Theory and Decision (2007) 63:41–51
DOI 10.1007/s11238-007-9030-x

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A RECURSIVE CORE FOR PARTITION FUNCTION FORM GAMES

ABSTRACT. We present a well-defined generalisation of the core to coalitional games with externalities, where the value of a deviation is given by an endogenous response, the solution (if nonempty: the core) of the residual game.

KEY WORDS: core, externalities, optimism, partition function, pessimism

1. INTRODUCTION

Externalities play a crucial role in many of the economic problems studied today, be those the formation of trade blocks (Yi, 1996), the use of public resources (Funaki and Yamato, 1999) or international environmental agreements (Eyckmans and Tulkens, 2003). Yet, most solutions in cooperative game theory do not directly generalise to games with externalities, such as to games in the partition function form (Thrall and Lucas, 1963), the form we use here.

The (coalition structure) core collects undominated imputations (outcomes): in TU-games the payoff of a deviating coalition is given by the characteristic function. Here: it depends on the reaction of the *residual* players. We allow an arbitrary, endogenously determined reaction and only assume that residual players play consistently. This model generalises the α-core (Aumann and Peleg, 1960), the optimistic approach by Shenoy (1979), the γ-core (Chander and Tulkens, 1997) or the status quo or the δ-approach (see also Hart and Kurz, 1983), each of which can arise in particular games. The r-core (Huang and Sjöström, 2003) also allows arbitrary reactions, but works via a characteristic function that is often undefined.

In the following, after the basic definitions we present our new concept. Then we discuss some properties and relate the recursive core to other models.

2. PRELIMINARIES

Let N be a set of players; its subsets are coalitions. The set of partitions of S is denoted $\Pi(S)$ with $\mathcal{P}_S$ being a typical element. A characteristic function form (CFF or TU) game is a pair (N, v), where $v: 2^N \longrightarrow \mathbb{R}$ is a characteristic function (CF). An outcome is a pair $(x, \mathcal{P})$, $x \in \mathbb{R}^N$ and $\mathcal{P} \in \Pi(N)$, such that (i) $x_i \geq v(\{i\})$ for all $i \in N$ and (ii) $x(S) = v(S)$ for all $S \in \mathcal{P}$, where $x(S) = \sum_{i \in S} x_i$. The coalition structure core (here simply: core) collects outcomes $(x, \mathcal{P})$ with $x(S) \geq v(S)$ for all $S \subseteq N$. Otherwise S deviates, to form a coalition and each of its members can benefit. We generalise deviation and the core to partition function form games.

A partition function (PF) $V: \Pi \longrightarrow (2^N \to \mathbb{R})$ assigns a CF to each partition. In effect, the payoff of a coalition may be different in each partition as it is not even given by the same CF. A partition function form (PFF) game is a pair (N, V). Here an outcome $(x, \mathcal{P})$ satisfies (i') $x_i \geq 0$ for all $i \in N$ and (ii) $x(S) = v(S)$ for all $S \in \mathcal{P}$. Condition (i') is a crude generalisation of (i), where we assume $\min_{\mathcal{P} \ni \{i\}} V(\{i\}, \mathcal{P}) \geq 0$. The two notions of outcome in the two settings should not lead to confusion. Let $\Omega(N, V)$ denote the set of outcomes in (N, V).

3. CONCEPTS

In CFF games, the value of a deviation does not depend on the reaction of the remaining residual players. As there are no externalities they may not react at all. In PFF games each residual partition might give a different value to the deviators, moreover the deviation may change the payoff of coalitions that are otherwise left intact. After a deviation we may expect widespread reshuffling of residual players. We make the following observation: *Given a deviation, the residual players face the problem of solving another, smaller PFF game.* We call this a *residual game*.

DEFINITION 1 *(Residual game)*. Let (N, V) be a game and consider $R \subsetneq N$. Assume $N\backslash R$ have committed to form partition $\mathcal{P}_{N\backslash R}$. Then the residual game $(R, V_{\mathcal{P}_{N\backslash R}})$ is the PFF game over the player set R and with the partition function given by $V_{\mathcal{P}_{N\backslash R}}(C, \mathcal{P}_R) = V(C, \mathcal{P}_R \cup \mathcal{P}_{N\backslash R})$.

The residual game is only *conditional* on $\mathcal{P}_{N\backslash R}$, it is a PFF game on its own. So *if* the core is the solution for (N, V), the core solves $(R, V_{\mathcal{P}_{N\backslash R}})$, too. Deviating coalitions must expect a residual core outcome to form. Should the core be empty this solution does not present a selection of the outcomes, and all possible responses must be considered. Even if the residual core is non-empty it may contain outcomes with different partitions. This gives rise the the following, double definition.

