The compactness of $2^{\mathbb{R}}$ and the axiom of choice

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Abstract

We show that for every well ordered cardinal number m the Tychonoff product 2^m is a compact space without the use of any choice but in Cohen's Second Model $2^{\mathbb{R}}$ is not compact.

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Definition 1 *$AC(\mathbb{R})$ (Form 79 in [1]) is the proposition:*
For every family $\mathcal{A} = \{A_i : i \in k\}$ of non empty subsets of $\mathbb{R}$ there exists a set $c = \{c_i : i \in k\}$ such that $c_i \in A_i$ *for all* $i \in k$.
Form 139 (see [1]) : The Tychonoff product $2^{\mathbb{R}}$ is compact.

Form 139 was introduced by J. Truss (see, [5]) and in [1], p. 352 it is asked whether it is provable in ZF^0 (= Zermelo-Fraenkel set theory ZF minus the axiom of regularity). The aim of this paper is to show that 139 is not provable in ZF^0 by demonstrating its failure in Cohen's Second Model, Model $\mathcal{M}7$ in [1].

In [3], Jan Mycielski proved that the statement:
$\mathcal{S}$. *For every cardinal number m the generalized Cantor set $\{0,1\}^m$ is compact,*
implies the statement:

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$\mathcal{C}$. *For every family of two element sets there exists a selector (a choice set).* In view of this implication, one would expect to prove that Form 139 is not deducible in ZF^0 by showing that every family $\mathcal{F}$ of two element subsets of $\mathbb{R}$ has a choice set. However, this fact is trivially true since from every pair of reals we can always choose the smallest one. Now if we replace families $\mathcal{F}$ of two element sets with families $\mathcal{A}$ of infinite sets then the proof of $\mathcal{S} \to \mathcal{C}$ as given in [3] does not go through and a different approach is required.

Theorem 2 *Form 139 implies "Every family $\mathcal{B} = \{B_i : i \in \omega\}$ of two element subsets of $\mathcal{P}(\mathbb{R})$ has a choice set".*

Proof. Fix $\mathcal{B} = \{B_i : i \in \omega\}$ a family of two element subsets of $\mathcal{P}(\mathbb{R})$. Without loss of generality we may assume that for each $i \in \omega$ and $p, q \in B_i, p \backslash q \neq \emptyset$ and $q \backslash p \neq \emptyset$ as otherwise we can choose $p \cap q$ from B_i. By replacing p with $p \backslash q$ and q by $q \backslash p$ we may also assume that each member of $\mathcal{B}$ is a disjoint set.

Fix a 1:1 and onto function $f_0 : \mathbb{R} \to (0,1)$ and let for every $i \in \omega \backslash \{0\}$ $f_i : \mathbb{R} \to (i, i+1)$ be the function given by $f_i(x) = f_0(x) + i$. It can be readily verified that the collection $\{\{f_i(p), f_i(q) : p, q \in B_i\} : i \in \omega\}$ is a family of two element subsets of $\mathcal{P}(\mathbb{R})$ such that:
(i) $f_i(p) \cap f_i(q) = \emptyset$ for all $i \in \omega$, and
(ii) $(f_i(p) \cup f_i(q)) \cap (f_j(p) \cup f_j(q)) = \emptyset$ for all $i, j \in \omega, i \neq j$.

Thus we may assume that the family $\mathcal{B}$ has the properties listed in (i), (ii) above.

For every $i \in \omega$ put

$$G_i = \{f \in 2^{\mathbb{R}} : f^{-1}(1),\ f^{-1}(0) \text{ separate the elements of } B_i\} \tag{1}$$

and let $G = \{G_i : i \in k\}$. We show first that each G_i is closed in $2^{\mathbb{R}}$. To this end, fix $g \in 2^{\mathbb{R}} \backslash G_i$. If $B_i = \{p, q\}$ then by (1) it follows that either p or q contains two distinct point x, y such that $g(x) = 1$ and $g(y) = 0$. Clearly $O = \pi_x^{-1}(1) \cap \pi_y^{-1}(0)$ is an open neighbourhood of g avoiding G_i or, $g|(p \cup q)$ is identically 1 or identically 0 in which case $O = \pi_x^{-1}(1) \cap \pi_y^{-1}(1), x \in p, y \in q$ ($O = \pi_x^{-1}(0) \cap \pi_y^{-1}(0), x \in p, y \in q$) is an open neighbourhood of g avoiding G_i.

Next we show that G has the finite intersection property (fip for abbreviation). Fix $Q = \{G_{i_n} : n < m\}, m \in \omega$ a finite subset of G. Let $B_{i_n} = \{p_{i_n}, q_{i_n}\}$. Define a function $h \in 2^{\mathbb{R}}$ by requiring:
$h|p_{i_n} = 1$ for all $n < m$ and $h|(\mathbb{R} \backslash \cup \{p_{i_n} : n < m\}) = 0$.

h is well defined because $\cup B_i \cap (\cup B_j) = \emptyset$ for all $i, j \in \omega, i \neq j$ (see the latter condition (ii)).

It can be readily verified that $h \in \cap Q$ and consequently G has the fip.

