19th century logic, the main issue concerning the singleton, or unit class, was the distinction between a and $\{a\}$, and this is closely connected to the emergence of the basic distinction between inclusion, $\subseteq$, and membership, $\in$, a distinction without which abstract set theory could not develop.

Set theory as a field of mathematics began, of course, with Cantor's [1874] result that the reals are uncountable. But it was only much later in [1891] that Cantor gave his now famous diagonal proof, showing in effect that for any set X the collection of functions from X into a two-element set is of a strictly higher cardinality than that of X. In retrospect the diagonal proof can be drawn out from the [1874] proof, but in any case the new proof enabled Cantor to dispense with the earlier topological trappings. Moreover, he could affirm [1891: para. 8] "the general theorem, that the powers [cardinalities] of well-defined sets have no maximum."

Cantor's diagonal proof is regarded today as showing how the power *set* operation leads to higher cardinalities. However, it would be misleading to assert that Cantor himself used power sets. Rather, he was expanding the 19th century concept of *function* by ushering in arbitrary functions. His theory of cardinality was based on one-to-one *correspondence* [Beziehung], and this had led him to the diagonal proof which in [1891] was first rendered in terms of *sequences* "that depend on infinitely many coordinates". By the end of [1891] he did deal explicitly with "all" *functions* with a specified domain L and range $\{0, 1\}$.

The recasting of Cantor's diagonal proof in terms of sets cannot be carried out without drawing the basic distinction between $\subseteq$, inclusion, and $\in$, membership. Surprisingly, neither this distinction nor the related distinction

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[4]See for example Carmichael [1943] [1943a].

[5]Something of the need of a place holder is illustrated by the use of {} in the computer typesetting program TeX. To render ${}^{\omega}2$, {} must be put in as a something to which ω serves as a superscript: `${}^\omega 2$`

between a class a and its unit class $\{a\}$ was generally appreciated in logic at the time of Cantor [1891]. This was symptomatic of a general lack of progress in logic on the traditional problem of the copula (how does "is" function?), a problem with roots going back at least to Aristotle. George Boole, like his British contemporaries, only had inclusion, not even distinguishing between proper and improper inclusion, in his part-whole analysis, and he conflated individuals with their unit classes in his notation. Cantor himself, working initially and mostly with real numbers and sets of real numbers, clearly distinguished inclusion and membership from the beginning for his sets, and in his *Beiträge* [1895], his mature presentation of his theory of cardinality, the distinction is clearly stated in the passages immediately following the oft-quoted definition of set [Menge]. The first to draw the inclusion vs. membership distinction generally in logic was Frege. Indeed, in his *Begriffsschrift* [1879] the distinction is manifest for concepts, and in his *Grundlagen* [1884] and elsewhere he emphasized the distinction in terms of "subordination" and "falling under a concept." The a vs. $\{a\}$ distinction is also explicit in his *Grundgesetze* [1893: §11], in a functional formulation. For Peirce [1885: III] the inclusion vs. membership distinction is unambiguous. Although Schröder drew heavily on Peirce's work, Schröder could not clearly draw the distinction, for he maintained (as we saw above) that a class is merely a collection of objects. Also, Schröder [1890:35] was concerned about "the representation [Vorstellung] of the representation of horse", and remained undecided about a traditional infinite regress problem. With an intensional approach there is an issue here, but if a is the extensional representation of horse, then $\{a\}$ is representation of the representation and hence avowedly distinct from a. In Zermelo's axiomatization paper [1908a] the inclusion vs. membership distinction is of course basic. In his Axiom II he posited the existence of singletons, and he pointed out in succeeding commentary that a singleton has no subset other than itself and the empty set. Thus, Zermelo would have understood that his Axiom I, the Axiom of Extensionality, ruled out having $a = \{a\}$, at least when a has more than one element.[6]

However acknowledged the a vs. $\{a\}$ distinction, there was still uncertainty and confusion in the initial incorporation of the distinction into the symbolization,[7] as we point out in describing in detail the writings of Dedekind and Peano:

[6]However, Zermelo's [1908a] axiomatization does not rule out having *some* sets a satisfying $\{a\} = a$; see our §4. Zermelo in the earliest notes about axiomatization found in his *Nachlass*, written around 1905, took the assertion $M \notin M$ as an axiom (see Moore [1982:155]), and this of course rules out having $\{a\} = a$ for any set a.

[7]While we use the now familiar notation $\{a\}$ to denote the singleton, or unit class, of a, it should be kept in mind that the notation varied through this period. Cantor [1895] wrote $M = \{m\}$ to indicate that M consists of members *typically* denoted by m, i.e., m was a variable ranging over the possibly many members of M. Of those to be discussed, Peano [1890:192] used ιa to denote the unit class of a. Russell [1903:517] followed suit, but from

Dedekind in *Was sind und was sollen die Zahlen?* [1888:(3)] appears to have identified the unit class $\{a\}$ with, or at least denoted it by, a—writing in that case (in modern notation) $a \subseteq S$ for $a \in S$.[8] This lack of clarity about the unit class would get strained: For a set [System] S and transformation [Abbildung] $\phi\colon S \to S$, he formulated what we would now call the closure of an $A \subseteq S$ under ϕ and denoted it by A_o. Then in the crucial definition of "simply infinite system"—one isomorphic to the natural numbers—he wrote $N = 1_o$, where 1 is a distinguished *element* of N. Here, we would now write $N = \{1\}_o$. In his *Grundgesetze* [1893:2–3] Frege roundly and soundly took Dedekind to task for his exclusion of the null class and for his confusion about the unit class, seeing this to be symptomatic of Dedekind's conflation of the inclusion vs. membership distinction. As in his criticism of Schröder, Frege railed against what he regarded as extensional sophistries, firmly convinced about the primacy of his intensional notion of concept.[9] Dedekind in a revealing note entitled "Dangers of the Theory of Systems" written around 1900 and found in his *Nachlass* drew attention to his own confused paragraph [1888:(3)] in order to stress the danger of confounding a with $\{a\}$:

> Suppose that a has distinct elements b and c, yet $\{a\} = a$; since the principle of extensionality had been assumed, it follows that $b = a = c$, which is a contradiction.

He mentioned raising the singleton problem in conversation, briefly with Felix Bernstein on 13 June 1897 and then with Cantor himself on 4 September 1899 pointing out the contradiction.[10] It is hard to imagine such a

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his [1908] on he used ι‘a. Frege [1893:§11] wrote a boldface backslash before a to denote its unit class. It was Zermelo [1908a:262] who introduced the now familiar use of $\{a\}$, having written just before: "The set that contains only the elements a, b, c, $\ldots$, r will often be denoted briefly by $\{a, b, c, \ldots, r\}$."

[8]At the beginning of the paragraph [1888:(3)] Dedekind wrote: "A system A is said to be *part* of a system S when every element of A is also an element of S. Since this relation between a system A and a system S will occur continually in what follows, we shall express it briefly by the symbol $A \ni S$." At the end, he wrote: "Since further every element s of a system S by (2) can itself be regarded as a system, we can hereafter employ the notation $s \ni S$." In the immediately preceding paragraph (2), Dedekind had written: "For uniformity of expression it is advantageous to include also the special case where a system S consists of a *single* (one and only one) element a, i.e., the thing a is an element of S, but no thing different from a is an element of S."

[9]Against Dedekind's description of a system (set) as different things "put together in the mind" Frege [1893:2] had written: "I ask, in whose mind? If they are put together in one mind but not in another, do they form a system then? What is supposed to be put together in my mind, no doubt must be in my mind: then do the things outside myself not form systems? Is a system a subjective figure in the individual soul? In that case is the constellation Orion a system? And what are its elements? The stars, or the molecules, or the atoms?"

[10]See Sinaceur [1971]. In the note Dedekind proposed various emendations to his essay to clarify the situation. Undercutting the exclusion of the empty set [Nullsystem] in the

conversation taking place between the two great pioneers of set theory, years after their foundational work on arithmetic and on the transfinite!

