NOTES AND REMARKS ON INFORMATION-SEEKING

1.

Soon after baby becomes wise enough to speak its mother tongue, it start asking questions. No matter in what amplitude, its questions are requests of information. At arms length or farther than that, the information the baby requests is never further from the truth of the matter. It is likely that its first requests are not only of information, but also of some practical needs. Regarding the fulfillment of the needs, a conscious being is constantly aware of the link between information and action in requests of information.

Primary sources of information are the people around and their language. Later on one gets familiar with sources of sources and sources with (publicly) hidden identities. How is that possible, is a good question, but it is simply actual. Everyone grows like that. Things get really interesting as soon as one is able to formulate questions like: Is it really the case that some people work harder than others and earn nothing? Or, questions like: is there a creator who can create a four sided triangle? It may be shocking when someone's oracle says only "yes" as a response.

Possibilities and impossibilities, implied by the information content one is interested in, are not too far away, as soon as one starts thought-experimenting, and looking for answers to one's questions.

I once saw a child amazed by seeing a truck carrying another truck on its back side. He was arguing that it was impossible that such a thing was happening. As a matter of fact, it was already happening. His mother sat laid back, letting the kid's questioning delve into the depths of the intuitable. What was so unacceptable in the mind of the child? Obviously, some presuppositions about properties like size, weight and power of trucks were (visually) blocking the limits of his thoughts.

2.

Generally, presuppositions impose limitations on requests of information. That is partly why, for instance, a baby's learning that it is snowing is one thing, but its mistaking snow flakes for butterflies is another.

Partly due to the limitations of presuppositions, making logical moves and asking timely questions are the backbone of building models for information. In logical theorizing, logicians generalize those moves and questionings. For the generalizations, what logic needs is a suitable language and its ways of codifying information, be it in words, or in symbols. (Codification of information is not for the purpose of disfiguring verbal content. Verbal information content can keep growing, in a variety of ways through coding. That is especially clear in the development of mathematical thought. Ordinary language and mathematical language are integrated.)

3.

One learns about mathematics just like one learns any other subject matter, by asking questions, reaching their answers and by figuring out the logical implications of the information that is acquired through questionings. Naturally, some of the answers one reaches might be uncertain and bracketed, after all, information-seeking is partly an elimination of uncertainties. People also seek for feedback information about answers, by asking further questions and figuring out further implications, so as to be able to make corrections by backtracking. All one needs for that purpose is answers of the questions and the logical consequences of the parts and pieces of the information that is acquired.

Either spoken, written, or implemented in some other way, information is codified in a language bit by bit. Therefore, the primary object of interest is the objective interpretations of mathematical language in mathematical information-seeking. That is not to suggest that there are no subjective elements involved in the process of such information-seeking. Only that, as soon as some information about mathematics is obtained, one can inquire into it, or into its relations to other subject matters, objectively.

One can inquire into a subject matter by reference to models that are specified by suitable codifications. In that sense, information-seeking in mathematics for example must be perceptive to variations between different specifications of models, including non-mathematical ones.

Some of the variations between different specifications of mathematical models can be envisaged even from an outsider's point of view. It is natural to find traces of different kinds of mathematical concepts in mathematical language. Although these concepts are studied in separate branches of mathematics, the overall view of mathematics suggest that all the separate branches are interrelated. The interrelatedness in question is by itself subject to mathematical investigation. Therefore, at least in principle, the so-called interrelations between separate branches of mathematics can be expressed in mathematical language.

Such expressibility calls for a structural approach to the multiplicity of mathematical concepts, which requires in turn a language to express different possible models and comparisons between models. As a result, the requirement concerning the study of the multiplicity of mathematical concepts in separate branches of mathematics amounts to the study of some abstract languages and the mathematical models they aim to specify.

Whatever abstract language it may be codified in, if no new information is added to the degree of information about a branch of mathematics, one will remain in the field that is intended to have been determined by the limitations of the basic assumptions that one inquires into about certain mathematical models. That will be information-seeking inside the boundaries of some already specified models or classes of models.

If one is interested in further information and its implications, by additional new information in the search, then one's logical conclusions and the answers of one's questions will be further from what has already been specified in the boundaries. One needs then new information in the premises.

If the boundaries themselves are in question, then the logical inquiry will also be carried out by delving into the specification of boundary conditions as well as their generalizations.

5.

By asking questions and seeking information with an eye on the logical implications of the answers of one's questions, one can become a lifelong seeker. However, one cannot become so, if one does not pay enough attention to the boundaries of logical inquiry.

It holds true in all types of information-seeking that some lines of thought may turn out to be pointless, unnecessary, or at most a waste of time. Some lines of thought, on the other hand, may turn out to be to the point, perhaps time consuming but necessary, or even possibly time saver. That is not to suggest, of course, that varying degrees of time consumption determine the boundaries of logical inquiry. The boundaries in question are determined rather by conclusiveness conditions of finding, evaluating and putting information in use. In that sense the ultimate boundaries, if there are any, should be determined rather by model building for information in real-time.

Mathematical models are an essential ingredient of scientific inquiry. The way people build them is by way of solving mathematical problems. Being questions about mathematical entities, mathematical problems are requests of information about those entities. Their general presupposition is that every well-defined mathematical problem has a conclusive solution, either in the form of a direct answer explaining how to solve it, or else, in the form of an indirect explanation why there is no possible solution as a direct answer to the question. In so far as such answers and explanations call for further implications and further questions, in principle, there is no end to the process. Nevertheless, besides the general unboundedness of mathematical inquiry, mathematicians are still interested in asking the following question: What are the conclusiveness conditions of a mathematical problem?

The conclusiveness conditions of a mathematical problem are usually assumed to be determined by the general characteristics of a mathematical proof. Such assumption brings about further questions concerning the mathematical proofs themselves, viz. What is it that makes a mathematical proof so conclusive about solutions to mathematical problems? But why does one have to be so strict about proving things; and hence about the conclusiveness of our answers and arguments? The answers are mainly related to human intentions to construct consistent sets of propositions and their consistent extensions and generalizations. Further from that, one's answers must be true of the entities that one seeks information about. In order for that to happen, the entities in question must exist as parts and pieces of mathematical models.

