

Hyperintensional Category Theory and Indefinite Extensibility

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Abstract

This essay endeavors to define the concept of indefinite extensibility in the setting of category theory. I argue that the generative property of indefinite extensibility for set-theoretic truths is identifiable with the Grothendieck Universe Axiom and Vopenka’s principle. The interaction between the interpretational and objective modalities of indefinite extensibility is defined via the epistemic interpretation of two-dimensional semantics. The semantics can be defined intensionally or hyperintensionally. By characterizing the modal profile of Ω -logical validity, and thus the generic invariance of mathematical truth, modal coalgebras are further capable of capturing the notion of definiteness for set-theoretic truths, in order to yield a non-circular definition of indefinite extensibility.

1 Introduction

This essay endeavors to provide a characterization of the defining properties of indefinite extensibility for set-theoretic truths in category theory, i.e. generation and definiteness. The concept of indefinite extensibility is introduced by Dummett (1963/1978), in the setting of a discussion of the philosophical significance of Gödel’s (1931) first incompleteness theorem. Gödel’s first incompleteness theorem can be characterized as stating that – relative to a coding defined over the signature of first-order arithmetic, a predicate expressing the property of provability, and a fixed point construction – the formula can be defined as not satisfying the provability predicate. Dummett’s concern is with the conditions on our grasp of the concept of natural number, given that the latter figures in a formula whose truth appears to be satisfied despite the unprovability – and thus non-constructivist profile – thereof (186). His conclusion is that the concept of natural number ‘exhibits a particular variety of inherent vagueness, namely indefinite extensibility’, where a ‘concept is indefinitely extensible if, for any definite characterisation of it, there is a natural extension of this

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characterisation, which yields a more inclusive concept; this extension will be made according to some general principle for generating such extensions, and, typically, the extended characterisation will be formulated by reference to the previous, unextended, characterisation' (195-196). Elaborating on the notion of indefinite extensibility, Dummett (1996: 441) redefines the concept as follows: an 'indefinitely extensible concept is one such that, if we can form a definite conception of a totality all of whose members fall under the concept, we can, by reference to that totality, characterize a larger totality all of whose members fall under it'. Subsequent approaches to the notion have endeavored to provide a more precise elucidation thereof, either by providing an explanation of the property which generalizes to an array of examples in number theory and set theory (cf. Wright and Shapiro, 2006), or by availing of modal notions in order to capture the properties of definiteness and extendability which are constitutive of the concept (cf. Fine, 2006; Linnebo, 2013; Uzquiano, 2015). However, the foregoing modal characterizations of indefinite extensibility have similarly been restricted to set-theoretic languages. Furthermore, the modal notions that the approaches avail of are taken to belong to a proprietary type which is irreducible to either the metaphysical or the logical interpretations of the operator.

The aim of this essay is to redress the foregoing, by providing a modal characterization of indefinite extensibility in the setting of category theory, rather than number or set theory. One virtue of the category-theoretic, modal definition of indefinite extensibility is that it provides for a robust account of the epistemological foundations of modal approaches to the ontology of mathematics. A second aspect of the philosophical significance of the examination is that it can serve to redress the lacuna noted in the appeal to an irreducible type of mathematical modality, which is argued (i) to be representational, (ii) still to bear on the ontological expansion of domains of sets, and yet (iii) not to range over metaphysical possibilities. By contrast to the latter approach, the category-theoretic characterization of indefinite extensibility is able to identify the functors of coalgebras with elementary embeddings and the modal properties of set-theoretic, Ω -logical consequence. Functors receive their values relative to two parameters, the first ranging over epistemically possible worlds and the second ranging over objective, though not metaphysical, possible worlds. Functors thus receive their values in an epistemic two-dimensional semantics.

In Section **2**, I examine the extant approaches to explaining both the property and the understanding-conditions on the concept of indefinite extensibility. In Section **3**, modal coalgebras are availed of to model Grothendieck Universes, and I define the notion of indefinite extensibility in the category-theoretic setting. I argue that the category-theoretic definition of indefinite extensibility yields an explanation of the generative property of indefinite extensibility for set-theoretic truths as well as of the notion of definiteness which figures in the definition. I argue that the generative property of indefinite extensibility for set-theoretic truths in category theory is identifiable with the Grothendieck Universe Axiom and the elementary embeddings in Vopenka's principle. I argue, then, that the notion of definiteness for set-theoretic truths can be captured by the role of modal coalgebras in characterizing the modal profile of Ω -logical

consequence, where the latter accounts for the absoluteness of mathematical truths throughout the set-theoretic universe. The category-theoretic definition is shown to circumvent the issues faced by rival attempts to define indefinite extensibility via extensional and intensional notions within the setting of set theory. Section 4 provides concluding remarks.

2 Indefinite Extensibility in Set Theory: Modal and Extensional Approaches

Characterizations of indefinite extensibility have so far occurred in the language of set theory, and have availed of both extensional and intensional resources. In an attempt to define the notion of definiteness, Wright and Shapiro (op. cit.) argue, for example, that indefinite extensibility may be characterized as follows (266).

Formally, let Π be a higher-order concept of type τ . Let P be a first-order concept falling under Π of type τ . Let f be a function from entities to entities of the same type as P . Finally, let X be a sub-concept of P . P is indefinitely extensible with respect to Π , if and only if:

$$\begin{aligned} \epsilon(P) &= f(X), \\ \epsilon(X) &= \neg[f(X)], \text{ and} \\ \exists X'[\Pi(X') &= (X \cup \{fX\})] \text{ (op. cit.).} \end{aligned}$$

The notion of definiteness is then defined as the limitless preservation of ‘ Π -hood’ by sub-concepts thereof ‘under iteration of the relevant operation’, f (269).

The foregoing impresses as a necessary condition on the property of indefinite extensibility. Wright and Shapiro note, e.g., that the above formalization generalizes to an array of concepts countenanced in first-order number theory and analysis, including concepts of the finite ordinals (defined by iterations of the successor function); of countable ordinals (defined by countable order-types of well-orderings); of regular cardinals (defined as occurring when the cofinality of a cardinal, κ – comprised of the unions of sets with cardinality less than κ – is identical to κ); of large cardinals (defined by elementary embeddings from the universe of sets into proper subsets thereof, which specify critical points measured by the ordinals); of real numbers (defined as cuts of sets of rational numbers); and of Gödel numbers (defined as codings by natural numbers of symbols and formulas) (266-267).

