THE ROLE OF STRUCTURAL ANALOGY IN PHYSICAL SCIENCES : A PHILOSOPHICAL PERSPECTIVE

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1. Introduction

Broadly speaking the concepts, feelings or explanations of certain phenomena in Nature, if cannot be put to a scientific test through experimentation, remain subjective in character in spite of their logical foundation. On the other hand, this is also not untrue that every concept, feeling or explanation can be put to a scientific test. May be large varieties but only a countable number of phenomena in Nature undergo this kind of test and thereby constitute the so called "objective" sciences. With regard to the understanding of these natural phenomena the use of models and analogies in recent years has become unavoidable in view of the degree of success achieved in their scientific tests. Sometimes in diverse fields and by using alternative methods. In view of the fact that these models and analogies constitute a scientific theory and work well only within certain set of assumptions or postulates (may be small or large in number). it remains questionable as to what extent they can provide the absolute (ultimate) reality in Nature even in the asymptotic limit.

As a matter of fact there remains an un-understood element (hidden in the murky depth of ideas) of truth in the complete understanding of these phenomena mainly due to the limitations either on the scientific laws or on the capabilities of human faculties of understanding. In this regard, while the first category of limitations can be attributed to the acceptance by a (scientific) community and the second to an individual, they both have a common origin in the background. It is only in this spirit the present philosophical survey of the

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role of structural analogy in human understanding of physical sciences is carried out. In fact these two categories of limitations. in general, respectively suggest[1] the concept of "outer" and "inner" domains with reference to the human facilities of understanding and subsequently that of the "objective" and "subjective" components of the ultimate absolute reality.

In general, the analogy means some kind of mental construct which is useful in understanding the "unknown" in terms of the "known". Literally, the word "analogy" means similarity or parallelism. This similarity could be of just a concept, or of a sequence of concepts, or of a description through a model or of the phenomenon as a whole. Further, these analogies could be well within the same discipline or in different disciplines of knowledge. As far as the meaning of the structural analogy is concerned, to some extent, the same can be attributed to the realization of space, time and geometry-which basically are[2] the creations of human consciousness. Note that since the human consciousness is not only different for different human beings but also of varying degree (i.e. of relative nature) even for the same individual, these realizations of space, time and geometry could be different at different levels of human faculties of understanding.

Once the concepts of space, time and geometry are created with reference to the "outer" world and at the level of senses of knowledge and action or at the level of biological body, they are settled or stored into the inner essences of life (i.e., at the level of mind, intellect and ego). These concepts once stored, are used again to understand or to describe the outer world through the principles of action -at-a-time and/or action-at-a-distance. Once the information about or the description of a natural phenomenon goes in, in the first stage, it comes out again in the second stage through the analogies as the degree of consciousness by this time has been changed due to the change either of the being (human or otherwise) or of the realizing faculty of the same being. In going information may correspond to the "known" situation where as the one coming out will generally correspond to the "unknown" situation. The first being can be termed as the "user" and the second one as the "analyser" of the structural analogy.

The human being, while realizing the realities of the inner and outer worlds, plays a dual role in the sense that it also acts as a describer of these worlds. As a matter of fact for the purposes of understanding the role of structural analogy a philosophical understanding of the human being himself is essential. The Indian philosophy has thrown a lot of light on this aspect and its modern version[2] could be further helpful in this regard.

The concept of structural analogy is intricately interwoven in different branches of science and engineering. This concept is capable of providing not only new laws but also new disciplines and horizons of knowledge. With a view to exploring the role of this important concept in objective sciences in general and in physics in particular, in this article we first make a brief survey of some representative examples to this effect and then suggest a possible framework within which the merit of these examples of structural analogy can be analysed in a quite general manner.

In the next two Sections, we highlight some typical demonstrative examples of structural analogy from the fields of mathematics and mathematical and physical sciences. In Sect. 4. we look for the basic contents of a structural analogy which help not only in defining its merit but also suggests a way to classify the examples of structural analogies at a somewhat deeper level. Sect. 5 is devoted to analyses, within this framework, the process of abstraction (i.e. the step by step effort to putting the available knowledge in a nut-shell) which has been prevailing through all these years in both mathematics and physics. The change of contents taking place in the process of using a structural analogy, is also discussed in this Section. The extent to which a given structural analogy can be exploited while using the same for an "unknown" situation, in fact, depends upon the degree of development of faculties of understanding of the analyser. This fact is demonstrated by analysing a few examples in Sect. 6. In Sect. 7, various manifestations of the structural analogy in the form of symmetries and models are presented. Finally, the ideas are summarized and the question of ultimate (absolute) truth is discussed in Sect. 8.

2. Examples of Structural Analogy in Mathematics

While the mathematical structures and constructs have an inherent beauty in their own right in the abstract sense, their applications to various disciplines in mathematical sciences are mainly through the concept of structural analogy. Here we cite some examples from both pure and applied mathematics and make some possible remarks towards their applications in sciences.

2.1 Pure Mathematics

In the language of set theory one can define an abstract system by introducing[3] in addition to the elements of the set, the concepts of relations, operation, postulates, definitions, theorems etc.. Depending upon the nature of these latter concepts, the given abstract system is identified with other mathematical systems like field, vector space, group, topology, ring etc. These constructs are liberally used in applied mathematics and in mathematical sciences mainly through the structural analogies and finally undergo to their experimental tests and that too sometimes in completely disconnected disciplines.

While these ideas are applicable as such to a variety of situations in various disciplines of mathematics through the structural analogy, some of the constructs provide foundations to several branches of mathematical sciences. For example, the applications of the concepts of group in the form of Lie groups, vector space in the form of Hilbert and Banach spaces are well known in quantum mechanics and (physical) field theories. For that matter, various types of algebras and geometries also have their origin in these abstract systems. In fact, the knowledge of groups developed in an abstract form, when applied to a physical situation through the structural analogy gives rise to a variety of new atomic and molecular spectral lines, nuclear states, crystal symmetries, elementary particles and resonances, and unification of fundamental interactions which are easily verifiable in the experiments performed in these disconnected areas. Sometimes even the gaps appearing in the systematics in the properties and features, can be filled on the basis of group theoretical predictions, While we postpone the details to the next Section, it should be noted here that indeed all this happens through the concept of structural analogy.