Nothing is further from mathematical truth in mathematics, and for that matter, we want nothing but mathematical models.

6.

One's presuppositions about the reachability of answers to mathematical questions bring about a further question: Are there elementary search procedures to build mathematical models such that one can produce conclusive answers to mathematical questions?

Notice that the conclusiveness conditions of elementary questions are generally determined by the initial presuppositions of a subject matter concerning some further non-elementary questions about boundary conditions. Thereof the requested information might already have gone beyond the reaches of the admitted search procedures as soon as the request is made. Such observation calls for a question concerning the existence of ultimate presuppositions that might put limitations to mathematical investigations.

Are there any such presuppositions, or is mathematics a science without presuppositions? To recall a related question: Is every possible mathematical problem solvable?

Mathematical models are presented as solutions to mathematical problems in some form of stylistic systematization of a proof. Their general outline is in the form of a series of implications.

There are two basic ingredients of a mathematical proof: 1. Questions about possible implications; 2. Implications of possible answers to questions, including uncertain answers. Both of them are subject to thought processes and further model building. In that sense, a study of the varieties of model building in mathematics is a study of the amplitudes of implication. Such study includes the amplitudes of intermediary thoughts between the two sides of an implication as a counterpart to their stylistic systematizations.

7.

Intermediary thoughts between two sides of an implication might involve subjective elements as much as objective ones. After all one's knowledge of mathematical models partly originates from several mediator activities such as imagining, picturing, comparing, analogizing, remembering, stipulating, restricting, selecting etc.

With their cognitive content all the activities in point are parts and pieces of non-mathematical activities. Their mathematical significance thereof is due to conceptualizations of their information content by building models for the entities they involve. The conceptualization in question takes place in mathematical language, written, spoken, or implemented in some other way. The novelty of using mathematical language after making observations on the non-mathematical activities is the intended information content objectively detached from empirical concerns. Hence the boundaries of mathematical activity are partly freed from presuppositions about the extent of reducibility to cognitive information content in mathematical language.

When one inquiresd into some information content in codified form, one realizes that the foremost need for inquiry is some criteria of consistency. When one realizes that need, both for the uses and against the abuses of information, one can generalize the particular need in order to reach correct information, so as to provide maps and guidelines in all the relevant practicable searches of one's theoretical contentment. One does that by embedding the consistency criteria into practicable possibilities of building models for the information in question. All that is for the purpose of seeking and finding varying degrees of information content about reality.

One's attention in seeking and finding information is usually directed against relatively small parts of reality, whose information content can be formulated as a separate objective question by itself. In that sense, the models that are built are not necessarily based on parts and pieces of an actual world of information that is flowing around. Most of the time they are rather parts of some alternative models that are compatible on the basis of one's consistency criteria with the information that one already has found out. Therefore, the particular objects of models are not limited by any domain of actually existing objects and their kinds. The objects in question can also be possible objects and possible kinds of objects, as is the case in mathematical and physical thought experiments.

8.

Naturally, information content is what guides people in their actions. The guidance in question includes people's acts of thinking as well. That is to say, one's actual thought processes and thought experimentations are special cases of being guided by information.

One chooses how to think in similar manner to choosing how to perform an action. Obviously, there are links between thoughts and actions, and the links in question are somewhat flexible. People are free, in principle, to evaluate their thoughts and actions, by disregarding some possible ways of thinking, or some possible ways of acting, which may or may not be compatible with the information content they acquire from their environment.

As a natural consequence of one's evaluations about the information content of thoughts and actions, often one wants to know how to move from thinking on something to acting on something. Whenever one does so, whether some new information is added to one's thinking makes a difference.

If new information enters one's thinking, it may either lead one to a consistent extension of one's thoughts, or else, it may lead to some inconsistent results. It may as well give a chance to correct some of the previous answers, or to make some uncertain answers certain or vice versa. In any case, deeper analyses and syntheses of the information content in question are needed.

In the initial case, where there is no new information added to one's thinking, there can be distinguished at least two kinds of consistency criteria for the logical analyses of the information content of thoughts. These are consequential and combinatorial criteria of consistency. Consequential criteria of consistency are needed for excluding inconsistent consequences from the implication content of logical moves completely. Combinatorial criteria are needed for testing the soundness of thought processes in combination with their consequential consistency. After all, the search must guarantee the existence of the models that are built through thought processes and thought experimentations. If both the complete exclusion of inconsistencies and the soundness of thought processes are satisfied, then deeper analyses of further consistent extensions will be a matter of syntheses of additional information content relative to the initial content. There is a variety of different ways to study such syntheses, for the purpose of which, axiomatization provides a general preliminary framework.

9.

In the axiomatization of a scientific theory, a class of models is studied by characterizing, in the first place, the system of things under mathematical investigation. Once a characterization is made, all one has to do is to find out what follows from the axioms as logical consequences.

A logical consequence is a proposition which cannot be false in the models of the premises of an argument. In that sense, mathematical arguments for instance, are not restricted to mere applications of a finite list of logical inference rules. That is so because logical inference rules are not explanatory of the reasons why an inference made leads to a logical consequence.

Axiomatic reasoning presupposes that the logical consequences of a group of axioms A, and possibly some additional assumptions B, are drawn without the input of any new information into the argument. That is to say, logical consequences from A and B, must follow tautologically. What is essential to logical consequences of A and B is, therefore, the source of information codified by the models of A and B is the only available source for the study of their boundaries. In that respect, there is no increase in the information content of A, B and their logical consequences, in the sense of producing new truths from the axioms. The only increase of information is inside the boundaries determined by the models themselves.

10.

In the axiomatization of a theory, the choice of the axioms is naturally a process of asking questions and answering them. Once some axioms are chosen, the rest of the work is carried out on a tautological basis, provided that a group of axioms A and possible additional assumptions B are consistent relative to each other. The additional assumptions B can be boundary conditions of particular applications. They can be captured within experience. Also different kinds of models of A can capture them.