As it stands, however, the definition might not be sufficient for the definition of indefinite extensibility, by being laconic about the reasons for which new sub-concepts – comprised as the union of preceding sub-concepts with a target operation defined thereon – are presumed interminably to generate. In response to the above desideratum, concerning the reasons for which indefinite extensibility might be engendered, philosophers have recently appealed to modal properties of the formation of sets. In his (2018a), Linnebo countenances both interpretational and metaphysical modalities, and he argues that the for-

mer also satisfy S4.2. Fine (2006) argues, e.g., that – in order to avoid the Russell property when quantifying over all sets – there are postulational interpretational modalities which induce a reinterpretation of quantifier domains, and serve as a mechanism for tracking the ontological inflation of the hierarchy of sets via, e.g., the power-set operation (2007). Reinhardt (1974) and Williamson (2007) argue that modalities are inter-definable with counterfactuals. While Williamson (2016) argues both that imaginative exercises take the form of counterfactual presuppositions and that it is metaphysically possible to decide propositions which are undecidable relative to the current axioms of extensional mathematical languages such as ZF, Reinhardt (op. cit.) argues that large cardinal axioms and undecidable sentences in extensional ZF can similarly be imagined as obtaining via counterfactual presupposition.

In an examination of the iterative hierarchy of sets, Parsons (1977/1983) notes that the notion of potential infinity, as anticipated in Book 3, ch. 6 of Aristotle’s *Physics*, may be codified in a modal set theory by both a principle which is an instance of the Barcan formula (namely, for predicates P and rigidifying predicates Q, $\forall x(Px \iff Qx) \wedge \Box\{\forall x(\Box Qx \vee \Box\neg Qx) \wedge \forall R[\forall x\Box(Qx \rightarrow Rx) \rightarrow \Box\forall x(Qx \rightarrow Rx)]\}$ (fn. 24), as well as a principle for definable set-forming operations (e.g., unions) for Borel sets of reals $\Box(\forall x)\Diamond(\exists y)[y=x \cup \{x\}]$ (528). The modal extension is argued to be a property of the imagination, or intuition, and to apply further to iterations of the successor function in an intensional variant of arithmetic (1979-1980).

Hellman (1990) develops the program intimated in Putnam (1967), and thus argues for an eliminativist, modal approach to mathematical structuralism as applied to second-order plural ZF. The possibilities at issue are taken to be logical – concerning both the consistency of a set of formulas as well as the possible satisfaction of existential formulas – and he specifies, further, an ‘extendability principle’, according to which ‘every natural model [of ZF] has a proper extension’ (421).

Extending Parsons’ project, Linnebo (2009, 2013) avails of a second-order, plural modal set theory in order to account for both the notion of potential infinity as well as the notion of definiteness. Similarly to Parsons’ use of the Barcan formula (i.e., $\forall\Box\phi \rightarrow \Box\forall\phi$), Linnebo’s principle for the foregoing is as follows: $\forall u(u \prec xx \rightarrow \Box\phi) \rightarrow \Box\forall u(u \prec xx \rightarrow \phi)$ (2013: 211). He argues, further, that the logic for the modal operator is S4.2, i.e. $K[\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)]$, $T(\Box\phi \rightarrow \phi)$, $4((\Box\phi \rightarrow \Box\Box\phi)$, and $G(\Diamond\Box\phi \rightarrow \Box\Diamond\phi)$. Studd (2013) examines the notion of indefinite extensibility by availing of a bimodal temporal logic. Uzquiano’s (2015) approach to defining the concept of indefinite extensibility argues that the height of the cumulative hierarchy is in fact fixed, and that indefinite extensibility can similarly be captured via the use of modal operators in second-order plural modal set theory. The modalities are taken to concern the possible reinterpretations of the intensions of the non-logical vocabulary – e.g., the set-membership relation – which figures in the augmentation of the theory with new axioms and the subsequent climb up the fixed hierarchy of sets (cf. Gödel, 1947/1964).

Author (ms₁) proffers a novel epistemology of mathematics, based on an

application of the epistemic interpretation of two-dimensional semantics in set-theoretic languages to the values of large cardinal axioms and undecidable sentences. Modulo logical constraints such as consistency and absoluteness in the extensions of ground models of the set-theoretic universe, the epistemic possibility that an undecidable proposition receives a value may serve, then, as a guide to the metaphysical possibility thereof. Finally, Author (blinded for review) argues that the modal profile of the consequence relation, in the Ω -logic defined in Boolean-valued models of set-theory, can be captured by coalgebras, and provides a necessary condition on the formal grasp of the concept of ‘set’.

The foregoing accounts of the metaphysics and epistemology of indefinite extensibility are each defined in the languages of number and set theory. In the following section, I examine the nature of indefinite extensibility in the setting of category theory, instead. One aspect of the philosophical significance of the examination is that it can serve to provide an analysis of the mathematical modality at issue. By contrast to Hellman’s approach, which takes the mathematical modality at issue to be logical (cf. Field, 1989: 37; Rayo, 2013), and Fine’s (op. cit.) approach, which takes the mathematical modality to be either interpretational or dynamic, I argue in the following sections that the mathematical modality can be captured by mappings in coalgebras, relative to two parameters, the first ranging over epistemically possible worlds and the second ranging over objective possible worlds.

The formal clauses for epistemic and objective mathematical modalities are as follows:

Let C denote a set of epistemically possibilities, such that $\llbracket \phi \rrbracket \subseteq C$;

(ϕ is a formula encoding a state of information at an epistemically possible world).

$\text{-pri}(x) = \lambda c. \llbracket x \rrbracket^{c,c}$;

(This is an epistemic intension, such that the two parameters relative to which x – a propositional variable – obtains its value are epistemically possible worlds).

$\text{-sec}(x) = \lambda w. \llbracket x \rrbracket^{w,w}$

(This is a subjunctive intension, such that the two parameters relative to which x obtains its value are objective possible worlds).

Epistemic mathematical modality is constrained by consistency, and the formal techniques of provability and forcing. A mathematical formula is metaphysically impossible, if it can be disproved or induces inconsistency in a model.

$\text{-2D}(x) = \lambda c \lambda w \llbracket x \rrbracket^{c,w} = 1$.