2.2 Applied Mathematics

Some of the abstract constructs of pure mathematics mentioned above are further reduced and specialized to the level of their wide applicability through various branches of applied mathematics. In a nut-shell the methods of applied mathematics can be projected as follows: Note that the vectors of the vector space, defined in the spirit of their magnitude and direction in finite space dimensions, have given birth to vector algebra and vector calculus at somewhat lower level of abstraction (cf. Sect. 5.) Such vectors along with the rules for their product which define tensors, have applications in different branches of mathematical sciences mainly through the structural analogy. A special class of transformations (namely the linear ones) defined on some abstract systems (such as vector spaces or groups) give rise to integral transforms, integral and differential equations through the concept of linear operators. Further, out of the varieties of each of these integral transforms, integral and differential equations, only some are indentified so far for the purposes of their applications to different physical situation. However, in recent time the importance of nonlinear transformation or operators have also been noticed in this context. The manifestation of these transformations and mappings respectively in the from of matrices and functions or functionals has also suggested[4] the study of these topics in applied mathematics. Also, the study of extrema of these functionals and/or of function through the calculus of variations or other similar methods basically conform to physical requirement of natural principles like the principle of least action or the conservation laws. In these studies however many side concepts like those of differentiation, integration or of analysis are introduced through definitions.

The use of tools of whole of the applied mathematics in sciences is mainly through the concept of structural analogy, and many examples can be cited[21] to this effect. However, for the purposes of a philosophical development of this concept at a later stage only a few representative cases are mentioned here. In general, it should be noted that while mathematics provides the same rules of the game, the symbols and subsequently the quantities or the expressions obtained by applying these rules of the game, appear to be quite different in terms of their physical and philosophical contents (cf. Sect. 4). Further, in mathematics rules of the game are important and not the symbols; in science, however, both symbols and rules of the game provide deep insight into the physical content of the natural phenomenon. Moreover, the symbols and also the relations amongst them in science speak a lot about the phenomenon itself.

No doubt, plenty of examples can easily be traced[21] from amongst the vectors, multiple integrals, linear and nonlinear differential equations (both ordinary and partial), power and exponential functions transforms etc., here however we remind the reader of only the following a few cases:

(i) Consider the case of the differential equation

$$p(x) \frac{d^2y}{dx^2} + q(x)\frac{}{dx} + r(x)\, y = 0, \qquad \ldots\ldots (1)$$

which has provided[5] a variety of special functions and orthogonal polynomials with appropriate forms of the functions p(x), q(x) and r(x) based on the symmetry of the coordinate system used, mainly on the basis of structural analogy. For q(x)= 0, again a variety of situations exists[6] where the resultant form is used on the basis of analogy.

(ii) Next consider the differential equation with contant coefficients,

$$\vec{\nabla} \cdot \vec{f} + \frac{\partial g}{\partial t} = 0 \qquad \ldots\ldots (2)$$

This peculiar type of partial differential equation has been of special interest. In adition that this equation represents a relation between a vector function $\vec{f}$ and this scalar function g, it speaks a lot of conceptual insight of various phenomena appearing in altogether different branches of mathematical sciences mainly on the basis of structural analogies of different order, for example, in fluid mechanics $\vec{f}$ = mass density (p_m) x vel. vector ($\vec{v}$), $g \equiv p_m$; in electrodynamics $\vec{f}$ = current density ($\vec{J}$), g = charge density (p_c) ; in quantum mechanics $\vec{f} \equiv$ probability current density ($\vec{s}$), g = probability density (P); in electromagnetic field $\vec{f} \equiv$ vector potential ($\vec{A}$), g = scalar potential (ϕ) in appropriate units. In fact, in fluid mechanics, electromagnetics and in quantum mechanics while eq.(2) represents the equation of continuity, in the case of electromagnetic field it however displays the Lorentz condition - a condition necessary to decouple the underlying differential equation satisfied by both scalar and vector potentials. On the other hand, with p as a distribution function of the phase points and $\vec{v}$ as the velocity vector of the flow in the phase space, eq. (2) offers a basis for the Liouville's equation and thereby suggests[7] a conceptual foundation of statistical mechanics. Another important aspect of the use of equation of continuity is in the realm of car traffic[8]. For a one-dimensional model, in this case, if p = the density of cars (measured in number of cars/km), $v \equiv$ velocity of cars (in Km/hr) ; then eq. (2) is satisfied with $g \equiv p$ and $f \equiv Q(p)$, describing the number of cars going past a given point in an interval of time.

(iii) While some analogous applications of the partial differential equation,

$$c_1 \nabla^2 \phi = \frac{\partial \phi}{\partial t} \quad (3)$$

where c_1 is a constant, in different disciplines are listed in Table 1, similar applications of Schrodinger equation

$$[c_2 \nabla^2 + f(x)]\phi = i\frac{\partial \phi}{\partial t} \quad (4)$$

where c_2 is a constant, can also be found [9] in the literature.

(iv) Analogous applications of the exponential function

$$f(x) = f_0 \exp(-\alpha x), \quad (5)$$

in different branches of science are listed in Table 2.

(v) The study of nonlinear dynamical systems in terms of mathematical and geometrical tools, particularly through the topological methods or phase space trajectories, has opened [10,12] the vistas of another interesting class of structural analogy. very recently, efforts have also been there to make [11] use of these tools in understanding the different aspects of human behaviour.

3. Examples of Structural Analogy in Physics

From the point of view of highlighting the examples of structural analogy in physics and allied fields we broadly categorize our survey here in terms of (A) Mathematical disciplines (consisting of classical, quantum, statistical and stochastic mechanics and also classical and quantum field theories), (B) conceptual disciplines (consisting of condensed matter, molecular, atomic, nuclear, quark and elementary particle physics, and astrophysics, (C) conventional disciplines (like heat, sound, optics, electricity and magnetism), and (D) the technological applications i.e. engineering, despite their intermixing. Although some examples of category (A) are already listed above in Sect.2 from the viewpoint of mathematical content, however, a few more cases will be mentioned here. While this category suggests a working (mathematical/conceptual) framework for the categories (B), (C) and (D), the latter ones, to some extent, appear as the modelling games as far as the understanding of Nature is concerned. It may be mentioned that the category (B) deals with micro-physics and (C) with macro-physics. Further as one traces back from the category (D) to the category (A) the abstraction in terms either of mathematical formulae or of concepts increases. The category (D), clearly as an applied version of (A), (B) and (C), deals however with the down-to-earth problems of day-to-day life, particularly in making the life more comfortable.

As a matter of fact, while the category (A) from the viewpoint of the knowledge content gets a feedback from the study of the systems in categories (B) - (D), it also prepares a ground for the structural analogies of different orders which later help in understanding the physical systems appearing in categories (B) - (D). In particular, for a given conceptual setting, the category (A) suggests a variety of mathematical formulae which are readily applicable to the systems in (B), (C) and (D). This happens with minor variations at the input-conceptual-level in (A)and with seemingly different major variations at the output-levels of experimentation and their subsequent applications in the practical life in (B)-(D).

