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AMENABLE EQUIVALENCE RELATIONS AND TURING DEGREES

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§1. Introduction. In [12] Slaman and Steel posed the following problem:

Assume ZF + DC + AD. Suppose we have a function assigning to each Turing degree d a linear order $<_d$ of d. Then must the rationals embed order preservingly in $<_d$ for a cone of d's?

They had already obtained a partial answer to this question by showing that there is no such $d \mapsto <_d$ with $<_d$ of order type $\zeta = \omega^* + \omega$ on a cone. Already the possibility that $<_d$ has order type $\zeta \cdot \zeta$ was left open.

We use here ideas and methods associated with the concept of amenability (of groups, actions, equivalence relations, etc.) to prove some general results from which one can obtain a positive answer to the above problem.

THEOREM. *Assume* ZF + DC + AD. *If $d \mapsto <_d$ is a function which assigns to each Turing degree d a linear order $<_d$ of d, then the rationals emded in $<_d$ for a cone of d's.*

The result holds "locally". For example, it can be proved, in ZF + DC only, that if $d \mapsto <_d$ is Borel (in an obvious sense to be explained below), then the rationals embed in $<_d$ for arbitrarily large d. Therefore it can be proved in ZF + DC + $\forall x\,(x^{\#}$ exists) only that, moreover, on a cone of d's the rationals embed in $<_d$. (It is not known if $\forall x\,(x^{\#}$ exists) is necessary here.) Similarly for projective $d \mapsto <_d$, using ZF + DC + PD, etc. Also, in ZF + DC alone, one can show that for a Borel $d \mapsto <_d$ the rationals embed in $<_d$ almost everywhere (with respect to the standard measure on 2^{ω}).

Some of the concepts and results discussed here could be of independent interest, and we have organized this paper in such a way that they can be read independently of their application to the proof of the above theorem.

In §2 we review the classical concept of amenability of countable groups and we introduce two concepts of amenability, one for *equivalence relations* and the other for *classes of structures* (in the sense of model theory).

For the first notion, let X be a Borel set in a Polish space and E a countable Borel equivalence relation on X (i.e. each equivalence class $[x]_E$ is countable and $E \subseteq X^2$ is Borel). We call *E amenable* if we can assign to each E-equivalence class C a finitely

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additive probability measure φ_C defined on all subsets of C so that the map $C \mapsto \varphi_C$ is universally measurable (in a precise sense that will be explained below). This definition carries over to the "pure" Borel context a concept introduced by Zimmer [14] in the context of "measured" countable Borel equivalence relations, i.e. when an appropriately related to E fixed Borel measure μ on X is present.

For the second notion, let $\mathscr{K}$ be a class of countable structures (in a fixed relational language) closed under isomorphism. We call $\mathscr{K}$ *amenable* if we can assign to each structure $\mathscr{A} = \langle A, \ldots \rangle \in \mathscr{K}$ a finitely additive probability measure $\mu_{\mathscr{A}}$ defined on all subsets of A, which is invariant under isomorphisms and such that the map $\mathscr{A} \mapsto \mu_{\mathscr{A}}$ is universally measurable.

Some basic facts concerning the relationships between these notions of amenability are also proved in §2. For example, if $\mathscr{K}$ is a class of structures, E is a countable Borel equivalence relation and there is a Borel assignment $C \mapsto \mathscr{A}_C$ which for each E-equivalence class C gives a structure $\mathscr{A}_C \in \mathscr{K}$ with universe C, then $\mathscr{K}$ amenable $\Rightarrow E$ amenable.

In §3 the key result of this paper, which asserts that the class $\mathscr{S}$ of countable scattered linear orders is amenable, is proved. The proof of this result requires ZFC + CH, since a basic step in the proof is the result of Mokobodzki (see [3]) asserting, under CH, the existence of universally measurable shift-invariant finitely additive probability measures defined on all subsets of $\mathbf{Z}$. (It is not known if Mokobodzki's result is provable in ZFC.)

According to the preceding comments this shows that a countable Borel equivalence relation each of whose equivalence classes is ordered, in a Borel way, by a scattered ordering must be amenable (under CH again). Finally, in §4 it is shown that the Turing equivalence relation $\equiv_T$ is not amenable, and with some additional work this is combined with the above to prove the above theorem. (Along the way one has to avoid the potential conflict of using a result proved in ZFC + CH to prove a result in ZF + DC + AD. This is done by standard metamathematical arguments concerning absolute consequences of CH.)

Some final comments:

1) There is possibly some question on whether the notion of amenability of countable Borel equivalence relations and classes of structures used here is ultimately the right one, one inconvenience of the present definition being the need to invoke the CH to establish the existence of amenable equivalence relations and classes of structures. It should be pointed out, however, that the present notions work smoothly because of the nice closure properties of universally measurable functions, especially their closure under composition, while the use of CH poses no problem in applications of these notions to various problems, since CH can often be eliminated by metamathematical arguments.

2) Foreman and Wehrung [5] have been motivated by some of the ideas used in §2 to prove the following result, solving a problem of Pincus, Solovay, and Luxemburg: ZF + Hahn-Banach $\Rightarrow$ there exists a nonmeasurable set.

3) We have recently found out that the concept of an equivalence relation with an ordering attached to each equivalence class has come up independently in a very interesting way in work of Muhly, Saito and Solel in operator algebras [8], [9]. In fact in [8] the authors establish that for a measured equivalence relation each of

whose equivalence classes is ordered (in a Borel way) by a scattered ordering, the associated von Neumann algebra (see [4]) is amenable. By known facts in operator algebras and ergodic theory this implies that the measured equivalence relation is amenable (in Zimmer's sense). This is the "measured" version of a "Borel theoretic" result (Theorem 4.1) that we prove in this paper. The set theoretic version implies immediately its "measured" counterpart (and CH can be avoided by standard metamathematical facts concerning absoluteness), so one has a different proof of the "measured" version, avoiding reference to von Neumann algebras.