Newton's proofs by experiment in his work on optics can be seen as a case in point. Likewise, non-Euclidean assumptions in geometry can be considered as other examples. Empirical conditions including experimental setups for building electronic components, amplifying their signals as well as transmitting the signals on the basis of Maxwell's equations, can be seen as some other cases in point. All such examples can be considered as parts and pieces of the models of axiomatizations and their boundaries. In general, all types of reasoning on how something is possible can be included in axiomatic inquiry and its boundaries. In that sense, the logic of axiomatization concerns how-possible explanations in addition to why-necessary explanations. That is not so much different from Hilbert's understanding of axiomatization as the study of necessary and sufficient conditions of the theorems of a theory.

This shows that the logic of axiomatization must be able to answer "how-possible" questions and not only "why-necessary" questions in order to seek information both inside and outside the boundaries of one's models. In fact, how-possible questions are often the most critical questions in thought experiments.

In attempts to exclude whatever is not practicable inside the boundaries of a thought experiment, let us say on whether A implies B, one tries to answer how it is possible that B is false, given A? (A moment's thought on such questioning shows that thought experimentation by means of how-possible questions can produce a logic of inventing new ideas.) The general presupposition of a how-possible question on whether B is false, given A is the existence of models in which B is false, given A. Therefore, building models for logical consequences in a thought experiment presuppose the explanation that an underlying how-possible metric exists as a measure of possible implication amplitudes.

11.

Partly --as a historical accident-- caused by Hilbert's formulation of inference rules for axiomatizations of logic, what is called here the implication amplitudes are identified with deepening structures of inference rules or function schemata. This, however, is superficial. It does not explain when a rule of inference or a function schema is able to give information on the models of implications.

When one considers a series of inferences so as to grant the provability of some proposition, one might take the applications of the inference rules as the final word. One might be tempted to take them as axioms in the sense of basic truths. No matter how complex parts an inferential structure may have, it can be shown, one might think, reducible to an axiom like "from A, infer A". On such basis, one can study infinitely many possible systems of logic. However the following questions remain open: What is it exactly that one thinks that can be done in such a way? Is it the possibilities of various algebraic manipulations and reductions of some inference patterns only? Or is there a deeper truth underlying such possibilities?

If an inference rule is supposed to take one from A to B, one must show, in order to explain its validity, that a model for A which is not a model of B cannot be built. How is that being done?

First, one needs a domain of particulars as building material. What they are is not important. They can be any objects. One uses the objects for imagining some models where A holds but B does not. In other words, one tries to build a counter model. Conditions of impossibility of such building validates the inference rules that are supposed to take one from A to B.

12.

When one is interested in the inner complexities of the models of A and B, one can start by analyzing the models for A and B from outside to inside, in terms of their consequential and combinatorial features. That is to say, one would admit their complexity initially unbounded, and move towards their components by building their bounded models. However, if one is not interested in the inner complexities of the models, one does not need such analysis, albeit by way of implicit synthesis. Therefore, the amount of information that one can extract from a given implication depends on how far one would like to go into the analyses of implicit syntheses.

One may consider the implicit synthesis in point as enveloped how-possible and why-necessary questionings. For example, is it possible at all, if A then B? If it is, how is it possible? If not, why not? Why is it necessary that it is not possible somehow? On the other hand, is it necessary that, if A then B? If it is, why is it necessary? If not, how is it not; how is it possible that it is not necessary?

Such questions need not turn our initial question into a question about the role of modal notions in logical reasoning. Nevertheless, their role in a deeper metric for propositional structures is clear enough from the twofold partition of the question concerning propositional implication.

If some transformation on the inner structural meanings of A and B, were possible and if $A \supset B$ was transformed from something necessarily true to something possibly false, then models of A and B would also have to be transformed to some degree into each other, for the purposes of a truthful search on the intended models. Same holds for their negations and combinations with other propositions. Why is that so? The reason can be summarized as that the ways we interpret $A \supset B$ as true or false have a dual nature with respect to the interpretations of A and B independently.

In the implication sense one can assume the models of A and B have things to do with each other. However, for the purpose of interpreting them one has to consider their models as dual separations of possible valuations. The transformations of the models in point can be called amplitude evaluations, or simply, evaluations.

The dual nature of evaluations gives a model-theoretical meaning, for example, to rules like *modus ponens*, viz. $((A \supset B) \ \& \ \ A) \ \supset B$. For instance, the varieties of assigning valuations, and the varieties of building models for a transformation from necessary truth to possible falsity of B on the basis of $A \supset B$ and A (interpreted non-monotonously), would admit some invariant particulars which do not change their hidden identity throughout the process of model building for different possible logical structures in the context of propositional implications. One can observe the same duality, in principle, with the meanings of connectives like 'and' and 'or'.

Suppose one wants to prove $A \supset B$ on the basis of a theory T. One's question then would be: how to prove $A \supset B$, or else, how to disprove it, from some premises included in T.

If $A \supset B$ had a disproof in T, then it would be possible that A is true and B is false in the models of T. That is, one could consider, as well, how to solve whether T & A has B as a logical consequence. For that purpose, one could first try to prove $A \supset B$ from T. If that was possible at all, then one would argue, on the basis of T & A, that, by modus ponens, $A \supset B$ and A proves B. In that case, no counter model would be possible to disprove $A \supset B$. On the other hand, if B had a disproof, on the basis of T and A, that would amount to the same thing as disproving $A \supset B$, on the basis of T. Hence proving $A \supset B$ from T, and proving B from T and A would be the same problem. That is so only on the basis of the dual character of possible evaluations of T, A and B.

How is it possible to codify the duality of possible evaluations of proofs and disproofs? Suppose $A \supset B$ is assumed to be true. Then either ~A would be true, or B would be true as implied by the assumption. On the other hand, suppose $A \supset B$ is assumed to be false. Then both ~A and B would be false as implied by the assumption.