The history of physics reveals that there has been a common practice to jump to newer disciplines of knowledge without really understanding the older ones in totality. Further, this has happened not only with the categories (B) - (D) which are prone to more frequent experimental tests but also with the category (A). For example, without understanding the classical mechanics completely[13] for three hundred years and more (after Newton), quantum concepts were floated only about 100 years ago and subsequently quantum field theory grew up in the same vein. Same is also the case with the development of conceptual disciplines of category (B) and to some extent with the conventional ones in category (C). As a matter of fact the degree of understanding of ctegories (B) and (C) turns out further one order less as compared to that of (A) since by and large it forms the basis of study for (B) - (D). Fundamental and challanging questions remain even today almost in each of these disciplines. This is how the physical sciences have progressed. In fact, the conceptual disciplines are basically the immediate applictions of the structures which appear in the mathematical disciplines along with certain types of modelling games. In spite of all this, note that both the actual physical concepts (which normally are the outcome of modelling games) and the concepts derived from rigorous mathematical tools (which can be attributed to some kind of structural analogy of mathematical nature) have come true under experimental tests. It appears that these concepts could well be the preferences of nature over the other. Also, the inquisitive nature of a scientific mind for these preferencial concepts forces one to these rather quicker jumps in the understanding of Nature without actually exploring the remaining other concepts. The subject of nuclear physics is one such immediate example [23] of this general trend.

While a huge list of examples of structural analogy from among these categories can be drawn[21], we list here only a few representative cases:

(i) Structural analogy exists[15] in the equations derived in particle and rigid-body mechanics by treating respectively the mass (m) and the moment of inertia (I) in an analogous manner. Other analogies to be noted in this context are through the role of velocity (v) and angular-velocity (w) in the definitions of linear momentum (p) and angular momentum (L), linear and rotational kinetic energies etc., respectively in linear and rotational motions by the relations

$$p = mv \quad ; \quad L = I\omega ,$$

$$E = 1/2\, mv^2 \quad ; \quad E_{rot} = 1/2\, I\omega^2 .$$

(ii) The concepts of specific heat and several other quantities introduced once basically in the context of thermal phenomena occuring in gases, have now been extended to various other apparantly disconnected situations like in describing[14] the thermodynamics of rods, magnetics, dielectrics, radiation, water, and of plasma, mainly on the basis of structural analogy of various physical quantities. In the same way, the analogy also exists in the literature regarding the concept of entropy in several disconnected fields such as in the theories of chaos and of fluids.

(iii) As a matter of fact the Weber-Fechner law, used in psychology in the form, S=K In R, (where S and R respectively are the intensities of sensation and stimulus and K is a constant), appears analogous to the famous Boltzmann entropy law, S=K In W.

(iv) The concept of a classical field variable is specifically discussed in and also borrowed from the theory of continuous systems. In fact, in the limit when the separation 'a' between the mass points becomes infinitesimal and the number of mass points (and hence the degrees of freedom) becomes infinitely large, the variable n_i, describing the position of the i-th mass point at any time t, is replaced[15] by the functional ϕ (x), called the field - variable. This replacement of the integer index i by the continuous position variable x leads to the spacederivative terms (in addition to the time-derivative ones) in the Lagrangian of the system and subsequently to the field equations satisfied by ϕ (x) instead of by n_i.

In some situations the quantized version of a field theory is developed independently of the classical version (e.g., the nonabelian gauge field theories). In that case one looks for the classical analogue[16] of these quantum theories. On the other hand, the problems of double-well and triple-well potentials studied at the classical level[17] have provided clue for several interesting mechanisms and phenomena (like Higgs mechanism and spontaneous symmetric breaking) at the quantum level, mainly on the basis of structural analogy.

(v) The Feynman diagrams found useful in field theory and elementary particle physics to reveal the schematic details of the underlying fundamental processes, are basically the analogue versions of the electrical circuits. The importance of the role of this diagramatic technique has also been noticed not only in many-body problems like nuclear and condensed - matter physics but also in other areas like biosciences and human behaviour[18] mainly on the basis of structural analogy.

(vi) Analogy of somewhat lower order worth mentioning here is that of the equivalence of the terminologies[19] used in mechanical or acoustical systems and also in electrical circuits. Here, charge ≡ displacement, current ≡ velocity, inductance ≡ mass are frequently used in the mathematical structures.

(vii) Following Achinstein[20] some cases from conventional disciplines can be listed here:

(a) With reference to the interference and diffraction phenomena, the analogy drawn by Huygens between waves of light, sound and water is worth noting.

(b) In connection with the kinetic theory of gases there exists an analogy between a gas and a container of billiard balls in which the molecules in the gas are linkened to perfectly elastic billiard balls striking the walls of the container as well as each other.

(c) The analogy drawn by Maxwell between the electric field and an imaginary incompressible fluid flowing through tubes of variable cross-section.

(d) The analogy noted by Kelvin between electrostatic attraction and the conduction of heat.

(viii) Some cases of structural analogy from engineering are the following:

(a) As far as the application of a physical principle and its converse is concerned, the analogy exists between the two different mathematically idealized situations, namely the case of a refrigerator and the heat engine and that of an electric generator and the motor. In the former case it is the mutual conversion of heat and mechanical energies whereas in the latter it is that of the electrical and mechanical energies.

(b) Analogy with respect to the working principle and the construction between the human eye and the camera (if one ignores the role of human consciousness) exists.

(c) Analogy with resepect to the working principle and the construction of a flying bird and the aeroplane (if one ignores the role of consciousness of the bird) exists.

(d) Certain degree of analogy exists (may be at the level of working principle) in a variety of laboratory instruments used in different contexts, e.g. different types of galvanometers which again work on the same principle as that of an electric motor, different types of oscillatory systems work again on the same principle as that of a pendulum.

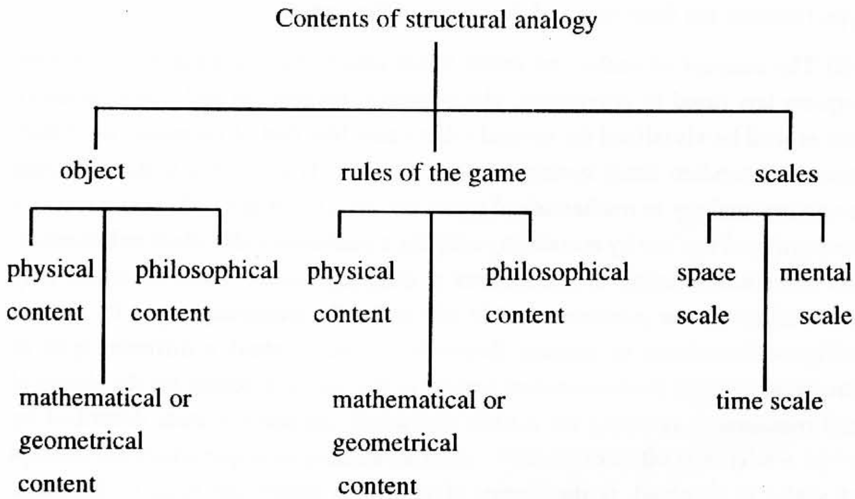
4. Contents and Classification of Structural Analogies

In this section we introduce in general and extract in particular, the essential contents of an analogy. This will not only allow the study of this 'important concept at a deeper level but also suggest a way to categorize these analogies in terms of their fineness. By and large most of the analogies presented in the previous Section can be understod in terms of only two contents, namely the "objects" and the "rules of the game" ; however, for a variety of situations one also needs the third content and that is the "scales". Let us analysee the process of using an analogy as such.

