Modality and Hyperintensionality in Mathematics

David Elohim*

Written: December 14, 2014. Revised: April 25, 2024

Abstract

This paper aims to contribute to the analysis of the nature of mathematical modality and hyperintensionality and to the applications of the latter to absolute decidability. Rather than countenancing the interpretational type of mathematical modality as a primitive, I argue that the interpretational type of mathematical modality is a species of epistemic modality. I argue, then, that the framework of two-dimensional semantics ought to be applied to the mathematical setting. The framework permits of a formally precise account of the priority and relation between epistemic mathematical modality and metaphysical mathematical modality. The discrepancy between the modal systems governing the parameters in the two-dimensional intensional setting provides an explanation of the difference between the metaphysical possibility of absolute decidability and our knowledge thereof. I also advance a topic-sensitive epistemic two-dimensional truthmaker semantics, if hyperintensional approaches are to be preferred to possible worlds semantics. I examine the relation between two-dimensional hyperintensional states and epistemic set theory, providing two-dimensional hyperintensional formalizations of epistemic set theory, large cardinal axioms, the modal axioms governing Ω-logic, and the Epistemic Church-Turing Thesis.

1 Introduction

This essay aims to contribute to the analysis of the nature of mathematical modality and hyperintensionality, and to the applications of the latter to absolute decidability. I argue that mathematical modality falls under at least four types; the interpretational, the metaphysical, the non-maximally objective, and the logical. The interpretational type of mathematical modality has traditionally been taken to concern possible reinterpretations of quantifier domains (cf. Fine, 2006, 2007; Linnebo, 2009, 2010, 2013; Studd, 2013), and the possible reinterpretations of the intensions of the concept of set (Uzquiano, 2015,a). The metaphysical type of modality concerns the ontological profile of abstracta and

*I changed my name, from Hasen Joseph Khudairi and Timothy Alison Bowen, to David Elohim, in April, 2024. Please cite this paper and my published book and articles under 'Elohim, David'.

mathematical truth. Abstracta are thus argued to have metaphysically necessary being, and mathematical truths hold of metaphysical necessity, if at all (cf. Fine, 1981). Metaphysical modality is the maximal objective modality.[1] However, the phenomenon of indefinite extensibility of the ordinals, cardinals, and reals is, I argue, possessed of two modalities whose interaction is captured by a two-dimensional semantics, and which consist of an epistemic modality characterizing reinterpretations of quantifier domains, and a non-maximal, hence non-metaphysical, yet still objective modality characterizing ontological expansion.[2] Another candidate for the non-maximal objective mathematical modality is the modal profile of forcing (cf. Kripke 1965; Hamkins and Löwe, 2008). Instances, finally, of the logical type of mathematical modality might concern the properties of consistency (cf. Field, 1989: 249-250, 257-260; Rayo, 2013: 50; Leng: 2007; 2010: 258), and can perhaps be further witnessed by the logic of provability (cf. Boolos, 1993).

The significance of the present contribution is as follows. (i) Rather than countenancing the interpretational type of mathematical modality as a primitive, I argue that the interpretational type of mathematical modality is a species of epistemic modality. (ii) I argue, then, that the framework of two-dimensional hyperintensional semantics ought to be applied to the mathematical setting. The framework permits of a formally precise account of the priority and relation between epistemic mathematical modality and metaphysical mathematical modality. I target, in particular, the modal axioms that the respective interpretations of the modal operator ought to satisfy. The discrepancy between the modal systems governing the parameters in the two-dimensional setting provides an explanation of the difference between the metaphysical possibility of absolute decidability and our knowledge thereof. (iii) I examine the application of the mathematical modalities beyond the issue of indefinite extensibility. As a test case for the two-dimensional approach, I investigate the interaction between epistemic and metaphysical mathematical modalities and both large cardinal axioms and Orey sentences which are undecidable relative to the axioms of ZFC, such as the generalized continuum hypothesis. The two-dimensional framework permits of a formally precise means of demonstrating how the metaphysical possibility of absolute decidability and the continuum hypothesis can be accessed by their epistemic-modal-mathematical profile. I argue that, in the absence of disproof, large cardinal axioms are epistemically possible, and thereby provide a sufficient guide to the objective mathematical possibility of determinacy claims and the continuum hypothesis. (iv) Finally, I define a novel, hyperintensional, topic-sensitive epistemic two-dimensional truthmaker semantics. I examine the relation between epistemic truthmakers and the axioms of epistemic set theory, large cardinal axioms, the Epistemic Church-Turing Thesis, as well as the verification-profile of Ω-logical consequence.

In Section **2**, I discuss how the properties of the epistemic mathematical

[1]For endorsements of this contention, see Kripke (1980: 99), Lewis (1986), Stalnaker (2003: 203), and Williamson (2016b: 459-460). For an argument in opposition, see Clarke-Doane (2021).

[2]See Author (ms) for further discussion.

modality and objective mathematical modality converge and depart from previous attempts to delineate the contours of similar notions. In Section **3**, I define the formal clauses and modal axioms governing the epistemic and metaphysical types of mathematical modality. I also advance a topic-sensitive epistemic two-dimensional truthmaker semantics, if hyperintensional approaches are to be preferred to possible worlds semantics. Section **4** extends the two-dimensional framework to the issue of mathematical knowledge; in particular, to the hyperintensional profile of large cardinal axioms and to the absolute decidability of the continuum hypothesis. Section **5** provides concluding remarks.

2 Departures from Precedent

Shoenfield (1967) writes: 'If we introduced symbols for new operations which cannot be defined in ZFC, we would increase our ability to describe sets and hence increase the power of the subset (and replacement) axioms. This appears to be a natural approach; but so far no one has been able to propose any suitable operations' (304). 'An approach which is more promising at the moment is to make fuller use of our principle of existence of stages: if we can imagine a situation in which all of the stages in a collection are completed, then there must be a stage after all the stages in the collection' (306).

Reinhardt (1974b) develops Shoenfield's comments, and countenances the following principle:

S If P is a property of stages, and if we can *imagine* a situation in which all the stages having P have been built up, then there *exists* a stage s beyond all the stages which have P' (5).

Imagination is defined thus: '[W]e choose here to pursue the sense of "imagine" according to which Existing = Real $\subseteq$ Imaginary rather than that according to which Imaginable = Visualizable $\subseteq$ (mathematically) Existing. To choose the latter would turn **S** into a tautology (or an exhortation to visualize!) and require for its usefulness a criterion of visualizability. One can read the axiom of **S** as such criteria, taking V to be the visualizable sets. Then [principle **S**] gives a condition under which ϕ will provide a "visualization" of a set. The axioms so read may be plausible for some kind of "visualizability" (such a concept of "visualizable" seems close to definable). However, I have trouble seeing that each $x \subseteq w$ is visualizable' (6).

Reinhardt writes of the relation between the imagination that a set-theoretic rank at which a property is satisfied and the existence of the rank at which the property is satisfied that: 'We remark at the outset that one can read **S** in either (i) a more or (ii) a less constructive way, namely (i) that the stage s exists mathematically because of ... the act of imagination, which is thus a sort of construction of s, or (ii) that what can be imagined is but an indication of what has mathematical existence, so that the latter can retain changeless Platonic impregnability or Cantorian absoluteness' (op. cit.).

Reinhardt develops the following theory for principle **S**. 'The principle **S** distinguishes between "imagine" and "exist". We shall do this formally for sets by treating "imagine" as the quantifier and "exist" as quantification relativized to a certain predicate. (Consequently we should not call $\exists$ the existential quantifier, but a generalized existential quantifier. We keep the usual logic for $\exists$, however. We do not consider the possibility of a non-classical logic for "imaginary" objects.) We suppose that we can imagine the set of all existing sets; V denotes this imaginary set. We have yet to explain the term "property". Generally, formulas correspond to properties of sets. Notice that the principle **S** only refers to existing properties, not imaginary ones (taking "is" not in the scope of "imagine" to have existential force). Formulas in which all parameters are existing sets will correspond to existing properties (we do not assume the correspondence is onto). Formulas involving merely imagined objects x, such as "$t \in x$", will not in general correspond to existing properties. Note in particular that we must not assume that V exists, and consequently the property P such that $t \in V \iff P(t)$ must not be assumed to exist either. / Our formalization of "there exists a set x such that ..." is "$\exists x(x \in V$ [$\wedge$] ...)' (7).