DEFINITION 2 *(Optimistic (pessimistic) recursive core)*.
Let (N, V) be a game.

(1) *Trivial game.* The *core* of $(\{1\}, V)$ is the only outcome with the trivial partition: $C(\{1\}, V) = \{(V(1, (1)), (1))\}$.
(2) *Inductive assumption.* Assume that the core $C(R, V)$ has been defined for all games with at most $k-1$ players. The *assumption about game* (R, V) is

$$A(R, V) = \begin{cases} C(R, V) & \text{if } C(R, V) \neq \varnothing \\ \Omega(R, V) & \text{otherwise.} \end{cases}$$

(3) *Dominance.* The outcome $(x, \mathcal{P})$ is *dominated via the coalition S forming partition* $\mathcal{P}_S$ if for **at least one (all)** $(y_{N\backslash S}, \mathcal{P}_{N\backslash S}) \in A(N \setminus S, V_{\mathcal{P}_S})$ there exists an outcome $((y_S, y_{N\backslash S}), \mathcal{P}_S \cup \mathcal{P}_{N\backslash S}) \in \Omega(N, V)$ such that $y_S > x_S$.
The outcome $(x, \mathcal{P})$ is *dominated* if it is dominated via a coalition.
(4) *Core.* The *core*, denoted $C(N, V)$, is the set of undominated outcomes.

The optimistic (pessimistic) core is denoted $C_{+}(N, V)$ (respectively $C_{-}(N, V)$).

The recursive cores are well-defined, though may be empty.

3.1. *Alternative models*

We begin with a survey of concepts in the literature. The first group reduces the PF to a CF by fixing the residual reaction. When deviators expect to be helped by the residuals (Shenoy, 1979) outcomes that are stable against such extremely optimistic deviations (that is, belong to what we call the ω-core) are clearly very stable. The α-characteristic function and core (Aumann and Peleg, 1960) seems to originate from the maximin-minimax rule (von Neumann and Morgenstern, 1944): residuals hurt deviators as much as possible. The following proposition requires no proof:

PROPOSITION 3. *The α-core contains the ω-core: $C_{\omega}(N, V) \subseteq C_{\alpha}(N, V)$.*

"Why should we expect that residual players act in such a bloodthirsty fashion as to hurt deviators to the maximum extent?" (Ray and Vohra, 1997) Punishments should be *reasonable* (Rosenthal, 1971), not punishing residuals, but only deviators. Already such mild restrictions have a substantial effect on the core (Richter, 1974). Residual players should act to maximise their *own* payoffs. The γ-approach (Chander and Tulkens, 1997) presumes residuals choose an *individually reasonable*, Nash response: in a PFF game this means breaking up to singletons. Note that the result is always an outcome, but it might be inferior to the status quo.

The r-theory (Huang and Sjöström, 2003) converts a normal form game (over a strategy space Δ and utility function u: $\Delta \to \mathbb{R}^N$) to a CF using the same concept for the response as for the original game. They define the worth $W(S|T, \mathcal{P}_{N\setminus T})$ of coalition S given that players in $N\setminus T$ have formed $\mathcal{P}_{N\setminus T}$, which will then help to determine the set of strategies $C(S|S, \mathcal{P}_{N\setminus S})$

that can possibly be played after the departure of $\mathcal{P}_{N\setminus T}$, using the solution, for example the core:

- Clearly $C(\{i\}\,|\,\{i\},\mathcal{P}_{N\setminus\{i\}}) \equiv E(\{i\},\mathcal{P}_{N\setminus\{i\}})$, the set of Nash equilibria.
- Assuming that $C(S|S,\mathcal{P}_{N\setminus S})$ is defined for all $|S|\leq s-1$ for $|S|=s$

$$W(T|S,\mathcal{P}_{N\setminus S})\equiv\min\left[\sum_{i\in T}u_i(\sigma),\sigma\in\begin{cases}E(S,\mathcal{P}_{N\setminus S}) & \text{if } T=S\\ C(S\setminus T|S\setminus T,T,\mathcal{P}_{N\setminus S}) & \text{if } T\subset S\end{cases}\right].$$

- $C(S|S,\mathcal{P}_{N\setminus S})$ is a set of strategies σ such that there exists an outcome $(x,\mathcal{P}_S)$ in the CFF game $(S,W(\cdot|S,\mathcal{P}_{N\setminus S}))$ such that $\exists\sigma\in E(\mathcal{P}_S,\mathcal{P}_{N\setminus S})$ we have $\sum_{i\in T}x_i=\sum_{i\in T}u_i(\sigma)\ \forall T\in\mathcal{P}_S$.