(This is a 2D intension. The intension determines a semantic value relative to two parameters, the first ranges over worlds from a first space and the second ranges over worlds from a distinct, second space. The value of the formula relative to the first parameter determines the value of the formula relative to the second.)

If one prefers hyperintensional semantics to possible worlds semantics – in order e.g. to avoid the situation in intensional semantics according to which all necessary formulas express the same proposition because they are true at all possible worlds – one can avail of the following epistemic two-dimensional

truthmaker semantics, which specifies a notion of exact verification in a state space and where states are parts of whole worlds (Fine 2017a,b; Hawke and Özgün, forthcoming). According to truthmaker semantics for epistemic logic, a modalized state space model is a tuple $\langle S, P, \leq, v \rangle$, where S is a non-empty set of states, i.e. parts of the elements in A in the foregoing epistemic modal algebra U , P is the subspace of possible states where states s and t comprise a fusion when $s \sqcup t \in P$, \leq is a partial order, and $v: \text{Prop} \rightarrow (2^S \times 2^S)$ assigns a bilateral proposition $\langle p^+, p^- \rangle$ to each atom $p \in \text{Prop}$ with p^+ and p^- incompatible (Hawke and Özgün, forthcoming: 10-11). Exact verification (\vdash) and exact falsification (\dashv) are recursively defined as follows (Fine, 2017a: 19; Hawke and Özgün, forthcoming: 11):

- $s \vdash p$ if $s \in \llbracket p \rrbracket^+$
(s verifies p , if s is a truthmaker for p i.e. if s is in p 's extension);
- $s \dashv p$ if $s \in \llbracket p \rrbracket^-$
(s falsifies p , if s is a falsifier for p i.e. if s is in p 's anti-extension);
- $s \vdash \neg p$ if $s \dashv p$
(s verifies not p , if s falsifies p);
- $s \dashv \neg p$ if $s \vdash p$
(s falsifies not p , if s verifies p);
- $s \vdash p \wedge q$ if $\exists t, u, t \vdash p, u \vdash q$, and $s = t \sqcup u$
(s verifies p and q , if s is the fusion of states, t and u , t verifies p , and u verifies q);
- $s \dashv p \wedge q$ if $s \dashv p$ or $s \dashv q$
(s falsifies p and q , if s falsifies p or s falsifies q);
- $s \vdash p \vee q$ if $s \vdash p$ or $s \vdash q$
(s verifies p or q , if s verifies p or s verifies q);
- $s \dashv p \vee q$ if $\exists t, u, t \dashv p, u \dashv q$, and $s = t \sqcup u$
(s falsifies p or q , if s is the state overlapping the states, t and u , t falsifies p , and u falsifies q);
- $s \vdash \forall x \phi(x)$ if $\exists s_1, \dots, s_n$, with $s_1 \vdash \phi(a_1), \dots, s_n \vdash \phi(a_n)$, and $s = s_1 \sqcup \dots \sqcup s_n$
[s verifies $\forall x \phi(x)$ "if it is the fusion of verifiers of its instances $\phi(a_1), \dots, \phi(a_n)$ " (Fine, 2017c)];
- $s \dashv \forall x \phi(x)$ if $s \dashv \phi(a)$ for some individual a in a domain of individuals (op. cit.)
[s falsifies $\forall x \phi(x)$ "if it falsifies one of its instances" (op. cit.)];
- $s \vdash \exists x \phi(x)$ if $s \vdash \phi(a)$ for some individual a in a domain of individuals (op. cit.)
[s verifies $\exists x \phi(x)$ "if it verifies one of its instances $\phi(a_1), \dots, \phi(a_n)$ " (op. cit.)];
- $s \dashv \exists x \phi(x)$ if $\exists s_1, \dots, s_n$, with $s_1 \dashv \phi(a_1), \dots, s_n \dashv \phi(a_n)$, and $s = s_1 \sqcup \dots \sqcup s_n$ (op. cit.)
[s falsifies $\exists x \phi(x)$ "if it is the fusion of falsifiers of its instances" (op. cit.)];
- s exactly verifies p if and only if $s \vdash p$ if $s \in \llbracket p \rrbracket$;
- s in exactly verifies p if and only if $s \triangleright p$ if $\exists s' \leq S, s' \vdash p$; and
- s loosely verifies p if and only if, $\forall t, s.t. s \sqcup t, s \sqcup t \vdash p$, where \sqcup is the relation of compatibility (35-36);

$s \vdash A\phi$ if and only if for all $t \in P$ there is a $t' \in P$ such that $t' \sqcup t \in P$ and $t' \vdash \phi$;

$s \dashv A\phi$ if and only if there is a $t \in P$ such that for all $u \in P$ either $t \sqcup u \notin P$ or $u \dashv \phi$, where $A\phi$ denotes the apriority of ϕ ;

$s \vdash A(\forall x\phi(x))$ if and only if for all $u \in P$ there is a $u' \in P$ such that $u \vdash [u' \vdash \exists s_1, \dots, s_n, \text{ with } s_1 \vdash \phi(a_1), \dots, s_n \vdash \phi(a_n), \text{ and } u' = s_1 \sqcup \dots \sqcup s_n]$;

$s \vdash A(\exists x\phi(x))$ if and only if or all $u \in P$ there is a $u' \in P$ such that $u \vdash [u' \vdash \phi(a)]$ for some individual a in a domain of individuals (op. cit.).

In order to account for two-dimensional indexing, we augment the model, M , with a second state space, S^* , on which we define both a new parthood relation, \leq^* , and partial function, V^* , which serves to map propositions in a domain, D , to pairs of subsets of S^* , $\{1,0\}$, i.e. the verifier and falsifier of p , such that $\llbracket P \rrbracket^+ = 1$ and $\llbracket p \rrbracket^- = 0$. Thus, $M = \langle S, S^*, D, \leq, \leq^*, V, V^* \rangle$. The two-dimensional hyperintensional profile of propositions may then be recorded by defining the value of p relative to two parameters, c, i : c ranges over subsets of S , and i ranges over subsets of S^* .

(*) $M, s \in S, s^* \in S^* \vdash p$ iff:

(i) $\exists c_s \llbracket p \rrbracket^{c,c} = 1$ if $s \in \llbracket p \rrbracket^+$; and

(ii) $\exists i_{s^*} \llbracket p \rrbracket^{c,i} = 1$ if $s^* \in \llbracket p \rrbracket^+$

(Distinct states, s, s^* , from distinct state spaces, S, S^* , provide a two-dimensional verification for a proposition, p , if the value of p is provided a truthmaker by s . The value of p as verified by s determines the value of p as verified by s^*).