'Imagine' and 'exist' can, too, be defined relative to ranks, with 'exists to mean "exists at a higher level", and "imagine" to mean "exists, but possibly only at higher levels"' (9).

The theory for principle **S** has the following axioms:

With $\exists x$ interpreted as "x can be imagined", "$x \in V$" as "x exists", and "$\phi(\mathrm{a}_1, \ldots, \mathrm{a}_n, t)$" as "$t$ has P", principle **S** is formalized as follows:

(i) $\langle \mathrm{a}_1, \ldots, \mathrm{a}_n \rangle \in V \wedge \exists x \forall t(\phi(\mathrm{a}_1, \ldots, \mathrm{a}_n, t) \rightarrow t \in x) \rightarrow \exists s[s \in V \wedge \forall t(\phi(\mathrm{a}_1, \ldots, \mathrm{a}_n, t) \rightarrow t \in s)]$, with ϕ a $\in$-formula '(i.e. any formula of L) with free variables $\mathrm{a}_1, \ldots, \mathrm{a}_n$, t, and x is distinct from all these' (8).

(ii) $\forall t(t \in x \iff t \in y) \rightarrow x = y$;

(iii) $t \in x \wedge x \in V \rightarrow t \in V$,

$t \subseteq x \wedge x \in V \rightarrow t \in V$,

$x \in V \wedge t \subseteq V \wedge t \equiv x \rightarrow t \in V$;

(iv) $\exists x \forall t(t \in x \iff \theta \wedge t \in x)$, where θ is a formula not involving x (7-8).

Reinhardt (1974a: §6) proposes the use of imaginary sets and classes as 'imaginary experiments' (204), in order to define imaginary projections corresponding to the universe of sets which define Reinhardt cardinals. An objection to the foregoing is advanced by Maddy (1988) who objects to the 'use of counterfactual situations to distinguish these new entities from sets' (754). Maddy writes: 'I think even those with strong modal intuitions will have trouble imagining how there might be more pure sets and ordinals than there are. After all, V is supposed to contain all the sets and ordinals there could possibly be' (op. cit.).

The approach to mathematical modality, according to which it yields a representation of the cumulative universe of sets, has been examined by Fine (2006), Uzquiano (2015), and Linnebo (2018a). Fine argues that the mathematical modality should be postulational and interpretational; and thus taken to concern the reinterpretation of the domain over which the quantifiers range, in order to avoid inconsistency. Uzquiano argues similarly for an interpretational construal of mathematical modality, where the cumulative hierarchy of sets is fixed, yet what is possibly reinterpreted is the non-logical vocabulary of the language, in particular the membership relation.[3]

In the setting of unrestricted quantification, suppose, e.g., that there is an interpretation for the domain over which a quantifier ranges. Fine writes that an interpretation 'I is exten[s]ible – in symbols, E(I) – if possibly some interpretation extends it, i.e. $\diamond\exists J(I \subset J)$' (2006: 30). Then, the interpretation of the domain over which the quantifier ranges is *extensible*, if '$\forall I.E(I)$'. The interpretation of the domain over which the quantifier ranges is *indefinitely extensible*, if '$\Box\forall I.E(I)$' iff '$\Box\forall I\diamond\exists J(I \subset J)$'. Fine's interpretational modality is taken as postulational and interpretational, although a natural thought might be to combine it with the dynamic postulational modality which he also countenances. On Fine (2005)'s approach, there are dynamic, postulational, and 'prescriptive' or imperatival modalities. The prescriptive element consists in the rule:

'Introduction: !x.C(x)',

such that one is enjoined to postulate, i.e. to 'introduce an object x conforming to the condition C(x)' (2005: 91; 2006: 38).

Then, possible reinterpretations of quantifier domains are induced via the prescriptive imperative to postulate the existence of a new object by the foregoing 'Introduction' rule (2006: 30-31; 38). Fine clarifies that the postulational approach is consistent with a 'realist ontology' of the set of reals. He refers to the imperative to postulate new objects, and thereby reinterpret the domain for the quantifier, as the 'mechanism' by which epistemically to track the cumulative hierarchy of sets (2007: 124-125).

In accord with Fine's approach, I will argue that epistemic mathematical modality has a similarly representational interpretation, and perhaps the postulational property is an optimal means of inducing a reinterpretation of the domain of the quantifier. However, the present approach avoids a potential issue with Fine's account, with regard to the the introduction of deontic modal properties of the prescriptive and imperatival rules that he mentions.[4] It is sufficient that the interpretational modalities are a species of epistemic modality, i.e. possibilities that are relative to agents' spaces of states of information.

Developing Parsons' (1983) program, Linnebo (2013) outlines a modalized version of ZF. [5] Linnebo argues that his modal set theory ought to be governed

[3]Compare Gödel, 1947; Williamson, 1998; and Fine, 2005.

[4]For an analysis of the precise interaction between the semantic values of epistemic and deontic modal operators, see Author (ms).

[5]Linnebo (2018b) discusses the differences between Putnam's and Parsons' accounts of the role of modality in mathematics. Berry (2022) also discusses the differences between the foregoing. Linnebo (op. cit.: 265-266) avails of two-dimensional indexing for the relation between interpretational and circumstantial modalities. In Linnebo (2018a), he characterizes the relation between interpretational and circumstantial modalities via a bimodal product logic, rather than a two-dimensional semantics. He countenances two commutativity principles – '$\Box\blacksquare\phi \iff \blacksquare\Box$' and '$\Diamond\blacklozenge\phi \iff \blacklozenge\Diamond$', with $\Box$ a circumstantial modality and $\blacksquare$ an interpretational modality – although details a counterexample to which they are susceptible. The present approach occurs in the setting of epistemic two-dimensional semantics, such that there is a one-way dependence of metaphysical profiles on epistemic profiles. The question of whether there might perhaps be other fruitful interaction principles between epistemic and metaphysical modality and hyperintensionality was raised in conversation with xx, and is a

by the system S4.2 – i.e. K $[\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)]$, T $(\Box\phi \rightarrow \phi)$, 4 $((\Box\phi \rightarrow \Box\Box\phi)$, and G $(\Diamond\Box\phi \rightarrow \Box\Diamond\phi)$ – the Converse Barcan formula, and a restricted version of the Barcan formula. However – rather than being either interpretational or epistemic – Linnebo deploys the mathematical modality in order to account for the notion of 'potential infinity', as anticipated by Aristotle.[6] The mathematical modality is thereby intended to provide a formally precise answer to the inquiry into the extent of the cumulative set-theoretic hierarchy; i.e., in order to precisify the answer that the hierarchy extends 'as far as possible' (2013: 205).[7]

Thus, Linnebo takes the modality to be constitutive of the actual ontology of sets; and the quantifiers ranging over the actual ontology of sets are claimed to have an 'implicitly modal' profile (2010: 146; 2013: 225). He suggests, e.g., that: 'As science progresses, we formulate set theories that characterize larger and larger initial segments of the universe of sets. At any one time, precisely those sets are actual whose existence follows from our strongest, well-established set theory' (2010: 159n21). However – despite his claim that the modality is constitutive of the actual ontology of sets – Linnebo concedes that the mathematical modality at issue cannot be interpreted metaphysically, because sets exist of metaphysical necessity if at all (2010: 158; 2013: 207). In order partly to allay the tension, Linnebo remarks, then, that set theorists 'do not regard themselves as located at some particular stage of the process of forming sets' (2010: 159); and this might provide evidence that the inquiry – concerning at which stage in the process of set-individuation we happen to be, at present – can be avoided.

