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Research Article

Global Dynamics and Bifurcations Analysis of a Two-Dimensional Discrete-Time Lotka-Volterra Model

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In this paper, global dynamics and bifurcations of a two-dimensional discrete-time Lotka-Volterra model have been studied in the closed first quadrant $\mathbb{R}^2$. It is proved that the discrete model has three boundary equilibria and one unique positive equilibrium under certain parametric conditions. We have investigated the local stability of boundary equilibria $O(0,0)$, $A((\alpha_1-1)/\alpha_3,0)$, $B(0,(\alpha_4-1)/\alpha_6)$ and the unique positive equilibrium $C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5),(\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$, by the method of linearization. It is proved that the discrete model undergoes a period-doubling bifurcation in a small neighborhood of boundary equilibria $A((\alpha_1-1)/\alpha_3,0)$, $B(0,(\alpha_4-1)/\alpha_6)$ and a Neimark-Sacker bifurcation in a small neighborhood of the unique positive equilibrium $C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5),(\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$. Further it is shown that every positive solution of the discrete model is bounded and the set $[0,\alpha_1/\alpha_3]\times[0,\alpha_4/\alpha_6]$ is an invariant rectangle. It is proved that if $\alpha_1<1$ and $\alpha_4<1$, then equilibrium $O(0,0)$ of the discrete model is a global attractor. Finally it is proved that the unique positive equilibrium $C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5),(\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$ is a global attractor. Some numerical simulations are presented to illustrate theoretical results.

1. Introduction

In this paper, we study the global dynamics and bifurcations analysis of a two-dimensional discrete-time Lotka-Volterra model in the closed first quadrant $\mathbb{R}^2$, which was proposed by Waltman [1]. In this model, if two populations are growing logistically without affecting each other, then their growth can be represented by the following system of two logistic equations:

$$\begin{aligned}\frac{dx}{dt}&=r_1x\left(1-\frac{x}{k_1}\right)\\ \frac{dy}{dt}&=r_2y\left(1-\frac{y}{k_2}\right),\end{aligned}\tag{1}$$

where $r_1,\ r_2,\ k_1,\ k_2$ and the initial conditions $x_0,\ y_0$ are positive real numbers. Now, assume that the carrying capacity is a shared resource-each population competes for the resource and thereby interferes with the other. Then the presence of each reduces the intrinsic rate of growth of the other. We refer the reader to [1–5] for detailed discussion on the above assumption. This phenomena can be represented as follows:

$$\begin{aligned}\frac{dx}{dt}&=r_1x\left(1-\frac{x}{k_1}-\zeta_1y\right)\\ \frac{dy}{dt}&=r_2y\left(1-\frac{y}{k_2}-\zeta_2x\right),\end{aligned}\tag{2}$$

where $\zeta_1,\ \zeta_2$ are positive constants. It is convenient to change the nondimensional variables by measuring x in units of k_1, y in units of k_2, and time in units of $1/r_1$. Then system (2) takes the following form:

$$\begin{aligned}\frac{dx}{dt}&=x\left(1-x-\lambda_1y\right)\\ \frac{dy}{dt}&=ry\left(1-y-\lambda_2x\right),\end{aligned}\tag{3}$$

where $\lambda_i=k_i\zeta_i$ for $i=1,2$ and $r=r_2/r_1$. It is clear that for all parametric values, system (3) has three boundary equilibria $O(0,0)$, $A(1,0)$, $B(0,1)$ and a unique positive equilibrium point $C((1-\lambda_1)/(1-\lambda_1\lambda_2),(1-\lambda_2)/(1-\lambda_1\lambda_2))$ if $\lambda_1 < 1$, $\lambda_2 < 1$, $\lambda_1\lambda_2 < 1$. According to continuous dynamical systems theory, it is easy to show that equilibrium $O(0,0)$ is a source but never sink and saddle; equilibrium $A(1,0)$ is a sink if $\lambda_2 > 1$ and saddle if $\lambda_2 < 1$, but it is never source; $B(0,1)$ is a sink if $\lambda_1 > 1$ and saddle if $\lambda_1 < 1$, but it is never source, and the unique positive equilibrium point $C((1-\lambda_1)/(1-\lambda_1\lambda_2),(1-\lambda_2)/(1-\lambda_1\lambda_2))$ is locally asymptotically stable.

A discrete dynamical system is defined as a system whose state evolves over state space in discrete-time steps according to a fixed rule. These systems are represented by a system of difference equations. This is a well-known fact that difference equations existed before differential equations and have played a fundamental role in the development of the latter. During the last fifty years the theory of difference equations received attention of both mathematicians and users of mathematics and developed greatly, because of its internal mathematical beauty and applicability in almost all branches of modern science such as ecology, population dynamics, queuing problems, statistical problems, stochastic time series, number theory, geometry, neural networks, quanta in radiation, genetics in biology, economics, psychology, sociology, physics, engineering, economics, combinatorial analysis, probability theory, electrical networks, and resource management [6, 7]. Dynamics of a discrete dynamical system is studied by analyzing the behavior of the solution of the system of difference equations representing the system under study. Analyzing the behavior of solutions of a higher-order nonlinear difference equation is very interesting and attracted many researchers in recent times. Behavior of solutions means studying the equilibrium point, boundedness and persistence, existence and uniqueness of positive equilibrium point, local and global stability, periodicity nature of such difference equations or systems of difference equations (see [8–16] and references cited therein).

Discrete dynamical systems described by difference equations are more appropriate for population dynamics as compared to continuous ones. Biologists believe that the equilibrium point and its stability analysis is important to understand the population dynamics [17, 18]. Therefore, in this paper, we study the behavior of the following discrete-time Lotka-Volterra model, which is obtained by discretization of continuous-time model (3) followed by forward Euler's method. Using forward Euler's method, continuous-time model (3) takes the following form:

$$\begin{aligned} \frac{x_{n+1}-x_n}{h} &= x_n - x_n x_{n+1} - \lambda_1 x_n y_n \\ \frac{y_{n+1}-y_n}{k} &= r y_n - r y_n y_{n+1} - r\lambda_2 x_n y_n. \end{aligned} \tag{4}$$

After some simplification, the above system becomes

$$\begin{aligned} x_{n+1} &= \frac{\alpha_1 x_n - \alpha_2 x_n y_n}{1+\alpha_3 x_n} \\ y_{n+1} &= \frac{\alpha_4 y_n - \alpha_5 x_n y_n}{1+\alpha_6 y_n}, \end{aligned} \tag{5}$$

where $\alpha_1 = 1+h$, $\alpha_2 = h\lambda_1$, $\alpha_3 = h$, $\alpha_4 = 1+kr$, $\alpha_5 = kr\lambda_2$, $\alpha_6 = kr$. It is also noted that the parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ and the initial conditions x_0, y_0 are positive real numbers.

The rest of the paper is organized as follows: In Section 2, we study the existence of equilibria of the discrete model. Section 3 deals with the study of local stability of three boundary equilibria and the unique positive equilibrium. Section 4 deals with the study of bifurcations analysis of boundary equilibria and the unique positive equilibrium. Section 5 discusses the boundedness character and the construction of invariant rectangle of the discrete model. Section 6 discusses the global behavior of $O(0,0)$ and the unique positive equilibrium, whereas Section 7 is about numerical simulation to verify the obtained theoretical results. In the last section a brief conclusion is given.

2. Existence of Equilibria

In this section, we will study the existence of equilibria of the discrete model (5) in the closed first quadrant $\mathbb{R}^2$. The results about the existence of equilibria are summarized as follows.

Lemma 1. *Under certain parametric conditions, system (5) has at least three boundary equilibria and one unique positive equilibrium in the closed first quadrant $\mathbb{R}^2$. More precisely,*

(i) *system (5) has a unique boundary equilibrium $O(0,0)$ if $\alpha_1 < 1$, $\alpha_4 < 1$, $\alpha_3\alpha_6 < \alpha_2\alpha_5$, $\alpha_6 < \alpha_2(\alpha_4-1)/(\alpha_1-1)$ and $\alpha_5 > \alpha_3(\alpha_4-1)/(\alpha_1-1)$;*

(ii) *system (5) has two boundary equilibrium $O(0,0)$, $A((\alpha_1-1)/\alpha_3,0)$ if $\alpha_1 > 1$, $\alpha_4 < 1$, $\alpha_3\alpha_6 < \alpha_2\alpha_5$, $\alpha_6 < \alpha_2(\alpha_4-1)/(\alpha_1-1)$ and $\alpha_5 > \alpha_3(\alpha_4-1)/(\alpha_1-1)$;*

(iii) *system (5) has three boundary equilibrium $O(0,0)$, $A((\alpha_1-1)/\alpha_3,0)$, $B(0,(\alpha_4-1)/\alpha_6)$ if $\alpha_1 > 1$, $\alpha_4 > 1$, $\alpha_3\alpha_6 < \alpha_2\alpha_5$, $\alpha_6 < \alpha_2(\alpha_4-1)/(\alpha_1-1)$ and $\alpha_5 > \alpha_3(\alpha_4-1)/(\alpha_1-1)$;*

(iv) *system (5) has three boundary equilibrium $O(0,0)$, $A((\alpha_1-1)/\alpha_3,0)$, $B(0,(\alpha_4-1)/\alpha_6)$ and one interior equilibrium $C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5),(\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$ if $\alpha_1 > 1$, $\alpha_4 > 1$, $\alpha_3\alpha_6 > \alpha_2\alpha_5$, $\alpha_6 > \alpha_2(\alpha_4-1)/(\alpha_1-1)$ and $\alpha_5 < \alpha_3(\alpha_4-1)/(\alpha_1-1)$. Additionally if $\alpha_1 > 1$, $\alpha_4 > 1$, $\alpha_3\alpha_6 > \alpha_2\alpha_5$, $\alpha_6 > \alpha_2(\alpha_4-1)/(\alpha_1-1)$ and $\alpha_5 < \alpha_3(\alpha_4-1)/(\alpha_1-1)$, then $C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5),(\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$ is the unique positive equilibrium of system (5).*

Proof. In order to find equilibria of system (5) in the interior of $\mathbb{R}^2$, we need to solve the following algebraic equations:

$$\begin{aligned} x &= \frac{\alpha_1 x - \alpha_2 xy}{1+\alpha_3 x} \\ y &= \frac{\alpha_4 y - \alpha_5 xy}{1+\alpha_6 y}. \end{aligned} \tag{6}$$

If $x = 0$, $y = 0$, then (6) are identically satisfied, and hence for all parametric values $O(0,0)$ is the unique equilibrium of this system. If $y = 0$ then the second equation of (6) is identically satisfied, and from the first equation we get $x = (\alpha_1 - 1)/\alpha_3$. Thus if $\alpha_1 > 1$, then $A((\alpha_1 - 1)/\alpha_3, 0)$ is one boundary equilibrium point of system (5). If $x = 0$, then first equation of (6) is identically satisfied, and from the second equation we get $y = (\alpha_4 - 1)/\alpha_6$. Thus if $\alpha_4 > 1$, then $B(0, (\alpha_4 - 1)/\alpha_6)$ is again a boundary equilibrium of system (5).