We say that p is hyper-rigid iff:

(**) $M, s \in S, s^* \in S^* \vdash p$ iff:

(i) $\forall c'_s \llbracket p \rrbracket^{c,c'} = 1$ if $s \in \llbracket p \rrbracket^+$; and

(ii) $\forall i_{s^*} \llbracket p \rrbracket^{c,i} = 1$ if $s^* \in \llbracket p \rrbracket^+$

The foregoing provides a two-dimensional hyperintensional semantic framework within which to interpret the values of a proposition. Further:

s is a two-dimensional exact truthmaker of p if and only if (*);

s is a two-dimensional inexact truthmaker of p if and only if $\exists s' \sqsubset S, s \rightarrow s', s' \vdash p$ and such that

$\exists c_{s'} \llbracket p \rrbracket^{c,c} = 1$ if $s' \in \llbracket p \rrbracket^+$, and

$\exists i_{s^*} \llbracket p \rrbracket^{c,i} = 1$ if $s^* \in \llbracket p \rrbracket^+$;

s is a two-dimensional loose truthmaker of p if and only if, $\exists t, s.t. s \sqcup t, s \sqcup t \vdash p$:

$\exists c_{s \sqcup t} \llbracket p \rrbracket^{c,c} = 1$ if $s' \in \llbracket p \rrbracket^+$, and

$\exists i_{s^*} \llbracket p \rrbracket^{c,i} = 1$ if $s^* \in \llbracket p \rrbracket^+$.

Epistemic (primary), subjunctive (secondary), and 2D hyperintensions can be defined as follows, where hyperintensions are functions from states to extensions, and intensions are functions from worlds to extensions:

- Epistemic Hyperintension:

$\text{pri}(x) = \lambda s. \llbracket x \rrbracket^{s,s}$, with s a state in the state space defined over the foregoing epistemic modal algebra, U ;

- Subjunctive Hyperintension:

$$\text{sec}_{v_{\otimes}}(x) = \lambda i. \llbracket x \rrbracket^{v_{\otimes}, i},$$
 with i a state in objective state space \mathbf{I} ;
- 2D-Hyperintension:

$$2D(x) = \lambda s \lambda w \llbracket x \rrbracket^{s, i} = 1.$$

Following the presentation of topic models in Berto (2018; 2019), Canavotto et al (2020), and Berto and Hawke (2021), atomic topics comprising a set of topics, \mathbf{T} , record the hyperintensional intentional content of atomic formulas, i.e. what the atomic formulas are about at a hyperintensional level. Topic fusion is a binary operation, such that for all $x, y, z \in \mathbf{T}$, the following properties are satisfied: idempotence ($x \oplus x = x$), commutativity ($x \oplus y = y \oplus x$), and associativity $[(x \oplus y) \oplus z = x \oplus (y \oplus z)]$ (Berto, 2018: 5). Topic parthood is a partial order, \leq , defined as $\forall x, y \in \mathbf{T} (x \leq y \iff x \oplus y = y)$ (op. cit.: 5-6). Atomic topics are defined as follows: $\text{Atom}(x) \iff \neg \exists y < x$, with $<$ a strict order. Topic parthood is thus a partial ordering such that, for all $x, y, z \in \mathbf{T}$, the following properties are satisfied: reflexivity ($x \leq x$), antisymmetry ($x \leq y \wedge y \leq x \rightarrow x = y$), and transitivity ($x \leq y \wedge y \leq z \rightarrow x \leq z$) (6). A topic frame can then be defined as $\{W, R, T, \oplus, t\}$, with t a function assigning atomic topics to atomic formulas. For formulas, ϕ , atomic formulas, p, q, r (p_1, p_2, \dots), and a set of atomic topics, $\text{Ut}\phi = \{p_1, \dots, p_n\}$, the topic of ϕ , $t(\phi) = \oplus \text{Ut}\phi = t(p_1) \oplus \dots \oplus t(p_n)$ (op. cit.). Topics are hyperintensional, though not as fine-grained as syntax. Thus $t(\phi) = t(\neg\neg\phi)$, $t\phi = t(\neg\phi)$, $t(\phi \wedge \psi) = t(\phi) \oplus t(\psi) = t(\phi \vee \psi)$ (op. cit.).

The diamond and box operators can then be defined relative to topics:

$$\begin{aligned} \langle M, w \rangle \Vdash \diamond^t \phi &\text{ iff } \langle R_{w,t} \rangle(\phi) \\ \langle M, w \rangle \Vdash \square^t \phi &\text{ iff } [R_{w,t}](\phi), \text{ with} \\ \langle R_{w,t} \rangle(\phi) &:= \{w' \in W \mid t' \in \mathbf{T} \mid R_{w,t}[w', t'] \cap \phi \neq \emptyset \text{ and } t'(\phi) \leq t(\phi)\} \\ [R_{w,t}](\phi) &:= \{w' \in W \mid t' \in \mathbf{T} \mid R_{w,t}[w', t'] \subseteq \phi \text{ and } t'(\phi) \leq t(\phi)\}. \end{aligned}$$

We can then combine topics with truthmakers rather than worlds, thus countenancing doubly hyperintensional semantics, i.e. topic-sensitive epistemic two-dimensional truthmaker semantics:

- Topic-sensitive Epistemic Hyperintension:

$$\text{pri}_t(x) = \lambda s \lambda t. \llbracket x \rrbracket^{s \cap t, s \cap t},$$
 with s a truthmaker from an epistemic state space.
- Topic-sensitive Subjunctive Hyperintension:

$$\text{sec}_{v_{\otimes} \cap t}(x) = \lambda w \lambda t. \llbracket x \rrbracket^{v_{\otimes} \cap t, w \cap t},$$
 with w a truthmaker from a metaphysical state space.
- Topic-sensitive 2D-Hyperintension:

$$2D(x) = \lambda s \lambda w \lambda t \llbracket x \rrbracket^{s \cap t, w \cap t} = 1.$$

In the section that follows, I examine the properties of indefinite extensibility in the category-theoretic setting.