In response, a solution to this dilemma might be by distinguishing, as above, between non-maximally objective and maximally objective modalities. Maximally objective modalities are metaphysical and the most general type of

topic for future research. Roberts (2019) countenances four interaction principles in a bimodal logic for interpretational and circumstantial modalities, similar to Linnebo's (op. cit.), and applied to the indefinite extensibility of possibilia. Vlach-operators in two of the principles simulate two-dimensional indexing. The principles are a bimodal version of the converse Barcan formula: $\blacksquare\Box\forall v\phi \rightarrow \forall v\blacksquare\Box\phi$ (1159); $\blacksquare(\Diamond A \rightarrow \Box\Diamond A)$ (1161); $\uparrow^{*1}\uparrow^{1}\blacklozenge\uparrow^{*2}\Diamond\uparrow^{2}\downarrow^{*1}\downarrow^{1}\Box\forall x([E(x) \rightarrow \downarrow^{*2}\downarrow^{2}E(x))]$ (1162); $\uparrow^{*1}\Box\uparrow^{1}\blacklozenge\uparrow^{*2}\Diamond\uparrow^{2}[\downarrow^{*1}\downarrow^{1}\forall x[E(x) \rightarrow \downarrow^{*2}\downarrow^{2}E(x)] \wedge \exists x[E(x) \wedge \downarrow^{*1}\downarrow^{1}\neg E(x)]]$ (op. cit.). $\uparrow^{*}A$ is a Vlach-operator on A selecting an interpretational modality from an ω-sequence comprising a set thereof which validates A, and $\uparrow A$ is a Vlach-operator on A selecting a metaphysical possibility from an ω-sequence comprising a set thereof which validates A. The appeal to epistemic two-dimensional semantics in order to account for interpretational as epistemic and circumstantial as objective modalities and their interaction in this essay was written in 2015 and pursued prior to knowledge of Linnebo's and Roberts' accounts. My approach differs, as well, by countenancing a hyperintensional, topic-sensitive epistemic two-dimensional truthmaker semantics and applying it to various phenomena in the philosophy of mathematics.

[6]Cf. Aristotle, *Physics*, Book III, Ch. 6.

[7]Precursors to the view that modal operators can be availed of in order to countenance the potential hierarchy of sets include Hodes (1984). Intensional constructions of set theory are further developed by Reinhardt (1974a,b); Parsons (op. cit.); Myhill (1985); Scedrov (1985); Flagg (1985); Goodman (1985); Hellman (1990); Nolan (2002); and Studd (2013). (See Shapiro (1985) for an intensional construction of arithmetic.) Chihara (2004: 171-198) argues that 'broadly logical' conceptual possibilities can be used to represent imaginary situations relevant to the construction of open-sentence tokens. The open-sentences can then be used to define the properties of natural and cardinal numbers and the axioms of Peano arithmetic.

modality. Non-maximally objective modalities, such as those figuring in the two-dimensional modal profile of indefinite extensibility, are not technically metaphysical though are still non-epistemic and non-deontic, are interpreted so as to concern reality, and are thus objective.

In his (2018a), Linnebo countenances both interpretational and metaphysical modalitities, and he argues that the former also satisfy S4.2.

Another distinction to note is that both Linnebo (op. cit.) and Uzquiano (op. cit.) avail of second-order plural quantification, in developing their primitivist and interpretational accounts of mathematical modality. By contrast to their approaches, the epistemic and metaphysical modalities defined in the next section are defined with second-order singular quantification over sets.

Linnebo and Uzquiano both suggest that their mathematical modalities ought to be governed by the G axiom; i.e. $\Diamond\Box\phi \rightarrow \Box\Diamond\phi$. The present approach eschews, however, the G axiom, in virtue of the following. Williamson (2009) demonstrates that an epistemic operator which validates the conjunction of the 4 axiom of positive introspection and the E axiom of negative introspection will be inconsistent with the condition of 'recursively enumerable conservativeness', although positive and negative introspection are individually consistent with r.e. (quasi-)conservativeness (30), where the conservativeness constraint defines recursively enumerable theories 'without reference to models' (12). 'A model M is r.e. if and only if $\Box^{-1}$M (which expresses what the agent cognizes in M) is an r.e. (recursively enumerable) theory in [a language of propositional modal logic,] $L_\Box$. In that sense, the agent's cognition in an r.e. model does not exceed the computational capacity of a sufficiently powerful Turing machine' (10). A theory is recursively enumerable if the valid strings in the theory can be enumerated by a Turing machine. A theory is recursive if the Turing machine halts on every input. 'Σ is the logic of an application on which $\Box^{-1}$M is a theory in $L_\Box$ for every intended model M' (8). Σ is r.e. conservative if and only if for every r.e. theory R in L, there is a maximal Σ-consistent set X such that $\Box^{-1}$X is r.e. and $L \cap \Box^{-1}X = R$. Σ is r.e. quasi-conservative if and only if for every consistent r.e. theory R in L, there is a maximal Σ-consistent set X such that $\Box^{-1}$X is r.e. and $L \cap \Box^{-1}X = R$' (12). S5 is not r.e. (quasi-)conservative (14). The G axiom is not r.e. quasi-conservative. K4 is r.e. conservative and GL is r.e. (quasi)-conservative (Williamson, 2009: 13, 26-27, 29). It is, however, an open question whether K4+GL is r.e. (quasi-)conservative.

My application of epistemic two-dimensional semantics to the epistemology of mathematics departs from full-blooded platonism, as well. According to full-blooded platonism, whatever mathematical objects can exist, do exist, and every consistent mathematical theory describes either a different part of the mathematical universe or distinct mathematical universes altogether (Balaguer, 1998). Thus, ZFC+CH and ZFC+¬CH both 'truly describe collections of mathematical objects', holding in distinct albeit equally real mathematical universes (Balaguer, 2001: 97: see also Hamkins, 2012).

Epistemic two-dimensionalism and full-blooded platonism differ, further, on both the nature of their target possibilities and on the status of the actuality of the possibilities. Epistemic two-dimensionalism avails of epistemic possibilities, whereas full-blooded platonism avails of logical possibilities. Further, not all epistemic possibilities are actual according to epistemic two-dimensionalism, whereas the objects of any logically consistent theory actually exist according to full-blooded platonism. One reason to prefer epistemic two-dimensionalism to full-blooded platonism is that the former can be formalized, whereas Restall (2003) has shown that there are significant challenges to formalizing the latter. Another reason to prefer epistemic two-dimensionalism is that – unlike full-blooded platonism – it avoids commitment to the existence of inconsistent universes of sets where e.g. both ZFC+V=L and ZFC+V≠L would obtain.

Waxman (ms) endeavors to account for the interaction between the imagination and mathematics. Whereas I avail in this paper of conceivability as defined in epistemic two-dimensional semantics – which I refer to in the mathematical setting as epistemic mathematical modality – in order to account for how the epistemic possibility of abstraction principles and large cardinal axioms relates to their metaphysical possibility, Waxman's aim is to account for how imagining a model of a mathematical theory entrains justification to believe its consistency (op. cit.). Unlike Waxman, epistemic mathematical modality is ideal, whereas imagination is, on his account, non-ideal (Waxman, op. cit.: 18; Chalmers, 2002), where ideal conceivability means true at the limit of apriori reflection unconstrained by finite limitations. Unlike Waxman, I believe, further, that imaginative contents are sensitive to hyperintensional subject-matters or topics (cf. Berto, 2018; Canavotto, Berto, and Giordani, 2020).

Further applications of hyperintensional semantics to the philosophy of mathematics include availing of epistemic two-dimensional hyperintensions (functions from topic-sensitive epistemic truthmakers to topic-sensitive metaphysical truthmakers to extensions) in order to capture the interaction between the epistemic and objective or metaphysical hyperintensional profiles of abstraction principles (Author, ms1), the access problem (Author, ms2), rational intuition (Author, ms3), and indefinite extensibility (Author, ms4).

3 Mathematical Modality

3.1 Metaphysical Mathematical Modality

A formula is a logical truth if and only if the formula is true in an intended model structure, M = <W, D, R, V>, where W designates a space of metaphysically possible worlds; D designates a domain of entities, constant across worlds; R designates an accessibility relation on worlds; and V is an assignment function mapping elements in D to subsets of W.

Metaphysical Mathematical Possibility
$[\![\Diamond\phi]\!]^{v,w} = 1 \Longleftrightarrow \exists w'[\![\phi]\!]^{v,w'} = 1$

Metaphysical Mathematical Necessity
$[\![\Box\phi]\!]^{v,w} = 1 \Longleftrightarrow \forall w'[\![\phi]\!]^{v,w'} = 1$,
with $\Diamond := \neg\Box\neg$

3.2 Epistemic Mathematical Modality

In order to accommodate the notion of epistemic possibility, we enrich M with the following conditions: M = <C, W, D, R, V>, where C, a set of epistemically possibilities, is constrained as follows:

Let $[\![\phi]\!]^c \subseteq$ C;

(ϕ is a formula encoding a state of information at an epistemically possible world).