The following is the preface to the third, 1911 edition of Dedekind [1888], with some words italicized for emphasis:[11]

> When I was asked roughly eight years ago to replace the second edition of this work (which was already out of print) by a third, I had misgivings about doing so, because in the mean time doubts had arisen about the reliability [Sicherheit] of important foundations of my conception. Even today I do not underestimate the importance, and to some extent the correctness of these doubts. But my trust in the inner harmony of our logic is not thereby shattered; *I believe that a rigorous investigation of the power* [Schöpferkraft] *of the mind to create from determinate elements a new determinate, their system, that is necessarily different from each of these elements, will certainly lead to an unobjectionable formulation of the foundations of my work.* But I am prevented by other tasks from completing such an investigation; so I beg for leniency if the paper now appears for the third time without changes—which can be justified by the fact that interest in it, as the persistent inquiries about it show, has not yet disappeared.

Thus, Dedekind did draw attention to the distinction between a system and its members, implicitly alluding to his confounding of a and $\{a\}$ in the text. This preface is remarkable for the misgivings it expresses, especially in a work that purports to be foundational for arithmetic.

It was Peano [1889] who first distinguished inclusion and membership with different signs, and it is to him that we owe "$\in$", taken as the initial for the Greek singular copula '$\varepsilon\sigma\tau\iota$.[12] In Part IV of his introduction of logical notations, Peano wrote: "The sign $\in$ signifies *is*. Thus $a \in b$ is read *a is a* [*est quoddam*] *b*; ...." In the preface he warned against confusing "$\in$" with the sign for inclusion. However, at the end of part IV he wrote, "Let s be a class and k a class contained in s; then we say that k is an individual of class s if k consists of just one individual." He then proceeded to give his formula

essay ([1888:(2)]) he also pointed out the importance of having the empty set [Nullsystem]. However, the emendations were never incorporated into later editions of [1888].

Interestingly, the second, 1887 draft of Dedekind [1888] had contained the following passage: "A system can consist in *one* element (i.e., in a *single* one, in one and *only* one), it can also (contradiction) be *void* (contain no element)." Ferreirós [1999:227ff] points this out and also that Dedekind in a manuscript written in the 1890's did introduce the empty set, denoting it by "0", and used "[a]" for the singleton of a (see Dedekind [1932:450–60]).

[11] cf. Ewald [1996:796]. This preface does not appear in Dedekind [1963], which is an English translation of the *second* edition of Dedekind [1888].

[12] Actually, the sign appearing in Peano [1889] is only typographically similar to our "$\in$", and only in Peano [1889a] was the Greek noted. The epsilon "ε" began to be used from then on, and in Peano [1891:fn. 8] the '$\varepsilon\sigma\tau\iota$ connection was made explicit.

56, which in modern notation is:

$$k \subseteq s \rightarrow (k \in s \leftrightarrow (k \neq \emptyset \,\&\, \forall x \in k\, \forall y \in k\, (x = y))).$$

Unfortunately, this way of having membership follow from inclusion undercuts the very distinction that he had so emphasized: Suppose that a is any class, and $s = \{a\}$. Then formula 56 implies that $s \in s$. But then $s = a$, and so $\{a\} = a$.[13] This was not intended by Peano:

In the paper that presented his now well-known existence theorems for ordinary differential equations, Peano introduced the sign "ι" as follows [1890:192]:

> Let us decompose in effect the sign = into its two parts *is* and *equal to*; the word *is* is already represented by ε; let us represent also the expression *equal to* by a sign, and let ι (initial of 'ισος) be that sign; thus instead of $a = b$ one can write $a \,\varepsilon\, \iota b$.

It is quite striking how Peano intended his mathematical notation to mirror the grammar of ordinary language. If indeed $a = b$ can be rendered as $a \,\varepsilon\, \iota b$, then the grammar dictates that ιb must be a class, the unit class of b. ι is the initial of 'ισος, "same", and any member of ιb is the same as b. That this is how "ιb" is autonomously used is evident from a succeeding passage, the first passage in print that drew the a vs. $\{a\}$ distinction (and also described the unordered pair) [1890:193]: "To indicate the class constituted of individuals a and b one writes sometimes $a \cup b$ (or $a + b$, following the more usual notation). But it is more correct to write $\iota a \cup \iota b$; ...." Peano had let "=" range widely from the equality of numerical quantities and of classes to the equivalence of propositions. He introduced "ι" to analyze his overworked "=", and this objectification of the unit class emerged from a remarkable analysis of one form of the copula, equality, in terms of another, membership.[14] While today we take equality and membership as primitive notions and proceed to define the singleton, in Peano's formulation the unit class played a fundamental role in articulating these notions, just as the null class did for "$\exists$" (as we saw above).

The lack of clarity about the distinction between inclusion and membership in the closing years of the 19th century reflected a traditional reluctance to comprehend a collection as a unity and was intertwined with the absence of the liberal, iterative use of the "set of" $\{\}$ operation. Of course, set theory

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[13] Van Heijenoort [1967:84] alluded specifically to formula 56, but did not point out how it leads to $\{a\} = a$ for every a.

[14] In [1894:§31] Peano again described the "ι" analysis of "=" and moreover emphasized the difference between "0" and "$\iota 0$" for arithmetic. Interestingly, Zermelo [1908a] reflected this Peano analysis of equality in his discussion of definite [definit] properties, those admissible for use in his Axiom (Schema) of Separation. Zermelo initially took $a \,\varepsilon\, b$ to be definite and *derived* that $a = b$ is definite: After positing singletons in Axiom II he wrote: "The question whether $a = b$ or not is definite (No. 4), since it is equivalent to the question of whether or not $a \,\varepsilon\, \{b\}$."

as a mathematical study of that operation could only develop after a sharp distinction between inclusion and membership had been made. This development in turn would depend increasingly on rules and procedures provided by axiomatization. On the logical side, the grasp of the inclusion vs. membership distinction was what spurred Bertrand Russell to his main achievements.

§3. **Russell.** The turn of the century saw Russell make the major advances in the development of his mathematical logic. As he later wrote in [1944]: "The most important year in my intellectual life was the year 1900, and the most important event in this year was my visit to the International Congress of Philosophy in Paris." There in August he met Peano and embraced his symbolic logic, particularly his use of different signs for inclusion and membership. During September Russell (see [1901]) extended Peano's symbolic approach to the logic of relations. Armed with the new insights, in the rest of the year Russell completed most of the final draft of *The Principles of Mathematics* [1903], a book he had been working on in various forms from 1898. However, the sudden light would also cast an abiding shadow, for by May 1901 Russell had transformed Cantor's diagonal proof into Russell's Paradox.[15] In reaction he would subsequently formulate a complex logical system of orders and types in Russell [1908] which multiplied the inclusion vs. membership distinction many times over and would systematically develop that system in Whitehead and Russell's *Principia Mathematica* [1910–3].

For Russell of *The Principles* mathematics was to be articulated in an all-encompassing logic, a complex philosophical system based on universal categories.[16] He had drawn distinctions within his widest category of "term" (but with "object" wider still[17]) among "propositions" about terms and "classes" of various kinds corresponding to propositions. Because of this, Russell's Paradox became a central concern, for it forced him to face the threat of both the conflation of his categories and the loss of their universality.

With the inclusion vs. membership distinction in hand, Russell had informatively recast Cantor's diagonal argument in terms of classes,[18] a formulation synoptic to the usual set-theoretic one about the power set. Cantor's

[15]See Grattan-Guinness [1978], Garciadiego [1992], Moore [1995] and Kanamori [1997] for more on the evolution of Russell's Paradox.

[16]Russell [1903:129] wrote: "The distinction of philosophy and mathematics is broadly one of point of view: mathematics is constructive and deductive, philosophy is critical, and in a certain impersonal sense controversial. Wherever we have deductive reasoning, we have mathematics; but the principles of deduction, the recognition of indefinable entities, and the distinguishing between such entities, are the business of philosophy. Philosophy is, in fact mainly a question of insight and perception."

[17]Russell [1903:55n] wrote: "I shall use the word *object* in a wider sense than *term*, to cover both singular and plural, and also cases of ambiguity, such as 'a man.' The fact that a word can be framed with a wider meaning than *term* raises grave logical problems."