On the other hand, we consider the existence of unique positive equilibrium of system (5) in the interior of $\mathbb{R}^2$. For this assume that if $x \neq 0$ and $y \neq 0$, then system (6) takes the following form:

$$1 = \frac{\alpha_1 - \alpha_2 y}{1 + \alpha_3 x}$$
$$1 = \frac{\alpha_4 - \alpha_5 x}{1 + \alpha_6 y}. \tag{7}$$

Solving system (7) for x and y, one gets $(x, y) = (((\alpha_1 - 1)\alpha_6 - \alpha_2(\alpha_4 - 1))/(\alpha_3\alpha_6 - \alpha_2\alpha_5), (\alpha_3(\alpha_4 - 1) + \alpha_5(1 - \alpha_1))/(\alpha_3\alpha_6 - \alpha_2\alpha_5))$. Hence if $\alpha_1 > 1$, $\alpha_4 > 1$, $\alpha_3\alpha_6 > \alpha_2\alpha_5$, $\alpha_6 > \alpha_2(\alpha_4 - 1)/(\alpha_1 - 1)$ and $\alpha_5 < \alpha_3(\alpha_4 - 1)/(\alpha_1 - 1)$, then

$$C\left(\frac{(\alpha_1 - 1)\alpha_6 - \alpha_2(\alpha_4 - 1)}{\alpha_3\alpha_6 - \alpha_2\alpha_5}, \frac{\alpha_3(\alpha_4 - 1) + \alpha_5(1 - \alpha_1)}{\alpha_3\alpha_6 - \alpha_2\alpha_5}\right) \tag{8}$$

is the unique positive equilibrium of system (5). □

Remark 2. The discrete model (5) has three boundary equilibria $O(0,0)$, $A((\alpha_1 - 1)/\alpha_3, 0)$, $B(0, (\alpha_4 - 1)/\alpha_6)$ and one interior equilibrium $C(((\alpha_1 - 1)\alpha_6 - \alpha_2(\alpha_4 - 1))/(\alpha_3\alpha_6 - \alpha_2\alpha_5), (\alpha_3(\alpha_4 - 1) + \alpha_5(1 - \alpha_1))/(\alpha_3\alpha_6 - \alpha_2\alpha_5))$ if $\alpha_1 > 1$, $\alpha_4 > 1$, $\alpha_3\alpha_6 > \alpha_2\alpha_5$, $\alpha_6 > \alpha_2(\alpha_4 - 1)/(\alpha_1 - 1)$ and $\alpha_5 < \alpha_3(\alpha_4 - 1)/(\alpha_1 - 1)$, where $\alpha_1 = 1 + h$, $\alpha_2 = h\lambda_1$, $\alpha_3 = h$, $\alpha_4 = 1 + kr$, $\alpha_5 = kr\lambda_2$, $\alpha_6 = kr$. Now, using the values α_i, $i = 1, \ldots, 6$, the equilibria of continuous-time model (3), $O(0,0)$, $A(1,0)$, $B(0,1)$, and $C((1 - \lambda_1)/(1 - \lambda_1\lambda_2), (1 - \lambda_2)/(1 - \lambda_1\lambda_2))$ can be recovered with the same conditions on the parameters $\lambda_1 < 1$, $\lambda_2 < 1$.

3. Local Stability

The Jacobian matrix $J_{(x,y)}$ of linearized system of (5) about equilibrium (x, y) is

$$J_{(x,y)} = \begin{pmatrix} \dfrac{\alpha_1 - \alpha_2 y}{(1 + \alpha_3 x)^2} & -\dfrac{\alpha_2 x}{1 + \alpha_3 x} \\ -\dfrac{\alpha_5 y}{1 + \alpha_6 y} & \dfrac{\alpha_4 - \alpha_5 x}{(1 + \alpha_6 y)^2} \end{pmatrix}. \tag{9}$$

3.1. Local Stability of Boundary Equilibria. Hereafter we will study the topological classification of the boundary equilibria. The results regarding the local stability of the boundary equilibria are summarized as follows.

Theorem 3. *For equilibrium point $O(0,0)$, the following statements hold:*

(i) *The equilibrium point $O(0,0)$ of system (5) is a sink if $\alpha_1 < 1$ and $\alpha_4 < 1$;*

(ii) *The equilibrium point $O(0,0)$ of system (5) is a source if $\alpha_1 > 1$ and $\alpha_4 > 1$;*

(iii) *The equilibrium point $O(0,0)$ of system (5) is a saddle if $\alpha_1 > 1$ and $\alpha_4 < 1$;*

(iv) *The equilibrium point $O(0,0)$ of system (5) is nonhyperbolic if $\alpha_1 = 1$ or $\alpha_4 = 1$.*

Theorem 4. *For equilibrium point $A((\alpha_1 - 1)/\alpha_3, 0)$, the following statements hold:*

(i) *The equilibrium $A((\alpha_1 - 1)/\alpha_3, 0)$ of system (5) is a sink if $\alpha_1 > 1$ and $\alpha_5 < \alpha_3(\alpha_4 + 1)/(\alpha_1 - 1)$;*

(ii) *The equilibrium $A((\alpha_1 - 1)/\alpha_3, 0)$ of system (5) is never source;*

(iii) *The equilibrium $A((\alpha_1 - 1)/\alpha_3, 0)$ of system (5) is a saddle if $\alpha_1 > 1$ and $\alpha_5 > \alpha_3(\alpha_4 + 1)/(\alpha_1 - 1)$;*

(iv) *The equilibrium $A((\alpha_1 - 1)/\alpha_3, 0)$ of system (5) is nonhyperbolic if $\alpha_5 = \alpha_3(\alpha_4 + 1)/(\alpha_1 - 1)$.*

Theorem 5. *For equilibrium point $B(0, (\alpha_4 - 1)/\alpha_6)$, the following statements hold:*

(i) *The equilibrium $B(0, (\alpha_4 - 1)/\alpha_6)$ of system (5) is a sink if $\alpha_4 > 1$ and $\alpha_2 < (\alpha_1 + 1)\alpha_6/(\alpha_4 - 1)$;*

(ii) *The equilibrium $B(0, (\alpha_4 - 1)/\alpha_6)$ of system (5) is never source;*

(iii) *The equilibrium $B(0, (\alpha_4 - 1)/\alpha_6)$ of system (5) is a saddle if $\alpha_4 > 1$ and $\alpha_2 > (\alpha_1 + 1)\alpha_6/(\alpha_4 - 1)$;*

(iv) *The equilibrium $B(0, (\alpha_4 - 1)/\alpha_6)$ of system (5) is nonhyperbolic if $\alpha_2 = (\alpha_1 + 1)\alpha_6/(\alpha_4 - 1)$.*

Now in the following we will study the local stability of the unique positive equilibrium $C(((\alpha_1 - 1)\alpha_6 - \alpha_2(\alpha_4 - 1))/(\alpha_3\alpha_6 - \alpha_2\alpha_5), (\alpha_3(\alpha_4 - 1) + \alpha_5(1 - \alpha_1))/(\alpha_3\alpha_6 - \alpha_2\alpha_5))$ by using Remark 1.3.1 of [7].

3.2. Local Stability of the Unique Positive Equilibrium

Theorem 6. *For the unique positive equilibrium $C(((\alpha_1 - 1)\alpha_6 - \alpha_2(\alpha_4 - 1))/(\alpha_3\alpha_6 - \alpha_2\alpha_5), (\alpha_3(\alpha_4 - 1) + \alpha_5(1 - \alpha_1))/(\alpha_3\alpha_6 - \alpha_2\alpha_5))$ of system (5), the following statements hold:*

(i) *The unique positive equilibrium point $C(((\alpha_1 - 1)\alpha_6 - \alpha_2(\alpha_4 - 1))/(\alpha_3\alpha_6 - \alpha_2\alpha_5), (\alpha_3(\alpha_4 - 1) + \alpha_5(1 - \alpha_1))/(\alpha_3\alpha_6 - \alpha_2\alpha_5))$ of system (5) is locally asymptotically stable if*

$$\Theta < (\alpha_2\alpha_3 - \alpha_2\alpha_3\alpha_4 - \alpha_2\alpha_5 + \alpha_1\alpha_3\alpha_6) \cdot (-\alpha_2\alpha_5 + \alpha_3\alpha_4\alpha_6 + \alpha_5\alpha_6 - \alpha_1\alpha_5\alpha_6); \tag{10}$$

(ii) *The unique positive equilibrium point* $C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5),(\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$ *of system (5) is unstable if*

$$\Theta > (\alpha_2\alpha_3-\alpha_2\alpha_3\alpha_4-\alpha_2\alpha_5+\alpha_1\alpha_3\alpha_6) \cdot (-\alpha_2\alpha_5+\alpha_3\alpha_4\alpha_6+\alpha_5\alpha_6-\alpha_1\alpha_5\alpha_6), \tag{11}$$

where

$$\begin{aligned}\Theta = {}& -2\alpha_2\alpha_3\alpha_6\alpha_5+{\alpha_3}^2{\alpha_6}^2(\alpha_1+\alpha_4+1) \\ &+{\alpha_6}^2\alpha_3\alpha_5(1-\alpha_1)+{\alpha_3}^2\alpha_6\alpha_2(1-\alpha_4) \\ &+{\alpha_2}^2\alpha_5^2((\alpha_1+1)\alpha_4+\alpha_1-2) \\ &+\alpha_5^2\alpha_2\alpha_6(2+\alpha_1(\alpha_1+1)) \\ &+{\alpha_2}^2\alpha_3\alpha_5(2+\alpha_4(\alpha_4+1)) \\ &+\alpha_2\alpha_3\alpha_6\alpha_5(3+(\alpha_1+2)\alpha_4+2\alpha_1).\end{aligned} \tag{12}$$

Proof. (i) The Jacobian matrix $J_{C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5),(\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))}$ of linearized system of (5) about $C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5),(\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$ is

$$\begin{aligned}&J_{C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5),(\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))} \\ &\quad= \begin{pmatrix} \dfrac{\alpha_3\alpha_6-\alpha_2\alpha_5}{\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5-\alpha_2\alpha_3\alpha_4+\alpha_2\alpha_3} & -\dfrac{\alpha_2\alpha_6(\alpha_1-1)-{\alpha_2}^2(\alpha_4-1)}{\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5-\alpha_2\alpha_3\alpha_4+\alpha_2\alpha_3} \\ -\dfrac{\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1)}{\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5-\alpha_6\alpha_5\alpha_1+\alpha_6\alpha_5} & \dfrac{\alpha_3\alpha_6-\alpha_2\alpha_5}{\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5-\alpha_6\alpha_5\alpha_1+\alpha_6\alpha_5} \end{pmatrix}.\end{aligned} \tag{13}$$