The interpretation of epistemic possibility which will here be at issue defines the notion in relation to logical reasoning (Jago, 2009; Bjerring, 2012), by contrast to a for all one knows operator (see MacFarlane, 2011) or as the dual of epistemic necessity i.e. apriority (see Chalmers, 2006, 2011). Bjerring writes: '[W]e can now spell out deep epistemic necessity and possibility by appeal to provability in n steps of logical reasoning using the rules in R. To that end, let a proof of A in n steps of logical reasoning be a derivation of A from a set Γ of sentences – potentially the empty set – consisting of at most n applications of the rules in R. Let a disproof of A in n steps of logical reasoning be a derivation of ¬A from A – or from the set Γ of sentences such that A∈Γ – consisting of at most n applications of the rules in R. Similarly, let a set Γ of sentences be disprovable in n steps of logical reasoning whenever there is a derivation of A and ¬A from Γ consisting of at most n applications of the rules in R. For simplicity, I will assume that agents can rule out sets of sentence that contain {A,¬A} non-inferentially. Finally, let '$\Box_n$' and '$\Diamond_n$' be metalinguistic operators, where '$\Diamond_n$' is defined as $\neg\Box_n\neg$. Read '$\Box_n$' as 'A is provable in n steps of logical reasoning using the rules in R', and read '$\Diamond_n$'as 'A is not disprovable in n steps of logical reasoning using the rules in R'. We can then define:

(Deep-Necn) A sentence A is deeply$_n$ epistemically necessary iff $\Box_n$.

(Deep-Posn) A sentence A is deeply$_n$ epistemically possible iff $\Diamond_n$' (op. cit.).

The interpretation of epistemic possibility which will here be at issue defines the notion as conceivability, the dual of epistemic necessity i.e. apriority (see Chalmers, 2006, 2011), instead of consistent logical reasoning (Jago, 2009; Bjerring, 2012) and a for all one knows operator (see MacFarlane, 2011). In the hyperintensional setting outlined below, the box and diamond operators are replaced by necessary and possible truthmakers which serve as verifiers for propositions. On the consistent logical reasoning interpretation of epistemic possibility, necessary truthmakers receive the same interpretation as $\Box_n$, i.e. that a proposition A is provable in n steps of logical reasoning using the rules in R. On their metaphysical interpretation, truthmakers verify the truth values of propositions and are orthogonal to the logical reasoning which figures in the interpretation of the epistemic truthmakers. The consistent logical reasoning interpretation ties truthmaking to provability and is of relevance to the discussion of epistemic possibility and hyperintensionality and their bearing on absolute decidability, but will not be here examined.

Intensions

-pri$(x) = \lambda c.[\![x]\!]^{c,c}$;

(This is a primary, or epistemic, intension. The two parameters relative to which x – a propositional variable – obtains its value are epistemically possible worlds).

-$\texttt{sec}(x) = \lambda c.[\![x]\!]^{w,w}$

(This is a secondary intension. The two parameters relative to which x obtains its value are metaphysically possible worlds).

-$\texttt{2D}(x) = \lambda c \lambda w [\![\mathrm{x}]\!]^{c,w} = 1$

(This is a 2D intension. A first parameter ranging over epistemic scenarios determines the value of the formula relative to a second parameter ranging over metaphysically possible worlds).

Then:

- **Epistemic Mathematical Necessity**

  $[\![\blacksquare\phi]\!]^{c,w} = 1 \iff \forall c' [\![\phi]\!]^{c',c'} = 1$

  (ϕ is true at all points in epistemic modal space).

- **Epistemic Mathematical Possibility**

  $[\![\blacklozenge\phi]\!] \neq \emptyset \iff [\![\neg\blacksquare\neg\phi]\!] = 1$

  (ϕ might be true if and only if it is not epistemically necessary for ϕ to be false).

Epistemic mathematical modality can be constrained by consistency, and the formal techniques of provability and forcing. A mathematical formula is false, and therefore metaphysically impossible, if it can be disproved or induces inconsistency in a model.

3.3 Interaction

- **Convergence**

  $\forall \mathrm{c} \exists \mathrm{w} [\![\phi]\!]^{c,w} = 1$

  (the value of x is relative to a parameter for the space of epistemically possible worlds. The value of x relative to the first parameter determines the value of x relative to the second parameter for the space of metaphysical possibility).

- **Super-rigidity**

  $[\![\phi]\!]^{c,w} = 1 \iff \forall \mathrm{w'},\mathrm{c'} [\![\phi]\!]^{c',w'} = 1$

  (ϕ is rigid in all points in epistemic and metaphysical modal space).

3.4 Modal Axioms

- Metaphysical mathematical modality is governed by the modal system KTE, as augmented by the Barcan formula and its Converse (cf. Fine, 1981).

K: $\Box[\phi \rightarrow \psi] \rightarrow [\Box\phi \rightarrow \Box\psi]$
T: $\Box\phi \rightarrow \phi$
E: $\neg\Box\phi \rightarrow \Box\neg\Box\phi$
Barcan: $\Diamond\exists xFx \rightarrow \exists x\Diamond Fx$
Converse Barcan: $\exists x\Diamond Fx \rightarrow \Diamond\exists xFx$

- Epistemic mathematical modality is governed by the modal system, K4+GL.[8]

K: $\blacksquare[\phi \rightarrow \psi] \rightarrow [\blacksquare\phi \rightarrow \blacksquare\psi]$
4: $\blacksquare\phi \rightarrow \blacksquare\blacksquare\phi$
GL: $\blacksquare[\blacksquare\phi \rightarrow \phi] \rightarrow \blacksquare\phi$

Note that, if one prefers a hyperintensional semantics to an intensional semantics, one can avail of the definitions of hyperintensions as functions from states in a state space to extensions instead of from whole epistemically and metaphysically possible worlds. See Chapter **2** for the relevant models and definitions.

3.5 Topic-sensitive Two-dimensional Truthmaker Semantics

If one prefers hyperintensional semantics to possible worlds semantics – in order e.g. to avoid the situation in intensional semantics according to which all necessary formulas express the same proposition because they are true at all possible worlds – one can avail of the following epistemic two-dimensional truthmaker semantics, which specifies a notion of exact verification in a state space and where states are parts of whole worlds (Fine 2017a,b; Hawke and Özgün, forthcoming). According to truthmaker semantics for epistemic logic, a modalized state space model is a tuple $\langle$S, P, $\leq$, v$\rangle$, where S is a non-empty set of states, P is the subspace of possible states where states s and t comprise a fusion when s $\sqcup$ t$\in$P, $\leq$ is a partial order, and v: Prop $\rightarrow$ (2^S x 2^S) assigns a bilateral proposition $\langle p^+, p^-\rangle$ to each atom p$\in$Prop with p^+ and p^- incompatible (Hawke and Özgün, forthcoming: 10-11). Exact verification ($\vdash$) and exact falsification ($\dashv$) are recursively defined as follows (Fine, 2017a: 19; Hawke and Özgün, forthcoming: 11):

s $\vdash$ p if s$\in[\![p]\!]^+$
(s verifies p, if s is a truthmaker for p i.e. if s is in p's extension);

[8] For further discussion of the properties of GL, see Löb (1955); Smiley (1963); Kripke (1965); and Boolos (1993). Löb's provability formula was formulated in response to Henkin's (1952) problem concerning whether a sentence which ascribes the property of being provable to itself is provable. (Cf. Halbach and Visser, 2014, for further discussion.) For an anticipation of the provability formula, see Wittgenstein (1933-1937/2005: 378). Wittgenstein writes: 'If we prove that a problem can be solved, the concept 'solution' must somehow occur in the proof. (There must be something in the mechanism of the proof that corresponds to this concept.) But the concept mustn't be represented by an external description; it must really be demonstrated. / The proof of the provability of a proposition is the proof of the proposition itself' (op. cit.). Wittgenstein contrasts the foregoing type of proof with 'proofs of relevance' which are akin to the mathematical, rather than empirical, propositions, discussed in Wittgenstein (2001: IV, 4-13, 30-31).