Notice that, for the purpose of formulating such evaluation assumptions, one does not have to assume the law of excluded middle. One can further extend the evaluations in point, and hence the possible varieties of model building on the basis of those evaluations, by adding a contradictory negation $\neg$, such that one's rules will include assumptions that are not false in addition to assumptions that are true or false. For instance, suppose $A \supset B$ is assumed to be not false. Then either $\sim A$ or B would not be false. One can express such implication as $\neg\sim(A \supset B) \supset (A \supset \leftarrow\sim B)$. Such extension is now an admittance of the law of excluded middle for evaluations of the extended formulas, since the added negation sign $\neg$ creates a complementary image for all applications of the dual negation $\sim$, and hence is a contradictory negation.

What one reaches by the additional contradictory negation, therefore, is the possibility of building models which are non-falsity models, and not truth models.

A formula which is not false can be expressed now as $\neg\sim A$.

In other words, $\neg$ has no role in the duality of evaluations. It is merely a tool for tracing sub-structural interpretations of evaluations. Being so, it blocks all possible evaluations of inner structural meanings of models. For example, if we are interested in the inner structural evaluation of proof figures, where A stands for a complex thought stating the provability of A, then $\sim A$ will mean that A is disprovable, whereas $\neg\sim A$ will mean that A is not disprovable. It is clear then, in what way, the so-called neither provable nor disprovable propositions of mathematics presuppose a logical basis, where the one and only negation obeys the law of excluded middle, and is not a dual negation that operates in pursuit of model building, by way of evaluations, albeit possibly implicitly.

13.

The notion of evaluation can be envisaged game theoretically:

Given a proposition A, the truth of A amounts to the existence of a winning strategy for the verification of A in a game played on possible evaluations of A. That would be a strategy that results for a win no matter which strategy for the purpose of falsification is used as attempts to show that A is false. Likewise, the falsity of A would be the existence of a winning strategy for the falsification of A.

Since language-games are not discriminate on the dualities of possible evaluations, verification and falsification attempts can be pursued in tandem and hence there can also be propositions that are neither true nor false as a result of some evaluations. That is to say, in a language-game, evaluations can take place independently of either procedure's information content, and leaving both of the attempts without a winning strategy with respect to the hidden dualities of the evaluation game. One implication of that, is the following: the information codified by game-theoretical means can be increased by way of logical consequence relations. All one has to do for that is to find out information on the models of independent evaluations, whose ingredients will be the particulars of non-falsity models.

A proposition B is not false, if and only if, for each strategy Φ, possibly chosen for the purpose of falsification, there exists a strategy ς, possibly chosen for the purpose of verification, that leads to a possible win for non-falsification. In other words, any interpreted counterexample Φ can be defeated by a suitable strategy ς. It means interpreting B as saying “I am not false” instead of “I am true”. The consistency of a proposition B, given some background information A, is expressed then, by using the two different negations, viz. $\neg\sim$B.

Being expressed as not false, B is supposed to give information about the models of A. The information is given on the basis of the tautological character of the underlying logic. Therefore, the information in question is given, provided that B is proved consistent relative to A. That is, it must really not be false in the models of A. In other words, in addition to the exclusion of possible falsities, and hence the application of consequential criteria of consistency, it must also apply and satisfy the combinatorial criteria of consistency by the existence of a model assuring its soundness.

Now, ¬~B can be proved in A by proving A ⊃ ¬~B. If the logic used has a complete proof procedure for this purpose, this means that the logic in question is a complete "how-possible" logic. A proof of ¬~B is a proof of the existence of models in which B is not false. It is not necessarily a proof in which B is true. Therefore, especially in the sense of how-possibility of non-falsity, logic can be defined as a science of thought experiments through such model buildings.

Similar observations explain the applicability of the same basic rules to why-necessary and to how-possible reasoning. In other words, the so-called amplitudes of implication can be interpreted so as to have varying probability densities to capture both types of questions and their answers in point. What one must be aware of is the conditions for the conclusiveness of how-possible arguments. The conditions in point define the conceptual completeness of the arguments aiming to reach a final solution. They determine whether a taken path is off the point or not in the practicable searches determined by the models built with respect to possible evaluations. Precisely in that sense how-possible reasoning can be seen as inventing or discovering new ideas. The same idea can be used in analyses and syntheses of logical proofs as well.

Let us take a simple example. Does

(1) Something identifies everything else,

logically imply

(2) Everything is identified by something?

As has been outlined, the steps to be taken starts with trying to build a model for (1), together with the negation of (2). It can assume the following as a negation of (2): Something is not identified by anything. Then, on the basis of the assumptions,

(3) Let a be an individual which identifies everything else.

In the counter-model,

(4) Let b be an individual which is not identified by anything.

From (3) and (4) one can say that

(5) a identifies b, if b is different from a

(6) b is not identified by a

At this point the model building splits into two branches. The first one includes the identification following from (5)

(7) b=a

The second branch involves

(8) a identifies b

Here (6) and (8) contradict each other, showing that, this construction branch is false. From these one can tell things about what the possibility is like that has been proved to exist. It turns out among other things, that the universal identifier a is not identified by anything. Likewise, it is seen that, a does not identify itself.

14.

On the basis of observations of logical thought processes, one can study the varieties of model building for different kinds of possibilities determining various kinds of inner complexity of thought processes or experiments. Assuming, for example, that one such level of complexity is propositional, one can formulate truth tables for combinations of propositions. However, the study of possible evaluations of propositional complexity would be a waste of time and space, in the case of a large number of propositional particulars that are combined, by means of connectives and operators. Instead, one can build partial models of search spaces connected with further complications, as is the case in mathematics or in physics.

One method for the study of variations in partial model building is the method of tableaux.

Tableaux can both be used as a basis for formal proof procedures, as well as a basis for developing various disproof procedures by model building. The main idea in those procedures is to first build spaces and subspaces for various kinds of searches in a given complexity domain, by generalizing disjunctive normal forms in the form of a tree. Every branch then will be a partial description of a model in searches for disproving some actual thought processes, by building counter models. A tableau then can be thought as a set of branches and a branch as a set of symbols or formulas that occur in it. In complete analysis, from outside to inside of formulas representing the potential structural complexity of a thought process or experiment, the set of all branches, with their inner sets of particulars, determine a disjunction of conjunctions, and hence possible generalizations for that matter. To consider a simple case (A & B) v (~A & B) v (A & ~B) v (~A & ~B) is a result of an evaluation analysis, where there are only two elementary propositions, A, and B, and their possible negations and connections. Assuming that A and B has no further inner structural complexity that may lead to further evaluations, the case in point exhausts all possibilities, and, as a matter of fact, do not give any new information, i.e. it is tautologous.