The characteristic polynomial of $J_{C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5),(\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))}$ about

$$C\left(\frac{(\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1)}{\alpha_3\alpha_6-\alpha_2\alpha_5}, \frac{\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1)}{\alpha_3\alpha_6-\alpha_2\alpha_5}\right) \tag{14}$$

is

$$P(\lambda)=\lambda^2-\Omega_1\lambda+\Omega_2, \tag{15}$$

where

$$\begin{aligned}
\Omega_1 &= A+B-C-D,\\
\Omega_2 &= E-F+G+H+I-J-K+L+M-N+O-P+Q+R,\\
A &= \frac{\alpha_3\alpha_6}{\alpha_2\alpha_3-\alpha_2\alpha_3\alpha_4-\alpha_2\alpha_5+\alpha_1\alpha_3\alpha_6},\\
B &= \frac{\alpha_3\alpha_6}{-\alpha_2\alpha_5+\alpha_3\alpha_4\alpha_6+\alpha_5\alpha_6-\alpha_1\alpha_5\alpha_6},\\
C &= \frac{\alpha_2\alpha_5}{\alpha_2\alpha_3-\alpha_2\alpha_3\alpha_4-\alpha_2\alpha_5+\alpha_1\alpha_3\alpha_6},\\
D &= \frac{\alpha_2\alpha_5}{-\alpha_2\alpha_5+\alpha_3\alpha_4\alpha_6+\alpha_5\alpha_6-\alpha_1\alpha_5\alpha_6},\\
E &= \frac{{\alpha_2}^2\alpha_3\alpha_5}{(\alpha_2\alpha_3-\alpha_2\alpha_3\alpha_4-\alpha_2\alpha_5+\alpha_1\alpha_3\alpha_6)(-\alpha_2\alpha_5+\alpha_3\alpha_4\alpha_6+\alpha_5\alpha_6-\alpha_1\alpha_5\alpha_6)},\\
F &= \frac{2{\alpha_2}^2\alpha_3\alpha_4\alpha_5}{(\alpha_2\alpha_3-\alpha_2\alpha_3\alpha_4-\alpha_2\alpha_5+\alpha_1\alpha_3\alpha_6)(-\alpha_2\alpha_5+\alpha_3\alpha_4\alpha_6+\alpha_5\alpha_6-\alpha_1\alpha_5\alpha_6)},\\
G &= \frac{{\alpha_2}^2\alpha_3{\alpha_4}^2\alpha_5}{(\alpha_2\alpha_3-\alpha_2\alpha_3\alpha_4-\alpha_2\alpha_5+\alpha_1\alpha_3\alpha_6)(-\alpha_2\alpha_5+\alpha_3\alpha_4\alpha_6+\alpha_5\alpha_6-\alpha_1\alpha_5\alpha_6)},
\end{aligned}$$

$$
\begin{aligned}
H &= \frac{\alpha_1 \alpha_2{}^2 \alpha_5^2}{\left(\alpha_2\alpha_3 - \alpha_2\alpha_3\alpha_4 - \alpha_2\alpha_5 + \alpha_1\alpha_3\alpha_6\right)\left(-\alpha_2\alpha_5 + \alpha_3\alpha_4\alpha_6 + \alpha_5\alpha_6 - \alpha_1\alpha_5\alpha_6\right)},\\
I &= \frac{\alpha_2{}^2 \alpha_4 \alpha_5^2}{\left(\alpha_2\alpha_3 - \alpha_2\alpha_3\alpha_4 - \alpha_2\alpha_5 + \alpha_1\alpha_3\alpha_6\right)\left(-\alpha_2\alpha_5 + \alpha_3\alpha_4\alpha_6 + \alpha_5\alpha_6 - \alpha_1 - \alpha_1\alpha_5\alpha_6\right)},\\
J &= \frac{\alpha_1 \alpha_2{}^2 \alpha_4 \alpha_5^2}{\left(\alpha_2\alpha_3 - \alpha_2\alpha_3\alpha_4 - \alpha_2\alpha_5 + \alpha_1\alpha_3\alpha_6\right)\left(-\alpha_2\alpha_5 + \alpha_3\alpha_4\alpha_6 + \alpha_5\alpha_6 - \alpha_1\alpha_5\alpha_6\right)},\\
K &= \frac{3\alpha_2\alpha_3\alpha_5\alpha_6}{\left(\alpha_2\alpha_3 - \alpha_2\alpha_3\alpha_4 - \alpha_2\alpha_5 + \alpha_1\alpha_3\alpha_6\right)\left(-\alpha_2\alpha_5 + \alpha_3\alpha_4\alpha_6 + \alpha_5\alpha_6 - \alpha_1\alpha_5\alpha_6\right)},\\
L &= \frac{\alpha_1\alpha_2\alpha_3\alpha_5\alpha_6}{\left(\alpha_2\alpha_3 - \alpha_2\alpha_3\alpha_4 - \alpha_2\alpha_5 + \alpha_1\alpha_3\alpha_6\right)\left(-\alpha_2\alpha_5 + \alpha_3\alpha_4\alpha_6 + \alpha_5\alpha_6 - \alpha_1\alpha_5\alpha_6\right)},\\
M &= \frac{\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6}{\left(\alpha_2\alpha_3 - \alpha_2\alpha_3\alpha_4 - \alpha_2\alpha_5 + \alpha_1\alpha_3\alpha_6\right)\left(-\alpha_2\alpha_5 + \alpha_3\alpha_4\alpha_6 + \alpha_5\alpha_6 - \alpha_1\alpha_5\alpha_6\right)},\\
N &= \frac{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6}{\left(\alpha_2\alpha_3 - \alpha_2\alpha_3\alpha_4 - \alpha_2\alpha_5 + \alpha_1\alpha_3\alpha_6\right)\left(-\alpha_2\alpha_5 + \alpha_3\alpha_4\alpha_6 + \alpha_5\alpha_6 - \alpha_1\alpha_5\alpha_6\right)},\\
O &= \frac{\alpha_2\alpha_5^2\alpha_6}{\left(\alpha_2\alpha_3 - \alpha_2\alpha_3\alpha_4 - \alpha_2\alpha_5 + \alpha_1\alpha_3\alpha_6\right)\left(-\alpha_2\alpha_5 + \alpha_3\alpha_4\alpha_6 + \alpha_5\alpha_6 - \alpha_1\alpha_5\alpha_6\right)},\\
P &= \frac{2\alpha_1\alpha_2\alpha_5^2\alpha_6}{\left(\alpha_2\alpha_3 - \alpha_2\alpha_3\alpha_4 - \alpha_2\alpha_5 + \alpha_1\alpha_3\alpha_6\right)\left(-\alpha_2\alpha_5 + \alpha_3\alpha_4\alpha_6 + \alpha_5\alpha_6 - \alpha_1\alpha_5\alpha_6\right)},\\
Q &= \frac{\alpha_1{}^2\alpha_2\alpha_5^2\alpha_6}{\left(\alpha_2\alpha_3 - \alpha_2\alpha_3\alpha_4 - \alpha_2\alpha_5 + \alpha_1\alpha_3\alpha_6\right)\left(-\alpha_2\alpha_5 + \alpha_3\alpha_4\alpha_6 + \alpha_5\alpha_6 - \alpha_1\alpha_5\alpha_6\right)},\\
R &= \frac{\alpha_3{}^2\alpha_6{}^2}{\left(\alpha_2\alpha_3 - \alpha_2\alpha_3\alpha_4 - \alpha_2\alpha_5 + \alpha_1\alpha_3\alpha_6\right)\left(-\alpha_2\alpha_5 + \alpha_3\alpha_4\alpha_6 + \alpha_5\alpha_6 - \alpha_1\alpha_5\alpha_6\right)}.
\end{aligned}
\tag{16}
$$

Assume that $\Theta < (\alpha_2\alpha_3 - \alpha_2\alpha_3\alpha_4 - \alpha_2\alpha_5 + \alpha_1\alpha_3\alpha_6)(-\alpha_2\alpha_5 + \alpha_3\alpha_4\alpha_6 + \alpha_5\alpha_6 - \alpha_1\alpha_5\alpha_6)$, and using Remark 1.3.1 of [7] one gets

$$
\begin{aligned}
&\left|\Omega_1\right| + \left|\Omega_2\right| \le A + B + C + D + E + F + G + H + I + J + K + L + M + N + O + P + Q + R,\\
&\quad < \frac{\Theta}{\left(\alpha_2\alpha_3 - \alpha_2\alpha_3\alpha_4 - \alpha_2\alpha_5 + \alpha_1\alpha_3\alpha_6\right)\left(-\alpha_2\alpha_5 + \alpha_3\alpha_4\alpha_6 + \alpha_5\alpha_6 - \alpha_1\alpha_5\alpha_6\right)} < 1.
\end{aligned}
\tag{17}
$$

Therefore $C(((\alpha_1 - 1)\alpha_6 - \alpha_2(\alpha_4 - 1))/(\alpha_3\alpha_6 - \alpha_2\alpha_5), (\alpha_3(\alpha_4 - 1) + \alpha_5(1 - \alpha_1))/(\alpha_3\alpha_6 - \alpha_2\alpha_5))$ of system (5) is locally asymptotically stable.

(ii) Similarly it is easy to show that $C(((\alpha_1 - 1)\alpha_6 - \alpha_2(\alpha_4 - 1))/(\alpha_3\alpha_6 - \alpha_2\alpha_5), (\alpha_3(\alpha_4 - 1) + \alpha_5(1 - \alpha_1))/(\alpha_3\alpha_6 - \alpha_2\alpha_5))$ of system (5) is unstable if $\Theta > (\alpha_2\alpha_3 - \alpha_2\alpha_3\alpha_4 - \alpha_2\alpha_5 + \alpha_1\alpha_3\alpha_6)(-\alpha_2\alpha_5 + \alpha_3\alpha_4\alpha_6 + \alpha_5\alpha_6 - \alpha_1\alpha_5\alpha_6)$. □

Hereafter we will compute the necessary and sufficient condition(s) for the unique positive equilibrium $C(((\alpha_1 - 1)\alpha_6 - \alpha_2(\alpha_4 - 1))/(\alpha_3\alpha_6 - \alpha_2\alpha_5), (\alpha_3(\alpha_4 - 1) + \alpha_5(1 - \alpha_1))/(\alpha_3\alpha_6 - \alpha_2\alpha_5))$ of system (5) to be locally asymptotically stable, repeller, saddle, and nonhyperbolic, respectively.