[18]See Russell [1903:365ff].

argument has a positive content in the generation of sets not in the range of functions $f: X \to \mathcal{P}(X)$ from a set into its power set. For Russell however, with U the class of all classes, $\mathcal{P}(U) \subseteq U$ so that the identity map is an injection, and so the Russellian $\{ x \in U \mid x \notin x \} \subseteq U$ must satisfy $\{ x \in U \mid x \notin x \} \in U$, arriving necessarily at a contradiction. Having absorbed the inclusion vs. membership distinction, Russell had to confront the dissolution of that very distinction for his universal classes.

Russell soon sought to resolve his paradox with his theory of types, adumbrated in *The Principles*. Although the inclusion vs. membership distinction was central to *The Principles*, the issues of whether the null class exists and whether a term should be distinct from its unit class became part and parcel of the considerations leading to types. Early on Russell had written [1903:23]:

> If x is any term, it is necessary to distinguish from x the class whose only member is x: this may be defined as the class of terms which are identical with x. The necessity for this distinction, which results primarily from purely formal considerations, was discovered by Peano; I shall return to it at a later stage.

But soon after, Russell [1903:68] distinguished between "class" and "class-concept" and asserted that "there is no such thing as the null-class, though there are null class-concepts ... [and] that a class having only one term is to be identified, contrary to Peano's usage, with that one term." Russell then distinguished between "class as one" and "class as many" and asserted [1903:76] "an ultimate distinction between a class as many and a class as one, to hold that the many are only many, and are not also one."

Next, in an early chapter (X, "The Contradiction") discussing his paradox Russell decided that propositional functions, while defining classes as many, do not always define classes as one, else they could participate *qua* terms for self-predication as in the paradox. There he first proposed a resolution by resorting to a difference in type [1903:104–5]:

> We took it to be axiomatic that the class as one is to be found wherever there is a class as many; but this axiom need not be universally admitted, and appears to have been the source of the contradiction. ... A class as one, we shall say, is an object of the same *type* as its terms .... But the class as one does not always exist, and the class as many is of a different type from the terms of the class, even when the class has only one term ...

Reversing himself again Russell concluded [1903:106] "that it is necessary to distinguish a single term from the class whose only member it is, and that consequently the null-class may be admitted."

Russell only discussed Frege's work in an appendix, having absorbed it relatively late. Of this Russell had actually forewarned the reader in the

Preface [1903:xvi] revealing moreover that he had latterly discovered errors in the text; "these errors, of which the chief are the denial of the null-class, and the identification of a term with the class whose only member it is, are rectified in the Appendices." In the appendix on Frege, Russell alluded to an argument, from Frege's critical review [1895] of Schröder's work, for why a should not be identified with $\{a\}$: In the case of a having many members, $\{a\}$ would still have only one member—the same and obvious concern Dedekind had, as we saw above. Russell observed [1903:514]: "... I contended that the argument was met by the distinction between the class as one and the class as many, but this contention now appears to me mistaken." Russell continued [1903:514]: "... it must be clearly grasped that it is not only the collection as many, but the collection as one, that is distinct from the collection whose only term it is." Russell went on to conclude [1903:515] that "the class as many is the only object that can play the part of a class", writing [1903:516]:

> Thus a class of classes will be many many's; its constituents will each be only many, and cannot therefore in any sense, one might suppose, be single constituents. Now I find myself forced to maintain, in spite of the apparent logical difficulty, that this is precisely what is required for the assertion of number.

Russell was then led to infinitely many types [1903:517]:

> It will now be necessary to distinguish (1) terms, (2) classes, (3) classes of classes, and so on *ad infinitum*; we shall have to hold that no member of one set is a member of any other set, and that $x \in u$ requires that x should be of a set of a degree lower by one than the set to which u belongs. Thus $x \in x$ will become a meaningless proposition; in this way the contradiction is avoided.

And he wrote further down the page:

> Thus, although we may identify the class with the numerical conjunction of its terms [class as many], wherever there are many terms, yet where there is only one term we shall have to accept Frege's range [Werthverlauf] as an object distinct from its only term.

Reading such passages from the *Principles* it does not seem too outlandish to draw connections to and analogies with the high scholastics, relating the "class as many" vs. "class as one" distinction to the traditional pluralism vs. monism debate. The a vs. $\{a\}$ distinction is inextricably connected for Russell, but it is now time to invoke a simple point. The following result is provable just in first-order logic with equality and the membership relation, with the usual definitions of $\subseteq$ and $\{a\}$.

Theorem. $\forall a\ (a = \{a\})$ *is equivalent to*:

$$(*) \qquad \forall a\, \forall b\ (a \subseteq b \longleftrightarrow a \in b).$$

Proof. Suppose first that $a = \{a\}$. Then for any b, $a \subseteq b$ if and only if $\{a\} \subseteq b$. But $\{a\} \subseteq b$ if and only if $a \in b$, and hence we can conclude that $a \subseteq b$ if and only if $a \in b$.

For the converse, first note that since $a \in \{a\}$, $(*)$ implies that $a \subseteq \{a\}$. But also, since $a \subseteq a$, $(*)$ implies that $a \in a$, so that $\{a\} \subseteq a$. Hence we can conclude that $a = \{a\}$. ⊣

In other words, the identification of classes with their unit classes is equivalent to the very dissolution of the inclusion vs. membership distinction that Russell so assiduously cultivated and exploited! And this is arguably near the surface of that by-now notorious paragraph from Dedekind [1888:(3)].

Years later, Russell was firm about the danger of confusing a class with its unit class, but still on the basis of a conflation of type. In his *A History of Western Philosophy* he wrote [1945:198]:[19]

> Another error into which Aristotle falls ... is to think that a predicate of a predicate can be a predicate of the original subject ... The distinction between names and predicates, or, in metaphysical language, between particulars and universals, is thus blurred, with disastrous consequences to philosophy. One of the resulting confusions was to suppose that a class with only one member is identical with that one member.

One direction of the above theorem is implicit here. However, Russell subsequently wrote in his retrospective *My Philosophical Development* [1959:66–7]:

> The enlightenment that I derived from Peano came mainly from two purely technical advances of which it is very difficult to appreciate the importance unless one has (as I had) spent years in trying to understand arithmetic ... The first advance consisted in separating propositions of the form 'Socrates is mortal' from propositions of the form 'All Greeks are mortal' [i.e., distinguishing membership from inclusion] ... neither logic nor arithmetic can get far until the two forms are seen to be completely different ... The second important advance that I learnt from Peano was that a class consisting of one member is not identical with that one member.

Russell may still have been unaware, as one might presume from this passage, that the a vs. $\{a\}$ distinction *follows from* the inclusion vs. membership distinction, let alone that they are equivalent, according to the above theorem. In view of the considerable mathematical prowess exhibited by Russell in the

[19] My thanks to Nimrod Bar-Am for bringing this passage to my attention.

Principia Mathematica this is surely an extreme case of metaphysical preoccupations beclouding developments based on "purely technical advances."

Whether here, or Peano's formula 56, or the analysis of the ordered pair to be discussed below, there is a simple mathematical argument that crucially informs the situation. It is simple for us today to find such arguments, since for us logic is mathematical, and we are heir to the development of set theory based on the iterated application of the "set of" $\{\}$ operation and axioms governing it. This development was first fostered by Zermelo; it is noteworthy that he discovered the argument for Russell's paradox independently and that it served as the beginning of a progress that in effect reverses Russell's:

§4. Classes and singletons. The first decade of the new century saw Zermelo at Göttingen make his major advances in the development of set theory.[20] His first substantial result in set theory was his independent discovery of Russell's Paradox. He then established [1904] the Well-Ordering Theorem, provoking an open controversy about this initial use of the Axiom of Choice. After providing a second proof [1908] of the Well-Ordering Theorem in response, Zermelo also provided the first full-fledged axiomatization [1908a] of set theory. In the process, he ushered in a new abstract, generative view of sets, one that would dominate in the years to come.