If such exhaustiveness is generalized to all possible finite complexities of propositional connectives, the limit of a totality of tautologies is reached at the depth of the interpretation of quantifiers as infinite extensions of propositional connectives, where $(\forall x)\ A(x)$ means $A(x_1)\ \&\ A(x_2)\ \&\ \ldots$, and $(\exists x)\ A(x)$ means $A(x_1) \vee A(x_2) \vee \ldots$. One is able to exhaust thereof all possible models of the inner complexity of a thought experiment in terms of potentially infinite mechanical computations. Nevertheless, how to do that is a separate question concerning the games played on evaluation models in a tableau.

At this point one may think of a variety of logics serving different computational purposes, each as an answer to a class of how possible questions. However, the real-time question turns around the varieties of model building in logic, and hence each alternative logical system is an instance of those varieties.

Tautologies requiring further and further analysis of depth of inner complexity of thought experiments requires therefore the introduction of new particulars, or new structures of particulars into models, for each new exhaustive description of possibilities. Introduction of new particulars in question determines new structural information, for each depth of exhaustiveness, in the sense of a totality of tautologies over a variety of 2^n particulars, n being the number of elementary propositions involved. If the exhaustiveness conditions are extended, so as to capture the potential varieties of model building for all possible evaluations, the number of particulars that one has to deal with will be 2^ω, i.e. all the countable well-orderings.

15.

Quantifiers ranging over infinite totalities is not an elementary basis for evaluations. Without an elementary basis, building more complicated logical systems and deductive axiomatizations for truth and proof hierarchies as answers to various how-possible questions would be a futile attempt to explain the role of models in logic. For one reason, a conclusive conceptual development of model building, beyond the reaches of a certain *a priori* defined limit, requires an explanation how it is possible in real time to introduce new particulars, in addition to the already defined ones, and how they lead to new multiplicities, without (*a priori*) necessitating the definition of a higher complexity domain.

In real-time questioning, such higher complexities exist only hypothetically. In addition to hypothetical complexities therefore, one needs structural complexities involving both known and unknown particulars in evaluations. The information content of such knowns and unknowns in point does not concern the existence of fixed domains of particulars, but rather the existence of possibly identifiable particulars. The only practicable way of approaching these particulars is model building by thought experimentations. Such approach should not only justify the hypothetical existence of higher complexities. It must also explain how to build models for discovering them in real-time possibilities of thought processes. Therefore, it should not be restricted to systems of inferences. It should be flexible enough to capture all possibilities of actual thought processes, including the various assessments of uncertain answers to one's questions, as well as the certain ones in the process of asking questions and answering them. Uncertain answers, together with all the other answers depending on them, as well as with all their logical consequences, can be corrected if wrong. Or else, they can be turned into certain ones, on the basis of some feedback information coming from a lengthened branch of a tableau. Lengthening of branches is by way of introducing new particulars anytime in the thought processes. Such corrections and certifications are made thereof on the basis of asking timely questions about timely chosen particulars and structures. Asking timely questions is therefore, a strategic objective. Hence even omitting some data can be included in them, by making use of strategic optimizations and approximations and by restricting the real-time model building to certain branches excluding some others. The pinpoint of such treatment lies only in its epistemic character of questioning, in the form of requesting information from a variety of sources, in order to be displaced in appropriate epistemic spaces. The displacement of information in point presupposes topological ingredients in the actual thought processes. They have been called here evaluations. Whether their role can be fully explicated by the means of elementary notions is a puzzling question. For simplicity, evaluations can be envisaged as varieties of questions on the models to be inquired, since presuppositions and conclusiveness conditions of elementary questions are dealt with by describing states of knowing the identity of a particular.

What evaluations ultimately presuppose is the existence of invariant particulars of the models. For more complicated tasks one needs to introduce independent states of knowing identities. In that sense, what one needs for an explication of the role of evaluations in thought processes is uniform strategies for approximation models.

Nevertheless, one can safely say that evaluations appear as evaluations of the results of an ongoing inquiry, distributed over varying degrees of information on the models one builds for our thought processes. As was pointed out earlier, such variations are due to introducing new information into one's arguments in the process of moving from one thought to another. The evaluations in point are therefore not possible without distinguishing between different degrees of information.

One has to provide logical explanations according to the appropriate degree. The logical explanations in question can be seen as part of thought experimentations in the form of analyses and syntheses, viz. varieties of model building in the form of how-possible and why-necessary arguments.

In a general setting, therefore, the method of tableaux must be able to promote a measure of information in the sense of eliminating possible uncertainties and inconsistencies, at varying intensities of implication amplitudes. It might seem that one way of approaching such measure of information-seeking is to study the varieties of model building relative to different systems of inference. Such an approach makes it a handy tool for computational purposes varying from one system to another. The idea of a logical system however, presuppose an inference oriented conception of axiomatization. No matter how the so-called logical systems highlight their own semantic features in well-defined ways, their boundaries separating each from other possible alternatives, as well as their possible interconnections and overlapping aspects are sharpened on the sentential criteria of consistency. Such criteria, although they point to an open-ended investigation of an apparent plurality of logics, are always structured under the limitations of the incompleteness of arithmetical coding. That is to say, it is ignored in the study of logical systems that characterization of a logical system is itself a special case of model building. The result of such ignorance is, in a sense, the infinite possibilities of developing new logical systems for different computational purposes, without specifying a unified model-based view of all those varieties from an elementary logical perspective. An inescapable consequence of such lack of unification is that, questions of discovering new possibilities of building logical system models for novel purposes are left out as if they are subject to a non-logical assessment of intuitions.