Theorem 7. *For equilibrium $C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5), (\alpha_3(\alpha_4 - 1) + \alpha_5(1 - \alpha_1))/(\alpha_3\alpha_6 - \alpha_2\alpha_5))$ of system (5), the following statements hold:*

(i) *Equilibrium $C(((\alpha_1 - 1)\alpha_6 - \alpha_2(\alpha_4 - 1))/(\alpha_3\alpha_6 - \alpha_2\alpha_5), (\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$ of system (5) is locally asymptotically stable if and only if*

$$\left|\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)(\alpha_3(\alpha_6(\alpha_1+\alpha_4)+\alpha_2(1-\alpha_4))+\alpha_5(\alpha_6(1-\alpha_1)-2\alpha_2))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))}\right| < 1-\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-\left(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1)\right)\left(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1)\right)}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))} < 2. \tag{18}$$

(ii) *Equilibrium* $C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5), (\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$ *of system (5) is a repeller if and only if*

$$\begin{gathered}\left|\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-\left(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1)\right)\left(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1)\right)}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))}\right| > 1, \\ \left|\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)(\alpha_3(\alpha_6(\alpha_1+\alpha_4)+\alpha_2(1-\alpha_4))+\alpha_5(\alpha_6(1-\alpha_1)-2\alpha_2))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))}\right| < \left|1-\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-\left(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1)\right)\left(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1)\right)}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))}\right|.\end{gathered} \tag{19}$$

(iii) *Equilibrium* $C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5), (\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$ *of system (5) is a saddle if and only if*

$$\begin{gathered}\left(\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)(\alpha_3(\alpha_6(\alpha_1+\alpha_4)+\alpha_2(1-\alpha_4))+\alpha_5(\alpha_6(1-\alpha_1)-2\alpha_2))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))}\right)^2 + 4\left(1-\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-\left(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1)\right)\left(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1)\right)}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))}\right) > 0, \\ \left|\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)(\alpha_3(\alpha_6(\alpha_1+\alpha_4)+\alpha_2(1-\alpha_4))+\alpha_5(\alpha_6(1-\alpha_1)-2\alpha_2))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))}\right| > \left|1-\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-\left(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1)\right)\left(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1)\right)}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))}\right|.\end{gathered} \tag{20}$$

(iv) *Equilibrium* $C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5), (\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$ *of system (5) is nonhyperbolic if and only if*

$$\left|\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)(\alpha_3(\alpha_6(\alpha_1+\alpha_4)+\alpha_2(1-\alpha_4))+\alpha_5(\alpha_6(1-\alpha_1)-2\alpha_2))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))}\right| = \left|1-\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-\left(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1)\right)\left(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1)\right)}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))}\right|. \tag{21}$$

Proof. (i) More precisely the characteristic equation of $J_{C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5),(\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))}$ about

$$C\left(\frac{(\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1)}{\alpha_3\alpha_6-\alpha_2\alpha_5}, \frac{\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1)}{\alpha_3\alpha_6-\alpha_2\alpha_5}\right) \tag{22}$$

is given by

$$\lambda^2-p\lambda+q=0, \tag{23}$$

where

$$\begin{gathered} p=\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)(\alpha_3(\alpha_6(\alpha_1+\alpha_4)+\alpha_2(1-\alpha_4))+\alpha_5(\alpha_6(1-\alpha_1)-2\alpha_2))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))} \\ q=\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-\left(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1)\right)\left(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1)\right)}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))}. \end{gathered} \tag{24}$$

Then, it follows from Theorem 1.1.1 of [12] that the unique positive equilibrium

$$C\left(\frac{(\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1)}{\alpha_3\alpha_6-\alpha_2\alpha_5}, \frac{\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1)}{\alpha_3\alpha_6-\alpha_2\alpha_5}\right) \tag{25}$$

of system (1) is locally asymptotically stable if and only if

$$\begin{gathered} \left|\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)(\alpha_3(\alpha_6(\alpha_1+\alpha_4)+\alpha_2(1-\alpha_4))+\alpha_5(\alpha_6(1-\alpha_1)-2\alpha_2))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))}\right| \\ <1-\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-\left(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1)\right)\left(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1)\right)}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))}<2. \end{gathered} \tag{26}$$

Similarly, one can prove (ii), (iii), and (iv). □

4. Bifurcations Analysis

In this section, we will study the bifurcation analysis of discrete model (5) about the equilibria $O(0,0)$, $A((\alpha_1-1)/\alpha_3,0)$, $B(0,(\alpha_4-1)/\alpha_6)$ and $C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5),(\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$. From theoretical results obtained in Section 3, we conclude the following:

(i) If condition (iv), that is, $\alpha_1=1$, of Theorem 3 holds, then one of the eigenvalues of $J_{O(0,0)}$ about $O(0,0)$ is 1 and so fold bifurcation may occur when parameters vary in a small neighborhood of $\alpha_1=1$. The condition (iv) of Theorem 3 can be rewritten as the following set:

$$F_{O(0,0)}=\{(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6):\alpha_1=1\}. \tag{27}$$

(ii) From Theorem 4, we can see that one of the eigenvalues of $J_{A((\alpha_1-1)/\alpha_3,0)}$ about $A((\alpha_1-1)/\alpha_3,0)$ is -1 and other is neither 1 nor -1 when the parameters of the discrete model (5) are located in the following set:

$$F_{A((\alpha_1-1)/\alpha_3,0)} = \left\{(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6):\alpha_5=\frac{\alpha_3(\alpha_4+1)}{\alpha_1-1}\right\}. \tag{28}$$

Therefore, $A((\alpha_1-1)/\alpha_3,0)$ can undergo flip or period-doubling bifurcation when all parameters of the discrete model (5) vary in a small neighborhood of $F_{A((\alpha_1-1)/\alpha_3,0)}$. When the parameters are in $F_{A((\alpha_1-1)/\alpha_3,0)}$, a center manifold of the discrete model (5) is $y=0$, and thus (5) restricted to this central manifold is

$$x_{n+1}=\frac{\alpha_1 x_n}{1+\alpha_3 x_n}. \tag{29}$$

This shows that the predator becomes extension and prey undergoes period-doubling bifurcation to chaos on choosing bifurcation parameter α_5.

(iii) From Theorem 5, it is easy to verify that one of the eigenvalues of $J_{B(0,(\alpha_4-1)/\alpha_6)}$ about $B(0,(\alpha_4-1)/\alpha_6)$ is -1 and the other is neither 1 nor -1 when the parameters of the discrete model (5) are located in the following set:

$$F_{B(0,(\alpha_4-1)/\alpha_6)} = \left\{(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6) : \alpha_2 = \frac{(\alpha_1+1)\alpha_6}{\alpha_4-1}\right\}. \tag{30}$$

Therefore, $B(0,(\alpha_4-1)/\alpha_6)$ undergoes period-doubling bifurcation when all parameters of the discrete model (5) vary in a small neighborhood of $F_{B(0,(\alpha_4-1)/\alpha_6)}$. When the parameters are in $F_{B(0,(\alpha_4-1)/\alpha_6)}$, a center manifold of the discrete model (5) is $x = 0$, and thus (5) restricted to this central manifold is

$$y_{n+1} = \frac{\alpha_4 y_n}{1+\alpha_6 y_n}. \tag{31}$$

This shows that the prey becomes extension and predator undergoes period-doubling bifurcation to chaos on choosing bifurcation parameter α_2.

Now before studying the bifurcation analysis of the discrete model (5) about the unique positive equilibrium point $C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5), (\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$, first we will prove that there exist different topological types of this equilibrium point. Recall that eigenvalues of $J_{C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5),(\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))}$ about

$$C\left(\frac{(\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1)}{\alpha_3\alpha_6-\alpha_2\alpha_5}, \frac{\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1)}{\alpha_3\alpha_6-\alpha_2\alpha_5}\right) \tag{32}$$

are

$$\lambda_{1,2} = \frac{-p\pm\sqrt{\Delta}}{2}, \tag{33}$$

where

$$\begin{aligned}\Delta &= p^2-4q,\\ &= \left(\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)(\alpha_3(\alpha_6(\alpha_1+\alpha_4))+\alpha_2(1-\alpha_4))+\alpha_5(\alpha_6(1-\alpha_1)-2\alpha_2))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))}\right)^2\\ &\quad-4\left(\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1))(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))}\right).\end{aligned} \tag{34}$$

Hereafter we state the topological classification about $C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5), (\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$ of system (5) as follows.

Theorem 8. *For $C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5), (\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$ of system (5), the following topological classification holds:*

(i) *$C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5), (\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$ is a locally asymptotically stable focus if $\Delta < 0$ and*

$$\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1))(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))} < 1; \tag{35}$$

(ii) *$C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5), (\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$ is an unstable focus if $\Delta < 0$ and*

$$\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1))(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))} > 1; \tag{36}$$

(iii) *$C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5),(\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$ is nonhyperbolic if $\Delta<0$ and*

$$\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-\left(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1)\right)\left(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1)\right)}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))}=1. \quad (37)$$

We can see that the eigenvalues of $J_{C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5),(\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))}$ are a pair of complex conjugates with modulus 1 when parameters of model (5) are located in the following set:

$$N_C=\left\{(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6):\Delta<0,\ \frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-\left(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1)\right)\left(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1)\right)}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))}=1\right\}. \quad (38)$$

Therefore, $C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5),(\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$ undergoes Neimark-Sacker bifurcation when all parameters of model (5) vary in a small neighborhood of $N_{C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5),(\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))}$.

Hereafter we will study the Neimark-Sacker bifurcation of model (5) about

$$C\left(\frac{(\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1)}{\alpha_3\alpha_6-\alpha_2\alpha_5},\frac{\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1)}{\alpha_3\alpha_6-\alpha_2\alpha_5}\right). \quad (39)$$

For given parameters $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6$, let

$$\begin{aligned}F(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6)&=(\alpha_3\alpha_6-\alpha_2\alpha_5)^2\\&-\left(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1)\right)\\&\cdot\left(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1)\right)\\&-(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))\\&\cdot(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1)).\end{aligned} \quad (40)$$

It is easy to verify that $\partial F(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6)/\partial\alpha_1<0$, then by implicit function theorem we obtain that $\alpha_1=\alpha_1(\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6)$ such that $F(\alpha_1(\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6),\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6)=0$, and therefore we can choose α_1 as a bifurcation parameter.

Now consider parameter α_1 in a small neighborhood of α_1^*; that is, $\alpha_1=\alpha_1^*+\epsilon$, where $\epsilon\ll 1$; then system (5) becomes

$$\begin{aligned}x_{n+1}&=\frac{(\alpha_1^*+\epsilon)x_n-\alpha_2x_ny_n}{1+\alpha_3x_n}\\y_{n+1}&=\frac{\alpha_4y_n-\alpha_5x_ny_n}{1+\alpha_6y_n}.\end{aligned} \quad (41)$$

The characteristic equation of $J_{C(((\alpha_1^*+\epsilon-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5),(\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1^*-\epsilon))/(\alpha_3\alpha_6-\alpha_2\alpha_5))}$ about

$$C\left(\frac{(\alpha_1^*+\epsilon-1)\alpha_6-\alpha_2(\alpha_4-1)}{\alpha_3\alpha_6-\alpha_2\alpha_5},\frac{\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1^*-\epsilon)}{\alpha_3\alpha_6-\alpha_2\alpha_5}\right) \quad (42)$$

of system (41) is

$$\lambda^2-p(\epsilon)\lambda+q(\epsilon)=0, \quad (43)$$

where

$$\begin{aligned}p(\epsilon)&=\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)(\alpha_3(\alpha_6(\alpha_1^*+\epsilon+\alpha_4)+\alpha_2(1-\alpha_4))+\alpha_5(\alpha_6(1-\alpha_1^*-\epsilon)-2\alpha_2))}{((\alpha_1^*+\epsilon)\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1^*-\epsilon))},\\q(\epsilon)&=\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-\left(\alpha_2\alpha_6(\alpha_1^*+\epsilon-1)-\alpha_2^2(\alpha_4-1)\right)\left(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1^*+\epsilon-1)\right)}{((\alpha_1^*+\epsilon)\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1^*-\epsilon))}.\end{aligned} \quad (44)$$