Zermelo's independent discovery of the argument for Russell's Paradox is substantiated in a note dated 16 April 1902 found in Edmund Husserl's *Nachlass*.[21] According to the note, Zermelo pointed out that any set M containing all of its subsets as members, i.e., $\mathcal{P}(M) \subseteq M$, is "inconsistent" by considering $\{ x \in M \mid x \notin x \}$. Schröder [1890:245] had argued that Boole's "class 1" regarded as consisting of everything conceivable is inconsistent, and Husserl in a review [1891] had criticized Schröder's argument for not distinguishing between inclusion and membership. That inclusion may imply membership is of course the same concern that Russell had to confront, but Zermelo did not push the argument in the direction of paradox as Russell had done. Also, Zermelo presumably came to his argument independently of Cantor's diagonal proof with functions. That $\mathcal{P}(M)$ has higher cardinality than M is evidently more central than $\mathcal{P}(M) \not\subseteq M$, but the connection between sub*sets* and characteristic *functions* was hardly appreciated then, and Zermelo was just making the first moves toward his abstract view of sets.[22]

[20]Peckhaus [1990] provides a detailed account of Zermelo's years 1897–1910 at Göttingen.

[21]See Rang-Thomas [1981].

[22]In Zermelo's axiomatization paper [1908a], the first result of his axiomatic theory was just the result in the Husserl note, that every set M has a subset $\{ x \in M \mid x \notin x \}$ not a member of M, with the consequence that there is no universal set. Modern texts of set theory usually take the opposite tack, showing that there is no universal set by *reductio* to Russell's Paradox. Zermelo [1908a] applied his first result *positively* to generate specific sets disjoint from given sets for his recasting of Cantor's theory of cardinality.

Reversing Russell's progress from Cantor's correspondences to the identity map inclusion $\mathcal{P}(U) \subseteq U$, Zermelo considered functions $F: \mathcal{P}(X) \to X$, specifically in the form of *choice functions*, those F satisfying $F(Y) \in Y$ for $Y \neq \emptyset$. This of course was the basic ingredient in Zermelo's [1904] formulation of what he soon called the Axiom of Choice, used to establish his Well-Ordering Theorem. Russell the metaphysician had drawn elaborate philosophical distinctions and was forced by Cantor's diagonal argument into a dialectical confrontation with them, as well as with the concomitant issues of whether the null class exists and whether a term should be distinct from its unit class. Zermelo the mathematician never quibbled over these issues for sets and proceeded to resolve the problem of well-ordering sets mathematically. In describing abstract functions Cantor had written in the *Beiträge* [1895:§4]: "... by a '*covering* [Belegung] of N with M,' we understand a law ... ", and thus had continued his frequent use of the term "law" to refer to functions. Zermelo [1904:514] specifically used the term "covering", but with his choice functions any residual sense of "law" was abandoned by him [1904]: "... we take an arbitrary covering γ and derive from it a definite well-ordering of the elements of M." It is here that *abstract* set theory began.

The generative view of sets based on the iteration of the "set of" $\{\}$ operation would be further codified by the assumption of the Axiom of Foundation in Zermelo's final axiomatization [1930], the source of the now-standard theory ZFC. Foundation disallows $a \in a$, and hence precludes $\{a\} = a$, for any set a. In any case, having any sets a such that $\{a\} = a$ is immediately antithetical to the emerging cumulative hierarchy view or "iterative conception" of set, and this scheme would become dominant in subsequent set-theoretic research. Nonetheless, having sets a satisfying $\{a\} = a$ has a striking simplicity in bearing the weight of various incentives, and the idea of having *some* such sets has been pivotal in the formal semantics of set theory.

Paul Bernays [1941:10] announced the independence of the Axiom of Foundation from the other axioms of his system and provided [1954:83] a proof based on having a's such that $\{a\} = a$. Ernst Specker in his Habilitationsschrift (cf. [1957]) also provided such a proof. Moreover, he coordinated such a's playing the role of individuals in his refinement of the Fraenkel-Mostowski method for deriving independence results related to the Axiom of Choice. Individuals, also known as urelemente or atoms, are objects distinct from the null set yet having no members and capable of belonging to sets. Zermelo [1908a][1930] had conceived of set theory as a theory of collections built on a base of individuals. However, Abraham Fraenkel [1921][1922:234ff] from the beginning of his articles on set theory emphasized that there is no need for individuals and generally advocated a minimalist approach as articulated by his Axiom of Restriction [Beschränktheit]. Individuals have since been dispensed with in the mainstream development of axiomatic set theory owing to considerations of elegance, parsimony, and canonicity: Retaining individuals would require a two-sorted formal system or at least predication for sets, with, e.g., the Axiom of Extensionality stated only for sets; set theory surrogates were worked out for mathematical objects; and the cumulative hierarchy view based on the empty set provided a uniform, set-theoretic universe. Nevertheless, individuals have continued to be used for various adaptations of the set-theoretic view. Fraenkel [1922] himself, in the Hilbertian axiomatic tradition and notwithstanding his own Axiom of Restriction, concocted a domain of individuals to argue for the independence of the Axiom of Choice, in the inaugural construction of the Fraenkel-Mostowski method.

Stipulating that all individuals are formally to satisfy $\{a\} = a$ has been a persistent theme in the set-theoretic work of the philosopher and logician Willard V. Quine, and is arguably a significant reflection of his philosophy. Already in [1936:50] Quine had provided a reinterpretation of a type theory that associated individuals with their unit classes. In his [1937] Quine formulated a set theory now known as New Foundations (NF), and in his book *Mathematical Logic* [1940] he extended the theory to include classes.[23] In §22 of [1940] Quine extended the "$x \in y$" notation formally to individuals ("non-classes") y by stipulating for such y that "$x \in y$" should devolve to "$x = y$". In §25 Quine gave a rationale in the reduction of primitive notions: He defined "$x = y$" in terms of $\in$ via extensionality, $\forall z\ (z \in x \leftrightarrow z \in y)$, and this now encompassed individuals x and y since via $\forall z\ (z = x \leftrightarrow z = y)$ one gets $x = y$. For individuals y, through the peculiarity of assimilating one form of the copula, "$x \in y$", into another, "$x = y$", there is a formal identification of y with $\{y\}$. While acknowledging that he himself had no occasion for entertaining "non-classes", Quine regarded this as a way of assimilating them into classes: Now every entity is a class, some are "individuals" satisfying $\{a\} = a$, and the Axiom of Extensionality serves all. Thus, Quine's pragmatism in simply stipulating that individuals satisfy $\{a\} = a$ served his extensionalism, with identity defined in terms of membership becoming universal, encompassing even individuals! In his later text *Set Theory and its Logic* Quine [1964:§4] carried out his stipulatory approach and emphasized how it leads to the seamless incorporation of individuals. Stimulated by Quine's discussion of his "individuals" Dana Scott [1962] showed how to transform models of NF into models having a satisfying $\{a\} = a$. Moreover, his technique has led to a spate of recent results.[24]

[23] As in von Neumann's set theory [1925] as recast by Bernays [1937], in Quine's system only some classes are membership-eligible, i.e., capable of appearing to the left of the membership relation, and these "elements" correspond to the sets of Quine's NF.

[24] See Forster [1995:92ff]. He wrote that the technique introduced by Scott is of great importance in set theories with a universal set, "for many of which it is the only independence technique available."

Many years later in *Pursuit of Truth* [1992:33] Quine wrote, diffusing ontology: "We could reinterpret 'Tabitha' as designating no longer the cat, but the whole cosmos minus the cat; or, again, as designating the cat's singleton, or unit class." This comes in the wake of his discussion of "proxy functions" as exemplifying the "inscrutability of reference." Such functions are one-to-one correspondences for the objects of the purported universe, correspondences that preserve predication and so provide different ontologies "empirically on a par." But a transposition of a with $\{a\}$ that keeps all other sets fixed is exactly what Scott [1962] had used to generate a's satisfying $\{a\} = a$ in models of NF. To paraphrase a paraphrase of Quine's, philosophy recapitulates mathematics! There is no longer a confounding of a and $\{a\}$, yet the *a posteriori* identification has served mathematical, even philosophical, purposes.