s $\dashv$ p if s$\in[\![p]\!]^-$
(s falsifies p, if s is a falsifier for p i.e. if s is in p's anti-extension);
s $\vdash \neg$p if s $\dashv$ p
(s verifies not p, if s falsifies p);
s $\dashv \neg$p if s $\vdash$ p
(s falsifies not p, if s verifies p);
s $\vdash$ p $\wedge$ q if $\exists$v,u, v $\vdash$ p, u $\vdash$ q, and s = v $\sqcup$ u
(s verifies p and q, if s is the fusion of states, v and u, v verifies p, and u verifies q);
s $\dashv$ p $\wedge$ q if s $\dashv$ p or s $\dashv$ q
(s falsifies p and q, if s falsifies p or s falsifies q);
s $\vdash$ p $\vee$ q if s $\vdash$ p or s $\vdash$ q
(s verifies p or q, if s verifies p or s verifies q);
s $\dashv$ p $\vee$ q if $\exists$v,u, v $\dashv$ p, u $\dashv$ q, and s = v $\sqcup$ u
(s falsifies p or q, if s is the fusion of the states v and u, v falsifies p, and u falsifies q);
s $\vdash \forall x\phi(x)$ if $\exists s_1, \ldots, s_n$, with $s_1 \vdash \phi(a_1), \ldots, s_n \vdash \phi(a_n)$, and $s = s_1 \sqcup \ldots \sqcup s_n$
[s verifies $\forall x\phi(x)$ "if it is the fusion of verifiers of its instances $\phi(a_1), \ldots, \phi(a_n)$" (Fine, 2017c)];
s $\dashv \forall x\phi(x)$ if s $\dashv \phi(a)$ for some individual a in a domain of individuals (op. cit.)
[s falsifies $\forall x\phi(x)$ "if it falsifies one of its instances" (op. cit.)];
s $\vdash \exists x\phi(x)$ if s $\vdash \phi(a)$ for some individual a in a domain of individuals (op. cit.)
[s verifies $\exists x\phi(x)$ "if it verifies one of its instances $\phi(a_1), \ldots, \phi(a_n)$" (op. cit.)];
s $\dashv \exists x\phi(x)$ if $\exists s_1, \ldots, s_n$, with $s_1 \dashv \phi(a_1), \ldots, s_n \dashv \phi(a_n)$, and $s = s_1 \sqcup \ldots \sqcup s_n$ (op. cit.)
[s falsifies $\exists x\phi(x)$ "if it is the fusion of falsifiers of its instances" (op. cit.)];
s exactly verifies p if and only if s $\vdash$ p if s$\in[\![p]\!]$;
s inexactly verifies p if and only if s $\triangleright$ p if $\exists$s'$\leq$S, s' $\vdash$ p; and
s loosely verifies p if and only if, $\forall$v, s.t. s $\sqcup$ v $\vdash$ p (35-36);
s $\vdash$ Aϕ if and only if for all u$\in$P there is a u'$\in$P such that u' $\sqcup$ u$\in$P and u' $\vdash \phi$, where Aϕ denotes the necessary truthmaker of ϕ; and
s $\dashv$ Aϕ if and only if there is a v$\in$P such that for all u$\in$P either v $\sqcup$ u$\notin$P or u $\dashv \phi$;
s $\vdash$ A(Aϕ) if and only if for all u$\in$P there is a u'$\in$P such that u' $\sqcup$ u $\in$P and u' $\vdash \phi$ and there is a u"$\in$P such that u' $\sqcup$ u"$\in$P and u" $\vdash \phi$;
s $\vdash$ A($\forall x\phi(x)$) if and only if for all u$\in$P there is a u'$\in$P such that u $\vdash$ [u' $\vdash \exists s_1, \ldots, s_n$, with $s_1 \vdash \phi(a_1), \ldots, s_n \vdash \phi(a_n)$, and u' = $s_1 \sqcup \ldots \sqcup s_n$];
s $\vdash$ A($\exists x\phi(x)$) if and only if or all u$\in$P there is a u'$\in$P such that u $\vdash$ [u' $\vdash \phi(a)$] for some individual a in a domain of individuals (op. cit.).

In order to account for two-dimensional indexing, we augment the model, M, with a second state space, S*, on which we define both a new parthood relation, $\leq$*, and partial function, V*, which serves to map propositions in a domain, D, to pairs of subsets of S*, {1,0}, i.e. the verifier and falsifier of p, such that $[\![p]\!]^+ = 1$ and $[\![p]\!]^- = 0$. Thus, M = ⟨S, S*, D, ≤, ≤*, V, V*⟩. The two-dimensional hyperintensional profile of propositions may then be recorded by defining the value of p relative to two parameters, c,i: c ranges over subsets of S, and i ranges over subsets of S*.

(*) M,s∈S,s*∈S* ⊢ p iff:
(i) $\exists c_s [\![p]\!]^{c,c} = 1$ if $s \in [\![p]\!]^+$; and
(ii) $\exists i_{s*} [\![p]\!]^{c,i} = 1$ if $s^* \in [\![p]\!]^+$

(Distinct states, s,s*, from distinct state spaces, S,S*, provide a multi-dimensional verification for a proposition, p, if the value of p is provided a truthmaker by s. The value of p as verified by s determines the value of p as verified by s*).

We say that p is hyper-rigid iff:

(**) M,s∈S,s*∈S* ⊢ p iff:
(i) $\forall c'_s [\![p]\!]^{c,c'} = 1$ if $s \in [\![p]\!]^+$; and
(ii) $\forall i_{s*} [\![p]\!]^{c,i} = 1$ if $s^* \in [\![p]\!]^+$

Epistemic (primary), subjunctive (secondary), and 2D hyperintensions can be defined as follows, where hyperintensions are functions from states to extensions, and intensions are functions from worlds to extensions. Epistemic two-dimensional truthmaker semantics receives substantial motivation by its capacity (i) to model conceivability arguments involving hyperintensional metaphysics, and (ii) to avoid the problem of mathematical omniscience entrained by intensionalism about propositions[9]:

- Epistemic Hyperintension:

  $\texttt{pri}(x) = \lambda s.[\![x]\!]^{s,s}$, with s a state in the epistemic state space S

- Subjunctive Hyperintension:

  $\texttt{sec}_{v_@}(x) = \lambda w.[\![x]\!]^{v_@,w}$, with w a state in metaphysical state space W

In epistemic two-dimensional semantics, the value of a formula or term relative to a first parameter ranging over epistemic scenarios determines the value of the formula or term relative to a second parameter ranging over metaphysically possible worlds. The dependence is recorded by 2D-intensions. Chalmers (2006: 102) provides a conditional analysis of 2D-intensions to characterize the dependence: "Here, in effect, a term's subjunctive intension depends on which epistemic possibility turns out to be actual. / This can be seen as a mapping from scenarios to subjunctive intensions, or equivalently as a mapping from (scenario, world) pairs to extensions. We can say: the two-dimensional intension of a statement S is true at (V, W) if V verifies the claim that W satisfies S. If $[A]_1$ and $[A]_2$ are canonical descriptions of V and W, we say that the two-dimensional intension is true at (V, W) if $[A]_1$ epistemically necessitates that

[9]See Author (ms_1) through (ms_n) for further discussion.

$[A]_2$ subjunctively necessitates S. A good heuristic here is to ask "If $[A]_1$ is the case, then if $[A]_2$ had been the case, would S have been the case?". Formally, we can say that the two-dimensional intension is true at (V, W) iff '$\Box_1([A]_1 \rightarrow \Box_2([A]_2 \rightarrow S))$' is true, where '$\Box_1$' and '$\Box_2$' express epistemic and subjunctive necessity respectively".

- 2D-Hyperintension:

  $\mathtt{2D}(x) = \lambda s \lambda w [\![\mathrm{x}]\!]^{s,w} = 1.$

Following the presentation of topic models in Berto (2018; 2019), Canavotto et al (2020), and Berto and Hawke (2021), atomic topics comprising a set of topics, T, record the hyperintensional intentional content of atomic formulas, i.e. what the atomic formulas are about at a hyperintensional level. Topic fusion is a binary operation, such that for all x, y, z∈T, the following properties are satisfied: idempotence ($x \oplus x = x$), commutativity ($x \oplus y = y \oplus x$), and associativity $[(x \oplus y) \oplus z = x \oplus (y \oplus z)]$ (Berto, 2018: 5). Topic parthood is a partial order, $\leq$, defined as $\forall x,y \in T(x \leq y \iff x \oplus y = y)$ (op. cit.: 5-6). Atomic topics are defined as follows: $\mathrm{Atom}(x) \iff \neg\exists y < x$, with $<$ a strict order. Topic parthood is thus a partial ordering such that, for all x, y, z∈T, the following properties are satisfied: reflexivity ($x \leq x$), antisymmetry ($x \leq y \wedge y \leq x \rightarrow x = y$), and transitivity ($x \leq y \wedge y \leq z \rightarrow x \leq z$) (6). A topic frame can then be defined as $\{W, R, T, \oplus, t\}$, with t a function assigning atomic topics to atomic formulas. For formulas, ϕ, atomic formulas, p, q, r (p_1, p_2, ...), and a set of atomic topics, $Ut\phi = \{p_1, \ldots\ p_n\}$, the topic of ϕ, $t(\phi) = \oplus Ut\phi = t(p_1) \oplus \ldots \oplus t(p_n)$ (op. cit.). Topics are hyperintensional, though not as fine-grained as syntax. Thus $t(\phi) = t(\neg\neg\phi)$, $t\phi = t(\neg\phi)$, $t(\phi \wedge \psi) = t(\phi) \oplus t(\psi) = t(\phi \vee \psi)$ (op. cit.).