4) Scatteredness seems intimately connected with amenability in the context of linear orderings. If one calls a single structure $\mathscr{A} = \langle A, \ldots \rangle$ *auto-amenable* if the class $\mathscr{K}_{\mathscr{A}} = \{\mathscr{B} : \mathscr{B} \cong \mathscr{A}\}$ is amenable (i.e. there is a finitely additive universally measurable probability measure defined on all subsets of A, which is invariant under automorphisms of $\mathscr{A}$), then Woodin has established a characterization of auto-amenable linear orderings which ties it very closely to scatteredness. In fact it may well be the case that Woodin's characterization reduces to the following:

A linear ordering $\mathscr{A} = \langle A, < \rangle$ is auto-amenable iff $\mathscr{A}$ has a scattered orbit, although this seems not to have been verified yet.

5) Another kind of structure where amenability has been studied is that of a tree. By a tree, we mean here an acyclic connected graph which is locally finite (i.e. every vertex has only finitely many immediate neighbors). Measured equivalence relations with trees attached (in a Borel way) to each equivalence class were studied first in S. Adams' Ph.D. thesis (see [1]), and a lot of the inspiration for our work here came from Adams' work. Dougherty and Kechris have afterwards identified various amenable classes of trees, and most recently Adams and Lyons have shown that the class of trees of branching number 1 (see Lyons [7]) is amenable, which includes all the earlier classes. Moreover, using results of C. Nebbia [10], they have characterized completely the auto-amenable trees. Their work will appear in a forthcoming paper.

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§2. Amenability. We first recall some standard notions. Given a *countable* set C a *finitely additive probability measure* (f.a.p.) on C is a function $\varphi\colon \{A : A \subseteq C\} \to [0, 1]$ such that $\varphi(A \cup B) = \varphi(A) + \varphi(B)$ if $A \cap B = \emptyset$, and $\varphi(C) = 1$. A *mean* on C is a continuous functional Φ on $l^\infty(C)$, the Banach space of bounded real functions on C, such that $\inf(f) \leq \Phi(f) \leq \sup(f)$. Means and f.a.p.'s on C are essentially the same thing: Given φ, define Φ by $\Phi(f) = \int f \, d\varphi$. Given Φ, define φ by $\varphi(A) = \Phi(\chi_A)$, where χ_A is the characteristic function of A. We will usually identify φ and Φ as above. (For more on f.a.p.'s the reader can consult, for example, J. Diestel, ***Sequences and series in Banach spaces***, Springer-Verlag, Berlin, 1984.)

A countable group G is *amenable* if there is a G-invariant f.a.p. (eq. mean) on G, i.e. an f.a.p. φ on G such that $\varphi(A) = \varphi(gA)$, for all $A \subseteq G$, $g \in G$. (eq. $\Phi(f) = \Phi(h \mapsto f(gh))$, $\forall h \in G$, $f \in l^\infty(G)$). The book Wagon [13] is a very good reference for this notion. Examples of amenable groups are the finite groups, the abelian (in fact the solvable) groups, etc. (To see, for example, that $\mathbf{Z}$ is amenable, let U be a nonprincipal ultrafilter on ω. Then, for each subset A of $\mathbf{Z}$, define $\varphi(A)$ to be the limit over U of the sequence card$(A \cap [-n, n])/2n + 1$.) The following is the standard nonamenable group.

Fact 2.1. *The free group with two generators F_2 is not amenable.*

Proof. Let a and b be the generators of F_2. Let $W(a)$, $W(a^{-1})$, $W(b)$, and $W(b^{-1})$ be the sets of words starting with the indicated elements. Since $F_2 = \{1\} \cup W(a) \cup W(a^{-1}) \cup W(b) \cup W(b^{-1}) = W(a) \cup aW(a^{-1}) = W(b) \cup bW(b^{-1})$, we immediately get a contradiction. ⊣

Now let X be a Borel set in a Polish space. We will consider equivalence relations E on X which are *Borel* and *countable*, i.e. each equivalence class $[x]_E$ is countable. By a *measure* on X we mean always a σ-finite Borel measure on X. Given an equivalence relation E as above and a measure μ on E, we call μ *quasi-invariant* (for E) if the *E-saturation* $[A]_E$ of each Borel set $A \subseteq X$ of μ-measure 0 has also μ-measure 0. A triple $\langle X, E, \mu\rangle$, where μ is quasi-invariant for X, is called a *measured equivalence relation.* In this measured context, there is a standard notion of amenability of $\langle X, E, \mu\rangle$ due to Zimmer [14]. We will now define, motivated by an equivalent form of Zimmer's definition (see [2]), a concept of amenability when no particular measure is present.

DEFINITION 2.2. Let X be a Borel set in a Polish space, E a countable Borel equivalence relation on X. We call E *amenable* if there is a map $C \mapsto \Phi_C$, assigning to each E-equivalence class $C = [x]_E$ of E a mean Φ_C on C which is universally measurable, in the following sense: if $F: X^2 \to \mathbf{R}$ is bounded and Borel, then the function $G: X \to \mathbf{R}$ given by $G(x) = \Phi_{[x]_E}(F_x)$ (where here and below F_x will denote the function $y \in [x]_E \mapsto F(x, y)$) is universally measurable. (Recall that a function $h: X \to Y$ between Borel subsets of Polish spaces is *universally measurable* if it is μ-measurable for every (probability) measure μ on X.)

There is a simple relation between amenability of groups and equivalence relations that we explain now.

A *Borel automorphism* of a Borel set X in a Polish space is a Borel bijection of X with itself. An *action* of a group G on X by *Borel* automorphisms, or simply a *Borel* action, is a homomorphism of G into the group of Borel automorphisms of X. We denote by $x \mapsto x \cdot g$ the Borel automorphism corresponding to $g \in G$. Thus $(x \cdot g) \cdot h = x \cdot (gh)$, $x \cdot 1 = x$.