Finally, despite all the experience with the iterative conception of set, ruminations as to the mysteriousness of the singleton and even the nonexistence of the null class have resurfaced in recent mereology, the reincarnated part-whole theory, like an opaque reflection of the debates of a century ago.[25]

§5. The ordered pair. We now make the final ascent, up to the ordered pair. In modern mathematics, the ordered pair is basic, of course, and is introduced early in the curriculum in the study of analytic geometry. However, to focus the historical background it should be noted that ordered pairs are not explicit in Descartes and in the early work on analytic geometry. Hamilton [1837] may have been the first to objectify ordered pairs in his reconstrual of the complex numbers as ordered "couples" of real numbers. In logic, the ordered pair is fundamental to the logic of relations and epitomized, at least for Russell, metaphysical preoccupations with time and direction. Given the formulation of relations and functions in modern set theory, one might presume that the analysis of the ordered pair emerged from the development of the logic of relations, and that relations and functions were analyzed in terms of the ordered pair. Indeed, this was how the historical development proceeded on the mathematical side, in the work of Peano and Hausdorff. However, the development on the logical side, in the work of Frege and Russell, was just the reverse, with functions playing a key initial role as a fundamental notion. The progression was *to* the ordered pair, and only later was the inverse direction from the ordered pair to function accommodated, after the reduction of the ordered pair itself to classes. While the approach of the logicians is arguably symptomatic of their more metaphysical preoccupations, the approach on the mathematical side was directed toward an increasingly extensional view of functions regarded as arbitrary correspondences.

[25]See Lewis [1991].

Frege [1891] had two fundamental categories, *function* and *object*, with a function being "unsaturated" and supplemented by objects as arguments. A *concept* is a function with two possible values, the True and the False, and a *relation* is a concept that takes two arguments. The extension of a concept is its graph or course-of-values [Werthverlauf], which is an object, and Frege [1893:§36] devised an iterated or double course-of-values [Doppelwerthverlauf] for the extension of a relation. In these involved ways Frege assimilated relations to functions.

As for the ordered pair, Frege in his *Grundgesetze* [1893:§144] provided the extravagant definition that the ordered pair of x and y is that class to which all and only the extensions of relations to which x stands to y belong.[26] On the other hand, Peirce [1883], Schröder [1895], and Peano [1897] essentially regarded a relation from the outset as just a collection of ordered pairs.[27] Whereas Frege was attempting an analysis of thought, Peano was mainly concerned about recasting ongoing mathematics in economical and flexible symbolism and made many reductions, e.g., construing a *sequence* in analysis as a *function* on the natural numbers. Peano from his earliest logical writings had used "(x, y)" to indicate the ordered pair in formula and function substitutions and extensions. In [1897] he explicitly formulated the ordered pair using "$(x; y)$" and moreover raised the two main points about the ordered pair: First, equation 18 of his Definitions stated the instrumental property which is all that is required of the ordered pair:

$$(*) \qquad \langle x, y\rangle = \langle a, b\rangle \quad \textit{if and only if} \quad x = a \text{ and } y = b.$$

Second, he broached the possibility of reducibility, writing: "The idea of a pair is fundamental, i.e., we do not know how to express it using the preceding symbols."

Peano's symbolism was the inspiration and Frege's work a bolstering for Whitehead and Russell's *Principia Mathematica* [1910–3], in which relations distinguished in intension and in extension were derived from "propositional" functions taken as fundamental and other "descriptive" functions derived from relations. They [1910:∗55] like Frege defined an ordered pair derivatively, in their case in terms of classes and relations, and also for a specific purpose.[28] Previously Russell [1903:§27] had criticized Peirce and Schröder for regarding a relation "essentially as a class of couples," although

[26]This definition, which recalls the Whitehead-Russell definition of the cardinal number 2, depended on Frege's famously inconsistent Basic Law V. See Heck [1995] for more on Frege's definition and use of his ordered pair.

[27]Peirce [1883] used "i" and "j" schematically to denote the components, and Schröder [1895:24] adopted this and also introduced "$i : j$". For Peano see below.

[28]Whitehead and Russell had first defined a cartesian product by other means, and only then defined their ordered pair $x{\downarrow}y$ as $\{x\} \times \{y\}$, a remarkable inversion from the current point of view. They [1910:∗56] used their ordered pair initially to define the ordinal number 2.

he overlooked this shortcoming in Peano.[29] Commenting obliviously on *Principia* Peano [1911,1913] simply reaffirmed an ordered pair as basic, defined a relation as a class of ordered pairs, and a function extensionally as a kind of relation, referring to the final version of his *Formulario Mathematico* [1905–8:73ff] as the source.

Capping this to and fro Norbert Wiener [1914] provided a definition of the ordered pair in terms of unordered pairs of classes only, thereby reducing relations to classes. Working in Russell's theory of types, Wiener defined the ordered pair $\langle x, y\rangle$ as

$$\{\{\{x\}, \Lambda\}, \{\{y\}\}\}$$

when x and y are of the same type and Λ is the null class (of the next type), and pointed out that this definition satisfies the instrumental property $(*)$ above. Wiener used this to eliminate from the system of *Principia* the Axiom of Reducibility for propositional functions of two variables; he had written a doctoral thesis comparing the logics of Schröder and Russell.[30] Although Russell praised Sheffer's stroke, the logical connective not-both, he was not impressed by Wiener's reduction. Indeed, Russell would not have been able to accept it as a genuine analysis. After his break with neo-Hegelian idealism, Russell insisted on taking relations to have genuine metaphysical reality, external to the mind yet intensional in character. On this view, order had to have a primordial reality, and this was part and parcel of the metaphysical force of intension. Years later Russell [1959:67] wrote:

> I thought of relations, in those days, almost exclusively as *intensions*. ... It seemed to me—as indeed, it still seems—that, although from the point of view of a formal calculus one can regard a relation as a set of ordered couples, it is the intension alone which gives unity to the set.

On the mathematical side, Hausdorff's classic text, *Grundzüge der Mengenlehre* [1914], broke the ground for a generation of mathematicians in both set theory and topology. A compendium of a wealth of results, it emphasized mathematical approaches and procedures that would eventually take firm root.[31] Making no intensional distinctions Hausdorff [1914:32ff, 70ff]

[29] In a letter accepting Russell's [1901] on the logic of relations for publication in his journal *Revista*, Peano had pointedly written "The classes of couples correspond to relations" (see Kennedy [1975:214]) so that relations are extensionally assimilated to classes. Russell [1903:§98] argued that the ordered pair cannot be basic and would itself have to be given sense, which would be a circular or an inadequate exercise, and "It seems therefore more correct to take an intensional view of relations ... ".

[30] See Grattan-Guinness [1975] for more on Wiener's work and his interaction with Russell.

[31] Hausdorff's mathematical attitude is reflected in a remark following his explanation of cardinal number in a revised edition [1937:§5] of [1914]: "This formal explanation says what the cardinal numbers are supposed to do, not what they are. More precise definitions have been attempted, but they are unsatisfactory and unnecessary. Relations between cardinal

defined an ordered pair in terms of unordered pairs, formulated functions in terms of ordered pairs, and ordering relations as collections of ordered pairs. (He did not so define an arbitrary relation, for which there was then no mathematical use, but he was first to consider general *partial* orderings, as in his maximality principle.[32]) Hausdorff thus made both the Peano [1911,1913] and Wiener [1914] moves *in* mathematical practice, completing the reduction of functions to sets.[33] This may have been congenial to Peano, but not to Frege nor Russell, they having emphasized the primacy of functions. Following the pioneering work of Dedekind and Cantor Hausdorff was at the crest of a major shift in mathematics of which the transition from an intensional, rule-governed conception of function to an extensional, arbitrary one was a large part, and of which the eventual acceptance of the Power Set Axiom and the Axiom of Choice was symptomatic.

In his informal setting Hausdorff took the ordered pair of x and y to be

$$\{\{x, 1\}, \{y, 2\}\}$$

where 1 and 2 were intended to be distinct objects alien to the situation. It should be pointed out that the definition works even when x or y is 1 or 2 to maintain the instrumental property $(*)$ of ordered pairs. In any case, the now-standard definition is the more intrinsic

$$\{\{x\}, \{x, y\}\}$$

due to Kazimierz Kuratowski [1921:171]. Notably, Kuratowski's definition is a by-product of his analysis of Zermelo's [1908] proof of the Well-Ordering Theorem. Not only does the definition satisfy $(*)$, but it only requires pairings of x and y.