3 Hyperintensional Coalgebras and Indefinite Extensibility

This section examines, finally, the reasons for which category theory provides a more theoretically adequate setting in which to define indefinite extensibility than do competing approaches such as the Neo-Fregean epistemology of mathematics. According, e.g., to the Neo-Fregean program, concepts of number in arithmetic and analysis are definable via implicit definitions which take the form of abstraction principles. Abstraction principles specify biconditionals in which – on the left-hand side of the formula – an identity is taken to hold between numerical term-forming operators from entities of a type to abstract objects, and – on the right-hand side of the formula – an equivalence relation on such entities is assumed to hold.

In the case of cardinal numbers, the relevant abstraction principle is referred to as Hume’s principle, and states that, for all x and y , the number of the x ’s is identical to the number of the y ’s if and only if the x ’s and the y ’s can be put into a one-to-one correspondence, i.e., there is a bijection from the x ’s onto the y ’s.

- $\forall \mathbf{A} \forall \mathbf{B} [\mathbf{N}x: \mathbf{A} = \mathbf{N}x: \mathbf{B} \equiv \exists R [\forall x [\mathbf{A}x \rightarrow \exists y (\mathbf{B}y \wedge Rxy \wedge \forall z (\mathbf{B}z \wedge Rxz \rightarrow y = z))] \wedge \forall y [\mathbf{B}y \rightarrow \exists x (\mathbf{A}x \wedge Rxy \wedge \forall z (\mathbf{A}z \wedge Rzy \rightarrow x = z))]]]$.

Abstraction principles for the concepts of other numbers have further been specified. Thus, e.g., Shapiro (2000: 337-340) specifies an abstraction principle for real numbers, which proceeds along the method of Dedekind’s definition of the reals (cf. Wright, 2007: 172). According to the latter method, one proceeds by specifying an abstraction principle which avails of the natural numbers, in order to define pairs of finite cardinals: ‘ $\forall x, y, z, w [\langle x, y \rangle = \langle z, w \rangle \iff x = z \wedge y = w]$. A second abstraction principle is defined which takes the differences of the foregoing pairs of cardinals, identifying the differences with integers: [$\text{Diff}(\langle x, y \rangle) = \text{Diff}(\langle z, w \rangle) \iff x + w = y + z]$. One specifies, then, a principle for quotients of the integers, identifying them subsequently with the rational numbers: [$\mathbf{Q}\langle m, n \rangle = \mathbf{Q}\langle p, q \rangle \iff n = 0 \wedge q = 0 \vee n \neq 0 \wedge q \neq 0 \wedge m \times q = n \times p]$. Finally, one specifies sets of rational numbers, i.e. the Dedekind cuts thereof, and identifies them with the reals: $\forall F, G [\text{Cut}(F) = \text{Cut}(G) \iff \forall r (F \leq r \iff G \leq r)]$.

The abstractionist program faces several challenges, including whether conditions can be delineated for the abstraction principles, in order for the principles to avoid entraining inconsistency¹; whether unions of abstraction principles can avoid the problem of generating more abstracts than concepts (Fine, 2002); and whether abstraction principles can be specified for mathematical entities in branches of mathematics beyond first and second-order arithmetic (cf. Boolos, 1997; Hale, 2000; Shapiro, op. cit.; and Wright, 2000). I will argue that the

¹Cf. Hodes (1984); Hazen (1985); Boolos (1990); Heck (1992); Fine (2002); Weir (2003); Cook and Ebert (2005); Linnebo and Uzquiano (2009); Linnebo (2010); and Walsh (2016).

last issue – i.e., being able to countenance definitions for the entities and structures in branches of mathematics beyond first and second-order arithmetic – is a crucial desideratum, the satisfaction of which remains elusive for the Neo-Fregean program while yet being satisfiable and thus adducing in favor of the hyperintensional platonist approach that is outlined in what follows.

One issue for the attempt, along abstractionist lines, to provide an implicit definition for the concept of set is that doing so with an unrestricted comprehension principle yields a principle identical to Frege’s (1893/2013) Basic Law V; and thus – in virtue of Russell’s paradox – entrains inconsistency. However, two alternative formulas can be defined, in order to provide a suitable restriction to the inconsistent abstraction principle. The first, conditional principle states that $\forall F, G[[\text{Good}(F) \vee \text{Good}(G)] \rightarrow [\{x|Fx\} = \{Gx\} \iff \forall x(Fx \iff Gx)]]$. The second principle is an unconditional version of the foregoing, and states that $\forall F, G[\{x|Fx\} = \{Gx\} \iff [\text{Good}(F) \vee \text{Good}(G) \rightarrow \forall x(Fx \iff Gx)]]$. Following von Neumann’s (1925/1967: 401-402) suggestion that Russell’s paradox can be avoided with a restriction of the set comprehension principle to one which satisfies a constraint on the limitation of its size, Boolos (1997) suggests that the ‘Good’ predicate in the above principles is intensionally isomorphic to the notion of smallness in set size, and refers to the principle as New V. However, New V is insufficient for deriving all of the axioms of ZF set theory, precluding, in particular, both the axioms of infinity and the power-set axiom (cf. Wright and Hale, 2005: 193). Further, there are other branches of number theory for which it is unclear whether acceptable abstraction principles can be specified. Wiles’ proof of Fermat’s Last Theorem (i.e., that, save for when one of the variables is 0, the Diophantine equation, $x^n = y^n = z^n$, has no solutions when $n > 2$; cf. Hardy and Wright, 1979: 190) relies, e.g., on both invariants and Grothendieck Universes in cohomological number theory (cf. McLarty, 2009: 4).

The foregoing issues with regard to the definability of abstracta in number theory, algebraic geometry (McLarty, op. cit.: 6-8), set theory, et al., can be circumvented in the category-theoretic setting; and in particular by coalgebras. In the remainder of this section, I endeavor to demonstrate how modal coalgebras are able to countenance two of the fundamental properties of indefinite extensibility. The first concerns the property of generation. The second includes the properties of intensional and extensional definiteness.

A labeled transition system is a tuple, LTS, comprised of a set of worlds, M ; a valuation, V , from M to its powerset, $\wp(M)$; and a family of accessibility relations, R . So $LTS = \langle M, V, R \rangle$ (cf. Venema, 2012: 7). Coalgebras can be thus characterized. Let a category C be comprised of a class $Ob(C)$ of objects and a family of arrows for each pair of objects $C(A, B)$ (Venema, 2007: 421). A functor from a category C to a category D , $\mathbf{E}: C \rightarrow D$, is an operation mapping objects and arrows of C to objects and arrows of D (422). An endofunctor on C is a functor, $\mathbf{E}: C \rightarrow C$ (op. cit.).