The following is a standard fact in the context of measured equivalence relations.

Fact 2.3. *Let X be a Borel set in a Polish space, G a countable group acting in a Borel way on X. Denote by E_G the corresponding equivalence relation on X, i.e.*

$$xE_G y \Leftrightarrow \exists g \in G\, (y = x \cdot g).$$

i) (CH) *If G is amenable, E_G is amenable.*

ii) *If G acts freely, i.e. $\forall g \neq 1\ \forall x(x \cdot g \neq x)$, and there exists a G-invariant probability measure μ on X, i.e. $\mu(A) = \mu(A \cdot g)$ for all Borel $A \subseteq X$* and *$g \in G$, then if E_G is amenable, G is amenable.*

Proof. i) As we will not use this part, we will be sketchy. By a result of Mokobodzki (see [3] and §3) and using the Følner condition of amenable groups (see [13]) it is easy to see that G admits a universally measurable G-invariant mean Φ. (To say that Φ is universally measurable means that $\Phi_1 = \Phi \upharpoonright [-1,1]^G$: $[-1,1]^G \to [-1,1]$ is universally measurable.) Now given an $E = E_G$-equivalence class $C = [x]_E$, define Φ_C as follows: For $f \in l^\infty(C)$, $\Phi_C(f) = \Phi(g \mapsto f(x \cdot g))$. By the G-invariance of Φ, Φ_C is well-defined independently of the choice of $x \in C$. To check the universal measurability of $C \mapsto \Phi_C$, fix $F: X^2 \to \mathbf{R}$ bounded and Borel, and let

$$H(x) = \Phi_{[x]_E}(F_x) = \Phi(g \mapsto F(x, x \cdot g)).$$

Assuming without loss of generality that $F: X^2 \to [-1,1]$, we have that $H(x) = \Phi_1(F'(x))$, where $F': X \to [-1,1]^G$ is given by $F'(x)(g) = F(x, x \cdot g)$. Clearly F' is Borel, and since universally measurable functions are closed under composition, H is universally measurable.

ii) For xE_Gy, let (by freeness) $g(x,y) =$ the unique $g \in G$ with $y = x \cdot g$. Given $C \mapsto \Phi_C$ that witnesses the amenability of $E = E_G$ define the following mean on G:

$$\Phi(f) = \int_X \Phi_{[x]_E}(y \in [x]_E \mapsto f(g(x,y)))\, d\mu(x).$$

Note that this formula becomes more comprehensible by going to the associated f.a.p.'s:

$$\varphi(A) = \int_X \varphi_{[x]_E}(x \cdot A)\, d\mu(x).$$

The above integral makes sense as the map $F(x,y) = f(g(x,y))$ is bounded and Borel for each fixed $f \in l^\infty(G)$; thus $H(x) = \Phi_{[x]_E}(F_x)$ is universally (hence μ-) measurable. The G-invariance of Φ follows from that of μ. (Notice that $g(x \cdot h^{-1}, y) = hg(x,y)$.) ⊣

We now define our final notion of amenability, that for a class of countable structures.

Let $\mathscr{L}$ be a relational language, which for simplicity we will assume to be finite, say $\mathscr{L} = \{R_1, \ldots, R_n\}$, where R_i is a k_i-ary relation symbol. By a *class $\mathscr{K}$ of countable structures* for $\mathscr{L}$ we mean a collection of countable $\mathscr{L}$-structures *closed under isomorphism*. We want also to encode countable $\mathscr{L}$-structures canonically by "reals". Let $X(\mathscr{L})$ denote the Polish space $2^\omega \times 2^{\omega^{k_1}} \times \cdots \times 2^{\omega^{k_n}}$. An element α of this space is a tuple $\alpha = \langle A, R_1, \ldots, R_n \rangle$, where $A \subseteq \omega$ and $R_i \subseteq \omega^{k_i}$. We associate with it the $\mathscr{L}$-structure $\mathscr{A}_\alpha = \langle A, R_1^{\mathscr{A}_\alpha} \cdots R_n^{\mathscr{A}_\alpha} \rangle$, where $R_i^{\mathscr{A}_\alpha} = A^{k_i} \cap R_i$. Every countable $\mathscr{L}$-structure is isomorphic to one of this form.

DEFINITION 2.4. A class $\mathscr{K}$ of countable structures in a language $\mathscr{L}$ is called *amenable* if there is a map $\mathscr{A} \mapsto \Phi_{\mathscr{A}}$ assigning to each $\mathscr{A} \in \mathscr{K}$ a mean $\Phi_{\mathscr{A}}$ on A, where $\mathscr{A} = \langle A, \ldots \rangle$, such that

(1) the assignment is invariant under isomorphisms, i.e. if $\pi: \mathscr{A} \to \mathscr{A}'$ is an isomorphism, then $\Phi_{\mathscr{A}}(f) = \Phi_{\mathscr{A}'}(f \circ \pi^{-1})$, and

(2) the assignment is universally measurable, i.e. for each Borel set $S \subseteq X(\mathscr{L}) \cap \mathscr{K}$ (i.e. $\alpha \in S \Rightarrow \mathscr{A}_\alpha \in \mathscr{K}$) the map $F_S: X(\mathscr{L}) \times [-1,1]^\omega \to [-1,1]$ given by

$$F_S(\alpha, f) = \begin{cases} \Phi_{\mathscr{A}_\alpha}(f \upharpoonright A) & \text{if } \alpha \in S, \\ 0 & \text{if } \alpha \notin S, \end{cases}$$

where $\alpha = \langle A, \ldots \rangle$, is universally measurable.

Note that this definition implies that F_S is universally measurable for all universally measurable S as well.