The diamond and box operators can then be defined relative to topics:
$\langle M,w \rangle \Vdash \Diamond^t \phi$ iff $\langle R_{w,t} \rangle(\phi)$
$\langle M,w \rangle \Vdash \Box^t \phi$ iff $[R_{w,t}](\phi)$, with
$\langle R_{w,t} \rangle(\phi) := \{w' \in W t' \in T \mid R_{w,t}[w', t'] \cap \phi \neq \emptyset$ and $t'(\phi) \leq t(\phi)$
$[R_{w,t}](\phi) := \{w' \in W t' \in T \mid R_{w,t}[w', t'] \subseteq \phi$ and $t'(\phi) \leq t(\phi)$.

We can then combine topics with truthmakers rather than worlds, thus countenancing doubly hyperintensional semantics, i.e. topic-sensitive epistemic two-dimensional truthmaker semantics:

- Topic-sensitive Epistemic Hyperintension:

  $\mathtt{pri}_t(x) = \lambda s \lambda t.[\![x]\!]^{s \cap t, s \cap t}$, with s a truthmaker from an epistemic state space.

- Topic-sensitive Subjunctive Hyperintension:

  $\mathtt{sec}_{v_@ \cap t}(x) = \lambda w \lambda t.[\![x]\!]^{v_@ \cap t, w \cap t}$, with w a truthmaker from a metaphysical state space.

- Topic-sensitive 2D-Hyperintension:

  $\mathtt{2D}(x) = \lambda s \lambda w \lambda t [\![\mathrm{x}]\!]^{s \cap t, w \cap t} = 1.$

4 Hyperintensional Epistemic Set Theory

4.1 Two-dimensional Hyperintensional Set Theory

Following the presentation in Scedrov (1986: 104), an epistemic truthmaker set theory can be defined as follows.

Logic

- Equality axioms, $x = y \wedge \phi(x) \rightarrow \phi(y)$

  All classical propositional tautologies
- From ϕ and $\phi \rightarrow \psi$ infer ψ
- $A(\phi) \rightarrow \phi$
- $A(\phi) \rightarrow AA(\phi)$
- $A(\phi) \wedge A(\phi \rightarrow \psi) \rightarrow A(\psi)$
- From ϕ infer $A(\phi)$
- $\forall\phi(x) \rightarrow \phi(y)$, where y is free for x in $\phi(x)$
- From $\phi \rightarrow \psi(x)$ infer $\phi \rightarrow \forall\psi(x)$, if x is not free in ϕ
- $\phi(y) \rightarrow \exists\phi(x)$, where y is free for x in $\phi(x)$
- From $\psi(x) \rightarrow \phi$ infer $\exists\psi(x) \rightarrow \phi$, if x is not free in A

Non-logical Axioms

- Epistemic Extensionality: $A[\forall z(z{\in}x \rightarrow z{\in}y) \rightarrow x = y]$
- Foundation: $\forall x[\forall y{\in}x\phi(y) \rightarrow \phi(x)] \rightarrow \forall x\phi(x)$
- Epistemic Foundation: $A[[\forall x[A[\forall y{\in}x\phi(y) \rightarrow \phi(x)]] \rightarrow A[\forall x\phi(x)]]$
- Pairing: $\exists zA(x{\in}z \wedge y{\in}z)$
- Union: $\exists zA[\forall w(\exists y{\in}xw{\in}y \rightarrow w{\in}z)]$
- Separation: $\exists zA[\forall y[y{\in}z \iff y{\in}x \wedge \phi(y)]]$, where z is not free in $\phi(y)$
- Epistemic Power Set: $\exists zA[\forall w[A(\forall y{\in}wy{\in}x \rightarrow w{\in}z)]]$
- Infinity: $\exists A[\exists yA(y{\in}z){\wedge} \forall u{\in}z\exists vA(v{\in}z \wedge u{\in}v)]$
- Collection: $\forall x{\in}u\exists y\phi(x, y) \rightarrow \exists z\forall x{\in}u\exists y{\in}z\phi(x, y)$ where z is not free in $\phi(x, y)$
- Epistemic Collection: $A[\forall x{\in}u\exists y\phi(x, y) \rightarrow \exists zA\forall x{\in}u\exists y[A(y{\in}z) \wedge \phi(x, y)]]$, where z is not free in $\phi(x, y)$.

Two-dimensional hyperintensions can then be defined for each of the foregoing axioms, such that each axiom would be defined relative to two parameters, the first ranging over topic-sensitive epistemic truthmakers, which determines the value of the axiom relative to a second parameter ranging over either non-maximally objective or maximally objective i.e. metaphysical truthmakers.

4.2 Two-dimensional Hyperintensional Large Cardinals

A provisional definition of large cardinal axioms is as follows.

∃xΦ is a large cardinal axiom, because:

(i) Φx is a Σ_2-formula, where 'a sentence ϕ is a Σ_2-sentence if it is of the form: There exists an ordinal α such that $V_\alpha \Vdash \psi$, for some sentence ψ' (Woodin, 2019);

(ii) if κ is a cardinal, such that $V \models \Phi(\kappa)$, then κ is strongly inaccessible, where a cardinal κ is regular if the cofinality of κ – comprised of the unions of sets with cardinality less than κ – is identical to κ, and a strongly inaccessible cardinal is regular and has a strong limit, such that if $\lambda < \kappa$, then $2^\lambda < \kappa$ (Cf. Kanamori, 2012: 360); and

(iii) for all generic partial orders $\mathbb{P} \in V_\kappa$, and all V-generics $G \subseteq \mathbb{P}$, $V[G] \models \Phi x$ (Koellner, 2006: 180).

The truthmaker 2D-intension for large cardinal axioms is then $\forall s \in S, i \in I [\![\Phi x]\!]^{s,i} = 1$ iff $\exists s' \in S, i' \in I [\![\Phi x]\!]^{s',i'} = 1$.

The intension states that the value of a large cardinal axiom relative to an epistemic truthmaker determines the value of the axiom relative to a metaphysical truthmaker.

4.3 Two-dimensional Hyperintensionality and the Epistemic Church-Turing Thesis

The Epistemic Church-Turing Thesis can receive a similar two-dimensional hyperintensional formalization. Carlson (2016: 132) presents the schema for the Epistemic Church-Turing Thesis as follows:

With $\Box$ interpreted as a knowledge operator, '$\Box\forall x\exists y\Box\phi \rightarrow \exists e\Box\forall x\exists y[E(e, x, y) \wedge \phi]$,

'where e does not occur free in ϕ and E is a fixed formula of L_{PA} [i.e the language of Peano Arithmetic] with free variables v_0, v_1, v_2 such that, letting N be the standard model of arithmetic,

'$N \Vdash E(e, x, y)[e, x, y \mid a, m, n]$

'iff on input m, the a^{th} Turing machine halts and outputs n. For convenience, we will write $\{t_1\}\{t_2\} \simeq t_3$ for $E(t_1, t_2, t_3)$ when t_1, t_2, t_3 are terms'. Carlson defines $(x_1, \ldots, x_n \mid (y_1, \ldots, y_1)$ as denoting the 'function which maps x_i to y_i for each $i = 1, \ldots, n$' (op. cit.: 130). Hyperintensionally reformalized, the Epistemic Church-Turing Thesis is then:

$A\forall x\exists yA\phi \rightarrow \exists eA\forall x\exists y[E(e, x, y) \wedge \phi]$.

The two-dimensional hyperintensional profile of the Epistemic Church-Turing Thesis can be countenanced by adding a topic-sensitive truthmaker from a metaphysical state space and making its value dependent on the value of the epistemically necessary truthmaker A(ϕ). Thus:

$A^{(w \cap t)}\forall x \exists y A^{(w \cap t)}\phi \rightarrow \exists e A^{(w \cap t)}\forall x \exists y[E(e, x, y) \wedge \phi]$.

