ON EINSTEIN'S 1905 ELECTRODYNAMICS PAPER*

W L Kennedy

Physics Department, University of Otago, Dunedin, New Zealand

Abstract

Examination of Einstein's 1905 landmark paper *'On the Electrodynamics of Moving Bodies'* reveals that his obtaining of the Lore ntz Transformation Equations contains errors apparently previously unnoticed or unremarked. Recognition of these errors reveals that there is an impasse in his procedure. Modifications of his analysis as well as extension of the procedures he has chosen allows his approach to be made into a complete proof, while still remaining within the unique framework of his method.

* Accepted for publication in Studies in the History and Philosophy of Modern Physics.

Keywords: Special relativity, Albert Einstein, Lorentz transformation equations, historic derivation.

Email address: wlk@physics.otago.ac.nz

June 1, 2004.

Einstein (1905a) in his most famous paper, *Zur Elektrodynamik bewegter Koerper* (ZEBK), introduced concepts that radically changed the path of physics. Examination of his method for obtaining the Lorentz transformation equations however reveals that because he claims wider validity for some of his equations than is intrinsic to their mode of derivation, his analysis is in fact deficient. For reference purposes, of the several English translations of ZEBK available we list just two, the one in the Dover reprint volume (Einstein, 1905b) and the translation given by Miller (1981, pp 392-415). In his paper Einstein, on his path to the Lorentz Transformation Equations in Section 3 of the Kinematic Part of ZEBK, takes two relatively at rest Cartesian inertial reference frames K and k with coincident corresponding spatial axes. Space-time coordinates are taken as x,y,z,t for K and ξ,η,ζ,τ for k. The frame k is then considered boosted so that it has a constant velocity $v = (v,0,0)$ with respect to K. Spatial origins are assumed coincident at a common zero of time so that the transformation equations between the space-time coordinates of the two frames are homogeneous.

The most striking aspect of Einstein's approach to the Lorentz Transformation Equations is that he first derives the substantive form of the time transformation equation independently of any considerations concerning the form of the spatial transformation equations. Application of his Clock Synchronization Hypothesis (CSH) to three thought examples involving a pair of k -frame clocks spatially separated, in turn, along the ξ, η, and ζ axes of the frame k produces three CSH equations involving the k-frame time coordinate function τ expressed as function of the K -frame space-time coordinates. The values of the K -frame arguments to be used in each value of τ are found from kinematic analysis of the light paths solely within the K -frame in conjunction with the second part (the light speed part) of his Relativity Principle (RP2). From these results Einstein produces a set of three partial differential equations for the function τ, integration of which yields

$$\tau = a\,(t - x'v/(c^2 - v^2)), \qquad (1\text{:E}3.4)$$

where Einstein has used x' to stand for the convenient combination, $x-vt$, and where the quantity a, being an integration constant, represents a function of v still to be determined. (Where appropriate, we use dual numbering of equations with the "E" part of the number referring to ZEBK equations, for which we follow the numbering that Miller (1981), pp 392-415, has added to his translation.) When τ is expressed as function of x,y,z,t, (1:E3.4) becomes

$$\tau = a\,\beta^2\,(t - x\,v/c^2)\,, \qquad \text{(2:E3.4')}$$

where

$$\beta = 1/\sqrt{(1-v^2/c^2)}\,. \qquad \text{(3:E3.15)}$$

Thus far Einstein's analysis has been elegant and exemplary. (From now onwards in ZEBK there is no longer algebraic advantage in the use of the quantity x' for the linear combination $x\text{-}vt$; consequently we abandon its use and in re-presenting Einstein's equations, freely substitute $x\text{-}vt$ for x' whenever necessary or convenient.)

Einstein now proposes to use RP2 in combination with his τ result to obtain the spatial k-frame coordinates ξ,η,ζ as functions of x,y,z,t. This proposal represents another unique feature of his method since the transformation equations for longitudinal and for transverse spatial coordinates are to arise in the same way. Three cases of light propagation are considered comprising one -way light signals sent out respectively along each of the ξ,η,ζ directions of frame k.