A basic fact that we will prove now is that an equivalence relation each of whose equivalence classes supports (in a Borel way) a structure in some amenable class is amenable. We first need to give one more definition.

DEFINITION 2.5. Let X be a Borel set in a Polish space, E a countable, Borel equivalence relation on X. Let $\mathscr{L} = \{R_1, \ldots, R_n\}$ be a language, with arity $(R_i) = k_i$. An assignment $C \mapsto \mathscr{A}_C$, which for each E-equivalence class C gives an $\mathscr{L}$-structure $\mathscr{A}_C = \langle C, R_1^{\mathscr{A}_C}, \ldots, R_n^{\mathscr{A}_C} \rangle$ with universe C, is called *Borel* if the relations

$$R_i(x, y_1, \ldots, y_{k_i}) \Leftrightarrow y_1, \ldots, y_{k_i} \in [x]_E \,\&\, R_i^{\mathscr{A}_{[x]_E}}(y_1, \ldots, y_{k_i})$$

are Borel.

PROPOSITION 2.6. *Let $\mathscr{K}$ be a class of countable structures. Let X be a Borel set in a Polish space, E a countable Borel equivalence relation on X. If there is a Borel assignment $C \mapsto \mathscr{A}_C$ which for each E-equivalence class C produces a structure $\mathscr{A}_C \in \mathscr{K}$ with universe C and $\mathscr{K}$ is amenable, E is amenable.*

PROOF. Assign to each E-equivalence class C the mean $\Phi_C =^{\text{def}} \Phi_{\mathscr{A}_C}$ on C, where $\mathscr{A} \mapsto \Phi_{\mathscr{A}}$ witnesses the amenability of $\mathscr{K}$. To verify the definition of amenability fix bounded Borel $F\colon X^2 \to \mathbf{R}$ and consider $G(x) = \Phi_{[x]_E}(F_x)$. We can of course assume $F\colon X^2 \to [-1, 1]$.

As E is Borel and countable, there is a sequence $\{F_i\}$ of Borel maps on X with $[x]_E = \{F_i(x)\colon i \in \omega\}$. (We can actually take these to form a group of Borel automorphisms—see Feldman and Moore [4]—but we will not need this here.) It follows easily that there are Borel maps $s\colon X \to X(\mathscr{L})$ and $i\colon X \to X^\omega$ such that $s(x) = \alpha_x$ with $\mathscr{A}_{\alpha_x} = \langle A_x, \ldots \rangle$ and $\mathscr{A}_{\alpha_x} \cong \mathscr{A}_{[x]_E}$ and $i(x) = f_x$, where $\pi_x = f_x \restriction A_x$: $\mathscr{A}_{\alpha_x} \cong \mathscr{A}_{[x]_E}$. Thus, by property (1) in Definition 2.4,

$$G(x) = \Phi_{[x]_E}(F_x) = \Phi_{\mathscr{A}_{\alpha_x}}(n \in A_x \mapsto F(x, \pi_x(n))).$$

Let $S = \text{range}(s)$. Then S is Σ_1^1, so universally measurable. In the notation of property (2) in 2.4 we have then

$$G(x) = F_S(s(x), n \mapsto F(x, i(x)(n))).$$

Now F_S is universally measurable, s is Borel and $f\colon X \to [-1, 1]^\omega$ given by $f(x)(n) = F(x, i(x)(n))$ is Borel too, so as universally measurably functions are closed under composition, G is universally measurable. ⊣

The referee has raised the question of the existence of a converse to 2.6. That is, if E is amenable can one assign in a Borel way to each equivalence class a structure in some amenable class?

Finally we state some simple closure properties of amenable equivalence relations.

PROPOSITION 2.7. *Let X be a Borel set in a Polish space, and E a countable Borel equivalence relation on X.*

(i) *If E is amenable and $A \subseteq X$ is Borel, then $E \restriction A$ is amenable.*

(ii) *If $A \subseteq X$ is Borel and* ***full****, i.e. $A \cap [x]_E \neq \varnothing$ for all $x \in X$, then if $E \restriction A$ is amenable, so is E.*

(iii) *If Y is a Borel set in a Polish space, F is a countable Borel equivalence relation on Y and $H: X \to Y$ is Borel such that $xEy \Leftrightarrow H(x)FH(y)$, then if F is amenable, so is E.*

(iv) *If $F \subseteq E$ is a Borel subequivalence relation of E and E is amenable, so is F.*

(v) (CH) *If $E_0 \subseteq E_1 \subseteq \cdots$ are amenable equivalence relations and each E_n is amenable, so is $E = \bigcup_n E_n$.*

PROOF. We will prove only (i)–(iv), since these are the only properties we will use later.

(i) Fix a sequence of Borel functions $\{F_i\}$ generating E, i.e. $xEy \Leftrightarrow \exists i(F_i(x) = y)$. Given an $E \restriction A$ equivalence class C', let $C = [C']_E$ be its E-saturation. For $x \in C$ let $i_x =$ least i with $F_i(x) \in C'$. Then $F(x) = F_{i_x}(x)$ maps C into C'. If $\{\Phi_C\}$ witnesses the amenability of E and we put $\Phi_{C'}(f) = \Phi_C(f \circ F)$, then $\{\Phi_{C'}\}$ witnesses the amenability of $E \restriction A$.

(ii) Given an E-equivalence class C, let $C' = C \cap A$ be the corresponding $E \restriction A$-equivalence class and put $\Phi_C(f) = \Phi_{C'}(f \restriction C')$.

(iii) Put $Y' = \bigcup\{[H(x)]_F: x \in X\}$. As H is countable-to-1, Y' is a Borel subset of Y. Consider the disjoint union $X \oplus Y' = \{\langle 0, x\rangle: x \in X\} \cup \{\langle 1, y\rangle: y \in Y'\}$ of X and Y', and define in it the countable Borel equivalence relation R whose equivalence classes are the sets $[x]_E \oplus [H(x)]_F$. Since $R \restriction Y'$ is just $F \restriction Y'$, we have by (i) that $R \restriction Y'$ is amenable. But Y' is full in R, so by (ii) R is amenable and so by (i) again $R \restriction X = E$ is amenable.