A \mathbf{E} -coalgebra is a pair $\mathbb{A} = (A, \mu)$, with A an object of C referred to as the carrier of \mathbb{A} , and $\mu: A \rightarrow \mathbf{E}(A)$ is an arrow in C , referred to as the transition map of \mathbb{A} (390). A Kripke coalgebra combines V and R into a Kripke functor, σ_s ; i.e. the set of binary morphisms from M to $\wp(M)$ (op. cit.: 7-8). Thus, for an $s \in M$,

$\sigma(s) := [\sigma_V(s), \sigma_R(s)]$ (op. cit.). $\sigma(s)$ can be interpreted both epistemically and metaphysically. Thus, $\sigma(s)^{s \cap t, w \cap t}$. Satisfaction for the system is defined inductively as follows: For a formula ϕ defined at a state, s , in M ,

$$\begin{aligned} \llbracket \phi \rrbracket^M &= V(s) \text{ }^2 \\ \llbracket \neg \phi \rrbracket^M &= S - V(s) \\ \llbracket \perp \rrbracket^M &= \emptyset \\ \llbracket \top \rrbracket^M &= M \\ \llbracket \phi \vee \psi \rrbracket^M &= \llbracket \phi \rrbracket^M \cup \llbracket \psi \rrbracket^M \\ \llbracket \phi \wedge \psi \rrbracket^M &= \llbracket \phi \rrbracket^M \cap \llbracket \psi \rrbracket^M \\ \llbracket \diamond \phi \rrbracket^M &= \langle R \rangle \llbracket \phi \rrbracket^M \\ \llbracket \Box \phi \rrbracket^M &= [R] \llbracket \phi \rrbracket^M, \text{ with} \\ \langle R \rangle(\phi) &:= \{s \in S \mid R[s] \cap \phi \neq \emptyset\} \text{ and} \\ [R](\phi) &:= \{s \in S \mid R[s] \subseteq \phi\} \text{ (9).} \end{aligned} \text{ }^3$$

Kripke coalgebras are the dual representations of Boolean-valued models of the Ω -logic of set theory (see Elohim, 2019). Modal coalgebras are able, then, to countenance the constitutive conditions of indefinite extensibility. Modal coalgebras are capable, e.g., of defining both the generative property of indefinite extensibility, as well as the notion of definiteness which figures therein. Further, the category-theoretic definition of indefinite extensibility is arguably preferable to those advanced in the set-theoretic setting, because modal coalgebras can account for both the modal profile and the epistemic tractability of Ω -logical consequence.

The *generative* property of indefinite extensibility for set-theoretic truths is captured by the Grothendieck Universe Axiom and the elementary embeddings in Vopenka's principle, $j: A \rightarrow B$, $\phi \langle a_1, \dots, a_n \rangle$ in A if and only if $\phi \langle j(a_1), \dots, j(a_n) \rangle$ in B . The large cardinals countenanced by Grothendieck Universes are restricted to strongly inaccessible cardinals. A cardinal κ is regular if the cofinality of κ – comprised of the unions of sets with cardinality less than κ – is identical to κ . Uncountable regular limit cardinals are weakly inaccessible (op. cit.). A strongly inaccessible cardinal is regular and has a strong limit, such that if $\lambda < \kappa$, then $2^\lambda < \kappa$ (Kanamori, 2012: 361). Indefinite extensibility follows from the Universe Axiom which states that for each set, the set belongs to a Grothendieck Universe such that the cardinality of inaccessible cardinals is unbounded. However, functors interpreted as elementary embeddings in category theory are such that Vopenka's principle can be satisfied, yielding Woodin cardinals. Vopenka's principle is secured via elementary embeddings between first-order structures interpreted as categories.

The notion of *definiteness* for set-theoretic truths is captured by the role of modal coalgebras in characterizing the modal profile of Ω -logical validity.

²Equivalently, $M, s \Vdash \phi$ if $s \in V(\phi)$ (9).

³Hamkins and Linnebo (2022) argue that the modal logic of Grothendieck potentialism has a lower bound of S4.3 [3: $(\diamond \phi \wedge \diamond \psi) \rightarrow \diamond[(\phi \wedge \diamond \psi) \vee (\psi \wedge \diamond \phi)]$] and an upper bound of S5 (KTE; E: $\neg \Box \phi \rightarrow \Box \neg \Box \phi$). The idea of accounting for indefinite extensibility with regard to category-theoretic Grothendieck Universes came to mind in January 2016, and this chapter was written in that month. Hamkins and Linnebo's paper was posted on Arxiv.org in 2017.

The absoluteness of set-theoretic truths in virtue of Ω -logical validity corresponds to a type of objective – perhaps maximal, and thus metaphysical – necessity. This characterization of definiteness for set-theoretic truths would thus satisfy Linnebo (2018b)’s characterizations of both intensional and extensional definiteness. According to Linnebo (op. cit.: 203), a concept is intensionally definite if it ‘has a sharp application condition’, and extensionally definite if ‘it has a fixed extension in all the circumstances in which the concept is available’. Linnebo (2018a: 210) argues that extensional definiteness is satisfied if set-membership is rigid: $\exists xx \Box \forall u [u \prec xx \iff \phi(u)]$, with \prec a plurality membership relation. With regard to extensional definiteness as secured via an absoluteness condition, Koellner (2010) writes of the invariance property of Ω -logical consequence: ‘[T]he logical consequence relation is not perturbed by passing to a generic extension. If we think of the models $V^{\mathbb{B}}$ as possible worlds, then this is tantamount to saying that the logical consequence relation is invariant across the possible worlds’.

Whereas the Neo-Fregean approach to comprehension for the concept of set relies on an unprincipled restriction of the size of the universe in order to avoid inconsistency, and one according to which the axioms of ZF still cannot all be recovered, modal coalgebras provide a natural means for defining the minimal conditions necessary for formal grasp of the concept set. The category-theoretic definition of indefinite extensibility is sufficient for uniquely capturing both the generative property as well as the notion of definiteness which are constitutive of the concept. The category-theoretic definition of indefinite extensibility avails of a notion of mathematical modality which captures both the epistemic property of possible interpretations of quantifiers, as well as the objective, circumstantial property of set-theoretic ontological expansion.