Slightly obscured by the complex notation used in normal form games, Huang and Sjöström (2003) have also used consistency, although in their model residual games are similar to, but not identical with an original game[1] and therefore the consistency argument is less natural. The r-core is only defined if *all* residual cores are non-empty – a demanding condition for large games, not even satisfied by all CFF games and hence the r-core is *not* a generalisation of the core.

Farsightedness (Chwe, 1994; Xue, 1998; Ray and Vohra, 1997) is often linked to our concept, but the similarity is superficial. Although deviators expect residuals to (re)shuffle themselves, their payoff would be undefined without this residual partition. No further deviations are considered and the one-step deviation must be an improvement. This is in contrast with farsighted models, where further deviations are considered and it is (only) the ultimate payoff that matters.

4. PROPERTIES

4.1. *Optimism versus pessimism*

It is somewhat unusual to have a *pair* of concepts. The two versions originate from the α- and ω-approaches, but we show that the recursive pair is less sensitive to behavioural assumptions.

PROPOSITION 4. $C_\omega(N, V) \subseteq C_\alpha(N, V)$.

Proof. Given outcome $(x, \mathcal{P})$ a deviation can only be profitable under the α-setting if it is under the ω-setting. Thus, $(x, \mathcal{P}) \in C_\omega(N, V)$ implies $(x, \mathcal{P}) \in C_\alpha(N, V)$. □

The following proposition is shown similarly.

PROPOSITION 5. $C_\omega(N,V) \subseteq C_+(N,V)$ *and* $C_-(N,V) \subseteq C_\alpha(N,V)$.

Now we show the relation of the optimistic and pessimistic recursive cores.

THEOREM 6. $C_+(N, V) \subseteq C_-(N, V)$.

Proof. The proof is by induction on the number of players and relies on two observations. First, over the same set of possible responses optimism leads to more deviations and hence less stability. Expanding this set will make optimistic players even more optimistic, pessimistic players even more pessimistic (as profitability must be guaranteed on a larger set).

For a single-player game, trivially, $C_+(\{1\}, V) = C_-(\{1\}, V)$.

Assuming $C_+(N_{k-1}, V) \subseteq C_-(N_{k-1}, V)$ for all games where $|N_{k-1}| \leq k-1$, we consider a deviation $\mathcal{P}_S$ of S from an outcome $(x, \mathcal{P})$ in a game of k players. As the deviation includes at least one player, the residual game consists of at most $k-1$ players. By assumption for the residual game, we have $C_+(N \backslash S, V_{\mathcal{P}_S}) \subseteq C_-(N \backslash S, V_{\mathcal{P}_S})$. Three cases are discussed: if none, if the optimistic and if both residual cores are empty. In each of these cases we show that profitability in the pessimistic case implies profitability in the optimistic case.

1. Both residual cores are non-empty. If the deviation is profitable under pessimism, it is profitable for all outcomes in $C_-(N \setminus S, V_{\mathcal{P}_S})$. By assumption $C_+(N \setminus S, V_{\mathcal{P}_S}) \subseteq C_-(N \backslash S, V_{\mathcal{P}_S})$ and hence also profitable under optimism.

2. Both residual cores are empty. Deviators form expectations with respect to the entire residual outcome set $\Omega(R, V_{\mathcal{P}_S})$. If deviation under pessimism is profitable, it is profitable for all residual outcomes, and therefore also under optimism.
3. The optimistic residual core is empty, the pessimistic is non-empty. Using the notation of Definition 2, $A_{-}(N \setminus S, V_{\mathcal{P}_S}) = C_{-}(N \setminus S, V_{\mathcal{P}_S})$, while $A_{+}(N \setminus S, V_{\mathcal{P}_S}) = \Omega(N \setminus S, V_{\mathcal{P}_S})$. If deviation under pessimism is profitable, it is profitable for all responses in $C_{-}(N \setminus S, V_{\mathcal{P}_S})$. As $C_{-}(N \setminus S, V_{\mathcal{P}_S}) \subseteq \Omega(N \setminus S, V_{\mathcal{P}_S})$, there exists a response such that an optimistic deviation is profitable.

We have discussed all cases and shown that if, for a given outcome, a pessimistic deviation would take place, then an optimistic would also and therefore an outcome is only stable against deviations under pessimism if it also under optimism.
□

Which approach is preferable? We see different uses of the two concepts. With optimism it is relatively easy to deviate; outcomes that belong to the optimistic recursive core are therefore rather stable. Pessimistic deviations are more difficult, so if an outcome is rejected by the pessimistic core it is rather unstable. Optimism is often dismissed on the grounds of "conservatism." In fact the pessimistic approach is the weaker one in rejecting unstable outcomes; the conservative scientist should use the optimistic recursive core to find stable outcomes.