(iv) Let C' be an F-equivalence class and C the unique E-equivalence class containing C. Now use the argument of the proof of (i). ⊣

§3. Scattered orders. Recall that a linear order $L = \langle L, < \rangle$ is *scattered* if the rationals do not embed in an order-preserving way into L. For the results about scattered orders that we will use below, see Rosenstein [11]. We denote by $\mathcal{S}$ *the class of countable scattered orders*. We now have

THEOREM 3.1 (CH). *The class $\mathcal{S}$ of countable scattered linear orders is amenable.*

PROOF. We will make use of the following basic result of Mokobodzki (see [3, pp. 102–108]).

THEOREM 3.2 (MOKOBODZKI). *Assume CH. Then there is a universally measurable shift-invariant mean $\Phi_{\mathbf{Z}}$ on $\mathbf{Z}$. Similarly there is a universally measurable shift-invariant mean $\Phi_{\mathbf{N}}$ on $\mathbf{N}$.*

Of course, universally measurable here means that $\Phi_{\mathbf{Z}} \restriction [-1, 1]^{\mathbf{Z}}$ is universally measurable. Shift-invariance means that $\Phi_{\mathbf{Z}}(f) = \Phi_{\mathbf{Z}}(f \circ s)$, for $s(n) = n + 1$. Similarly for $\Phi_{\mathbf{N}}$.

REMARK. It is well known (see [13]) that such Φ cannot have the property of Baire. It is not known if Mokobodzki's theorem can be proved in ZFC alone.

If now $L = \langle L, < \rangle$ is a countable scattered linear order, denote by c^α, for $\alpha < \omega_1$, its αth iterated Hausdorff condensation—denoted by c_F^α in [11]. Thus

$$c^1(x) = \{y \in L: \text{the interval between } x \text{ and } y \text{ is finite}\},$$

$$c^\lambda(x) = \bigcup_{\alpha < \lambda} c^\alpha(x), \quad \text{for } \lambda \text{ limit.}$$

The sets $c^\alpha(x)$ are intervals in L and partition L; denote by $<_\alpha$ the order on these intervals induced by $<$. Then $c^{\alpha+1}(x) = \bigcup\{c^\alpha(y)$: the $<_\alpha$- interval between $c^\alpha(x)$ and

$c^\alpha(y)$ is finite}. There is a least countable ordinal α_0 such that $c^{\alpha_0}(x) = L$ for all $x \in L$. Call it the *rank* of L, in symbols $r(L)$—it is called F-rank and denoted by r_F in [11].

We assign now to each $L \in \mathscr{S}$ a mean Φ_L on L by induction on $r(L)$ as follows:

(1) $r(L) = 1$: then $L \cong n \in \omega$ or $L \cong \omega$ or $L \cong \omega^*$ or $L \cong \mathbf{Z}$.

If $L = \{x_0 < x_1 < \cdots < x_{n-1}\}$, put $\Phi_L(f) = f(x_0)$.

If $L = \{x_0 < x_1 < \cdots\}$, put again $\Phi_L(f) = f(x_0)$.

If $L = \{\cdots < x_1 < x_0\}$, put $\Phi_L(f) = f(x_0)$.

Finally if $\pi: L \to \mathbf{Z}$ is an isomorphism put $\Phi_L(f) = \Phi_{\mathbf{Z}}(f \circ \pi^{-1})$.

Since $\Phi_{\mathbf{Z}}$ is shift-invariant, it is obviously invariant under order-preserving automorphisms $\delta: \mathbf{Z} \to \mathbf{Z}$, so this definition is independent of π.

(2) $r(L) = \alpha + 1$: Consider then the linear order

$$L^\alpha = \langle c^\alpha[L] = \{c^\alpha(x): x \in L\}, <_\alpha \rangle.$$

We must have $L^\alpha \cong n \in \omega$, $L^\alpha \cong \omega$, $L^\alpha \cong \omega^*$ or $L^\alpha \cong \mathbf{Z}$. Since $y \in c^\alpha(x) \Leftrightarrow c^\alpha(x) = c^\alpha(y)$, it follows that $r(c^\alpha(x)) \leq \alpha$, so we have already assigned $\Phi_{c^\alpha(x)}$ for each $x \in L$—here $c^\alpha(x)$ means $\langle c^\alpha(x), < \restriction c^\alpha(x)\rangle$.

We consider again cases:

If $L^\alpha = \{c^\alpha(x^0) <_\alpha c^\alpha(x_2) <_\alpha \cdots <_\alpha c^\alpha(x_n)\}$, put $\Phi_L(f) = \Phi_{c^\alpha(x_0)}(f \restriction c^\alpha(x_0))$.

If $L^\alpha = \{c^\alpha(x_0) <_\alpha c^\alpha(x_1) <_\alpha \cdots\}$, put $\Phi_L(f) = \Phi_{c^\alpha(x_0)}(f \restriction c^\alpha(x_0))$.

If $L^\alpha = \{\cdots <_\alpha c^\alpha(x_1) <_\alpha c^\alpha(x_0)\}$, put $\Phi_L(f) = \Phi_{c^\alpha(x_0)}(f \restriction c^\alpha(x_0))$.

Finally, if $\pi: L^\alpha \to \mathbf{Z}$ is an isomorphism, say with $\pi^{-1}(n) = c^\alpha(x_n)$ for $n \in \mathbf{Z}$, put

$$\Phi_L(f) = \Phi_{\mathbf{Z}}(n \in \mathbf{Z} \mapsto \Phi_{\pi^{-1}(n)}(f \restriction \pi^{-1}(n))).$$

Again since $\Phi_{\mathbf{Z}}$ is shift-invariant this is independent of the choice of π.