Whenever an analogy - a given "known" situation, is going to be used in another 'unknown" situation, the former is bound to consist not only of "objects" but also of their mutual interactions or correlations or the criterion for their compatibility. These correlations are termed as the "rules of the game". Again note that both "objects" and "rules of the game" can be charaterized by or qualified for according to their merit with respect to the human faculties of understanding. In fact, the "known" situation can exist any where, at any time, in any form and at any level in the spectrum of understanding of Nature with reference to these faculties. On the other hand, the "unknown" situation for which the "known" situation is going to be used, can also belong to these domains of understanding which, in fact, are very well covered by the realm of objective sciences but only in parts[21] and not in totality since the subjective components[22] of human faculties of understanding start entering at this stage.

From the point of view of "understanding", the "unknown" situation, no doubt, is expected to be more difficult for the human faculties than the "known" one. However this is not always the case mainly because the understanding of one is used into the other (or *vice versa*) only in a limited sense, often in the form of models or in terms[21] of suppositions. Also, the question of quality of these models demands further classification of these contents..

In view of the fact that both user and analyser of a structural analogy have to use their inner faculties[1,2] of understanding for the purpose and the quantity of understanding vis-a-vis the development of these faculties of the individual also come on the way, the contents of structural analogy can further be categorised. It is also not difficult to analyse the role of these sub-contents (namely physical, mathematical or geometrical and philosophical with reference to the objects and the rules of the game, whereas of space, time and mental with reference to the scales) in various examples listed in Sects.2 and 3. Schematically, these contents can be represented as follows:



To elaborate the feeling for these contents the following examples would suffice at this point.

(i) The knowledge of groups of pure mathematics is used to describe "unknown" situations in physics, chemistry and biology. Clearly, the group elements form the examples of "objects" and the basic axioms regarding the group elements (namely, associativity, closure, existences of inverse and identity) form the example of "rules of the game". In another case, in the use of Pythagoras theorem, the three numbers a,b,c form the example of "objects" and the relation, $a^2+b^2 =c^2$, forms the example of the "rule of the game". An appropriate identification (depending upon the "unknown" situation) of these objects in both the cases form the example of suppositions in using the analogy.

(ii) In the analogy of the atomic nucleus[23] with a liquid drop, while the drop as a whole could be the example of "objects", the set of some of its physical properties (like mass, volume and surface) can also form the example of "objects". On the other hand, the Coulombic (as the drop is charged) and other interactions among the constituents of the drop also form the example of "objects". With regard to the rules of the game, the first category of objects offers one type of "rules of the game" whereas the second category offers another type (perhaps the finer ones) of the "rules of the game".

(iii) The concept of scales, no doubt holds clearly for the example of inverse sequare law (used in gravitation, electrostatics, magnetism and quark physics), can as well be visualized for several other cases like that of harmonic oscillator[5] (time independent case) system $x^{oo}(t) + w^2_0\ x(t) = 0$. This system, besides involving analogy in mathematical terms (cf. eq. (1) for $q(x) = 0$, $p(x) = 1$, $r(x) =$ constant) and that too by accounting only for a mathematically idealized situation, has described a variety of phenomena at different scales. Here the scales may differ either for the position variable $x(t)$ or for the frequency w_0 or for both in different disciplines of science Recently, in this context a different type of structural analogy (to some extent similar to the one as it occurs for the electrical and mechanical systems) for a time dependent oscillator system described by $x^{oo}(t) + w^2(t)\,x(t) = 0$, is explored[24]. Another striking example where the concept of scales is involved, is the theory of collisions where the rules of the game remain more or less the same but only objects and scales keep on changing for mega, macro and micro systems alongwith some conceptual modifications. At least, for the scattering process of micro systems like molecules, atoms, nuclei and elementary particles, there exists[25] a common analogous machinery.

It may be noted that such a scheme of contents of a structural analogy also helps in the classification of analogies in terms of its fineness or the order. First stage classification could be way of the presence or the domination of either one, two or of all the three contents, out of objects, rules of the game and scales. At second stage, however finer classification arises from the study of subcontents. As it will be argued later, the search of these contents in an analogy will very much depend on the degree of development of faculties of understanding of an individual.

5. The Process of Abstraction and Mental Constructs

Another context in which the structrual analogy plays a dominant role in the physical sciences (or in general in objective sciences[21]) is the process of abstraction. In fact, in both mathematics and physics there have been consistent efforts, as the histories of these subjects reveal, to condense the available knowledge at a given point of time in a nut-shell if possible, by way of finding a generalization of both " objects" and "rules of the game" in this proccess, while new vistas and horizons of knowledge are very often opened, however the contents of the underlying structural analogies change more or less at every stage of abstraction. Also, the analogy becomes finer in terms of its content. The following examples from both mathematics and physics would elaborate these intricacies of the abstraction process.

5.1 *Mathematics*

The well known arithmetical rules of playing with numerals, no doubt, taught in abstract manner, are however frequently used in different contexts only through the examples of structural analogies. The algebraic relization of the numbers and the rules is a first step in the direction of abstraction. A variety of number games and numerical problems finally culminate or condense in to an algebraic formula. Though such formulae, in their particular form are ready for applications to various disciplines through structural analogy, they are normally subjected to further abstraction through possible generalizations.

Example : Consider the numbers and the underlying operations :

2×3 , $2^2 = (2 \times 2)$, $3^3 = (3 \times 3 \times 3)$, ... They are the special cases of xy , $(x + y)^2$ or $(x + y)^3$, where x and y are the appropriate numbers. The latter two are further special cases of $(x + y)^n$ for $n = 2 , 3$ and so is the first one of $x^m y^n$ for $m = n = 1$.

Other examples of algebraic realization of numerals could be in the form of equations, inequalities, sequences, series, and what not. Note that at the level of algebraic realization not only the objects become different but also again new rules of the game need to be framed. It is needless to emphasize that these algebraic structures are used frequently not only in science but also in other disciplines[21] (like economics, commerce, social science etc.) by virtue of structural analogies.