It is as if one wants to make sense of a discovery in logic (or in mathematics) by means of some inspirational extensions of our justificatory possibilities. Such an extension of possibilities is nothing but confessing that one cannot do better than appealing to set expansions for the purpose of measuring information on the basis of a mysterious realm of higher-order entities. What is wrong with such confession? Nothing, under the limitations of sentential criteria of consistency, for computational purposes. However, in reality it is not a genuine confession at all. It is only a paradox delay through invented set sizes, either by limitation or by forcing at a long distance call. The logical basis of such invention must be studied truthfully, without any appeal to mysterious notions like non-logical intuitions. Otherwise, the whole apparent variations from system to system of model building will be a variation from one phenomenological mystery to another. [Of course, that does not mean that there is no phenomenological metric for implicit information-seeking.]

The key to understanding the source of the problem at hand with the mysterious origin of different logical systems is to generalize the notion of tautological implication. For that purpose one needs to characterize varying degrees of tautological implication on the basis of exhaustiveness conditions. Such conditions are determined by way of generalizing disjunctive normal forms on a non-compositional framework of evaluations, in order to study their outside-in complexity. Therefore, one need to know the conclusiveness conditions of each possible evaluation. Those are impossibility conditions localized by the existence of inconsistencies at each depth of thought experimentation. Even though it may seem that conditions for the conclusiveness of the obtained results are logical system dependent, at bottom they depend only on the existence of invariant particulars and their dualities with respect to possible evaluations. Such observation is of course possible by way of certain compatibility and completability conditions over the temporal stages of a thought experiment, with additional new elements to its logical structure. In any case, one is free to choose new elements at any depth and keep building one's models for new exhaustiveness conditions. Thus the conclusiveness of information-seeking depends only on the varieties of model building, and not on the different meanings of tautological implication in different logical systems, for the same formula.

16.

Generalization of disjunctive normal forms as exhaustiveness conditions can also be studied by blind expansion of consistency sets for model building, as is the case in classical sequent calculus. There a sequent is of the form $X_1,X_2,...,X_n \supset Y_1,Y_2,...,Y_n$ where $X_1,X_2,...,X_n$, $Y_1,Y_2,...,Y_n$ are formulas. They are connected by $\supset$ in the sense that conjunction of $X_1,X_2,...,X_n$ implies disjunction of $Y_1,Y_2,...,Y_n$. Consistency sets expand with some operational rules, applied to each sub-formula up to a point where logical identities in the $A \supset A$ form are reached. Complexity analysis in that kind of expansion of consistency sets has predetermined ending points thereof.

Dualities of evaluations are assumed to have been distributed as widely as possible. Therefore not always a formula appearing before the application of a set expansion rule is a sub-formula of the one appearing after a rule application. In that sense there are non-elementary assumptions violating the constructive ways of expanding model sets from the very beginning. It is known that such assumptions can be eliminated. However, the elimination after assuming the existence of expanded sets is parasitic on the model theoretical meaning of rules like *modus ponens*.

In one's actual thought processes, using non-elementary assumptions to give a model theoretical meaning to inference rules is like using short cut explanations for our exhaustiveness conditions. Those shortcuts are like lemmas in a mathematical proof, and they can be eliminated. However, the streamline of actual thought experimentations does not have to include such (roundabout) use of shortcuts. In the actual mathematical thought experimentations, the possibility of model building must be prior to using lemmas. Otherwise, one has to admit proofs by chance. Lemmas are used to shorten the unbearably rich flow of thoughts in mathematical evaluations. After all, they are not the only shortcuts possible. Expectedly, there are various hidden strategic aspects in actual thought experimentations. [Varying degrees of probability densities in the amplitudes of mathematical implications can be seen as evidence for the existence of such aspects.]

Similar to the study of sequent calculus, intuitionists suggested studying the components of complex formulas as separately provable ones. For example, proving A or B means, proving A or proving B, according to intuitionists. Such separation comes to the same point as eliminating shortcuts. The idea is correct but the direction of implementation is not. What they suggest amounts to admitting mathematical thought processes mainly as recursive enumerations of some topologically complete descriptions. It does not capture thought experiments with topologically complete but recursively not enumerable descriptions. It seems to restrict the domain of epistemically possible thought processes, by means of some causal accessibility assumptions. Hence it is a limitation on the variations of topological ingredients of model building. However, the elementary basis of thought experiments admits causally inaccessible but dually identifiable variations of model building, without any restrictions prescribed for epistemic possibilities, other than the ones determined by consequential and combinatorial criteria of consistency.

In general terms, moving from one thought to another presupposes a space of models. Someone who knows how to move from A to B has the needed information to restrict his or her attention to a subspace so as to eliminate uncertainties. Propositionally, such attention restriction, in order to move from models of A to models of B, with a view of possible evaluations and their invariants determined by certain exhaustiveness conditions, is the implementation of a topological closure operation. Therefore, it is somewhat very close to what intuitionists had in mind. However, since the intended applications of information-seeking are to relatively small parts of reality, it can also take place in language, by generalizing the information content of mathematical and logical thought processes. Particularly in that general sense, causal accessibility is not sufficient for filling in the received gap between form and content in mathematical language. It does not disfigure content however at the expense of its discontentment with the form...

Dually identifiable particulars of mathematical language is a further step towards erasing the gap between form and content. The underlying theoretical contentment of such erasure is in line with Hilbert's 1900 claim that “arithmetical symbols are written diagrams, and geometrical figures are graphic formulas”. Since both signs and figures have dual roles with respect to possible evaluations in the actual codification of mathematical information, the easiest way to envisage their hidden identities is by way of building models for the information contents capturing their dualities. One general example can be the dual role of continuity assumptions in geometry. They can be assumed either from the very beginning as is the case in the construction of invariant properties of a space under group theoretical transformations, or at the very end in order to render the invariant properties of axioms for the complete characterization of a space.

17.