(3) $r(L) = \lambda$, limit: Fix any $x \in L$. Again for any $\alpha < \lambda$, $r(c^\alpha(x)) < \lambda$, so we have already defined $\Phi_{c^\alpha(x)}$. Fix once and for all for each limit $\lambda < \omega_1$ a sequence $\alpha_n = \alpha_n(\lambda) \uparrow \lambda$, $n \in \mathbf{N}$. Then put

$$\Phi_L(f) = \Phi_{\mathbf{N}}(n \in \mathbf{N} \mapsto \Phi_{c^{\alpha_n}(x)}(f \restriction c^{\alpha_n}(x))).$$

Let us note that this is independent of x. Indeed, if $y \in L$ is another point, then $c^{\alpha_n}(x) = c^{\alpha_n}(y)$ for all large enough $n \in \mathbf{N}$, so by the shift-invariance of $\Phi_{\mathbf{N}}$ we are done.

We now verify properties (1) and (2) of Definition 2.4.

(1) is a routine induction on the rank.

To verify (2), fix a Borel set $S \subseteq X(\mathscr{L}) \cap \mathscr{S}$, where $\mathscr{L} = \{<\}$.

LEMMA 3.3. *If $T \subseteq X(\mathscr{L}) \cap \mathscr{S}$ is Σ^1_1, then, for some $\xi < \omega_1$, $r(L_\alpha) < \xi$, $\forall \alpha \in S$. Here L_α is the linear order coded by α.*

PROOF. For each scattered L_α consider the equivalence relations

$$xE_\theta y \Leftrightarrow c^\theta(x) = c^\theta(y) (\Leftrightarrow x \in c^\theta(y)),$$

for $\theta < \omega_1$. Starting with $E_0 = \{(x, x): x \in L_\alpha\}$, we have $E_\lambda = \bigcup_{\theta<\lambda} E_\theta$, if λ is limit, and

$$\begin{aligned} xE_{\theta+1}y \Leftrightarrow [x < y \;\&\; \exists x_1 \cdots \exists x_n (x < x_1 < \cdots < x_n < y \\ \& \; \forall z \in [x, y](zE_\theta x_1 \vee \cdots \vee zE_\theta x_n))] \\ \vee [x > y \wedge \cdots]. \end{aligned}$$

Thus $\{E_\theta\}$ is given by a positive elementary induction on $L_\alpha \times L_\alpha$ leading to the fixed point $E_\infty = L_\alpha \times L_\alpha$. It follows that $r(L_\alpha)$, which is the closure ordinal of this induction, is $< \omega_1^\alpha$. By a standard boundedness argument we can now find $\xi < \omega_1$ such that $\forall \alpha \in T\ (r(L_\alpha) < \xi)$.

(Equivalently one can check that $X(\mathscr{L}) \cap \mathscr{S}$ is Π_1^1 and the map $\alpha \mapsto r(L_\alpha)$ is a Π_1^1-norm on $X(\mathscr{L}) \cap \mathscr{S}$, so the above fact follows from the boundedness theorem for Π_1^1-norms.)

So in order to show that the function F_S of Definition 2.4 is universally measurable, it is enough to show for each $\xi < \omega_1$ that the function

$$F_\xi(\alpha, f) = \begin{cases} \Phi_{L_\alpha}(f \restriction L_\alpha) & \text{if } r(L_\alpha) \leq \xi, \\ 0 & \text{otherwise} \end{cases}$$

(where $\alpha \in X(\mathscr{L}) \cap \mathscr{S}$ and $f \in [-1,1]^\omega$) is universally measurable. This can be done easily by induction on ξ, using repeatedly that $\Phi_{\mathbf{Z}}$ and $\Phi_{\mathbf{N}}$ are universally measurable and simple closure properties of universally measurable functions, particularly closure under compositions. ⊣

It is interesting to consider also the following "local" notion of amenability.

DEFINITION 3.4. Let $\mathscr{A}$ be a countable structure in a relational language $\mathscr{L}$. We call $\mathscr{A}$ *auto-amenable* if the class $\mathscr{K}(\mathscr{A}) = \{\mathscr{B}: \mathscr{B} \cong \mathscr{A}\}$ is amenable. Spelled out, this means that if $\mathscr{A} = \langle A, \ldots \rangle$ there is a universally measurable mean Φ on A which is $\mathscr{A}\!ut(\mathscr{A})$-invariant, i.e. $\Phi \restriction [-1,1]^{\mathscr{A}}$ is universally measurable and for any automorphism π of $\mathscr{A}$, $\Phi(f) = \Phi(f \circ \pi)$.

Recall that the *orbits* of a structure $\mathscr{A} = \langle A, \ldots \rangle$ are the equivalence classes of the following equivalence relation on A:

$$a \sim b \Leftrightarrow \exists \pi \in \mathscr{A}\!ut(\mathscr{A})[\pi(a) = b].$$

It is clear that if an orbit of a structure $\mathscr{A}$, viewed as a substructure of $\mathscr{A}$, is auto-amenable, so is $\mathscr{A}$. Thus from Theorem 3.1 it follows (from CH) that if a countable linear order has a scattered orbit then it is auto-amenable. Woodin has established a characterization of the auto-amenable orders which relates them closely to scatteredness. In fact it may be that Woodin's characterization reduces to this: L is auto-amenable iff some orbit of L is scattered, but this has not been verified yet.

Finally, we point out that although each rigid (i.e. having no nontrivial automorphisms) linear order is auto-amenable, the class of rigid linear orders is not amenable (we will prove this in §4).

§4. Orderings on equivalence relations. We now combine the preceding results to provide proofs of the theorem in the Introduction and related facts.