An alternative step in the direction of abstraction is in terms of set theory. As mentioned before, one can define an abstract system by introducing in addition to the elements of the set the eoncepts of relations, operations, postulates, theorems etc. Operations on two sets further bring in the concept of mappings and subsequently that of functions and functionals. While these ideas are applicable as such to a variety of situations in various disciplines through the sructural analogy, some special classes of the abstract systems in the form of fields, vector spaces, rings, groups etc. have very wide applicability in mathematical sciences, again through the structural analogy (cf. Sect.2.1).

5.2 Physics

Next we turn to the discussion of philosophical component of the physical concepts from the point of view of abstraction process.

Look at the Newton's laws and the corresponding equations of motion. While the discussion of the first and the third law can easily be extended to the philosophical domain of human behaviour through the structural analogy the second law can be considered as a special case of Euler-Lagrange's equation of motion. Note that this latter equation is basically the special case of a generalized equation derived by extremizing a necessary functional (in case of dynamics this functional is the action integral) in the calculus of variations. The use of the variational principle as the basis of formulation to express the " equations of motion" (whether they be Newtonian equations, Maxwell's equations, Einstein equations or the Schrodinger equation) in diverse fields offers the example of a much deeper analogy[21]. In fact, just an alteration of the physical content in the theory of one field gives rise to the results which are verifiable by all together different sets of experiments. Perhaps, the Nature likes such structural analogies.

While all this is possible through structural analogy in mathematical terms, the concept of extremising the functional (in dynamics) is again extended to the philosophical domain through the Hamilton's principle of least action - a fact well known and in-built in the human nature as well. Thus, all the three laws of motion have their origin in philosophy in one way or the other. Another important analogy is through the Noether's therorem where the objects become the field variables and the rules of the game remain more or less the same. The consideration and an account of space time symmetries lead, in a naive manner, to several fundamental laws in physics (like energy, momentum and angular momentum conservation laws) which are not only useful in classical mechanics but also in the domain of quantum physics and quantum field theories. Interestingly, an account of gravity by modifying the metric in the action integral and subsequently its extremization through the same structural analogy of mathematical nature leads to what is known as Einstein's equation in the theory of gravitation. This all happens through the structural analogies of different orders in mathematical, physical and philosophical terms and covers in fact all three domains of physics namely macro-micro- and mega-physics.

Another striking example of structural analogy which offers a back-bone for several branches of physics and starts from the mathematical level is the inverse square law of forces in the form $F = CQ_1 Q_2/r^2$, where C is a constant; Q_1, Q_2 are the measurable physical quantities characterizing the physical objects separated at a distance r units apart. To a fair degree of accuracy this law works in disconnected fields like in gravitation, electrostatics and magnetism where Q_1, Q_2 respectively are identified with masses, electric charges and magnetic pole strengths and the constant C takes appropriate values in these branches. This law found to work in quark physics at short distances and with some modifications in the field of photometry, however, again has mathematical origin. In fact, for the central field problems the closure requirement of the phase space trajectories demands the solution of the Euler-Lagrange equation only in this power law form. Experimental verification of this law in different contexts further adds to the Nature's liking for the structural analogy.

In the first case while both objects and the rules of the game keep on changing during the process of abstraction by maintaining certain common feature (may be at the philosophical level) in the later example of inverse sequare law however objects keep on changing by maintaining the same rules of the game. Another situation where objects remain more or less the same but the rules of the game keep on changing in different fields is the case of electrical circuits. Only on the basis of structural analogy Richard Feynman suggested the use of these circuit diagrams in field theory where they not only speak much more than the necessary from the point of view of exploring new ideas in the field of particle physics but have also brought out the inherent intricacies of Nature to an understandable form.

6. Use of a Structural Analogy vis-a-vis Human Understanding

6.1 Change of Contents

Whether it is the process of abstraction or the use of structural analogy, in both the contents change. This change can occur not only in all the three major contents (namely, objects, rules of the game and scales) but also in their finer components[23]. As a matter of fact the changes in the contents can be stated precisely only if the "unknown" situation is best (exactly) known either form the direct or indirect experimentation. The description of changes in contents is partly subjective in the sense that it has bearing on the development of essences of life of the individual analyser. To support these statements following examples will be sufficient:

(i) Consider the case of nuclear models, particularly the liquid drop model (cf. Ref. (23)). While the structural analogy used for the volume and surface terms in the mass formula is of somewhat lower order, it is of finer level for the Coulomb, asymmetry and pairing terms. Note that in the first three terms rules of the game (with reference to their mathematical and physical contents) remain essentially the same, however the objects in the corresponding terms change in additon to the space (geometry) content of the scales. The structural analogy in this case, could be good at the physical livel. This model although explains so many observed facts, however, seems to be far from reality.

With regard to the shell structure of micro-systems (atom, nucleus and nucleon) note that the objects (like quantum numbers characterizing the energy levels) more or less remain the same but the rules of the game (i.e.,-the nature of the potential) keep on changing. In addition to this, space- and time-contents of the scales for atom, nucleus and nucleon also change without affecting much the mental (conceptual) scale. As far as the physical, mathematical and philosophical contents in both objects and rules of the game are concerned they remain more or less the same in all the three systems.

With reference to the Fermi gas model[23] of the nucleus and its comparison with the electron gas, the rules of the game alongwith all the three contents remain the same. However, both the objects and the scales with some of their contents change.

(ii) Look at the example of analogy in the definitions of linear and angular monents (or for that matter of kinetic energy expressions for linear and rotational motions) in eq.(6). From the first look objects (in terms of mathematical symbols) are different, rules of the game (i.e. arithmetical operation of multiplication) are the same. In the second look, while the mathematical content of the objects changes and it remains the same for the rules of the game, the physical content (in the sense that p relates to linear motion and L to rotational motion) of both apparantly changes. With regard to the philosophical content note that one can open the Pandora's box. In fact, p can be visualized for a point particle of mass m moving with velocity v and L for a rigid body having the moment of inertia I about a direction in accordance with that of the angualr velocity vector w. For further finer detains I turns out to be the principal value of a second rank tensor - a mathematically complicated object. On the other hand, during the motion, at a particular instant and in the infinitesimal limit of the arc length, the angular motion can be considered as the linear motion for point particles of which the rigid body is constituted.

Thus, in the above analyses the structural analogy appears only at a few levels of understanding i.e. at the level of the rules of the game in the first look and at the level or philosophical content in the second look. Accordingly, the merit and the category of the analogy could have different meanings for the school-level, college-level and research-level users/analysers. Moreover, this was the example when "known" and "unknown" situations are the best known to the author.