A full scale implementations of similar ideas pointed out above has been thought to have taken place by way of generalized algorithms. In that case formulas involving large number of variables are the source of the challenge for computer scientists, whose tasks have to be accomplished in limited time and space. Not surprisingly, lengths of proofs are seen as Turing's road maps when he claims in his “Computing machinery” paper that “we can see only a short distance ahead, but we can see plenty there that needs to be done”. The real-time challenge in such seeing however, is not so much different from a problem with some lengthier bits of history, viz. why is it that the road seems longer to us when we don't know how long it is? Even propositional complexity is hence a real challenge for the thought experimenter. As soon as one conceptualizes about a theory for such complexity tasks, one realizes that the punchline of the general attitude towards accomplishment lies somewhere further from mechanical inference.

One has to study evaluations of propositional variables by quantifying over them in clever ways, in order to save space and time. Whatever logical approach one is trying to develop as a solution to a decision problem, the truth of the matter lies neither in propositional variable evaluations (where large number of variables makes the longest routes), nor in the undecidable validity of first-order quantifications (where the study of algorithms and decision procedures are open-ended, being a challenge for shortening the longest routes).

Partial interpretations have been thought as escape tunnels from the false prison of undecidability. But still partial functions are needed in order to deal with partial interpretations. In that sense, one has to deal with how partial functions act on formulas and on possible evaluations, hence presupposing a quantificational framework from the very beginning.

The real-time question then is how to update information about partially interpreted quantifiers themselves, at the same rate as the information about their dependencies on each other is received.

18.

From a meta-logical point of view, a full account of all complications about the actual thought processes should consider the theory of quantification in relation to the interplay of quantifiers with epistemic concepts. Therefore, it is a natural inclination to think that searches for an embedding of a logical space into spaces of modalities is an inevitable goal for a formal theory of logic. Although there is some truth in such inclination, a closer examination of epistemic concepts shows that whatever means of reasoning are embedded in whatever framework of modalities, logic at bottom deals with particulars and structures of particulars, such as information about what, where, who, when etc. something or someone is. As such, particulars are positioned in different kinds of spaces with other particulars. The actual complexity of possible ways of positioning particulars in various structures is the source of our different ways of coding and tracing the spaces they belong. Logical reasoning ideally forgets no detail unexamined in those ways of codings and tracings, and hence no object unidentified. However, it may disregard futile lines of thought in information-seeking.

The information that one is looking for may include information about non-falsity models as much as about truth models. That is why asking timely questions in a variety of ways gives somewhat a pragmatic shift to information-seeking. One can inquire into that shift in game-theoretical terms, assuming that building models for relatively small parts of reality presupposes some hidden assumptions concerning the applicability of mathematical models.

As Einstein put sharply, in his "Geometry and experience" lecture, "in so far as the laws of mathematics refer to reality they are not certain, and in so far as they are certain they do not refer to reality". Models are built for various kinds of applications, including applications to mathematics itself and applications to physical systems. The question concerning the applicability of mathematics to reality is thus a very puzzling one, for the application of models presuppose a compatibility of thought and reality. The main purpose of building models is to apply them primarily on the models that are studied in mathematics and physics. Nobody knows, what it means to apply them directly to reality. On the other hand, it is often (mistakenly) assumed that the idea of such direct application to reality makes sense.

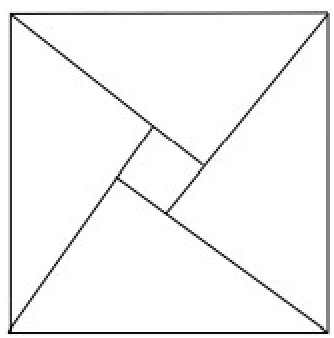
19.

The actual situation with thought experimentations however is not as puzzling as it appears in their applications. In actual thought processes, one can build non-falsity models, on the basis of one's consistency criteria, by assuming, explicitly or implicitly, the relevant boundary conditions not to be false, as well as assuming them to be true about mathematical entities. In that sense, mathematical models have no ultimate presupposition concerning correspondence with an actual reality. If there is a question about correspondence, it should be posed rather in the context of a presupposed compatibility of thought and reality. As soon as some inner organization for the theorizing process is completed, one is in the possible domains of intended models for possible inquiries. Hence non-falsity models become, in a sense, truth models for the inner complexities of some possible evaluations that are compatible with the totality of one's information. That is what is actually presupposed about the compatibility of thought and reality, viz. deciding whether something is not false, rather than deciding whether true or false. Otherwise, imposing truth definitions on the models that one inquires into, by reference to a truth hierarchy obeying some supposed correspondence relation with an actual realm, without building non-falsity models for the purpose of assuring consistency strength of one's theories, is only a hypothetical wandering, which ignores the possibilities of background evaluations. In that respect, intuitions about new possible axioms of a theory have no genuine logical backup. They are not self-reflective thought processes, unless the self-reflection demand is tacitly made through hypotheses underlying the so-called compatibility of thought and reality assumption. Therefore, for conclusive solutions to mathematical problems, there is no way out from non-falsity models, albeit implicitly. That is simply due to the same sense of truth and existence underlying the realm of mathematical models.

One might feel tempted to ask: Where do all those mathematical models come from? The answer is that they are built by finding out the hidden assumptions of the presuppositions concerning the compatibility of thought and reality. The conclusiveness of a mathematical solution then lies in the ultimate dualities between completed evaluations for the compatibility of thought and reality. The hidden assumptions in question are the invariants of those evaluations. As soon as a complete set of such hidden invariants are built into the solution of a mathematical problem, they can generate further evaluations in order to define new problems, hence new amplitudes of implications. That is why some theorems always play a central role in figuring out the general structure of an axiomatic theory, as well as in figuring out the relations between different axiomatic theories.

The Pythagorean theorem plays a central role in the development of geometrical and algebraic methods, in order to have a glimpse of how hidden invariants and dualities, as well as the varieties of model building can play a role in actual thought experimentations. The role in question acts also on the actual presuppositions about the compatibility of thought and reality, hence in the applications of mathematical thought.

Suppose A means that the angle ACB of a triangle is a right angle, and B means that $a^2 + b^2 = c^2$, where a, b, c are the lengths of the sides BC, AC and AB of the triangle, respectively. One of the oldest proof of A $\supset$ B goes simply as follows: Look and see below!