COROLLARY 7. *The recursive cores, as a pair, are a refinement of the α- and ω-core pair.* $C_{\omega}(N, V) \subseteq C_{+}(N, V) \subseteq C_{-}(N, V) \subseteq C_{\alpha}(N, V)$.

Recursive cores are less sensitive to the behavioural assumptions of optimism and pessimism. In fact, the optimistic and pessimistic recursive cores often coincide. Even if this is not the case, the recursive approach reduces the "gray zone" to accept/reject outcomes as stable in the core sense.

Funaki and Yamato (1999) introduce a common-pool resource game, describe the unique equilibrium and show that the tragedy of commons can be avoided (the core is nonempty). Their results rely on using the α-core and observe that for particular production functions, with optimism the results do not hold. Using the recursive core the contradiction can often be reconciled. If we use a production function $f(l_N) = 1 - e^{-l_N}$, we get that the tragedy of commons can only be avoided in 3, 4 and 5-player games if the cost of labour is greater than 0.212, 0.471 and 0.471 (with pessimism), respectively. The recursive core of the optimistic 5-player or any larger game does not contain outcomes with the grand coalition. The results for optimism and pessimism mostly coincide making our conclusions robust.

4.2. *Partitional deviations*

In non-cooperative games, the absence of communication implies single player deviations; in cooperative games coalitional deviations are common. We allow partitional deviations; deviations, where more than one coalition deviates simultaneously, allowing them to internalise some of the mutual positive externalities they exert on each other. This way, a deviation, which, when done coalition-by-coalition, would not be profitable, can be destabilising.

EXAMPLE 8. Consider the 4-player game (N, V) with V such that[2]: $V(1234) = (8)$, $V(1,2,3,4) = (1,1,1,1)$, $V(ij,k,l) = (0,4,4)$, $V(ijk,l) = (6,1)$, $V(ij,kl) = (6,6)$ where $\{i,j,k,l\} = N$. Although $((2,2,2,2),\{N\})$ is (strictly!) Pareto dominated (e.g. by outcome $((3,3,3,3),\{\{1,2\},\{3,4\}\})$, it is immune to coalitional deviations. If partitional deviations are permitted, this cannot occur.

PROPOSITION 9. *Outcomes in the recursive cores are Pareto-efficient.*

Proof. Assume the contrary: there exists an outcome $(x,\mathcal{P}) \in C(N,V)$, such that there is another outcome $(y,\mathcal{P}')$ with $y > x$. Consider the – profitable – deviation by coalition N forming partition $\mathcal{P}'$. Outcome $(x,\mathcal{P})$ is dominated. Hence it cannot belongs to the core. Contradiction. Note that here there was no need to make a reference to optimism or pessimism. □

4.3. *Generalisation of the core*

The term *core* is justified if our concept returns the coalition structure core for characteristic function form games.

LEMMA 10. *Let* (N, v) *and* (N, V) *be CFF and PFF games, such that* $V(C, \mathcal{P}) = v(C)$ *for all* $\mathcal{P}$ *and* $C \in \mathcal{P}$. *Then* $C(N, v) = C(N, V)$.

Proof. First show $C(N, v) \subseteq C(N, V)$ and let $(x, \mathcal{P}) \in C(N, V) \setminus C(N, v)$. Then $\exists S$, such that $\sum_{i \in S} < v(S)$. Deviate by S in (N, V) to get $V(S, \mathcal{P}_{N \setminus S}) = v(S)$ for the residual reaction $\mathcal{P}_{N \setminus S}$, which is an improvement. Contradiction. □

5. SUMMARY

We introduce a new concept to solve PFF games. It is less sensitive to optimism/pessimism than the ω- and α-cores; allows a rational residual response as the γ-core, but this response is general, endogenous and consistent with the solution of the main game. The recursive cores are well-defined for all games and generalise the coalition structure core. Results on the non-cooperative implementation of the recursive core are presented in a companion paper (Kóczy, 2006).

6. ACKNOWLEDGEMENTS

Discussions with Johan Eyckmans, Luc Lauwers, Carmen Bevía, Parkash Chander and Henry Tulkens lead to considerable improvements of the paper. The author also thanks the funding of the Catholic University Leuven, the Netherlands Organisation for Scientific Research and the European Union (MEIF-CT-2004-011537).

NOTES

1. The original game is cohesive and hence when nonempty, its core contains outcomes with the grand coalition. For residual games this is not true.
2. In a simplified way: e.g. $V(1,234) = (V(\{1\}, \{\{1\}, \{2,3,4\}\}), V(\{2,3,4\}, \{\{1\}, \{2,3,4\}\}))$.

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