6.2 Structural Invariance

Before extending the concept of structural analogy in the philosophical domain, the invariance of structutres with respect to scales needs a discussion. No doubt, several physical phenomena in nature are understood in terms of analogies of different order but sometimes the structures appear to be the same with respect to a certain content inspite of their different underlying meanings. For example, look at the following formulae used in physics

$$\text{(a)}\ \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}\ ; \quad \text{(b)}\ \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \quad \text{(c)}\ \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \qquad ...7$$

or

(a') $\bar{R} = R_1 + R_2$; (b') $\bar{M} = m_1 + m_2$; (c') $C = C_1 + C_2$, ...8

In terms of the rules of the game and with reference to the mathematical content in mind all the three cases in (7) and (8) are separately identical. However, form the point of view of the physical content the 3 cases of (7) differ not only mutually but also differ from (8). Also in (a) and (c) (or in (a') and (c')) R_i and C_i (for i = 1,2) are the resistances and capacitances belonging to the field of electricity whereas in b (or in (b')) m_i are the masses of two particles in the area of mechanics. Thus, in some sense there exists a structural invariance with respect to the mathematical content and the use of the analogy in different disciplines. However, it is not so at the level of physical content or in terms of mental scales as the objects carry different meanings. Further note that for the same definition of symbols on the right hand side of (7), formulae in (8) also make sense.

Now consider another class of examples in this category from optics i.e.

(a) $\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$; (b) $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$,.... (9)

for which the rules of the game (especially the mathematical content) are the same as for the type (7) above but there is no analogous counter part of the type (8) for the same set of objects mainly from the point of view of physical content of the objects.

One more class of examples in this category is from mathematics. In algebra, if a, b, c are in harmonic progression, then (1/a), (1/b), (1/c) will be in arithmatic progression and the corresponding arithmatical mean be expressed

(a) $\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$ .. (10)

For this structure again there does not exist the counter part $b = a + c$ for the same definition of symbols (or for the same physical content of the objects). On the other hand, at the mental scale, the philosophical content of the rules of the game in all the cases, namely the cases (a), (b), (c) of (7), (a) and (b) of (9) and (a) of (10) is the same i.e., they all represent the addition of inverses of two quantities. This latter realization represents the case of a structural invariance at a higher level and leads[21] to the construction of a content-matrix to make these changes more transparent.

7. Manifestations of Structural Analogy: Symmetries and Models

In general the models and/or symmetries, as used all over in objective sciences and helped a lot in understanding Nature, basically are the incarnations of the structural analogy. As a matter of fact after understanding the subtlities of structural analogy in terms of its contents, it is clear that only a fraction of structral analogy manifests through these models, symmetries or through both. Although now and then a mention of models is made in the above survey but their birth alongwith that of symmetries can be understood as follows:

With regard to the role of human being in understanding Nature, two (complementary in reality but alternatives in practice) approaches in Indian philosophy are well known namely, "One in all" ("Vyaṣṭi" in "Samaṣṭi") or "all in one" ("Samaṣṭi" in "Vyaṣṭi") i.e. by considering either the human being as placed in the universe or else the universe as exists in the human being. It is the second approach which scientists (particularly, the objective scientists) have been following all along. Perhaps most of them are not properly trained to think even of the first approach in which the philosophical/subjective component of "understanding" Nature has to dominate the scene. And it is within the framework of this approach Einstein's words[27] . "What does the fish know about water? After all, it spends all its life in there" will come true. It is also true that none of these approaches is superior to other, rather complementary to each other. For a complete understanding of Nature an amalgamation of the two in the human being is essential. Perhaps for this reason only all great scientists in the last part of their active life start realizing the individual's limitations as far as the understanding of Nature in totality is concerned.

For the objective scientists, in the second approach, Nature appears highly complex in spite of the fact that it works on the basis of some well defined (perhaps only for Nature itself) principles. These principles are so subtle and real in the absolute sense that they always remain asymptotic as well as beyond the domain of any kind of objective experimentation- a basis for the objective science. The objective scientists, on the other hand, try to understand the "infinite" with their finite potential, capabilities and the development of faculties of understanding with reference to the essences of life. Alternatively, they want to project the "infinite" onto their "finite" and that too with "finite" tools. One but important of these "finite" tools is that of the structural analogy. Logically speaking, since the projection of an object cannot be greater than the object itself (at the most it can be equal in the limiting case), the understanding of Nature by the objective scientists remains limited or the absolute truth for them remains asymptotic, implying thereby the birth of models, symmetries or in general that of structural analogies. As a matter of fact, in such a situation, no alternative is left for the objective scientists except for using models, symmetries, analogies and what not, to understand nature even in parts.

Here we first comment on the mutual relationship between models and symmetries and then discuss their place in the spectrum of analogies.

Apparently visible irregularities during the study of a natural phenomenon in totality, in spite of having their roots in some much deeper and subtle, but precise and systematic principles (which perhaps are not within the reach of objective methods of present sciences), however turn again into regularities when the phenomenon is looked into in parts by objective methods. The symmetries in physical sciences basically are the manifestations of these partial regulatrities (or the order) in Nature. On the other hand, the models are the tools to investigate these underlying partial systematics/symmetries in Nature. While the search for these symmetries gives birth to models by way of framing postulates/assumptions for this purpose, both (models and symmetries) have their origin in structural analogy. As a result, both models and symmetries can also be analysed in terms of the contents of Sect.4.

It may be mentioned that the symmetries (or some sort of orderliness) in the objects and the scales are no doubt often transparent, their search in the rules of the game however has not always been easy; once if it is found, then it provides a better way of understanding the corresponding phenomenon. As a matter of fact by using symmetry one needs to compute every thing or else one computes only a few and derives the conclusions about whole (every thing).

The symmetries and also the tools of their study (namely the models), can further be analysed at the level of sub contents of a structural analogy. In summary, the availability even of these partial symmetries at any level of study of a phenomenon provides a glimpse of the underlying beauty in Nature and thereby encourages the scientists of different specialization to search them further. This is how the objective sciences progress.

8. Summary and the Question of Ultimate Truth

With a view to exploring the role of the fundamental concept of structural analogy in the studies of objective sciences in general, and that of physical and mathematical sciences in particular, a survey of modern advancements in mathematics and physics is carried out at a philosophical level. From the cases of analogous structures cited in Sects.2 and 3, it appears that this concept is distributed all over in these disciplines and only a few glimpses of them are presented in this article. While only a brief account of examples from mathematics is given, a variety of examples from physics are cited under four categories, nemely, mathematical. conceptual, conventional and engineering. Further, a philosophical basis is saught to understand these varieties of examples of structural analogy in terms of a few basic contents and subcontents. Also, a way to classify these analogies within this framework is pointed out. It is noticed that in general the contents of a structural analogy change in both the situations, namely in its use and aslo in the process of abstraction. This latter process (also known as generalization in literature) however has been prevailing in both mathematics and physics through all these years as the histories of these disciplines reveal. In spite of the changes, some features of these contents remain unaffected during these processes and this leads to what is termed as "structural invariance". It is argued as well as emphasized that the concepts of symmetries and models (also intricately interwoven in the objective sciences) not only are the manifestations of the structural analogy but also often appear as special cases within this general framework of study.