For the first case a light ray emitted from the common space-time origins of the two frames satisfies

$$\xi - c\tau = 0,\quad \eta = 0,\quad \zeta = 0 \text{ in k,} \quad \text{and } x - ct = 0,\quad y = 0,\quad z = 0 \text{ in K,} \qquad (4)$$

respectively. Einstein, using (1:E3.4), (i.e. (2:E3.4')), writes the k-frame equation, $\xi = c\tau$, as

$$\xi = c\tau = ac\,(t - x'v/(c^2 - v^2))\,, \qquad \text{(5:E3.5)}$$

i.e. $$\xi = c\tau = a\beta^2\,(ct - x\,v/c)\,, \qquad \text{(6:E3.5')}$$

and then uses the K-frame light ray equation, $x=ct$, to obtain

$$\xi = a\,\beta^2 x' = a\,\beta^2 (x - vt)\,. \qquad \text{(7:E3.7)}$$

Equation (7:E3.7) doesn't constitute a general coordinate transformation equation *in the context in which it appears*, since its truth has only been established for events along a single ray of the light cone. Equation (7:E3.7) is of course true not only along the chosen ray but also everywhere in space-time, both on and off the light cone, but Einstein has not proved it so. In order to obtain an expression for ξ as function of K coordinates which is true both on and off the chosen light ray, one uses linearity of the desired transformation equations together with coincidence of the zeros of the two linear forms $\xi - c\tau$ and $x - ct$ to write

$$\xi - c\tau = f_1(v)\,(x - ct), \tag{8}$$

with $f_1(v)$ a function of v yet to be determined. Substituting from (2:E3.4') gives

$$\xi = x(f_1 - a\beta^2 v/c) + ct(a\beta^2 - f_1). \tag{9}$$

The impasse in Einstein's method is thus readily apparent. The essential dependence of the function ξ on the K-frame coordinates x,y,z,t cannot be determined without knowledge of f_1. Einstein's use of $x=ct$ which is only true on the light ray means in effect that Einstein stays on the light ray where f_1 may remain arbitrary so that an uncountable infinity of possible linear combinations of x and ct is allowed as in the RHS of (9). To determine f_1 one needs to go beyond the material presented in Einstein's section 3. Equivalently from a graphical viewpoint we have that the level lines of $c\tau$ in the $(x,\ ct)$ plane, being already known from (2:E3.4'), provide $\xi = c\tau$ values along the ray $x=ct$. However level lines of ξ cannot yet be drawn unless ξ values are known along some other line of the $(x,\ ct)$ plane, for example along the line $x = -ct$. Thus, keeping within the framework of Einstein's approach, one takes a second light ray along the negative ξ and x axes. Equations analogous to (8) and (9) then result:

$$\xi + c\tau = f_2(v)\,(x + ct), \tag{10}$$

$$\xi = x(f_2 + a\beta^2 v/c) + ct(f_2 - a\beta^2), \tag{11}$$

where $f_2(v)$ is also a function of v yet to be determined. Equivalence of (9) and (11) at all space-time points means that corresponding coefficients of x and ct can be equated: Thus

$$f_2(-v) = f_1(v) = a\beta^2(1+v/c)\,. \tag{12}$$

Substitution of f_1 into (9), or f_2 into (11), gives

$$\xi = a\,\beta^2(x-vt) \tag{13:E3.7}$$

as a transformation equation now demonstrated to be true at any world point as a consequence of its mode of derivation. It must seem remarkable that Einstein appears to have adopted a deliberate plan to obtain the form of this equation by purely light ray considerations in conjunction with RP2.

Einstein's intention to use the same kind of light ray technique for determining the function $\eta\,(x.y,z,t)$ however seems even more remarkable. Many post -1905 derivations of the transformation equations for transverse coordinates make use of plausibility, symmetry, reciprocity etc. Einstein's purely algebraic light ray approach would surely still be considered innovative if it were freshly introduced today. A ray, emitted from the common space-time origins of the two frames along the positive η direction of frame k, satisfies the equations in K

$$x-vt = 0, \quad y-t\sqrt{c^2-v^2} = 0, \quad z = 0, \tag{14:E3.9}$$

using RP2, whereas in frame k, the ray satisfies

$$\xi = 0, \qquad \eta - c\tau = 0, \qquad \zeta = 0\,. \tag{15}$$

Using (2:E3.4') in $\eta = c\,\tau$, and substituting from (14:E3.9), Einstein obtains