The roots of characteristic equation of are $J_{C(((\alpha_1^*+\epsilon-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5),(\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1^*-\epsilon))/(\alpha_3\alpha_6-\alpha_2\alpha_5))}$ about

$$C\left(\frac{(\alpha_1^*+\epsilon-1)\alpha_6-\alpha_2(\alpha_4-1)}{\alpha_3\alpha_6-\alpha_2\alpha_5},\right.\\ \left.\frac{\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1^*-\epsilon)}{\alpha_3\alpha_6-\alpha_2\alpha_5}\right) \tag{45}$$

$$\begin{aligned}\lambda_{1,2} &= \frac{p(\epsilon)\pm\iota\sqrt{4q(\epsilon)-p^2(\epsilon)}}{2},\\ &= \frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)(\alpha_3(\alpha_6(\alpha_1^*+\epsilon+\alpha_4)+\alpha_2(1-\alpha_4))+\alpha_5(\alpha_6(1-\alpha_1^*-\epsilon)-2\alpha_2))}{2((\alpha_1^*+\epsilon)\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1^*-\epsilon))}\pm\frac{\iota}{2}\sqrt{\Gamma(\varepsilon)},\end{aligned} \tag{46}$$

where

$$\begin{aligned}\Gamma(\epsilon) = {}&4\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-(\alpha_2\alpha_6(\alpha_1^*+\epsilon-1)-\alpha_2^2(\alpha_4-1))(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1^*+\epsilon-1))}{((\alpha_1^*+\epsilon)\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1^*-\epsilon))}\\ &-\left(\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)(\alpha_3(\alpha_6(\alpha_1^*+\epsilon+\alpha_4)+\alpha_2(1-\alpha_4))+\alpha_5(\alpha_6(1-\alpha_1^*-\epsilon)-2\alpha_2))}{((\alpha_1^*+\epsilon)\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1^*-\epsilon))}\right)^2.\end{aligned} \tag{47}$$

Moreover $|\lambda_{1,2}| = \sqrt{q(\epsilon)}$ and $(d|\lambda_{1,2}|/d\epsilon)|_{\epsilon=0} \neq 0$. Additionally, we required that when $\epsilon = 0$, $\lambda_{1,2}^m \neq 1$, $m = 1,2,3,4$, which corresponds to $p(0) \neq -2,0,1,2$. Since $p(0)^2-4q(0) < 0$ and $q(0) = 1$, then $q(0)^2 < 4$ and hence $q(0) \neq \pm 2$. Thus, we only require that $q_1(0) \neq 0,1$, which holds true by computation.

Let $u_n = x_n - x^*$, $v_n = y_n - y^*$; then $C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5), (\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$ of system (5) transforms into $O(0,0)$. By calculation, we obtain

$$\begin{aligned}u_{n+1} &= \frac{(\alpha_1^*+\epsilon)(u_n+x^*)-\alpha_2(u_n+x^*)(v_n+y^*)}{1+\alpha_3(u_n+x^*)}\\ &\quad - x^*\\ v_{n+1} &= \frac{\alpha_4(v_n+y^*)-\alpha_5(u_n+x^*)(v_n+y^*)}{1+\alpha_6(v_n+y^*)}-y^*,\end{aligned} \tag{48}$$

where $x^* = ((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5)$ and $y^* = (\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5)$. Hereafter when $\epsilon = 0$, normal form of system (48) is studied. Expanding (48) up to second-order about $(u_n, v_n) = (0,0)$ by Taylor series, we get

$$\begin{aligned}u_{n+1} &= m_{11}u_n + m_{12}v_n + m_{13}u_n^2 + m_{14}u_nv_n\\ &\quad + o\left((|u_n|+|v_n|)^2\right)\\ v_{n+1} &= m_{21}u_n + m_{22}v_n + m_{23}u_nv_n + m_{24}v_n^2\\ &\quad + o\left((|u_n|+|v_n|)^2\right),\end{aligned} \tag{49}$$

where

$$\begin{aligned}m_{11} &= \frac{\alpha_1^*-\alpha_2y^*}{(1+\alpha_3x^*)^2},\\ m_{12} &= -\frac{\alpha_2x^*}{1+\alpha_3x^*},\\ m_{13} &= -\frac{\alpha_3(\alpha_1^*-\alpha_2y^*)}{(1+\alpha_3x^*)^3},\\ m_{14} &= -\frac{\alpha_2}{(1+\alpha_3x^*)^2},\\ m_{21} &= -\frac{\alpha_5y^*}{1+\alpha_6y^*},\\ m_{22} &= \frac{\alpha_4-\alpha_5x^*}{(1+\alpha_6y^*)^2},\\ m_{23} &= -\frac{\alpha_5}{(1+\alpha_6y^*)^2},\\ m_{24} &= -\frac{\alpha_6(\alpha_4-\alpha_5x^*)}{(1+\alpha_6y^*)^3}.\end{aligned} \tag{50}$$

Now, let

$$\eta = \frac{(\alpha_3\alpha_6 - \alpha_2\alpha_5)(\alpha_3(\alpha_6(\alpha_1^* + \alpha_4) + \alpha_2(1-\alpha_4)) + \alpha_5(\alpha_6(1-\alpha_1^*) - 2\alpha_2))}{2(\alpha_1^*\alpha_3\alpha_6 - \alpha_2\alpha_5 + \alpha_2\alpha_3(\alpha_4 - 1))(\alpha_3\alpha_6\alpha_4 - \alpha_2\alpha_5 + \alpha_6\alpha_5(1-\alpha_1^*))},$$
$$\zeta = \frac{1}{2}\sqrt{\Gamma(0)}, \tag{51}$$

and invertible matrix T is defined by

$$T = \begin{pmatrix} m_{12} & 0 \\ \eta - m_{11} & -\zeta \end{pmatrix}. \tag{52}$$

Using the following translation

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} m_{12} & 0 \\ \eta - m_{11} & -\zeta \end{pmatrix}\begin{pmatrix} X_n \\ Y_n \end{pmatrix}, \tag{53}$$

(49) gives

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} \eta & -\zeta \\ \zeta & \eta \end{pmatrix}\begin{pmatrix} X_n \\ Y_n \end{pmatrix} + \begin{pmatrix} F(X_n, Y_n) \\ G(X_n, Y_n) \end{pmatrix}, \tag{54}$$

where

$$F(X_n, Y_n) = l_{11}X_n^2 + l_{12}X_nY_n + o\left((|X_n| + |Y_n|)^2\right),$$
$$G(X_n, Y_n) = l_{21}X_n^2 + l_{22}X_nY_n + l_{23}Y_n^2 + o\left((|X_n| + |Y_n|)^2\right),$$

$$l_{11} = m_{12}m_{13} + m_{14}(\eta - m_{11}),$$
$$l_{12} = -\zeta m_{14}$$
$$l_{21} = \frac{(\eta - m_{11})}{\zeta}\left[m_{12}\left(m_{13} + \frac{m_{14}(\eta - m_{11})}{m_{12}} - m_{23}\right) - m_{24}(\eta - m_{11})\right], \tag{55}$$
$$l_{22} = -m_{12}\left(\frac{m_{14}(\eta - m_{11})}{m_{12}} - m_{23}\right) + m_{24}(\eta - m_{11}),$$
$$l_{23} = -\zeta m_{24}.$$

In addition,

$$F_{X_nX_n}\big|_{(0,0)} = 2l_{11},$$
$$F_{X_nY_n}\big|_{(0,0)} = l_{12},$$
$$F_{Y_nY_n}\big|_{(0,0)} = 0,$$
$$F_{X_nX_nX_n}\big|_{(0,0)} = F_{X_nX_nY_n}\big|_{(0,0)} = F_{X_nY_nY_n}\big|_{(0,0)} = F_{Y_nY_nY_n}\big|_{(0,0)} = 0,$$
$$G_{X_nX_n}\big|_{(0,0)} = 2l_{21},$$
$$G_{X_nY_n}\big|_{(0,0)} = l_{22},$$
$$G_{Y_nY_n}\big|_{(0,0)} = 2l_{23},$$
$$G_{X_nX_nX_n}\big|_{(0,0)} = G_{X_nX_nY_n}\big|_{(0,0)} = G_{X_nY_nY_n}\big|_{(0,0)} = G_{Y_nY_nY_n}\big|_{(0,0)} = 0. \tag{56}$$

To guarantee the supercritical Neimark-Sacker bifurcation for (54), we require the following discriminatory quantity, that is, $\Psi < 0$ (see [19–23]):

$$\Psi = -\operatorname{Re}\left[\frac{(1-2\lambda)\bar{\lambda}^2}{1-\lambda}\tau_{11}\tau_{20}\right] - \frac{1}{2}\|\tau_{11}\|^2 - \|\tau_{02}\|^2 + \operatorname{Re}(\bar{\lambda}\tau_{21}), \tag{57}$$

where

$$\begin{aligned}
\tau_{02} &= \frac{1}{8}\left[F_{X_nX_n} - F_{Y_nY_n} + 2G_{X_nY_n} + \iota\left(G_{X_nX_n} - G_{Y_nY_n} + 2F_{X_nY_n}\right)\right]\Big|_{(0,0)}, \\
\tau_{11} &= \frac{1}{4}\left[F_{X_nX_n} + F_{Y_nY_n} + \iota\left(G_{X_nX_n} + G_{Y_nY_n}\right)\right]\Big|_{(0,0)}, \\
\tau_{20} &= \frac{1}{8}\left[F_{X_nX_n} - F_{Y_nY_n} + 2G_{X_nY_n} + \iota\left(G_{X_nX_n} - G_{Y_nY_n} - 2F_{X_nY_n}\right)\right]\Big|_{(0,0)}, \\
\tau_{21} &= \frac{1}{16}\left[F_{X_nX_nX_n} + F_{X_nY_nY_n} + G_{X_nX_nY_n} + G_{Y_nY_nY_n} + \iota\left(G_{X_nX_nX_n} + G_{X_nY_nY_n} - F_{X_nX_nY_n} - F_{Y_nY_nY_n}\right)\right]\Big|_{(0,0)}.
\end{aligned} \tag{58}$$

A calculation reveals

$$\begin{aligned}
\tau_{02} &= \frac{1}{4}\left[l_{11} + l_{22} + \iota\left(l_{21} - l_{23} + l_{12}\right)\right], \\
\tau_{11} &= \frac{1}{2}\left[l_{11} + \iota\left(l_{21} + l_{23}\right)\right], \\
\tau_{20} &= \frac{1}{4}\left[l_{11} + l_{22} + \iota\left(l_{21} - l_{23} - l_{12}\right)\right], \\
\tau_{21} &= 0.
\end{aligned} \tag{59}$$

Based on this analysis and Neimark-Sacker bifurcation Theorem discussed in [19, 20], we reach the following Theorem.