The general adoption of the Kuratowski pair proceeded through the major developments of mathematical logic: Von Neumann initially took the ordered pair as primitive but later noted [1928:338] [1929:227] the reduction via the Kuratowski definition. Gödel in his incompleteness paper [1931:176] also pointed out the reduction.[34] Tarski in his [1931:n. 3], seminal for its precise, set-theoretic formulation of a first-order definable set of reals, pointed out the reduction and acknowledged his compatriot Kuratowski. In his

numbers are merely a more convenient way of expressing relations between sets; we must leave the determination of the 'essence' of the cardinal number to philosophy."

[32]Before Hausdorff and going beyond Cantor, Dedekind was first to consider non-linear orderings, e.g., in his remarkably early, axiomatic study [1900] of lattices.

[33]As to historical precedence, Wiener's note was communicated to the Cambridge Philosophical Society, presented on 23 February 1914, while the preface to Hausdorff's book is dated 15 March 1914. Given the pace of book publication then, it is arguable that Hausdorff came up with his reduction first.

[34]In footnote 18, Gödel blandly remarked: "Every proposition about relations that is provable in [*Principia Mathematica*] is provable also when treated in this manner, as is readily seen." This stands in stark contrast to Russell's labors in *Principia* and his antipathy to Wiener's reduction of the ordered pair.

recasting of von Neumann's system, Bernays [1937:68] also acknowledged Kuratowski [1921] and began with its definition for the ordered pair. It is remarkable that Nicolas Bourbaki in his treatise [1954] on set theory still took the ordered pair as primitive, only later providing the Kuratowski reduction in the [1970] edition.[35]

As with the singleton, we end here with a return to philosophy through Quine. In *Principia Mathematica* Whitehead and Russell had developed the theory of monadic (unary) and dyadic (binary) relations separately and only indicated how the theory of triadic (ternary) relations and so forth was to be analogously developed. Quine in his 1932 Ph. D. dissertation showed how to develop the theory of n-ary relations for all n simultaneously, by defining ordered n-tuples in terms of the ordered pair. Again, a simple technical innovation was the focus, but for Quine such moves would be crucial in an economical, extensional approach to logic and philosophy, already much in evidence in his dissertation.[36] In its published version Quine acknowledged in a footnote [1934:16ff] having become aware of the reduction of the ordered pair itself, which he had taken to be primitive, to classes. Significantly, Quine wrote of the "Wiener-Kuratowski" ordered pair, though their definitions were quite different; the possibility, rather than the particular implementation, was for Quine the point. And unlike for Russell, for whom a reduction was metaphysically unacceptable, for Quine it was merely a question of how the reduction could be worked in.

Quine in his major philosophical work *Word and Object* [1960:§53] declared that the reduction of the ordered pair is a paradigm for philosophical analysis. Perhaps for Quine the Wiener-Kuratowski reduction loomed particularly large because of his encounter with it after the labors of his dissertation. After describing the Wiener ordered pair, Quine wrote:

> This construction is paradigmatic of what we are most typically up to when in a philosophical spirit we offer an "analysis" or "explication" of some hitherto inadequately formulated "idea" or expression. We do not claim synonymy. We do not claim to make clear and explicit what the users of the unclear expression had unconsciously in mind all along. We do not expose hidden meanings, as the words 'analysis' and 'explication' would suggest; we supply lacks. We fix on the particular functions of the unclear expression that make it worth troubling about, and then devise a substitute, clear and couched in terms to our liking, that fills those functions. Beyond those conditions of partial agreement, dictated by our interest and purposes, any traits of the explicans come under the head of "don't cares" ...

[35]See Mathias [1992] for the ignorance of Bourbaki.

[36]See Dreben [1989] for more on Quine and his dissertation.

Quine went on to describe the Kuratowski ordered pair but emphasized that as long as a definition satisfies the instrumental property $(*)$, the particular choice to be made falls under the "don't care" category. Quine then wrote:

> A similar view can be taken of every case of explication: *explication is elimination*. We have, to begin with, an expression or form of expression that is somehow troublesome. It behaves partly like a term but not enough so, or it is vague in ways that bother us, or it puts kinks in a theory or encourages one or another confusion. But also it serves certain purposes that are not to be abandoned. Then we find a way of accomplishing those same purposes through other channels, using other and less troublesome forms of expression. The old perplexities are resolved.
>
> According to an influential doctrine of Wittgenstein's, the task of philosophy is not to solve problems but to dissolve them by showing that there were really none there. This doctrine has its limitations, but it aptly fits explication. For when explication banishes a problem it does so by showing it to be in an important sense unreal; viz., in the sense of proceeding only from needless usages.

The ordered pair may be a simple device in set theory, but it is now carrying the weight of what "analysis" is to mean in philosophy.

REFERENCES

[1937] PAUL BERNAYS, *A system of axiomatic set theory – Part I*, ***The Journal of Symbolic Logic***, vol. 2, pp. 65–77.

[1941] ———, *A system of axiomatic set theory – Part II*, ***The Journal of Symbolic Logic***, vol. 6, pp. 1–17.

[1954] ———, *A system of axiomatic set theory – Part VII*, ***The Journal of Symbolic Logic***, vol. 19, pp. 81–96.

[1847] GEORGE BOOLE, ***The mathematical analysis of logic, being an essay towards a calculus of deductive reasoning***, Macmillan, Barclay, and Macmillan, Cambridge, reprinted in Ewald [1996], vol. 1, pp. 451–509.

[1854] ———, ***An investigation of the laws of thought, on which are founded the mathematical theories of logic and probabilities***, Macmillan, London.

[1954] NICOLAS BOURBAKI, ***Eléments de mathématique. I. Théorie des ensembles***, Hermann, Paris.

[1970] ———, ***Eléments de mathématique. I. Théorie des ensembles***, combined ed., Hermann, Paris.

[1874] GEORG CANTOR, *Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen*, ***Journal für die reine und angewandte Mathematik***, vol. 77, pp. 258–262; reprinted in Cantor [1932] below, pp. 115–118; translated in Ewald [1996], vol. 2, pp. 839–843.

[1880] ———, *Über unendliche, lineare Punktmannigfaltigkeiten. II*, ***Mathematische Annalen***, vol. 17, pp. 355–358; reprinted in Cantor [1932] below, pp. 145–148.

[1891] ———, *Über eine elementare Frage der Mannigfaltigkeitslehre*, ***Jahresbericht der Deutschen Mathematiker-Vereinigung***, vol. 1, pp. 75–78; reprinted in Cantor [1932] below, pp. 278–280; translated in Ewald [1996], vol. 2, pp. 920–922.

[1895] ———, *Beiträge zur Begründung der transfiniten Mengenlehre. I*, ***Mathematische Annalen***, vol. 46, pp. 481–512; translated in Cantor [1915] below; reprinted in Cantor [1932] below, pp. 282–311.

[1915] ———, ***Contributions to the founding of the theory of transfinite numbers***, including translations of Cantor [1895] and Cantor [1897] above with introduction and notes by Philip E. B. Jourdain, Open Court, Chicago; reprinted Dover, New York, 1965.

[1932] ———, ***Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*** (Ernst Zermelo, editor), Julius Springer, Berlin; reprinted Springer-Verlag, Berlin, 1980.

[1943] PETER A. CARMICHAEL, *The null class nullified*, ***Philosophical Review***, vol. 52, pp. 61–68.

[1943a] ———, *Animadversion on the null class*, ***Philosophy of Science***, vol. 10, pp. 90–94.

[1888] RICHARD DEDEKIND, ***Was sind und was sollen die Zahlen?***, F. Vieweg, Braunschweig; sixth, 1930 edition reprinted in Dedekind [1932] below, pp. 335–390; second, 1893 edition translated in Dedekind [1963] below, pp. 29–115 and also translated in Ewald [1996], vol. 2, pp. 795–833.

[1900] ———, *Über die von drei Moduln erzeugte Dualgruppe*, ***Mathematische Annalen***, vol. 53, pp. 371–403; reprinted in Dedekind [1932], vol. 2, pp. 236–271.