First we have as an immediate corollary of 3.1 and 2.6.

THEOREM 4.1. *Assume CH. Let X be a Borel set in a Polish space, E a countable Borel equivalence relation on X, and let $C \mapsto <_C$ be a Borel assignment which for each E-equivalence class C produces a scattered linear order $<_C$ of C. Then E is amenable.*

On the other hand, a key property of the Turing equivalence relation $\equiv_{\mathrm{T}}$ on 2^ω is the following.

THEOREM 4.2. *The Turing equivalence relation $\equiv_{\mathrm{T}}$ on 2^ω is not amenable.*

PROOF. By 2.7(iv) it is enough to find a nonamenable subequivalence relation of $\equiv_T$. We will give two different proofs of that.

1) We use here the following lemma of Slaman and Steel [12].

LEMMA 4.3. *There is a free action of the free group F_2 of 2 generators on 2^ω by Lipschitz recursive homeomorphisms.*

(This means that for each $g \in F_2$, $x \mapsto x \cdot g$ is a Lipschitz recursive homeomorphism on 2^ω, i.e. $(x \cdot g) \upharpoonright n$ depends only on $x \upharpoonright n$ and in a recursive way.)

Now notice that if λ is the standard probability measure on 2^ω, then λ is invariant under Lipschitz homeomorphisms. Thus if $x \mapsto x \cdot g$ is the group action of F_2 given in Lemma 4.3, λ is invariant under this action. If E_{F_2} is the equivalence relation induced by this action, then clearly $E_{F_2} \subseteq \equiv_T$ and by 2.3(ii) E_{F_2} is not amenable.

2) Consider the space 2^{F_2} with the canonical F_2-action given by $(x \cdot f)(g) = x(fg)$. Let λ be the standard probability measure on 2^{F_2}. Although this action of F_2 is not free it is λ-a.e. free, i.e. $\{x \in 2^{F_2}: \exists g \neq 1 (x \cdot g = x)\}$ has λ-measure 0. Since λ is invariant under this action, the proof of 2.3(ii) applies as well to show that the equivalence relation E'_{F_2} induced by this action is not amenable. By a standard recursive identification of F_2 with ω we can identify 2^{F_2} with 2^ω, and then clearly $E'_{F_2} \subseteq \equiv_T$, so we are done. ⊣

For further reference let us note the following strengthening of 4.2.

THEOREM 4.4. *Let $X \subseteq 2^\omega$ be Borel and $\equiv_T$-invariant. If λ is the standard measure on 2^ω and $\lambda(X) > 0$, then $\equiv_T \upharpoonright X$ is not amenable.*

PROOF. Apply the preceding argument to X and $\lambda \upharpoonright X$ instead of λ. ⊣

COROLLARY 4.5 (ZF + DC). *Let E be a countable Borel equivalence relation on 2^ω extending $\equiv_T$, i.e. $\equiv_T \subseteq E$. Let $C \mapsto <_C$ be a Borel map which assigns to each E-equivalence class C a linear order of C. If λ is the standard measure on 2^ω, then, for λ-a.e. x, $\langle [x]_E, <_{[x]_E} \rangle$ is not scattered.*

PROOF. Since the statement we want to prove is Π^1_2, it is enough to prove it from ZFC + CH. So assume this below.

If this corollary fails, let $X \subseteq 2^\omega$ be Borel and E-invariant with $\lambda(X) > 0$ such that for $x \in X$, $\langle [x]_E, <_{[x]_E} \rangle$ is scattered. From 2.6 and 3.1 it follows that $E \upharpoonright X$ and thus $\equiv_T \upharpoonright X$ is amenable, contradicting 4.4. ⊣

We now have

THEOREM 4.6 (ZF + DC + AD). *Let $d \mapsto <_d$ be a map assigning to each Turing degree d a linear order $<_d$ of d. Then for a cone of d's, the rationals embed order-preservingly in $\langle d, <_d \rangle$.*

PROOF. Otherwise, by Turing determinacy, on a cone of d's $\langle d, <_d \rangle$ is scattered. Fix $x_0 \in 2^\omega$ so that, for all $x \geq_T x_0$, $\langle [x]_T, <_{[x]_T} \rangle$ is scattered. Define the following countable Borel equivalence relation E on 2^ω:

$$xEy \Leftrightarrow \langle x, x_0 \rangle \equiv_T \langle y, x_0 \rangle$$

Clearly $\equiv_T \subseteq E$. Define for each E-equivalence class C a linear order $<_C$ of C as follows:

$$x <_C y \Leftrightarrow \langle x, x_0 \rangle <_{[<x,x_0>]_T} \langle y, x_0 \rangle$$

Clearly $\langle C, <_C \rangle$ is scattered for every C. We would now like to apply 4.5, but $C \mapsto <_C$ might not be Borel. However we claim that there is an E-invariant Borel set $X \subseteq 2^\omega$ of λ-measure 1 on which $C \mapsto <_C$ is Borel, and this completes the proof.

Indeed, let $\{F_i\}$ be a sequence of Borel functions on 2^ω such that $[x]_E = \{F_i(x): i \in \omega\}$. Since $[x]_E$ is always infinite, we can easily arrange to have $F_i(x) \neq F_j(x)$ for $i \neq j$. Put

$$f(x) = \{\langle i, j\rangle: F_i(x) <_{[x]_E} F_j(x)\}.$$

Then $f: 2^\omega \to 2^\omega$ is λ-measurable (since we have AD), so there is Borel $g: 2^\omega \to 2^\omega$ such that $f = g$ λ-a.e., say $f(x) = g(x)$ for $x \in X'$, where $\lambda(X') = 1$, X' Borel. Let $X = [X']_E$ be the E-saturation of X'. Then, since E is countable Borel, X is Borel with $\lambda(X) = 1$. We check that $C \mapsto <_C$ is Borel on X. Indeed, for $x, y, z \in X$

$$y <_{[x]_E} z \Leftrightarrow yEzEx \wedge \exists x'[x'Ex \wedge x' \in X' \wedge \exists i, j(F_i(x') = y \wedge F_j(x') = z \wedge g(x')(\langle i, j\rangle) = 1)]. \quad \dashv$$

The previous proof is "local". For example it shows in ZF + DC that if $d \mapsto <_d$ is a Borel assignment of a linear order to each Turing degree, then, for arbitrarily large d's, $\langle d, <_d\rangle$ is not scattered. Moreover, assuming also $\forall x\,(x^\#$ exists), "arbitrarily large" can be replaced by "on a cone of". Similarly, for projective $d \mapsto <_d$ under PD, etc.