Theorem 9. *If $\Psi \neq 0$, then discrete model (5) undergoes a Neimark-Sacker bifurcation about*

$$C\left(\frac{(\alpha_1 - 1)\alpha_6 - \alpha_2(\alpha_4 - 1)}{\alpha_3\alpha_6 - \alpha_2\alpha_5}, \frac{\alpha_3(\alpha_4 - 1) + \alpha_5(1 - \alpha_1)}{\alpha_3\alpha_6 - \alpha_2\alpha_5}\right) \tag{60}$$

as the parameters $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$ go through $N_{C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5),(\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))}$. Additionally, attracting (resp., repelling) invariant closed curve bifurcates from the unique positive equilibrium $C(((\alpha_1-1)\alpha_6 - \alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5), (\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$ if $\Psi < 0$ (resp., $\Psi > 0$).

5. Boundedness and Construction of Invariant Rectangle

In this section we will study boundedness character and construction of invariant rectangle of positive solution of the discrete model (5).

Theorem 10. *For every positive solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of the discrete model (5), the following holds:*

(i) *Every positive solution of the discrete model (5) is bounded.*

(ii) *The set $[0, \alpha_1/\alpha_3] \times [0, \alpha_4/\alpha_6]$ is an invariant rectangle.*

Proof. (i) Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be any positive solution of the discrete model (5). From (5), we have

$$\begin{aligned}
x_{n+1} &= \frac{\alpha_1 x_n - \alpha_2 x_n y_n}{1 + \alpha_3 x_n} < \frac{\alpha_1 x_n}{1 + \alpha_3 x_n} < \frac{\alpha_1}{\alpha_3}, \\
y_{n+1} &= \frac{\alpha_4 y_n - \alpha_5 x_n y_n}{1 + \alpha_6 y_n} < \frac{\alpha_4 y_n}{1 + \alpha_6 y_n} < \frac{\alpha_4}{\alpha_6}.
\end{aligned} \tag{61}$$

Hence, for every solution $\{(x_n, y_n)\}$ of the discrete model (5), one has

$$\begin{aligned}
0 &\le x_n < \frac{\alpha_1}{\alpha_3}, \\
0 &\le y_n < \frac{\alpha_4}{\alpha_6}.
\end{aligned} \tag{62}$$

(ii) For any positive solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of the discrete model (5) with initial conditions $x_0 \in [0, \alpha_1/\alpha_3]$ and $y_0 \in [0, \alpha_4/\alpha_6]$, we have from (5)

$$\begin{aligned}
0 \le x_1 &= \frac{\alpha_1 x_0 - \alpha_2 x_0 y_0}{1 + \alpha_3 x_0} < \frac{\alpha_1 x_0}{1 + \alpha_3 x_0} < \frac{\alpha_1}{\alpha_3}, \\
0 \le y_1 &= \frac{\alpha_4 y_0 - \alpha_5 x_0 y_0}{1 + \alpha_6 y_0} < \frac{\alpha_4 y_0}{1 + \alpha_6 y_0} < \frac{\alpha_4}{\alpha_6}.
\end{aligned} \tag{63}$$

Hence, $x_1 \in [0, \alpha_1/\alpha_3]$ and $y_1 \in [0, \alpha_4/\alpha_6]$. Similarly, one can show that if $x_k \in [0, \alpha_1/\alpha_3]$ and $y_k \in [0, \alpha_4/\alpha_6]$, then $x_{k+1} \in [0, \alpha_1/\alpha_3]$ and $y_{k+1} \in [0, \alpha_4/\alpha_6]$. □

6. Global Stability

Now in the following we will investigate global dynamics of the discrete model (5) about $O(0,0)$ and the unique positive equilibrium $C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5), (\alpha_3(\alpha_4-1) + \alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$.

Theorem 11. *If $\alpha_1 < 1$ and $\alpha_4 < 1$, then equilibrium $O(0,0)$ of the discrete model (5) is globally asymptotically stable.*

Proof. According to the conclusion (i) in Lemma 1, discrete model (5) has a unique equilibrium $O(0,0)$ in the first quadrant $\mathbb{R}^2$ and is a sink by Theorem 3. Moreover every positive solution of the discrete model (5) in the first quadrant $\mathbb{R}^2$ satisfies

$$\begin{aligned}0 &< x_{n+1} \leq \frac{\alpha_1 x_n}{1+\alpha_3 x_n} = x_n\left(1-\frac{\alpha_3 x_n + 1 - \alpha_1}{1+\alpha_3 x_n}\right) \\ &< x_n, \\ 0 &< y_{n+1} \leq \frac{\alpha_4 y_n}{1+\alpha_6 y_n} = y_n\left(1-\frac{\alpha_6 y_n + 1 - \alpha_4}{1+\alpha_6 y_n}\right) \\ &< y_n,\end{aligned} \tag{64}$$

which leads to $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 0$. Hence the boundary equilibrium $O(0,0)$ of the discrete model (5) is globally asymptotically stable in the first quadrant $\mathbb{R}^2$. □

Hereafter we use Theorem 1.16 of [14] to determine the global dynamics of the discrete model (5) about $C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5), (\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$.

Theorem 12. *The unique positive equilibrium point* $C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5), (\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$ *of the discrete model (5) is a global attractor.*

Proof. Let $f(x,y) = (\alpha_1 x - \alpha_2 xy)/(1+\alpha_3 x)$ and $g(x,y) = (\alpha_4 y - \alpha_5 xy)/(1+\alpha_6 \overline{y})$. Then, it is easy to see that $f(x,y)$ is nondecreasing in x and nonincreasing in y for every $(x,y) \in [0,\alpha_1/\alpha_3]\times[0,\alpha_4/\alpha_6]$. Moreover, $g(x,y)$ is nonincreasing in x and nondecreasing in y for each $(x,y) \in [0,\alpha_1/\alpha_3]\times[0,\alpha_4/\alpha_6]$. Let (m_1, M_1, m_2, M_2) be a positive solution of system

$$\begin{aligned}m_1 &= f(m_1, M_2), \\ M_1 &= f(M_1, m_2), \\ m_2 &= g(M_1, m_2), \\ M_2 &= g(m_1, M_2).\end{aligned} \tag{65}$$

Then, one has

$$\begin{aligned}m_1 &= \frac{\alpha_1 m_1 - \alpha_2 m_1 M_2}{1+\alpha_3 m_1}, \\ M_1 &= \frac{\alpha_1 M_1 - \alpha_2 M_1 m_2}{1+\alpha_3 M_1},\end{aligned} \tag{66}$$

$$\begin{aligned}m_2 &= \frac{\alpha_4 m_2 - \alpha_5 M_1 m_2}{1+\alpha_6 m_2}, \\ M_2 &= \frac{\alpha_4 M_2 - \alpha_5 m_1 M_2}{1+\alpha_6 M_2}.\end{aligned} \tag{67}$$

From (66), one has

$$\begin{aligned}1+\alpha_3 m_1 &= \alpha_1 - \alpha_2 M_2, \\ 1+\alpha_3 M_1 &= \alpha_1 - \alpha_2 m_2.\end{aligned} \tag{68}$$

From (67), one has

$$\begin{aligned}1+\alpha_6 m_2 &= \alpha_4 - \alpha_5 M_1, \\ 1+\alpha_6 M_2 &= \alpha_4 - \alpha_5 m_1.\end{aligned} \tag{69}$$

On subtracting (68), one gets

$$\alpha_3(m_1 - M_1) = \alpha_2(m_2 - M_2). \tag{70}$$

Similarly, subtracting (69) one gets

$$\alpha_6(m_2 - M_2) = \alpha_5(m_1 - M_1). \tag{71}$$

From (70) and (71) one gets $(\alpha_6\alpha_3 - \alpha_2\alpha_4)(m_1 - M_1) = 0$. Hence, $m_1 = M_1$, and similarly one has $m_2 = M_2$. Hence, from Theorem 1.16 of [14] the equilibrium $C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5), (\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$ of the discrete model (5) is a global attractor. □

Corollary 13. *Under the conditions (10) and (18), the unique positive equilibrium point*

$$C\left(\frac{(\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1)}{\alpha_3\alpha_6-\alpha_2\alpha_5}, \frac{\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1)}{\alpha_3\alpha_6-\alpha_2\alpha_5}\right) \tag{72}$$

of the discrete model (5) is globally asymptotically stable.

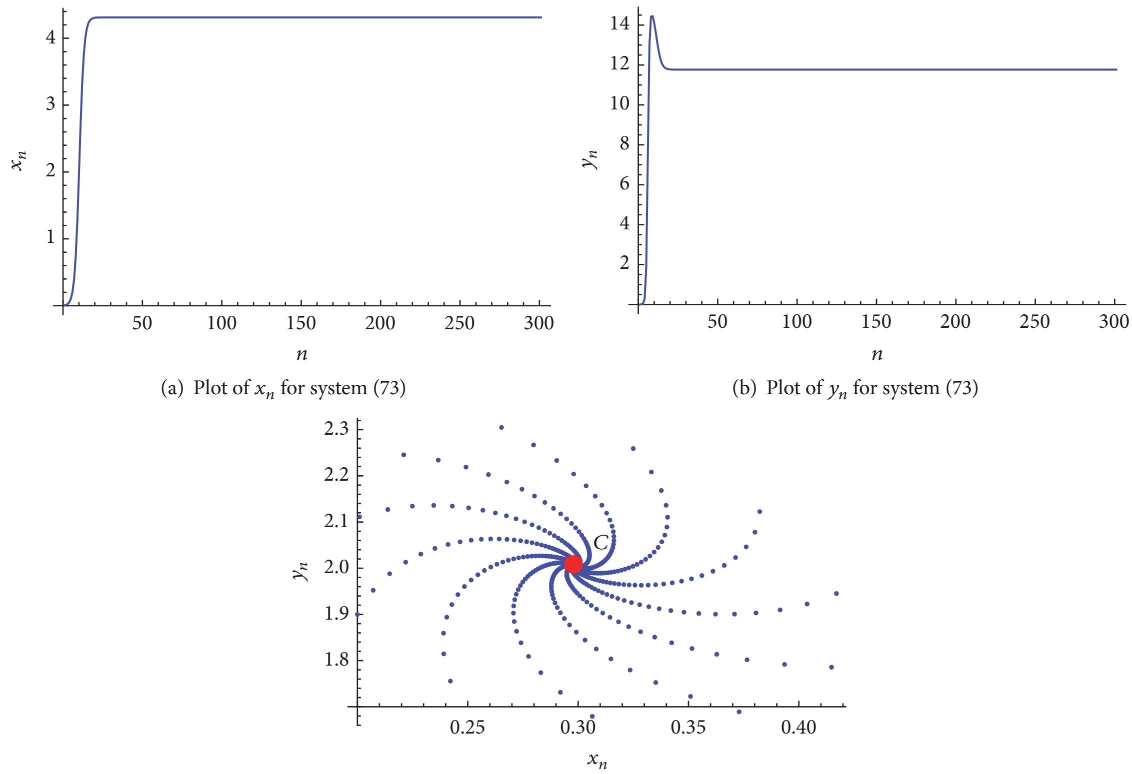
7. Numerical Simulations

In the following we present numerical simulations that represent different types of qualitative behavior of the discrete model (5).