4.4 Two-dimensional Hyperintensionality and Ω-logic

Finally, the hyperintensional formalization of Ω-logical consequence in set theory can be defined as follows. For partial orders, $\mathbb{P}$, let $V^{\mathbb{P}} = V^{\mathbb{B}}$, where $\mathbb{B}$ is the regular open completion of ($\mathbb{P}$).[10] $M_a = (V_a)^M$ and $M_a^{\mathbb{B}} = (V_a^{\mathbb{B}})^M = (V_a^{M^{\mathbb{B}}})$. *Sent* denotes a set of sentences in a first-order language of set theory. $T \cup \{\phi\}$ is a set of sentences extending ZFC. *c.t.m* abbreviates the notion of a countable transitive ∈-model. *c.B.a.* abbreviates the notion of a complete Boolean algebra.

Define a *c.B.a.* in V, such that $V^{\mathbb{B}}$. Let $V_0^{\mathbb{B}} = \emptyset$; $V_\lambda^{\mathbb{B}} = \bigcup_{b<\lambda} V_b^{\mathbb{B}}$, with λ a limit ordinal; $V_{a+1}^{\mathbb{B}} = \{f\colon X \rightarrow \mathbb{B} \mid X \subseteq V_a^{\mathbb{B}}\}$; and $V^{\mathbb{B}} = \bigcup_{a \in On} V_a^{\mathbb{B}}$.

ϕ is true in $V^{\mathbb{B}}$, if its Boolean-value is $1^{\mathbb{B}}$, if and only if

$V^{\mathbb{B}} \models \phi$ iff $[\![\phi]\!]^{\mathbb{B}} = 1^{\mathbb{B}}$.

Thus, for all ordinals, a, and every *c.B.a.* $\mathbb{B}$, $V_a^{\mathbb{B}} \equiv (V_a)^{V^{\mathbb{B}}}$ iff for all $x \in V^{\mathbb{B}}$, $\exists y \in V^{\mathbb{B}} [\![x = y]\!]^{\mathbb{B}} = 1^{\mathbb{B}}$ iff $[\![x \in V^{\mathbb{B}}]\!]^{\mathbb{B}} = 1^{\mathbb{B}}$.

Then, $V_a^{\mathbb{B}} \models \phi$ iff $V^{\mathbb{B}} \models$ '$V_a \models \phi$'.

Ω-logical validity can then be defined as follows:

For $T \cup \{\phi\} \subseteq Sent$,

$T \models_\Omega \phi$, if for all ordinals, a, and *c.B.a.* $\mathbb{B}$, if $V_a^{\mathbb{B}} \models T$, then $V_a^{\mathbb{B}} \models \phi$.

Supposing that there exists a proper class of Woodin cardinals and if $T \cup \{\phi\} \subseteq Sent$, then for all set-forcing conditions, $\mathbb{P}$:

$T \models_\Omega \phi$ iff $V^T \models$ '$T \models_\Omega \phi$',

where $T \models_\Omega \phi \equiv \emptyset \models$ '$T \models_\Omega \phi$'.

The Ω-Conjecture states that $V \models_\Omega \phi$ iff $V^{\mathbb{B}} \models_\Omega \phi$ (Woodin, ms). Thus, Ω-logical validity is invariant in all set-forcing extensions of ground models in the set-theoretic universe.

The soundness of Ω-Logic is defined by universally Baire sets of reals. For a cardinal, e, let a set A be e-universally Baire, if for all partial orders $\mathbb{P}$ of cardinality e, there exist trees, S and T on ω X λ, such that $A = p[T]$ and if $G \subseteq \mathbb{P}$ is generic, then $p[T]^G = \mathbb{R}^G - p[S]^G$ (Koellner, 2013). A is universally Baire, if it is e-universally Baire for all e (op. cit.).

Ω-Logic is sound, such that $V \vdash_\Omega \phi \rightarrow V \models_\Omega \phi$. However, the completeness of Ω-Logic has yet to be resolved.

Leach-Krouse (ms) defines the modal logic of Ω-consequence as satisfying the following axioms. The interaction between hyperintensional necessary truthmakers and the axioms is as follows:

For a theory **T** and with A($\Box\phi$) := for all t∈P there is a t'∈P such that t' ⊔ t ∈P and t' ⊢ '$\mathbf{T}_\alpha^{\mathbb{B}} \Vdash$ ZFC $\Rightarrow \mathbf{T}_\alpha^{\mathbb{B}} \Vdash \phi$', where $\Box$ is interpreted as $\mathbf{T}_\alpha^{\mathbb{B}} \Vdash$ ZFC $\Rightarrow \mathbf{T}_\alpha^{\mathbb{B}} \Vdash \phi$,

[10] The definitions in this section follow the presentation in Bagaria et al. (2006).

$\text{ZFC} \vdash \phi \Rightarrow \text{ZFC} \vdash \text{A}(\Box\phi)$
$\text{ZFC} \vdash \text{A}[\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)]$
$\text{ZFC} \vdash \text{A}(\Box\phi) \rightarrow \phi \Rightarrow \text{ZFC} \vdash \phi$
$\text{ZFC} \vdash \text{A}(\Box\phi) \rightarrow \text{A}(\Box\Box\phi)$
$\text{ZFC} \vdash \text{A}[\Box(\Box\phi \rightarrow \phi)] \rightarrow \text{A}(\Box\phi)$

$\text{A}[\Box(\Box\phi \rightarrow \psi) \vee \Box(\Box\psi \wedge \psi \rightarrow \phi)]$, where this clause added to GL is the logic of 'true in all V_κ for all κ strongly inaccessible' in ZFC. As with the two-dimensional hyperintensional profile of the Epistemic Church-Turing Thesis, the two-dimensional hyperintensional profile of Ω-logical consequence can be countenanced by adding a topic-sensitive truthmaker from a metaphysical state space and making its value dependent on the value of the epistemically necessary truthmaker $\text{A}(\phi)$.

5 Knowledge of Absolute Decidability

Williamson (2016a) examines the extension of the metaphysically modal profile of mathematical truths to the question of absolute decidability. A statement is decidable if and only if there is an algorithm for deciding it or its negation. Statements are absolutely undecidable if and only if they are 'undecidable relative to any set of axioms that are justified' rather than just relative to a system (Koellner, 2006: 153), and they are absolute decidable if and only if they are not absolutely undecidable. In this section, I aim to extend Williamson's analysis to the notion of epistemic mathematical modality that has been developed in the foregoing sections. The extension provides a crucial means of witnessing the significance of the two-dimensional approach for the epistemology of mathematics.

Williamson proceeds by suggesting the following line of thought. Suppose that A is a true interpreted mathematical formula which eludes present human techniques of provability; e.g. the continuum hypothesis (op. cit.). Williamson argues that mathematical truths are metaphysically necessary (op. cit.). Williamson then enjoins one to consider the following scenario: It is metaphysically possible that there is a species which finds A primitively compelling in virtue of their brain states and the evolutionary history thereof. Further, the species 'could not easily have come to believe ¬A or any other falsehood in a relevantly similar way'. He writes: 'In current epistemological terms, their knowledge of A meets the condition of safety: they could not easily have been wrong in a relevantly similar case. Here the relevantly similar cases include cases in which the creatures are presented with sentences that are similar to, but still discriminably different from, A, and express different and false propositions; by hypothesis, the creatures refuse to accept such other sentences, although they may also refuse to accept their negations ... Therefore A is absolutely provable, because the creatures can prove it in one line' (11). Williamson writes then that: 'The claim is not just that A *would* be absolutely provable *if* there were such creatures. The point is the stronger one that A is absolutely *provable* because there *could* in principle be such creatures.'

One way that Williamson's argument might be improved is by endeavoring to accommodate epistemic possibilities in a two-dimensional setting, such that the epistemic possibility of deciding Orey sentences such as CH can be a guide to the metaphysical possibility of deciding Orey sentences. Woodin (2010) discusses a number of results, e.g., with regard to the maximality of an inner model for a supercompact cardinal, and takes such results to comprise evidence for the axiom that the set-theoretic universe, V, is Ultimate-L.[11] The axiom implies the truth of CH. This is thus one case in which the evidence for the epistemic possibility of CH can provide a guide to its metaphysical possibility.

The relation between the Epistemic Church-Turing Thesis and absolute undecidability is complicated, however, by there being results pointing to two opposing conclusions.