The concept of contents of a structural analogy, no doubt, is derived in this survey from the examples in mathematics and physics, but it is quite general in the sense that the same can be used to study the role of structural analogy in other hard disciplines like chemical and biosciences and also in the soft disciplines like literabure, humanities, social and economic sciences. As a matter of fact the concept of structural analogy in literature already exists and manifests through the use of figures of speech (particularly through the use of simile and metaphor), examples (used to elaborate the deeper philosophical meanings pertaining to life and human behaviour), quotations etc. On the other hand, the structural analogy in the forms of models also dominates[2)] the modern studies of both social and economic sciences. It may be mentioned that in the case of literature while mathematical (or geometrical) contents of objects and rules of the game are often absent, physical content of both of them corresponds to the practical aspect of life and human behaviour unlike their philosophical content. Further, here and also in social and economics scienies, the space and time contens of scales should be considered, respectively in the contexts of geographical and historical situations prevailing in the phenomenon under study. Some of these studies are under way and will appear elsewhere[21].

Achinstein[20] while discussing analogies in the concepts of science, classifies models into four categories, namely (i) Analogue models, (ii) representation models, (iii) theoretical models, and (iv) imaginary models. To some extent, this classification by Achinstein is phenomenon- based in the sense that the merit of a model cannot be assessed at an absolute scale. On the other hand, it is also not difficult to rank these models and their finer points of distinction among themselves, in the present rather broad (and perhaps deep too) framework of contents and subcontents of a structural analogy. Such details however we postpone for future studies[21].

The order of an analogy in the sense of its fineness can be understood better in terms of the patomic model[2] of the buman being. As mentioned in Sect. 1, it is the human consciousness which creates the space, time and geometry and makes the physical world realizable to the human being, normally through the analogies of different orders. For example, for an event taking place in the domain[1,2)] of wordly objects WO as a result of its space-time creation (and in the presence of consciouness) and subsequently its realization by the faculty[10] of memory E (of course through the roles of the biological body B, the sense of knowledge and action SE, the mind M and the intellect I), the interpretations for the same (in the form of an anology) can be advanced again from different levels (i.e. at the levels of SE, M, I and E) of the human faculties of understanding, depending on their development in the analyser.

In view of the fact that objective scientists have been following the "all in one" approach (cf. Sect. 7) with regard to understanding Nature and "one in all" component is either missing or followed to its minimal in the wholistic approach, the role of structural analogies become important not only in the studies of objective sciences but also that of soft disciplines in such situation. When one looks at the scientific advancements which have taken place in the modern times and their utility to the human race, one cannot believe that this all is the role of structural analogy in general, or of models in particular, besides a few fundamental principles distributed all over in different branches of objective sciences. On the other hand, similar is the case with the varieties of human (or of other beings) experiences and behaviour in the understanding of which the structural analogies again play[21] the important role. Indeed, the former hints towards the objective reality in Nature and the latter gives a clue towards the subjective reality through the same game of analogies. These two types of realities basically are the two sides of the same coin, i.e., of the absolute reality or the ultimate truth in Nature. To some extent, the analogies for the human being do not appear more than the means of communication with Nature. As far as the search of ultimate truth is concerned it remains the matter of realization only and perhaps much beyond the language of analogies and that too for a "man of perfection" only.

NOTES

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1. Radhey Shyam Kaushal, "Modern Physics in the Philosophical Domain : The Role of Structrual Analogy " *Presented* in National Level Seminar on Philosophy of Science, Jaipur, March 5-7, 1998: With reference to the inner world of human being the faculties of understanding are categorized besides the worldly objects (WO) as biological body (B), sense organs responsible for knowledge and human action (SE), mind (M), intellect (I), ego (E) and above them all is the soul (SO). in accordance with their fineness and subtle character. With these essences of life as ingredients a philosophical atomtype model (called in brief the patomic model) of the human being is proposed in Ref. (2) below. In this framework the being (b) while "revolves" around the nucleus WO. also is free to perform transitions both upward and downward (depending on the value of the meditation parameter μ) in the frequency states B, SE, M, I, E and SO. This kind of modelling of the human being which has strong bearing on the Vedantic philosophy, explains a variety of human actions and behaviour at its most fundamental and absolute level by way of describing the interactions among the human beings [cf. Ref.(18)]. For further details of the patomic model, see Ref. (2) below.

2. Radhey Shyam Kaushal. "*The Philosophy of the Vedanta : A Modern Scientific Perspective*". Sri Garib Das Oriental Series No. 179. Shri Satguru Publication. Indian Book Centre. Delhi - 7, 1994: Also see, Journ. SC & Ind. Res. (New Delhi) 49 (1990) 78.

3. See, for example, D.T. Finkbeiner II, " *Introduction to Matrices and Linear transformations*" (D. B. Taraporewala Sons & Co. Pvt. Ltd., Bombay, 1968).

4. Here the concept of mapping should be understood for sets and that of linear transformations for other abstract systems.

5. R.S. Kaushal, " *Classical and Quantum Mechanics of Noncentral Potentials: A Survey of two Dimensional system*" (*Jointly Published by* Narosa Publishing House, New Delhi and Springer-Verblag, Heidelberg 1998.)

6. See. for example, L.A. Pipes and L.R. Harvill, "*Applied Mathematics for Engineers and Physicists*" (McGraw - Hill Book Co., Int. Student ed., 1970) p.783.

7. L.D. Landau and E.M.Lipshitz. " *Statistical Physics*" (Pergamon Press, 1980).

8. A.J. Roberts, " *A One Dimensional Introduction to Continuum Mechanics*"

(World Scientific Pub. Co. Ltd., Singapore. 1994) p. 16

9. A. Ghatak and K. Thiagarajan in *"Progress in Optics"* ed. by E. Wolf (North Holland, Amsterdam, 1980) : M. Hashimoto, Opt. Commun. 32 (1980) 383.

10. See, for example, Ref. (9) and D.W. Jordan and P. Smith, " *Nonlinear Ordinary Differential Equations"* (Clarendon Press. Oxford, 1988).

11. See. for example. " *Nonlinear Dynamics in Human Behaviour"* (Studies of Nonlinear Phenomena in Life Sciences-Vol.5), ed. by W. Sulis and A. Combs. (World Scientific. Singapore, 1996).