One objection to the two-dimensional characterization of indefinite extensibility might be that modality is itself indefinitely extensible, such that it would be circular to define indefinite extensibility via modal notions.⁴ In response, the modal characterization of indefinite extensibility can be illuminating, even if modality is itself indefinitely extensible. The explanations required for each phenomenon are distinct. Thus, indefinite extensibility can have interpretational and objective modalities, expressing epistemically possible reinterpretations of quantifier domains and objectively possible ontological expansion. However, the explanation for modality being itself indefinitely extensible can proceed via, e.g., Fritz (2017)’s puzzle. Fritz proffers two modal principles which are inconsistent. The first states that $\Box \uparrow_1 \Diamond \exists x (Ax \wedge \downarrow_1 \neg Ax)$ (op. cit.: 551). The second principle states that $\Diamond \uparrow_1 \Box \forall x (Ax \rightarrow \downarrow_1 Ax)$ (550). The first principle claims that possibilia are indefinitely extensible (557; see also Rayo, 2020).⁵ So the indefinite extensibility of modality would not have to be explained by further modalities, but is rather explained by a principle which shows that it is always

⁴Thanks to Justin Clarke-Doane for the objection.

⁵Fritz suggests that the notion of metaphysical possibility might not be in good standing, in light of the inconsistency of the two foregoing principles. By contrast, I take the modalities at issue to be objective, though not maximal and thus not metaphysical, where non-maximal objective modalities can be indefinitely extensible.

possible to expand the domain of possibilia.

4 Concluding Remarks

In this essay, I outlined a number of approaches to defining the notion of indefinite extensibility, each of which restricts the scope of their characterization to set-theoretic languages. I endeavored, then, to define indefinite extensibility in the setting of category-theoretic languages, and examined the benefits accruing to the approach, by contrast to the extensional and modal approaches pursued in ZF.

The extensional definition of indefinite extensibility in ZF was shown to be insufficient for characterizing the generative property in virtue of which number-theoretic concepts are indefinitely extensible. The generative property of indefinite extensibility for set-theoretic truths in the category-theoretic setting was argued, by contrast, to be identifiable with the elementary embeddings by which large cardinal axioms can be specified. The modal definitions of indefinite extensibility in ZF were argued to be independently problematic, in virtue of endeavoring simultaneously to account for the epistemic properties of indefinite extensibility – e.g., possible reinterpretations of quantifier domains and mathematical vocabulary – as well as the objective properties of indefinite extensibility – i.e., the ontological expansion of the target domains, without providing an account of how this might be achieved. Coalgebraic functors can secure these two dimensions, by having both epistemic and objective interpretations. The mappings are interpreted both epistemically and objectively, such that the mappings are defined relative to two parameters, the first ranging over epistemically possible worlds and the second ranging over objective possible worlds. The mappings thus receive their values in a hyperintensional epistemic two-dimensional semantics.

Finally, against the Neo-Fregean approach to defining concepts of number, and the limits thereof in the attempt to define concepts of mathematical objects in other branches of mathematics beyond arithmetic, I demonstrated how – by characterizing the modal profile of Ω -logical validity and thus the generic invariance and absoluteness of mathematical truths concerning large cardinals throughout the set-theoretic universe – modal coalgebras are capable of capturing the notion of definiteness within the concept of indefinite extensibility for set-theoretic truths in category theory.

References

- Adamek, J., and J. Rosicky. 1994. *Locally Presentable and Presentable Categories*. Cambridge University Press.
- Aristotle. 1987. *Physics*, tr. E. Hussey (Clarendon Aristotle Series, 1983), text: W.D. Ross (Oxford Classical Texts, 1950). In J.L. Ackill (ed.), *A New Aristotle Reader*. Oxford University Press.
- Avigad, J. 2021. Foundations. <https://arxiv.org/pdf/2009.09541.pdf>
- Bagaria, J., N. Castells, and P. Larson. 2006. An Ω -logic Primer. *Trends in Mathematics: Set Theory*. Birkhäuser Verlag.
- Berry, S. 2022. *A Logical Foundation for Potentialist Set Theory*. Cambridge University Press.
- Berto, F. 2018. Aboutness in Imagination. *Philosophical Studies*, 175.
- Berto, F. 2019. The Theory of Topic-Sensitive Intentional Modals. In I. Sedlar and M. Blichta (eds.), *The Logica Yearbook, 2018*. College Publications.
- Berto, F., and P. Hawke. 2021. Knowledge relative to Information. *Mind*, 130: 517.
- Boolos, G. 1990. The Standard of Equality of Numbers. In Boolos (ed.), *Meaning and Method*. Cambridge University Press.
- Boolos, G. 1997. Is Hume’s Principle Analytic? In R. Heck (ed.), *Language, Thought, and Logic*. Oxford University Press.
- Canavotto, I., F. Berto, and A. Giordani. 2020. Voluntary Imagination: a Fine-grained Analysis. *Review of Symbolic Logic*.
- Cook, R., and P. Ebert. 2005. Abstraction and Identity. *Dialectica*, 59:2.
- Dummett, M. 1963/1978. The Philosophical Significance of Gödel’s Theorem. In Dummett (1978), *Truth and Other Enigmas*. Harvard University Press.
- Dummett, M. 1996. What is Mathematics about? In Dummett, *The Seas of Language*. Oxford University Press.
- Field, H. 1989. *Realism, Mathematics, and Modality*. Blackwell Publishing.
- Fine, K. 2002. *The Limits of Abstraction*. Oxford University Press.
- Fine, K. 2005. Our Knowledge of Mathematical Objects. In T. Gendler and J. Hawthorne (eds.), *Oxford Studies in Epistemology, Volume 1*. Oxford University Press.
- Fine, K. 2006. Relatively Unrestricted Quantification. In A. Rayo and G. Uzquiano (eds.), *Absolute Generality*. Oxford University Press.
- Fine, K. 2007. Response to Weir. *Dialectica*, 61:1.
- Fine, K. 2017a. A Theory of Truthmaker Content I: Conjunction, Disjunction, and Negation. *Journal of Philosophical Logic*, 46:6.
- Fine, K. 2017b. A Theory of Truthmaker Content II: Subject-matter, Common Content, Remainder, and Ground. *Journal of Philosophical Logic*, 46:6.