By Form 139 $\cap G \neq \emptyset$. Fix $t \in \cap G$ and let t_i be the unique element of B_i such that $t_i \subset t^{-1}(1)$. Clearly $c = \{t_i : i \in \omega\}$ is a choice set for the family $\mathcal{B}$ finishing the proof of the theorem. □

Corollary 3 *Form 139 is not provable in ZF.*

Proof. In Cohen's Second Model, Model $\mathcal{M}7$ in [1], the statement
"*Every family $\mathcal{B} = \{B_i : i \in \omega\}$ of two element subsets of $\mathcal{P}(\mathbb{R})$ has a choice set*"
fails. In $\mathcal{M}7$ there are two countably infinite sets $\mathcal{P} = \{p_i : i \in \omega\}$ and $\mathcal{Q} = \{q_i : i \in \omega\}$ such that for each $i \in \omega$, p_i and q_i cannot be distinguished. Thus the family $\mathcal{B} = \{\{p_i, q_i\} : i \in \omega\}$ has no choice set and consequently Form 139 fails finishing the proof of the corollary. □

It has been shown in [2] and independently in [4] that the Boolean Prime Ideal theorem BPI is equivalent to the statement:
The product of compact T_2 spaces is compact
Finally, J. Mycielski proved (see, [3]) that BPI is equivalent to the statement $\mathcal{S}$ (For every cardinal number m the generalized Cantor set 2^m is compact). It is a curious fact that for m a well ordered cardinal, "2^m is compact" is deducible in ZF^0. We demonstrate this result in the next theorem.

Theorem 4 *If m is a well ordered cardinal number then the Tychonoff product 2^m is a compact space.*

Proof. Fix $\mathcal{B} = \{B_i : i \in k\}$ a family of closed subsets of 2^m having the fip and let

$$C = \{\pi_x^{-1}(1), \pi_x^{-1}(0) : x \in m\}.$$

Since for every $x \in m$,

$$\pi_x^{-1}(1) \cup \pi_x^{-1}(0) = 2^m,$$

it follows that for every family $\mathcal{F} \subset \mathcal{P}(2^m)$ having the fip either $\mathcal{F} \cup \{\pi_x^{-1}(1)\}$ or $\mathcal{F} \cup \{\pi_x^{-1}(0)\}$ has the fip. We construct via an easy transfinite induction on m a set

$$\{F_{x_i} : i \in m\}$$

such that for all $i \in m$ the set $\mathcal{B} \cup \{F_{x_j} : j \in i\}$ has the fip.

For $n=0$ let F_{x_0} be the first element of $\{\pi_{x_0}^{-1}(1), \pi_{x_0}^{-1}(0)\}$ such that $\mathcal{B} \cup \{F_{x_0}\}$ has the fip.

For $n = k+1$ a non limit ordinal of m, we let F_{x_n} be the first element of $\{\pi_{x_n}^{-1}(1), \pi_{x_n}^{-1}(0)\}$ such that $\mathcal{B} \cup \{F_{x_i} : i \leq k\} \cup \{F_{x_n}\}$ has the fip.

For n a limit ordinal of m we let F_{x_n} be the first element of $\{\pi_{x_n}^{-1}(1), \pi_{x_n}^{-1}(0)\}$ such that $\mathcal{B} \cup \{F_{x_i} : i \in n\} \cup \{F_{x_n}\}$ has the fip.

Clearly, $\mathcal{B} \cup \{F_{x_i} : i \in m\}$ is a family of closed subsets of 2^m having the fip. Define a function $f : m \to 2$ by requiring:

$$f(x) = \begin{cases} 1 \text{ if } F_x = \pi_x^{-1}(1) \\ 0 \text{ if } F_x = \pi_x^{-1}(0) \end{cases} . \tag{2}$$

In order to complete the proof of the theorem it suffices to show:

Claim. $f \in \cap\mathcal{B}$.

Proof of the Claim. It suffices to show that every basic neighbourhood O_f of f meets non trivially every $B \in \mathcal{B}$. Fix such a $B \in \mathcal{B}$ and let

$$O_f = \pi_{x_{n_1}}^{-1}(f(x_{n_1})) \cap \pi_{x_{n_2}}^{-1}(f(x_{n_2})) \cap ... \cap \pi_{x_{n_v}}^{-1}(f(x_{n_v}))$$

be a basic neighbourhood of f. In view of (2) it follows that

$$O_f = F_{x_{n_1}} \cap F_{x_{n_2}} \cap ... \cap F_{x_{n_v}}.$$

Since $\mathcal{B} \cup \{F_{x_i} : i \in m\}$ has the fip and

$$B, F_{x_{n_j}} \in \mathcal{B} \cup \{F_{x_i} : i \in m\}, j \leq v$$

it follows that $O_f \cap B \neq \emptyset$ finishing the proof of the claim and of the theorem. □

Since Form 79 is equivalent to the statement "$\mathbb{R}$ is well ordered" and Form 79 is true in permutation models we have as a corollary to Theorem 4:

Corollary 5 *(i) Form 79 implies $2^{\mathbb{R}}$ is compact.*
(ii) Form 139 is true in permutation models.

References

[1] P. Howard and J. E. Rubin, *Consequences of the Axiom of Choice*, Math. Surveys and Monographs, AMS **59** (1998).

[2] J. Łoś, C. Ryll-Nardzewski, *Effectiveness of the representation theory for Boolean algebras,* Fund. Math., **41**, 1954, 49-56.

[3] J. Mycielski, *Two remarks on Tychonoff's Product Theorem*, Bull. Acad. Polon. Sci., Vol. XII, No **8**, (1964) 439-441.

[4] H. Rubin and D. Scott, *Some topological theorems equivalent to the Boolean prime ideal theorem*, Bull. AMS **60** (1954), 389.

[5] J. K. Truss, *Cancellation laws for surjective cardinals*, Ann. Pure Appl. Logic, **27**, (1984) 165-207.