Example 1. If $\alpha_1 = 2,\ \alpha_2 = 0.0007,\ \alpha_3 = 0.23,\ \alpha_4 = 7,\ \alpha_5 = 0.3,\ \alpha_6 = 0.4$, then discrete model (5) with initial values $x_0 = 0.007,\ y_0 = 0.0009$ can be written as

$$\begin{aligned}x_{n+1} &= \frac{2x_n - 0.0007 x_n y_n}{1+0.23 x_n}, \\ y_{n+1} &= \frac{7y_n - 0.3 x_n y_n}{1+0.4 y_n}.\end{aligned} \tag{73}$$

In this case $\alpha_1 = 2 > 1$, $\alpha_4 = 7 > 1$, $\alpha_3\alpha_6 = 0.092 > \alpha_2\alpha_5 = 0.00021$, $\alpha_6 = 0.4 > \alpha_2(\alpha_4-1)/(\alpha_1-1) = 0.0042$, $\alpha_5 = 0.3 < \alpha_3(\alpha_4-1)/(\alpha_{11}-1) = 1.38$. This shows the correctness of the conditions for the unique positive equilibrium. A straightforward computation shows that condition (10) of Theorem 6, that is, $\Theta = 0.07435317508000001 < (\alpha_2\alpha_3 - \alpha_2\alpha_3\alpha_4 - \alpha_2\alpha_5 + \alpha_1\alpha_3\alpha_6)(-\alpha_2\alpha_5 + \alpha_3\alpha_4\alpha_6 + \alpha_5\alpha_6 - \alpha_1\alpha_5\alpha_6) = 0.0957613829600004$, under which the unique positive equilibrium point is locally asymptotically stable, holds. Moreover for these arbitrary chosen values of parameters, the necessary and sufficient condition, under which the unique positive equilibrium point is locally asymptotically stable, is also satisfied; that is, $|(\alpha_3\alpha_6 - \alpha_2\alpha_5)(\alpha_3(\alpha_6(\alpha_1+\alpha_4)+\alpha_2(1-\alpha_4))+\alpha_5(\alpha_6(1-\alpha_1)-2\alpha_2))/$



(a) Plot of x_n for system (73)

(b) Plot of y_n for system (73)

(c) Attractor of system (73)

Figure 1: Plots for system (73).

$(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))| = 0.6702268987306942 < 1-((\alpha_3\alpha_6-\alpha_2\alpha_5)^2-(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1))(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1)))/(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1)) = 0.9138643327940412 < 2$. This verifies the condition for which the unique positive equilibrium is locally asymptotically stable. Also $0 \le x_n < \alpha_1/\alpha_3 = 8.695652173913043$, $0 \le y_n < \alpha_4/\alpha_6 = 17.5$, and hence the parametric conditions under which every positive solution is bounded hold true. Moreover, in Figure 1 the plot of x_n is shown in Figure 1(a), the plot of y_n is shown in Figure 1(b), and attractor of system (73) is shown in Figure 1(c).

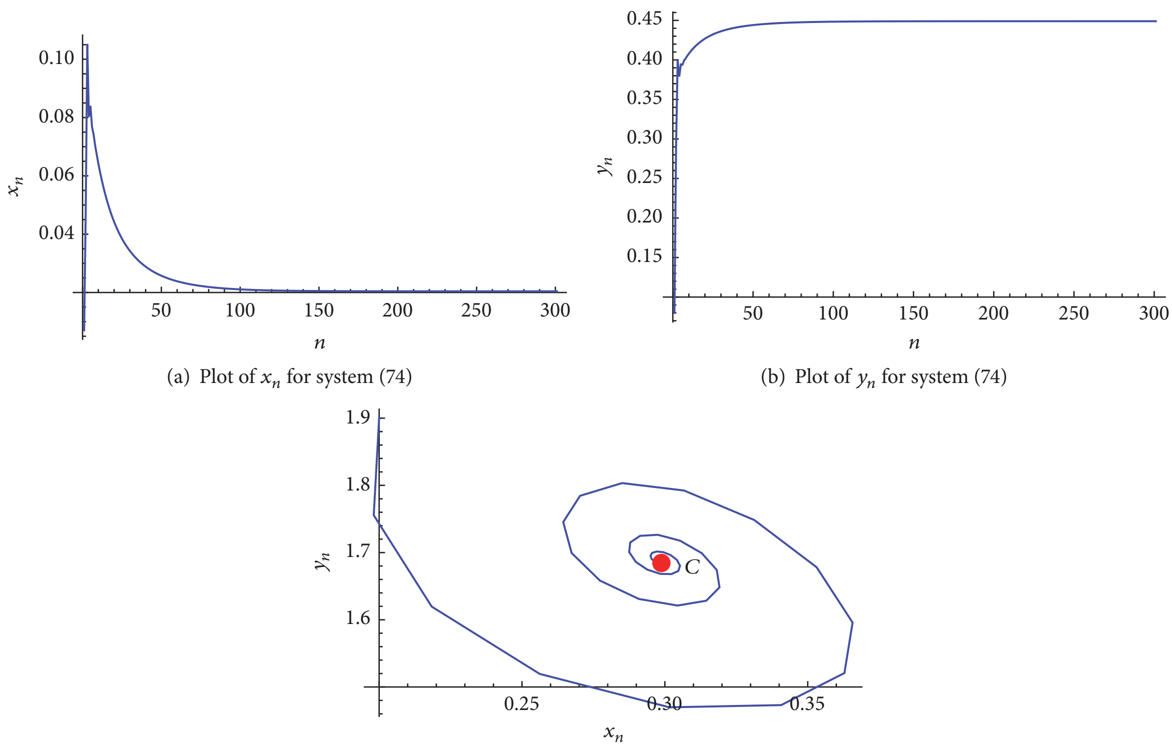
Example 2. If $\alpha_1 = 8$, $\alpha_2 = 0.00007$, $\alpha_3 = 3.23$, $\alpha_4 = 7$, $\alpha_5 = 1.3$, $\alpha_6 = 3.4$, then discrete model (5) with initial values $x_0 = 0.007$, $y_0 = 0.08$ can be written as

$$x_{n+1} = \frac{8x_n - 0.00007x_ny_n}{1+3.23x_n}$$
$$y_{n+1} = \frac{7y_n - 1.3x_ny_n}{1+3.4y_n}. \quad (74)$$

In this case $\alpha_1 = 8 > 1$, $\alpha_4 = 7 > 1$, $\alpha_3\alpha_6 = 10.982 > \alpha_2\alpha_5 = 0.000091$, $\alpha_6 = 3.4 > \alpha_2(\alpha_4-1)/(\alpha_1-1) = 0.00006$, $\alpha_5 = 1.3 < \alpha_3(\alpha_4-1)/(\alpha_{11}-1) = 2.76857$. This shows the correctness of the conditions of the unique positive equilibrium point. A straightforward computation shows that condition (10) of Theorem 6, that is, $\Theta = 4035.5030151773317 < (\alpha_2\alpha_3-\alpha_2\alpha_3\alpha_4-\alpha_2\alpha_5+\alpha_1\alpha_3\alpha_6)(-\alpha_2\alpha_5+\alpha_3\alpha_4\alpha_6+\alpha_5\alpha_6-\alpha_1\alpha_5\alpha_6) = 4035.5030151773317$, holds. Moreover the necessary and sufficient condition under which the unique positive equilibrium point of the system is locally asymptotically stable is also satisfied; that is, $|(\alpha_3\alpha_6-\alpha_2\alpha_5)(\alpha_3(\alpha_6(\alpha_1+\alpha_4)+\alpha_2(1-\alpha_4))+\alpha_5(\alpha_6(1-\alpha_1)-2\alpha_2))/(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))| = 0.36407043421653185 < 1-((\alpha_3\alpha_6-\alpha_2\alpha_5)^2-(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1))(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1)))/(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1)) = 0.9701211132646083 < 2$. This verifies the condition for which the unique positive equilibrium is locally asymptotically stable. Also $0 \le x_n < \alpha_1/\alpha_3 = 2.476780185758514$, $0 \le y_n < \alpha_4/\alpha_6 = 2.058823529411765$, and hence the parametric conditions under which every positive solution is bounded hold true. Moreover, in Figure 2 the plot of x_n is shown in Figure 2(a), the plot of y_n is shown in Figure 2(b), and attractor of system (74) is shown in Figure 2(c).

Example 3. If $\alpha_1 = 15$, $\alpha_2 = 0.057$, $\alpha_3 = 6.23$, $\alpha_4 = 7$, $\alpha_5 = 1.3$, $\alpha_6 = 3.4$, then discrete model (5) with initial values $x_0 = 0.008$, $y_0 = 0.009$ can be written as

$$x_{n+1} = \frac{15x_n - 0.057x_ny_n}{1+6.23x_n}$$
$$y_{n+1} = \frac{7y_n - 1.3x_ny_n}{1+3.4y_n}. \quad (75)$$



(a) Plot of x_n for system (74)

(b) Plot of y_n for system (74)

(c) Attractor of system (74)

Figure 2: Plots for system (74).

In this case $\alpha_1 = 15 > 1$, $\alpha_4 = 7 > 1$, $\alpha_3\alpha_6 = 21.182 > \alpha_2\alpha_5 = 0.0741$, $\alpha_6 = 3.4 > \alpha_2(\alpha_4 - 1)/(\alpha_1 - 1 = 0.0244286$, $\alpha_5 = 1.3 < \alpha_3(\alpha_4 - 1)/(\alpha_1 - 1) = 2.67$. This shows the correctness of the conditions of the unique positive equilibrium point. A computation shows that condition (10) of Theorem 6, that is, $\Theta = 9280.610848888002 < (\alpha_2\alpha_3 - \alpha_2\alpha_3\alpha_4 - \alpha_2\alpha_5 + \alpha_1\alpha_3\alpha_6)(-\alpha_2\alpha_5 + \alpha_3\alpha_4\alpha_6 + \alpha_5\alpha_6 - \alpha_1\alpha_5\alpha_6) = 27236.10716427601$, holds. Moreover for these arbitrary chosen values of parameters the necessary and sufficient condition, under which the unique positive equilibrium point is locally asymptotically stable, is also satisfied; that is, $|(\alpha_3\alpha_6-\alpha_2\alpha_5)(\alpha_3(\alpha_6(\alpha_1+\alpha_4))+\alpha_2(1-\alpha_4))+\alpha_5(\alpha_6(1-\alpha_1)-2\alpha_2))/(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))| = 0.30727881013351915 < 1-((\alpha_3\alpha_6-\alpha_2\alpha_5)^2-(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1))(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1)))/(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4 - \alpha_2\alpha_5 + \alpha_6\alpha_5(1-\alpha_1)) = 0.9862925899775977 < 2$. This verifies the condition for which the unique positive equilibrium is locally asymptotically stable. Also $0 \le x_n < \alpha_1/\alpha_3 = 2.407704654895666$, $0 \le y_n < \alpha_4/\alpha_6 = 2.058823529411765$, and hence the parametric conditions under which every positive solution is bounded hold true. Moreover, in Figure 3 the plot of x_n is shown in Figure 3(a), the plot of y_n is shown in Figure 3(b), and attractor of system (75) is shown in Figure 3(c).