$$\eta = a\,\beta\,y\,. \tag{16:E3.10a}$$

It is clear that Einstein's equation (16:E3.10a) also is not a general coordinate transformation equation *in the context in which it appears*, since it has only been derived to be true for events along a single ray of the light cone. Equation (16:E3.10a) is true in all space-time of course, but Einstein's substitutions have not proved it so. To complete the η-derivation a second ray also needs to be taken and this ray may be chosen along the negative η-axis. In place of the ray equation *sets* (14) and (15), and their analogues for the second ray, it is preferable to construct *single* equations representing light speed conditions for each ray and for each frame, so that the light -speed equivalence condition between the frames becomes

$$\eta \mp c\tau = 0 \text{ iff } y\sin\theta \mp (ct - x\cos\theta) = 0\,, \tag{17}$$

where sin θ = 1/β, and where $(\cos\theta, \pm\sin\theta, 0)$ gives direction cosines for the two rays in K. From (17) we get off-ray equations of the form,

$$\eta \mp c\tau = g_{1,2}(v)\ (y\sin\theta \mp (ct - x\cos\theta))\,, \tag{18}$$

the two new unknown functions being determined as in the ξ–derivation: $g_2(v) = g_1(v) = a\beta^2$. Thence

$$\eta = a\,\beta\, y \tag{19:E3.10a}$$

is obtained rigorously as a transformation equation true throughout spacetime. Einstein's equation (E3.10b), i.e. (3.14) below, for ζ may be de rived in similar rigorous fashion so that Einstein's approach when supplemented as described above now properly yields his equation set,

$$\tau = a\,\beta^2\,(t - x\,v/c^2)\,, \tag{E3.11}$$

$$\xi = a\,\beta^2(x - vt)\,, \tag{E3.12}$$

$$\eta = a\,\beta\, y\,, \tag{E3.13}$$

$$\zeta = a\,\beta\, z\,. \tag{E3.14}$$

From this point the rest of his procedure, namely to show that the function $a(v)$ satisfies $a\beta = 1$, enables one to arrive at the Lorentz Transformation Equations (E3.26-29) of ZEBK.

The author has extensively searched for any prior direct appearance in print concerning the considerations with respect to ZEBK raised in this paper. One point pos sibly worth noting is the existence of the footnote, apparently due to Einstein (McCausland 1984), which appears in the various reprints and translations of ZEBK (e.g. Einstein, 1905b, p 46), and that reads, in translation, as "the equations of the Lorentz transformation may be more simply deduced directly from [invariance of the light cone]". How is this statement to be understood? Is it a disclaimer, or is it a comment? Since Einstein had ample opportunity from 1905 until the first appearance of that footnote in 1913 in which to send a corrigendum for ZEBK to Annalen der Physik, that footnote must, I believe, be taken at face value without reading more into it than it says, even if it is due to Einstein.

It seems that Einstein never revisited, in print, this particular method of his fundamental 1905 paper. However in a simple derivation of the Lorentz Transformation Equations in his popular Special and General Relativity text (Einstein (1920), Appendix, pp113-118) he did use a two-ray starting point to limit the general form of a 2-by-2 transformation matrix. The matrix was then completely fixed by using the specified relative velocity of the two frames, and by calculating and equating reciprocal length contraction results.

Acknowledgment

The author is most grateful to the chair and faculty of the Physics Department of the University of Otago for their generous hospitality.

References

Einstein, A. (1905a). Zur Elektrodynamik bewegter Koerper. *Annalen der Physik 17*, 891-921

Einstein, A. (1905b). *On the Electrodynamics of Moving Bodies*, in H. A. Lorentz et al, *The Principle of Relativity*. London: Methuen, 1923 (also, New York: Dover – no date, pp 45-65).

Einstein, A. (1920). *Relativity. The Special and the General Theory* . London: Methuen (original German edition, Braunschweig, 1917).

McCausland, I (1984). Einstein and Special Relativity: Who wrote the added footnotes? *Brit J Phil Sci 35*, 60-61.

Miller, A. I. (1981). *Albert Einstein's Special Theory of Relativity: Emergence 1905 and early interpretation 1905-1911*. Reading, Mass: Addison-Wesley.