The first result is by Leitgeb (2009). Leitgeb endeavors to argue for the convergence between the notion of informal provability – countenanced as an epistemic modal operator, K – and mathematical truth. Availing of Hilbert's (1923/1996: ¶18-42) epsilon terms for propositions, such that, for an arbitrary predicate, **C**(x), with x a propositional variable, the term 'ϵp.**C**(p)' is intuitively interpreted as stating that 'there is a proposition, x(/p), s.t. the formula, that p satisfies **C**, obtains' (op. cit.: 290). Leitgeb purports to demonstrate that $\forall$p(p $\rightarrow$ Kp), i.e. that informal provability is absolute; i.e. truth and provability are co-extensive. He argues as follows. Let Q(p) abbreviate the formula 'p $\wedge$ $\neg$K(p)', i.e., that the proposition, p, is true while yet being unprovable. Let K be the informal provability operator reflecting knowability or epistemic necessity, with $\langle$K$\rangle$ its dual.[12] Then:

1. $\exists$p(p $\wedge$ $\neg$Kp) $\iff$ ϵp.Q(p) $\wedge$ $\neg$Kϵp.Q(p).

By necessitation,

2. K[$\exists$p(p $\wedge$ $\neg$Kp)] $\iff$ K[ϵp.Q(p) $\wedge$ $\neg$Kϵp.Q(p)].

Applying modal axioms, KT, to (1), however,

3. $\neg$K[ϵp.Q(p) $\wedge$ $\neg$Kϵp.Q(p)].

Thus,

4. $\neg$K$\exists$p(p $\wedge$ $\neg$Kp).

Leitgeb suggests that (4) be rewritten

5. $\langle$K$\rangle\forall$p(p $\rightarrow$ Kp).

Abbreviate $\forall$p(p $\rightarrow$ Kp) by $\mathtt{B}$. By existential introduction and modal axiom K, both

6. $\mathtt{B} \rightarrow \exists$p[K(p $\rightarrow \mathtt{B}$) $\vee$ K(p $\rightarrow \neg\mathtt{B}$) $\wedge$ p], and

7. $\neg\mathtt{B} \rightarrow \exists$p[K(p $\rightarrow \mathtt{B}$) $\vee$ K(p $\rightarrow \neg\mathtt{B}$) $\wedge$ p].

Thus,

8. $\exists$p[K(p $\rightarrow \mathtt{B}$) $\vee$ K(p $\rightarrow \neg\mathtt{B}$) $\wedge$ p].

[11]The axiom states that '(i) There is a proper class of Woodin cardinals, and (ii) For each Σ_2-sentence ϕ, if ϕ holds in V then there is a universally Baire set A $\subseteq \mathbb{R}$ such that HOD$^{L(A,\mathbb{R})} \Vdash \phi$, where a set is universally Baire if for all topological spaces Ω and for all continuous functions $\pi : \Omega \rightarrow \mathbb{R}^n$, the preimage of A by π has the property of Baire in the space Ω' (Woodin, 2019).

[12]See Section 5, for further discussion of the duality of knowledge, and its relation to doxastic operators.

Abbreviate (8) by $\mathtt{C}$(p). Introducing epsilon notation,
9. $[K(\epsilon p.\mathtt{C}(p) \rightarrow \mathtt{B}) \vee K(\epsilon p.\mathtt{C}(p) \rightarrow \neg \mathtt{B})] \wedge \epsilon p.\mathtt{C}(p)$.
By K,
10. $[K(\epsilon p.\mathtt{C}(p) \rightarrow K\mathtt{B}) \vee K(\epsilon p.\mathtt{C}(p) \rightarrow K\neg \mathtt{B})]$.
From (9) and necessitation, one can further derive
11. $K\epsilon p.\mathtt{C}(p)$.
By (10) and (11),
12. $K\mathtt{B} \vee K\neg\mathtt{B}$.
From (5), (12), and K, Leitgeb derives
13. $K\mathtt{B}$.
By, then, the T axiom,
14. $\forall p(p \rightarrow Kp)$ (291-292).

Leitgeb takes the proof to demonstrate that formulas of epistemic logic and the epsilon calculus are not logical truths (292). If they are, however, then Leitgeb's proof witnesses the collapse between informal provability and mathematical truth.

The second result is by Marfori and Horsten (2016: 260-261), who prove that if the Epistemic Church-Turing Thesis is true, then there are absolutely undecidable propositions in the language of Epistemic Arithmetic. They prove the following theorem:

'If ECT restricted to Π_1 arithmetical relations ϕ(x, y) holds, then there are absolutely undecidable Π_3 sentences of L_{EA}'.

They proceed by proving the contrapositive: If there are no Π_3 absolute undecidable sentences of L_{EA}, then ECT restricted to Π_1 arithmetical relations is false. They write: 'Suppose that there are no absolutely undecidable Π_3 sentences in L_{EA}:

'$\Box\Psi \iff \Psi$ for all Π_3 $\Psi \in L_{EA}$.

'Choose a Turing-uncomputable total functional Π_1 arithmetical relation ϕ(x, y); from elementary recursion theory we know that such ϕ(x, y) exist.

'Then, $\forall x \exists y \phi(x, y)$. But then we also have that $\forall x \exists y \Box\phi(x, y)$. The reason is that $\Pi_1 \subset \Pi_3$, so for every m and n, ϕ(m, n), being a Π_1 statement, entails $\Box\phi$(m, n). However, $\forall x \exists y \Box\phi(x, y)$ is now a Π_3 statement of L_{EA}, so again from our assumption it follows that $\Box\forall x \exists y \Box\phi(x, y)$.

'Therefore, for the chosen ϕ(x, y) the antecedent of ECT is true whereas its consequent is false. Therefore, for the chosen ϕ(x, y), ECT is false.'

Leitgeb's result demonstrates that informal provability converges with truth, and thus corroborates that mathematical truths are absolutely decidable, whereas Marfori and Horsten's result demonstrates the inconsistency of the Epistemic Church-Turing Thesis and absolute decidability. The consistency of these results is innocuous, and vindicates Gödel's (1951) disjunction: 'Either mathematics is incompletable in this sense, that its evident axioms can never be comprised in a finite rule, that is to say, the human mind (even within the realm of pure mathematics) infinitely surpasses the powers of any finite machine, or else there exist absolutely unsolvable diophantine problems of the type specified (where the case that both terms of the disjunction are true is not excluded, so that there are, strictly speaking, three alternatives)' (Gödel, 1951/1995: 310, §13).

When epistemic possibility is interpreted as informal provability rather than as a type of mechanism, mathematical truths are absolutely decidable. Epistemic-modally constrained computability as in the Epistemic Church-Turing Thesis is, however, inconsistent with absolute decidability.

Note that the two-dimensional intensions and hyperintensions of epistemic two-dimensional semantics account as well for the linking between what Cantor refers to as intrasubjective i.e. immanent reality and transsubjective i.e. transient reality (Cantor, 1883/1996: §8). Immanent reality concerns the reality of mathematical objects relative to the 'understanding', whereas transient reality concerns the reality of mathematical objects relative to the 'external world' (op. cit.). Cantor attributes the relation between the two realities as owing to the 'unity of the all to which we ourselves belong' (op. cit.). However, the existence of functions, i.e. hyperintensions, from topic-sensitive epistemic state spaces to topic-sensitive objective or metaphysical state spaces to extensions provides a more illuminating explanation of the relation between concepts and metaphysics than does the contention that all entities can figure as members of sets or classes in set theory.

The significance of the two-dimensional intensional framework outlined in the foregoing is that it provides an explanation of the discrepancy between metaphysical mathematical modality and epistemic mathematical modality. Metaphysical mathematical modality is governed by the system S5, the Barcan formula, and its Converse, whereas epistemic mathematical modality is governed by GL.Thus, epistemic mathematical modality figures as the mechanism, such that it can provide a guide to the metaphysical possibility of mathematical truth. In the hyperintensional setting, the relation between epistemic and objective states was defined by way of a 2D-hyperintensions.

6 Concluding Remarks

In this paper, I have endeavored to delineate the types of mathematical modality, and to argue that the epistemic interpretation of topic-sensitive two-dimensional truthmaker semantics can be applied in order to explain, in part, the two-dimensional status of large cardinal axioms and the decidability of Orey sentences. The formal constraints on hyperintensional conceivability adumbrated in the foregoing can therefore be considered a guide to our possible knowledge of objective mathematical truth.

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