12. N. Minorsky, " *Nonlinear Oscillations* " (D. Van Nostrand Co. INC., 1962, Indian ed.).

13. Particularly, the systems admitting nonlinear equations of motion or those involving either an harmonicity or noncentrality or both are not yet understood completely. Moreover, the systems involving explicit dependence on time alongwith these features also require special investigation even at present times. For further details see Ref. (5) above.

14. See, for example, Y.B. Rumer and M.S. Ryvkin. *" Thermodynamics, Statistical Physics and Kinetics"* (Mir Publishers. Moscow 1980) Chap. 1

15. H. Goldstein, " *Classical Mechanics* " (Addison-Wesley Pub, Co., 1981) 2nd ed., Chap. 12.

16. See, for example, G.K. Savvidy, Nucl. Phys. *B246* (1984) 302: J. Villarroel, J. Math. Phys. *29* (1988) 2132. and the Appendix I in Ref. (5) above.

17. S. Coleman, *"Aspects of Symmetry",* Selected Erice Lectures (Cambridge Univ. Press, 1988) p.234: R. Rajaraman, " *Solitons and Instantons"* (North Holland Pub. Co.. 1982) Chap. 2 and 5.

18. R.S. Kaushal, "Human Communication and Cognition : A Scientific Outlook in Vedanta Philosophy", *Int. Jour. Common.* (Delhi) 5 (1995) p. 111-124.

19. H.F. Olson, *" Solution of Engineering Problems by Dynamical Analogues"* (Von Nostrand. New York. 1966).

20. Peter Achinstein, " *Concepts of Science : A Philosophical Analysis"* (The John

Hopkins Press. Baltimore, Maryland, 1968).

21. Radhey Shyam Kaushal, " *Structural Analogy : A Useful Concept in Science and Philosophy*" (in preparation).

22. Roger Penrose, " *Shadows of the Mind: A Search for Missing Science of Consciousness*" (Oxford Univ. Press, 1994).

23. R.S. Kaushal, "Understanding the Atomic Nucleus through Structural Analogies: A Philosophical Survey " *Presented* in the National Seminar on Nuclear Physics and Engineering held at I.I.T. Kanpur (India), April 17-18, 1997; also in Physics News (Bombay) *29* (1998) *127-135, Sept.-Dec.* Issue.

24. R.S. Kaushal and D. Parashar. Journ. Phys. *A:* Math. & Gen. 29 (1996) 889.

25. M.L. Goldberger and K.M. Watson, " *Collision Theory*" (Johj Wiley & Sons, Inc,. 1964).

26. Newton's first law of motion, in fact. defines the law of inertia. It states that if a body is at rest or in motion it will remain in the same situation unless an external force acts on it. Same is true as far as the human nature is concerned. Human activities in terms of its routine work remain unaffected unless some external guidance is brought in the routine. It may be noted that while "physical" body can work under only contact or action-at-a-distance type forces (no question of memory), the "human or living" body can work under all three forces namely contact, action-at-a-distance and action-at-a-time type forces (for details see Chap. 6 of Ref. (2)). Newton's third law states that action and reaction on a body in the same inertial frame are equal and opposite. This is also true for human nature. There is a reaction for every action on a human or living being provided the inner essences of life of the being are in a matching tone with the external (living or nonliving) agents. As far as the equality in Newton's third law is concerned it is again a mathematically idealized situation in the sense that the variations in the "physical conditions" during the acting or reacting times are being neglected.

27. Albert Einstein, *" The World as I see It "* (1935).

Table 1 : Some analogous applications of eq. (3)

S. No.	Constant c_1	Dependent Variable (ϕ)	Remarks about eq. (3)
1	(k/cp_m)	temperature (T)	Heat flow equation describing the distribution of temperature in solids (c ≡ specific heat: p_m ≡ mass density : K ≡ thermal conductivity).
2	(1/μό)	current-density vector $\vec{J}$	Skin-effect equation used in electromagnetic theory and electrical engineering (μ ≡ magnetic inductive capacity: equation also holds for the electric and magnetic field vectors E and R. respectively.
3	K	concentration (U) (in gm/cm^3)	Used in physical chemistry (K = diffusivity constant measured in cm^2 /sec.)
4	c_v	ecxess hydrostatic pressure (U)	Used in the theory of consolidation of soil. (C_v ≡ coefficient of consolidation)
5	(1/RC)	potential (e) and current (i) along an electrical cable	Equation governing the propagation of potential (e) and current (i) along an electric cable. (R ≡ resistance per unit length: C ≡ capacitance per unit length).

Table 1 (contd) : Some analogous applications of eq. (3)

S. No.	Constant C_1	Dependent Variable (ϕ)	Remarks about eq. (3)
6	D	relative concentration c	Study of diffusion in gases. (D ≡ diffusion coefficient : C ≡ no. of gas molecules in unit Vol./no. of gas molecules in total Vol.).
7	C_1	?	Stochastic quantum mechanics. ($c_1 = h^2/2m$: h = Planck constant /2π. m ≡ particle mass)
8	μ	p	Describing an aggregation of slime mold Amoebae

Table 2 : Some analogous applications of the function (5)

S. No.	independent variable x	constant α	function f (x)	remarks about the function (5)
1	time t	decay constant λ	No. of atoms present at any time t:N (t)	Describes the law of radioactive disintegration : ($f_0 \equiv N_0$. the no. of atoms present at t = o).
2	time t	inverse of the capacitative time const. Y_c (= RC)$^{-1}$	charge on the condensor at any time t: Q(t)	Describes the discharge of a condensor in the RC -circuit. ($f_0 \equiv Q_0$. the charge at t = o).
3	time t	some constant	.temprerature at any time t:Q (t)	Describes the Newton's law of cooling with surro-unding temperature (θ) as zero. ($f_0 \equiv \theta_0$ some constant temprature at t = 0).
4	distance x	absorption coefficient (y)	intensity at the point x	Describes the law of absorption of sound or light ($f_0 \equiv 1_0$. the intensity at the point x = 0)
5	height z	inverse of the characteristic lenght h of the Boltzmann distribution in gravitation field (=kT/mg)	density at height z : *p* (z)	Barometric height formula $p(z)=p_0$ exp (z/h). which describes the variation of density with height in gravitational field :($f_0 \equiv p_0$ the density at z = o).
6	energy E(x,y,z,)	α ≡(κT) –^{1} k≡Boltzmann constant and T≡abs. temperature	probability per unit volume: n(x.y,z,)	Describes the Boltzmann distribution in classical statistics: ($f_0 \equiv n_0$, the normalization constant)
7	inverse of temperature (=T^{-1})	(E/R) with E=activation energy R≡gas cons.	rate constant (k)	Arrhenius eqn,. describing the behaviour of reaction rate as a function of temperature.