§5. Further results and problems. If $d \mapsto <_d$ assigns to each Turing degree d a linear order of d, what kind of order type can it have on a cone? If $\langle d, <_d\rangle$ has the same order type on a cone, one has the following assertion.

THEOREM 5.1 (WOODIN). *Let $d \mapsto <_d$ assign in a Borel way to each Turing degree d a linear order of d, and assume, for some fixed linear order L, that $L \cong \langle d, <_d\rangle$ on a cone. Then no orbit of L can be scattered.*

*Similarly for **any** $d \mapsto <_d$ in ZF + DC + AD.*

PROOF. Consider first the case of Borel $d \mapsto <_d$. Again by absoluteness we can assume CH. Let $M \subseteq L$ be a scattered orbit, towards a contradiction. Say $\langle d, <_d\rangle \cong L$ for $d \geq d_0$, and for such d put

$$X_d = \{x \in d: \exists \pi: L \cong \langle d, <_d\rangle\, (x \in \pi[M])\}.$$

Notice that also $X_d = \{x \in d: \forall \pi: L \cong \langle d, <_d\rangle\, (x \in \pi[M])\}$ and $\langle X_d, <_d \restriction X_d\rangle \cong M$. Put $X = \bigcup_{d \geq d_0} X_d$. Notice that X is Borel, since $d \mapsto <_d$ is Borel. Look at $\equiv_T \restriction X$. Since $\langle X_d, <_d \restriction X_d\rangle \cong M$ is scattered, $\equiv_T \restriction X$ is amenable. So, by 2.7(ii), $\equiv_T \restriction \{x: [x]_T \geq d_0\}$ is amenable. If $d_0 = [x_0]_T$ put $xEy \Leftrightarrow \langle x, x_0\rangle \equiv_T \langle y, x_0\rangle$. Then, by 2.7(iii), E is amenable and as $\equiv_T \subseteq E$ we have a contradiction.

For the ZF + DC + AD case note that X is cofinal in the Turing degrees, so by Martin's proof of Turing determinacy (see e.g. [6]) there is a recursively pointed perfect tree S with $[S] = \{x: x \text{ in a path through } S\} \subseteq X$. Let $h_S: 2^\omega \to [S]$ be the canonical homeomorphism. Put $xEy \Leftrightarrow h_S(x) \equiv_T h_S(y)$. Then $\equiv_T \subseteq E$ (as S is pointed). Also we can define the scattered order $<_C$ on each E-equivalence class $C = [z]_E$ by

$$x <_C y \Leftrightarrow h_S(x) <_{[h_S(z)]_T} h_S(y).$$

A contradiction can be attained now as in the proof of 4.6. $\dashv$

It would be interesting to find a characterization of such L.

For the case where $\langle d, <_d \rangle$ is allowed to have different order types (on a cone), Woodin has also pointed out $\langle d, <_d \rangle$ can be actually rigid. In fact, given a countable Borel equivalence E extending the tail equivalence relation on 2^ω

$$x \sim_t y \Leftrightarrow \exists n \exists m \forall k (x(n+k) = y(m+k))$$

a Borel assignment $C \mapsto <_C$ can be constructed which gives for each E-equivalence class C a linear order $<_C$ of C which is rigid. This is done as follows. Consider the ordinal ω^2. For $x \in 2^\omega$ define a linear order $<_x$ as follows: If $x(n) = 0$, replace by ω every point in the nth copy of ω in ω^2. If $x(n) = 1$, replace by ω^* every point in the nth copy of ω in ω^2. View $<_x$ in some canonical way as having universe ω. Clearly $<_x$ is scattered rigid, and $x \neq y \Rightarrow <_x \not\cong <_y$. Given now an E-equivalence class C for which we can assume without loss of generality that it does not contain the constant sequences 0 and 1, define $<_C$ as follows: For $x \in 2^\omega$ not constant, let x' be defined by $x = 1^n 0 x'$ and put $x_0 = n$. Notice that in the lexicographical order $<_{\text{lex}}$, $\{x' : x \in C\}$ has the order type η of the rationals. For $x, y \in C$, put

$$x <_C y \Leftrightarrow x' <_{\text{lex}} y' \vee (x' = y' \wedge x_0 <_{x'} y_0).$$

Then $<_C$ is isomorphic to an order obtained by replacing each rational by a distinct scattered rigid order, so $<_C$ is rigid.

We can use this example to show that the class $\mathscr{R}$ of rigid countable orders *is not amenable*: Indeed, if $X = 2^\omega$ consider the equivalence relation $\equiv_T$. Since to each $\equiv_T$-equivalence class C one can assign in a Borel way a rigid order $<_C$, if $\mathscr{R}$ was amenable, so would be $\equiv_T$, a contradiction.

Finally, let us point out that in Slaman and Steel [12], the authors pose the following additional problem: Assume ZF + DC + AD. If $d \mapsto <_d$ assigns to each Turing degree d a linear order of d, is there a linear order $<$ of all of 2^ω such that $<_d = < \restriction d$ on a cone? They point out that this would imply the theorem of the Introduction. We do not know the answer to this question.

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