[1932] ———, ***Gesammelte mathematische Werke*** (Robert Fricke, Emmy Noether, and Öystein Ore, editors), vol. 3, F. Vieweg, Braunschweig; reprinted Chelsea, New York, 1969.

[1963] ———, ***Essays on the theory of numbers*** (Wooster W. Beman, translator), Dover, New York; reprint of original edition, Open Court, Chicago, 1901.

[1989] BURTON S. DREBEN, *Quine*, in ***Quine in perspective*** (Robert Barrett and Roger Gibson, editors), Blackwell, Oxford.

[1996] WILLIAM EWALD, ***From Kant to Hilbert: A source book in the foundations of mathematics***, Clarendon Press, Oxford.

[1999] JOSÉ FERREIRÓS, ***Labyrinth of thought: A history of set theory and its role in modern mathematics***, Birkhauser Verlag, Basel.

[1995] THOMAS E. FORSTER, ***Set theory with a universal set***, second ed., Oxford Logic Guides, vol. 31, Clarendon Press, Oxford.

[1921] ABRAHAM FRAENKEL, *Über die Zermelosche Begründung der Mengenlehre*, ***Jahresbericht der Deutschen Mathematiker-Vereinigung***, vol. 30, pp. 97–98.

[1922] ———, *Zu den Grundlagen der Cantor-Zermeloschen Mengenlehre*, ***Mathematische Annalen***, vol. 86, pp. 230–237.

[1879] GOTTLOB FREGE, *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*, Nebert, Halle; reprinted Hildesheim, Olms, 1964; translated in van Heijenoort [1967], pp. 1–82.

[1884] ———, ***Die Grundlagen der Arithmetik, eine logisch-mathematische Untersuchung über den Begriff der Zahl***, Wilhelm Köbner, Breslau; translated with German text by John L. Austin as ***The foundations of arithmetic, A logico-mathematical enquiry into the concept of number***, Blackwell, Oxford, 1950; later editions without German text, Harper, New York.

[1891] ———, ***Function und Begriff***, Hermann Pohle, Jena; translated in Frege [1952] below, pp. 21–41.

[1893] ———, ***Grundgesetze der Arithmetik, Begriffsschriftlich abgeleitet***, vol. 1, Hermann Pohle, Jena; reprinted Olms, Hildesheim, 1962.

[1895] ———, *Kritische Beleuchtung einiger Punkte in E. Schröders Vorlesungen über die Algebra der Logik*, ***Archiv für systematische Philosophie***, vol. 1, pp. 433–456; translated in Frege [1952] below, pp. 86–106.

[1952] ———, ***Translations from the philosophical writings of Gottlob Frege*** (Peter Geach and Max Black, translators and editors), Blackwell, Oxford; second, revised edition 1960; latest edition, Rowland & Littlewood, Totowa, 1980.

[1992] ALEJANDRO R. GARCIADIEGO, ***Bertrand Russell and the origins of the set-theoretic "paradoxes"***, Birkhäuser, Boston.

[1931] KURT GÖDEL, *Über formal unentscheidbare Sätze der* ***Principia Mathematica*** *und verwandter Systeme I*, ***Monatshefte für Mathematik und Physik***, vol. 38, pp. 173–198; reprinted, together with a translation incorporating minor emendations by Gödel, in Gödel [1986] below, pp. 144–195.

[1986] ———, ***Collected works*** (Solomon Feferman et al., editors), vol. 1, Oxford University Press, New York.

[1975] IVOR GRATTAN-GUINNESS, *Wiener on the logics of Russell and Schröder. An account of his doctoral thesis, and of his subsequent discussion of it with Russell*, ***Annals of Science***, vol. 32, pp. 103–132.

[1978] ———, *How Bertrand Russell discovered his paradox*, ***Historia Mathematica***, vol. 5, pp. 127–137.

[1837] WILLIAM R. HAMILTON, *Theory of conjugate functions, or algebraic couples: with a preliminary and elementary essay on algebra as the science of pure time*, ***Transactions of the Royal Irish Academy***, vol. 17, pp. 293–422; reprinted in ***The mathematical papers of Sir William Rowan Hamilton*** (H. Halberstam and R. E. Ingram, editors), Cambridge University Press, Cambridge, pp. 1931–1967.

[1914] FELIX HAUSDORFF, ***Grundzüge der Mengenlehre***, de Gruyter, Leipzig; reprinted Chelsea, New York, 1965.

[1937] ———, ***Mengenlehre***; third, revised edition of Hausdorff [1914]; translated by John R. Auman as ***Set theory***, Chelsea, New York, 1962.

[1995] RICHARD G. HECK, JR., *Definition by induction in Frege's* ***Grundgesetze der Arithmetik***, in ***Frege's philosophy of mathematics*** (William Demopoulos, editor), Harvard University Press, Cambridge, pp. 295–333.

[1891] EDMUND HUSSERL, *Besprechung von E. Schröder*, ***Vorlesungen über die Algebra der Logik (exakte Logik), Leipzig, 1890***, vol. I, Göttingische Gelehrte Anzeigen; reprinted in Edmund Husserl, ***Aufsätze und Rezensionen (1890–1910), Husserliana, vol. XXII***, Nijhoff, The Hague, 1978, pp. 3–43; translated by Dallas Willard in ***Edmund Husserl, Early writings in the philosophy of logic and mathematics. Collected works V***, Kluwer, Dordrecht, 1994, pp. 52–91, pp. 243–278.

[1997] AKIHIRO KANAMORI, *The mathematical import of Zermelo's well-ordering theorem*, this BULLETIN, vol. 3, pp. 281–311.

[1975] HUBERT C. KENNEDY, *Nine letters from Giuseppe Peano to Bertrand Russell*, ***Journal of the History of Philosophy***, vol. 13, pp. 205–220.

[1921] KAZIMIERZ KURATOWSKI, *Sur la notion de l'ordre dans la théorie des ensembles*, ***Fundamenta Mathematicae***, vol. 2, pp. 161–171; reprinted in Kuratowski [1988] below, pp. 1–11.

[1988] ———, ***Selected papers*** (Karol Borsuk et al., editors), Państwowe Wydawnictwo Naukowe, Warsaw.

[1991] DAVID LEWIS, ***Parts of classes***, Basil Blackwell, Cambridge.

[1992] ADRIAN R. D. MATHIAS, *The ignorance of Bourbaki*, ***Mathematical Intelligencer***, vol. 14, pp. 4–13.

[1982] GREGORY H. MOORE, ***Zermelo's axiom of choice: Its origins, development, and influence***, Springer-Verlag, New York.

[1995] ———, *The origins of Russell's paradox: Russell, Couturat, and the antinomy of infinite number*, in ***From Dedekind to Gödel*** (Jaakko Hintikka, editor), Synthese Library, vol. 251, Kluwer, Dordrecht, pp. 215–239.

[1889] Giuseppe Peano, ***Arithmetices principia nova methodo exposita***, Bocca, Turin; reprinted in Peano [1957–9] below, vol. 2, pp. 20–55; partially translated in van Heijenoort [1967], pp. 85–97; translated in Peano [1973] below, pp. 101–134.

[1889a] ———, ***I principii di geometrica logicamente esposití***, Bocca, Turin; reprinted in Peano [1957–9] below, vol. 2, pp. 56–91.

[1890] ———, *Démonstration de l'intégrabilité des équations différentielles ordinaires*, ***Mathematische Annalen***, vol. 37, pp. 182–228; reprinted in Peano [1957–9] below, vol. 1, pp. 119–170.

[1891] ———, *Principii di logica matematica*, ***Rivista di matematica***, vol. 1, pp. 1–10; reprinted in Peano [1957–9] below, vol. 2, pp. 92–101; translated in Peano [1973] below, pp. 153–161.

[1894] ———, ***Notations de logique mathématique (introduction au formulaire de mathématiques)***, Guadaguini, Turin.

[1897] ———, *Studii di logica matematica*, ***Atti della Accademia delle Scienze di Torino, Classe di Scienze Fisiche, Matematiche e Naturali***, vol. 32, pp. 565–583; reprinted in Peano [195–9] below, vol. 2, pp. 201–217; translated in Peano [1973] below, pp. 190–205.