8. Conclusion

This work is related to the global dynamics and bifurcations analysis of a two-dimensional discrete-time Lotka-Volterra model in the closed first quadrant $\mathbb{R}^2$. We proved that the discrete model (5) has three boundary equilibria $O(0,0)$, $A((\alpha_1-1)/\alpha_3, 0)$, $B(0,(\alpha_4-1)/\alpha_6)$ and the unique positive equilibrium

$$C\left(\frac{(\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1)}{\alpha_3\alpha_6-\alpha_2\alpha_5}, \frac{\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1)}{\alpha_3\alpha_6-\alpha_2\alpha_5}\right) \tag{76}$$

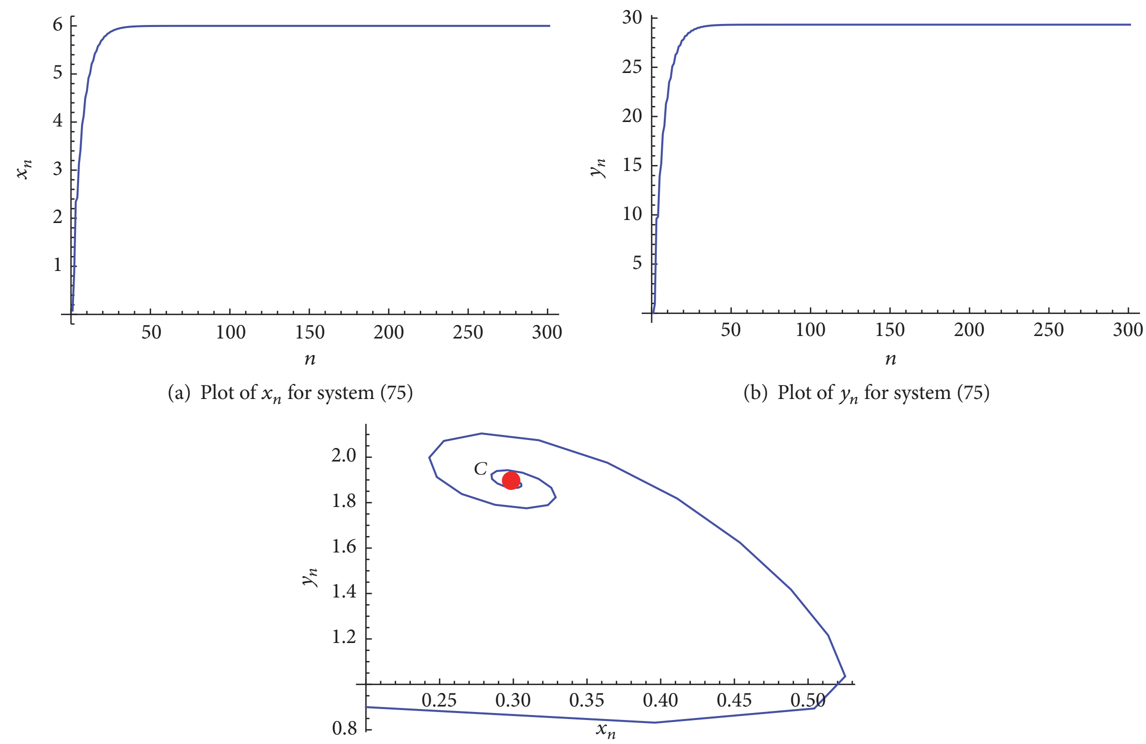
under certain parametric conditions. The method of linearization is used to prove the local asymptotic stability of these equilibria, and conclusions are presented in Table 1. We proved that boundary equilibrium $O(0,0)$ undergoes fold bifurcation when parameters vary in a small neighborhood of $\alpha_1 = 1$ and both $A((\alpha_1-1)/\alpha_3, 0)$ and $B(0,(\alpha_4-1)/\alpha_6)$ undergo period-doubling bifurcation when parameters of the discrete model (5) are, respectively, located in the following sets:

$$\begin{aligned} &F_{A((\alpha_1-1)/\alpha_3,0)} \\ &\quad = \left\{(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6) : \alpha_5 = \frac{\alpha_3(\alpha_4+1)}{\alpha_1-1}\right\}, \\ &F_{B(0,(\alpha_4-1)/\alpha_6)} \\ &\quad = \left\{(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6) : \alpha_2 = \frac{(\alpha_1+1)\alpha_6}{\alpha_4-1}\right\}. \end{aligned} \tag{77}$$

We have also shown that $C(((\alpha_1-1)\alpha_6-\alpha_2(\alpha_4-1))/(\alpha_3\alpha_6-\alpha_2\alpha_5), (\alpha_3(\alpha_4-1)+\alpha_5(1-\alpha_1))/(\alpha_3\alpha_6-\alpha_2\alpha_5))$ undergoes

Table 1: Number of equilibria along with qualitative behavior.

EP	Corresponding behavior
O	Sink if $\alpha_1 < 1$ and $\alpha_4 < 1$; source if $\alpha_1 > 1$ and $\alpha_4 > 1$; saddle if $\alpha_1 > 1$ and $\alpha_4 < 1$; nonhyperbolic if $\alpha_1 = 1$ or $\alpha_4 = 1$.
A	Sink if $\alpha_1 > 1$ and $\alpha_5 < \alpha_3(\alpha_4+1)/(\alpha_1-1)$; never source; saddle if $\alpha_1 > 1$ and $\alpha_5 > \alpha_3(\alpha_4+1)/(\alpha_1-1)$; nonhyperbolic if $\alpha_5 = \alpha_3(\alpha_4+1)/(\alpha_1-1)$.
B	Sink if $\alpha_4 > 1$ and $\alpha_2 < (\alpha_1+1)\alpha_6/(\alpha_4-1)$; never source; saddle if $\alpha_4 > 1$ and $\alpha_2 > (\alpha_1+1)\alpha_6/(\alpha_4-1)$; nonhyperbolic if $\alpha_2 = (\alpha_1+1)\alpha_6/(\alpha_4-1)$.
C	Locally asymptotically stable if $\Theta < (\alpha_2\alpha_3 - \alpha_2\alpha_3\alpha_4 - \alpha_2\alpha_5 + \alpha_1\alpha_3\alpha_6)(-\alpha_2\alpha_5 + \alpha_3\alpha_4\alpha_6 + \alpha_5\alpha_6 - \alpha_1\alpha_5\alpha_6)$. Unstable if $\Theta > (\alpha_2\alpha_3 - \alpha_2\alpha_3\alpha_4 - \alpha_2\alpha_5 + \alpha_1\alpha_3\alpha_6)(-\alpha_2\alpha_5 + \alpha_3\alpha_4\alpha_6 + \alpha_5\alpha_6 - \alpha_1\alpha_5\alpha_6)$. Locally asymptotically stable if and only if $\left\lvert \frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)(\alpha_3(\alpha_6(\alpha_1+\alpha_4)+\alpha_2(1-\alpha_4))+\alpha_5(\alpha_6(1-\alpha_1)-2\alpha_2))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))} \right\rvert < 1 - \frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1))(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))} < 2.$ Repeller if and only if $\left\lvert \frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1))(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))} \right\rvert > 1,$ $\left\lvert \frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)(\alpha_3(\alpha_6(\alpha_1+\alpha_4)+\alpha_2(1-\alpha_4))+\alpha_5(\alpha_6(1-\alpha_1)-2\alpha_2))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))} \right\rvert < \left\lvert 1 - \frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1))(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))} \right\rvert.$ Saddle if and only if $\left(\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)(\alpha_3(\alpha_6(\alpha_1+\alpha_4)+\alpha_2(1-\alpha_4))+\alpha_5(\alpha_6(1-\alpha_1)-2\alpha_2))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))} \right)^2 + 4\left(1 - \frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1))(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))} \right) > 0,$ $\left\lvert \frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)(\alpha_3(\alpha_6(\alpha_1+\alpha_4)+\alpha_2(1-\alpha_4))+\alpha_5(\alpha_6(1-\alpha_1)-2\alpha_2))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))} \right\rvert > \left\lvert 1 - \frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1))(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))} \right\rvert.$ Nonhyperbolic if and only if $\left\lvert \frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)(\alpha_3(\alpha_6(\alpha_1+\alpha_4)+\alpha_2(1-\alpha_4))+\alpha_5(\alpha_6(1-\alpha_1)-2\alpha_2))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))} \right\rvert = \left\lvert 1 - \frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1))(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))} \right\rvert.$ Locally asymptotically stable focus if $\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1))(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))} < 1.$ Unstable focus if $\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1))(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))} > 1.$ Nonhyperbolic (under which the eigenvalues are a pair of complex conjugates with modulus 1) if $\frac{(\alpha_3\alpha_6-\alpha_2\alpha_5)^2-(\alpha_2\alpha_6(\alpha_1-1)-\alpha_2^2(\alpha_4-1))(\alpha_3\alpha_5(\alpha_4-1)-\alpha_5^2(\alpha_1-1))}{(\alpha_1\alpha_3\alpha_6-\alpha_2\alpha_5+\alpha_2\alpha_3(\alpha_4-1))(\alpha_3\alpha_6\alpha_4-\alpha_2\alpha_5+\alpha_6\alpha_5(1-\alpha_1))} = 1.$



(a) Plot of x_n for system (75)

(b) Plot of y_n for system (75)

(c) Attractor of system (75)

FIGURE 3: Plots for system (75).

Neimark-Sacker bifurcation when parameters of the discrete model (5) are located in the following set:

$$N_C = \left\{ \left(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\right) : \Delta < 0, \frac{\left(\alpha_3\alpha_6 - \alpha_2\alpha_5\right)^2 - \left(\alpha_2\alpha_6\left(\alpha_1 - 1\right) - \alpha_2^2\left(\alpha_4 - 1\right)\right)\left(\alpha_3\alpha_5\left(\alpha_4 - 1\right) - \alpha_5^2\left(\alpha_1 - 1\right)\right)}{\left(\alpha_1\alpha_3\alpha_6 - \alpha_2\alpha_5 + \alpha_2\alpha_3\left(\alpha_4 - 1\right)\right)\left(\alpha_3\alpha_6\alpha_4 - \alpha_2\alpha_5 + \alpha_6\alpha_5\left(1 - \alpha_1\right)\right)} = 1 \right\}. \tag{78}$$

It is proved that every positive solution of the discrete model (5) is bounded and the set $[0, \alpha_1/\alpha_3] \times [0, \alpha_4/\alpha_6]$ is an invariant rectangle. The most interesting aspect in the theory of dynamical systems is to predict the global dynamics about equilibria. In this paper, we proved that if $\alpha_1 < 1$ and $\alpha_4 < 1$, then equilibrium $O(0,0)$ of the discrete model (5) is globally asymptotically stable. Furthermore, we have investigated the global stability of the unique positive equilibrium point

$$C\left(\frac{\left(\alpha_1 - 1\right)\alpha_6 - \alpha_2\left(\alpha_4 - 1\right)}{\alpha_3\alpha_6 - \alpha_2\alpha_5}, \frac{\alpha_3\left(\alpha_4 - 1\right) + \alpha_5\left(1 - \alpha_1\right)}{\alpha_3\alpha_6 - \alpha_2\alpha_5}\right) \tag{79}$$

of the discrete model (5). Some numerical examples are provided to support our theoretical results. These examples provide experimental verifications of the theoretical discussions.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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