Suppose here that we have already imagined a triangle, created its 3 clones, rotated and translated them so as to be displaced from their original positions, and completed the figure as is shown above. There would have been a lot of things going on in drawing figures as such by following one's attention here and there in order to have a focus on the relevant part of reality. One can summarize that kind of transformation process by reference to Euclid's awareness of the problem with how to draw geometrical figures, in the *Elements*. Such awareness is both about the boundary conditions of geometry problems, as well as it is an awareness of certain epistemic boundaries concerning the links between thought and action. In a sense, continuous transformations and evaluations provide enough information for how to introduce new particulars and structures of particulars into thought experiments and processes, at varying distances from the original displacement. Nevertheless, one can rather follow partial transformations in the streamlines of one's thought processes in order to produce such figures. Therefore, one's attention is not necessarily directed towards the topological ingredients of the thinking process immediately. It is rather directed to ingredients of partial transformations of possible evaluations as topological ingredients. However, how to bridge the gaps in between partial transformations can be considered a principal question.

Based on such observations one can follow an algebraic formulation of the figure above. One can do that without any observational basis as well, i.e. algebraic transformations can be completely independent from any preconceived design for possible evaluations. Not because they are telling a completely different story, but because the choice of particulars and structures of particulars for each questioning step can be either dependent on previous moves, or else it can be independent from them. What is possibly the case is an informational dependence or independence of the choice of particulars in point. Assuming that $a \geq b$, $c^2 = 4ab/2 + (a - b)^2 = a^2 + b^2$. That is the situation with the needed invariants, since it is what the figure tells in part. That is to say, in a game played on $A \supset B$, someone who brings in that algebraic information on the problem description somehow, will have proved the existence of a winning strategy against possible falsifications in any case, with respect to possible evaluations determining the underlying geometry or more generally the underlying topological ingredients. In other words, there possibly goes on a real-time evaluation process updating information content at the same rate as what the language game in point provides as conclusiveness conditions of the problem.

If it were the case that a = b then the figure would have only crossing diagonals and no little square inside. Envisaging the figure inside a circle with varying sizes of inner squares and triangles as well as envisaging it in a three dimensional cone relates the figure to different varieties of possible generalizations. Not surprisingly, as extremity conditions of the possible underlying topological ingredients, which suggest that in such cases of thought experimentation identities and identifications play a role in understanding the boundaries of the topology as well as of the underlying thought processes.

Henceforth one can ask the following question: Where is the non-falsity assumptions here in this implication $A \supset B$? It turns out that on an axiomatic basis, as was the case in the *Elements,* the theorem in question had no proof without the parallel postulate. The postulate says that if the interior angles of two parallel lines intersected by a third line add up to less than two right angles, then the two parallel lines meet at some distant point. For sure, when Euclidean definitions, axioms and postulates are considered as true propositions about the specification of the models for space, a different picture in mind from certain non-falsity models in compatibility with the reality of the space around us appears. However, such a picture rests on an inference-oriented misconception of the axiomatic method.

20.

Suppose the axiomatization is described by using first-order quantification, as is the case, for example, in Tarski's “What is elementary geometry?” paper. The truth of a geometrical proposition then is not definable in the same system of deductive axiomatization, due to Tarski's undefinability theorem. Even though the truth in question can be defined on a higher-order level, such definition will only push away the paradoxes of higher-order reasoning towards higher and higher-order languages aiming to describe more complex geometrical structures than the so-called elementary models, at the limit to higher cardinalities of sets of geometrical objects, and ultimately to an indescribable absolute infinity as had been indicated by Cantor. On the other hand, in the ancient mathematical works, there is no such assumptions concerning infinity. Their model building was rather based on epistemic evaluations. In that sense, the ancients must have been aware of other possibilities than what the parallel postulate specifies as models of geometry. The invariants that they could build into the solutions of mathematical problems at the time however, included only the elements of epistemic constructions up to a horizon determined by what we know today as the intended Euclidean geometry. In that regard, Euclid and other mathematicians of the time were most likely aware of what Einstein among others observed concerning the characteristics of mathematical certainty and its reference in reality. So that when $A \supset B$ is a propositional codification of the so-called Pythagorean theorem, the inner complexities and possible evaluations of A, B and $A \supset B$ are determined by permissive principles concerning how to draw a triangle, how to find the area of a triangle, a square etc. All such possible evaluations admitted boundary conditions mainly bordered by the parallel postulate, in addition to other elements of geometry, independently of the rest of the framework. Hence it should be no surprise in that sense that the Pythagorean identity is provable only on the assumption that the parallel postulate holds, either as a truth of the *Elements*, or else as a proposition which is not false in its compatibility with reality.

The assumption in point is an actual presupposition concerning the possible links between thought and action as well as thought and reality. The compatibility of thought and reality in the context of Euclidean geometry thereof does not point to any correspondence between geometrical propositions and space. Rather the propositions are actual non-falsity assumptions in the same context concerning the possibility of building models for the presupposed compatibility of thought and reality.

In terms of the inner complexity of possible codifications of Euclidean propositions and implications, and hence their possible evaluations, Euclidean proofs and truths can be studied and has been studied as some truths of geometry and mathematical space. In that regard, their truth is internal to axiomatization, and hence epistemic due to being a production of asking questions about bounded models in an unbounded realm and finding their answers in the bounded models. Likewise, when the Euclidean model was treated meta-theoretically, the long-run problematic attitude towards the parallel postulate was replaced by further non-falsity assumptions, as is the case in Riemanian and other non-Euclidean model building.

When Hilbert systematized the interrelated aspects of all such assumptions in his study of the foundations of geometry, all the previous non-falsity models and model buildings and their interrelated aspects added up to the problem of proving the consistency of geometry axioms, including the varieties of different possible boundary conditions.

The case is very similar in the use of certain ideal elements, for example, in building models for projective planes and spaces with points at infinity, as well as building models for the complex planes and spaces.

In general, the developments in abstract mathematics as a whole can be studied as exemplars of further studies in non-falsity models as extensions of the previously internal truth models.