[1905–8] ———, ***Formulario mathematico***, Bocca, Torin; reprinted Edizioni Cremonese, Rome, 1960.

[1911] ———, *Sulla definizione di funzione*, ***Atti della Accademia Nazionale dei Lincei, Rendiconti, Classe di Scienze Fisiche, Matematiche e Naturali***, vol. 20-I, pp. 3–5.

[1913] ———, *Review of: A. N. Whitehead and B. Russell,* ***Principia Mathematica***, *vol. I, II*, ***Bollettino di bibliografia e storia delle scienze matematiche (Loria)***, vol. 15, pp. 47–53, 75–81; reprinted in Peano [1957–9] below, vol. 2, pp. 389–401.

[1957–9] ———, ***Opera scelte***, three volumes (U. Cassina, editor), Edizioni Cremonese, Rome.

[1973] ———, ***Selected works of Giuseppe Peano*** (Hubert C. Kennedy, translator and editor), University of Toronto Press, Toronto.

[1990] Volker Peckhaus, *"Ich habe mich wohl gehütet, alle Patronen auf einmal zu verschiessen". Ernst Zermelo in Göttingen*, ***History and Philosophy of Logic***, vol. 11, pp. 19–58.

[1883] Charles S. Peirce, *A theory of probable inference. Note B. The logic of relatives*, ***Studies in logic by members of the John Hopkins University*** (Charles S. Peirce, editor), Little-Brown, Boston; reprinted in Peirce [1931–58] below, vol. 3, pp. 195–209, pp. 187–203.

[1885] ———, *On the algebra of logic: A contribution to the philosophy of notation*, ***The American Journal of Mathematics***, vol. 7, pp. 180–202; reprinted in Peirce [1931–58] below, vol. 3, pp. 210–238, and in Ewald [1996], vol. 1, pp. 608–32.

[1931–58] ———, ***Collected papers of Charles Sanders Peirce*** (Charles Hartshorne and Paul Weiss, editors), Harvard University Press, Cambridge.

[1934] Willard V. Quine, ***A system of logistic***, Harvard University Press, Cambridge.

[1936] ———, *Set-theoretic foundations for logic*, ***The Journal of Symbolic Logic***, vol. 1, pp. 45–57; reprinted in Quine [1966] below, pp. 83–99.

[1937] ———, *New foundations for mathematical logic*, ***American Mathematical Monthly***, vol. 44 (1937), pp. 70–80.

[1940] ———, ***Mathematical logic***, Norton, New York.

[1960] ———, ***Word and object***, MIT Press, Cambridge.

[1963] ———, ***Set theory and its logic***, Harvard University Press, Cambridge; revised edition, 1969.

[1966] ———, ***Selected logic papers***, Harvard University Press, Cambridge; enlarged edition, 1995.

[1992] ———, ***Pursuit of truth***, revised ed., Harvard University Press, Cambridge.

[1981] Bernhard Rang and Wolfgang Thomas, *Zermelo's discovery of the "Russell paradox"*, ***Historia Mathematica***, vol. 8, pp. 15–22.

[1901] Bertrand A. W. Russell, *Sur la logique des relations avec des applications à la théorie des séries*, ***Revue de mathématiques (Rivista di matematica)***, vol. 7, pp. 115–148; partly reprinted in ***The collected papers of Bertrand Russell***, vol. 3 (Gregory H. Moore, editor), Routledge, London, 1993, pp. 613–627; translated in same, pp. 310–349.

[1903] ———, ***The principles of mathematics***, Cambridge University Press, Cambridge; later editions, George Allen & Unwin, London.

[1908] ———, *Mathematical logic as based on the theory of types*, ***American Journal of Mathematics***, vol. 30, pp. 222–262; reprinted in van Heijenoort [1967], pp. 150–182.

[1918] ———, ***Mysticism and logic, and other essays***, Longmans, Green & Co., New York.

[1944] ———, *My mental development*,***The philosophy of Bertrand Russell*** (Paul A. Schilpp, editor), The Library of Living Philosophers, vol. 5, Northwestern University, Evanston, pp. 3–20.

[1945] ———, ***A history of Western philosophy***, Simon Schuster, New York.

[1959] ———, ***My philosophical development***, George Allen & Unwin, London.

[1890] Ernst Schröder, ***Vorlesungen über die Algebra der Logik (exakte Logik)***, vol. 1, B. G. Teubner, Leipzig; reprinted in Schröder [1966] below.

[1895] ———, ***Vorlesungen über die Algebra der Logik (exakte Logik)***, vol. 3: Algebra und Logik der Relative, B. G. Teubner, Leipzig; reprinted in Schröder [1966] below.

[1966] ———, ***Vorlesungen über die Algebra der Logik***, three volumes, Chelsea, New York.

[1962] Dana S. Scott, *Quine's individuals*, ***Logic, methodology and the philosophy of science*** (Ernst Nagel, editor), Stanford University Press, Stanford, pp. 111–115.

[1971] Mohammed A. Sinaceur, *Appartenance et inclusion: un inédit de Richard Dedekind*, ***Revue d'Histoire des Sciences et de leurs Applications***, vol. 24, pp. 247–255.

[1957] Ernst Specker, *Zur Axiomatik der Mengenlehre (Fundierungs- und Auswahlaxiom)*, ***Zeitschrift für mathematische Logik und Grundlagen der Mathematik***, vol. 3, pp. 173–210.

[1931] Alfred Tarski, *Sur les ensembles définissables de nombres réels*, ***Fundamenta Mathematicae***, vol. 17, pp. 210–239; translated in ***Logic, semantics, metamathematics. Papers from 1923 to 1938*** (J. H. Woodger, translator), second ed., Hackett, Indianapolis, 1983, pp. 110–142.

[1967] Jean van Heijenoort, ***From Frege to Gödel: A source book in mathematical logic, 1879–1931***, Harvard University Press, Cambridge; reprinted 2002.

[1925] John von Neumann, *Eine Axiomatisierung der Mengenlehre*, ***Journal für die reine und angewandte Mathematik***, vol. 154, pp. 219–240; Berichtigung in *155*, p. 128; reprinted in von Neumann [1961] below, pp. 34–56; translated in van Heijenoort [1967], pp. 393–413.

[1928] ———, *Über die Definition durch transfinite Induktion und verwandte Fragen der allgemeinen Mengenlehre*, ***Mathematische Annalen***, vol. 99, pp. 373–391; reprinted in von Neumann [1961] below, pp. 320–338.

[1929] ———, *Über eine Widerspruchfreiheitsfrage in der axiomaticschen Mengenlehre*, ***Journal für die reine und angewandte Mathematik***, vol. 160, pp. 227–241; reprinted in von Neumann [1961] below, pp. 494–508.

[1961] ———, ***John von Neumann. Collected works*** (Abraham H. Taub, editor), vol. 1, Pergamon Press, New York.

[1910–3] Alfred N. Whitehead and Bertrand A. W. Russell, ***Principia Mathematica***, three volumes, Cambridge University Press, Cambridge.

[1914] Norbert Wiener, *A simplification of the logic of relations*, ***Proceedings of the Cambridge Philosophical Society***, vol. 17, pp. 387–390; reprinted in van Heijenoort [1967], pp. 224–227.

[1904] Ernst Zermelo, *Beweis, dass jede Menge wohlgeordnet werden kann (Aus einem an Herrn Hilbert gerichteten Briefe)*, ***Mathematische Annalen***, vol. 59, pp. 514–516; translated in van Heijenoort [1967], pp. 139–141.

[1908] ———, *Neuer Beweis für die Möglichkeit einer Wohlordnung*, ***Mathematische Annalen***, vol. 65, pp. 107–128; translated in van Heijenoort [1967], pp. 183–198.

[1908a] ———, *Untersuchungen über die Grundlagen der Mengenlehre I*, ***Mathematische Annalen***, vol. 65, pp. 261–281; translated in van Heijenoort [1967], pp. 199–215.

[1930] ———, *Über Grenzzahlen und Mengenbereiche: Neue Untersuchungen über die Grundlagen der Mengenlehre*, ***Fundamenta Mathematicae***, vol. 16, pp. 29–47; translated in Ewald [1996], vol. 2, pp. 1208–1233.

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