- Fine, K. 2017c. Truthmaker Semantics. In B. Hale, C. Wright, and A. Miller (eds.), *A Companion to Philosophy of Language*. Blackwell.
- Frege, G. 1893/2013. *Basic Laws of Arithmetic, Vol. I-II*, tr. and ed. P. Ebert, M. Rossberg, C. Wright, and R. Cook. Oxford University Press.
- Fritz, P. 2017. A Purely Recombinatorial Puzzle. *Nous*, 51:3.
- Gödel, K. 1931. On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems I. In Gödel (1986), *Collected Works, Volume I*, eds. S. Feferman, J. Dawson, S. Kleene, G. Moore, R. Solovay, and J. van Heijenoort. Oxford University Press.
- Gödel, K. 1947/1964. What is Cantor's Continuum Problem? . In Gödel (1990), *Collected Works, Volume II*, eds. S. Feferman, J. Dawson, S. Kleene, G. Moore, R. Solovay, and J. van Heijenoort. Oxford University Press.
- Hale, B. 2000. Abstraction and Set Theory. *Notre Dame Journal of Formal Logic*, 41:4.
- Hale, B. and C. Wright. 2005. Logicism in the Twenty-first Century. In S. Shapiro (ed.), *The Oxford Handbook of Philosophy of Mathematics and Logic*. Oxford University Press.
- Hamkins, J., and {O. Linnebo. 2022. The Modal Logic of Set-theoretic Potentialism and the Potentialist Maximality Principles. *Review of Symbolic Logic*, 15:1.
- Hardy, G., and E.M. Wright. 1979. *An Introduction to the Theory of Numbers*, 5th ed. Oxford University Press.
- Hawke, P., and A. Özgün. Forthcoming. Truthmaker Semantics for Epistemic Logic. In F. Faroldi and F van de Putte (eds.), *Outstanding Contributions to Logic: Kit Fine*.
- Hazen, A. 1985. Review of Crispin Wright's *Frege's Conception of Numbers as Objects*. *Australasian Journal of Philosophy*, 63:2.
- Heck, R. 1992. On the Consistency of Second-order Contextual Definitions. *Nous*, 26.
- Hellman, G. 1990. Toward a Modal-Structural Interpretation of Set Theory. *Synthese*, 84.
- Hodes, H. 1984. Logicism and the Ontological Commitments of Arithmetic. *Journal of Philosophy*, 81:3.
- Kanamori, A. 2012. Large Cardinals with Forcing. In D. Gabbay, A. Kanamori, and J. Woods (eds.), *Handbook of the History of Logic: Sets and Extensions in the Twentieth Century*. Elsevier.
- Koellner, P. 2010. Strong Logics of First and Second Order. *Bulletin of Symbolic Logic*, 16:1.
- Koellner, P., and W.H. Woodin. 2010. Large Cardinals from Determinacy. In M. Foreman and A. Kanamori (eds.), *Handbook of Set Theory, Volume 3*. Springer.

- Lawvere, F.W. 2005. An Elementary Theory of the Category of Sets. *Reprints in Theory and Applications of Categories*, 11.
- Leach-Krouse, G. ms. Ω -Consequence Interpretations of Modal Logic.
- Linnebo, Ø. 2009. Bad Company Tamed. *Synthese*, 170.
- Linnebo, Ø. 2010. Some Criteria on Acceptable Abstraction. *Notre Dame Journal of Formal Logic*, 52:3.
- Linnebo, Ø. 2013. The Potential Hierarchy of Sets. *Review of Symbolic Logic*, 6:2.
- Linnebo, Ø. 2018a. Putnam on Mathematics as Modal Logic. In G. Hellmann and R. Cook (eds), *Hilary Putnam on Logic and Mathematics*. Springer.
- Linnebo, Ø. 2018b. Dummett on Indefinite Extensibility. *Philosophical Issues*, 28.
- Linnebo, Ø., and G. Uzquiano. 2009. What Abstraction Principles Are Acceptable? Some Limitative Results. *British Journal for the Philosophy of Science*, 60.
- McLarty, C. 2008. Introduction to Categorical Foundations of Mathematics.
- McLarty, C. 2009. What Does It Take to Prove Fermat's Last Theorem? Grothendieck and the Logic of Number Theory. *Bulletin of Symbolic Logic*.
- Parsons, C. 1977/1983. What is the Iterative Conception of Set? In P. Benacerraf and H. Putnam (eds.), *Philosophy of Mathematics*, 2nd ed. Cambridge University Press.
- Parsons, C. 1979-1980. Mathematical Intuition. *Proceedings of the Aristotelian Society*, New Series, 80.
- Pettigrew, R. ms. An Introduction to Toposes.
- Putnam, H. 1967. Mathematics without Foundations. *Journal of Philosophy*, 64.
- Rayo, A. 2013. *The Construction of Logical Space*. Oxford University Press.
- Rayo, A. 2020. On the Open-Endedness of Logical Space. *Philosophers' Imprint*, 20:4.
- Reinhardt, W. 1974. Remarks on Reflection Principles, Large Cardinals, and Elementary Embeddings. In T. Jech (ed.), *Proceedings of Symposia in Pure Mathematics, Vol. 13, Part 2: Axiomatic Set Theory*. American Mathematical Society.
- Roberts, A. 2019. Modal Expansionism. *Journal of Philosophical Logic*, 48.
- Shapiro, S. 2000. Frege Meets Dedekind: A Neologicist Treatment of Real Analysis. *Notre Dame Journal of Formal Logic*, 41:4.
- Shapiro, S., and C. Wright. 2006. All Things Indefinitely Extensible. In A. Rayo and G. Uzquiano (eds.), *Absolute Generality*. Oxford University Press.

- Studd, J. 2013. The Iterative Conception of Set: a (Bi-)Modal Characterisation. *Journal of Philosophical Logic*, 42.
- Uzquiano, G. 2015. Varieties of Indefinite Extensibility. *Notre Dame Journal of Formal Logic*, 58:1.
- Venema, Y. 2007. Algebras and Coalgebras. In P. Blackburn, J. van Benthem, and F. Wolter (eds.), *Handbook of Modal Logic*. Elsevier.
- Venema, Y. 2012. Lectures on the Modal μ -Calculus.
- Venema, Y. 2020. Lectures on the Modal μ -Calculus.
- von Neumann. 1925/1967. An Axiomatization of Set Theory (trans. S. Bauer-Mengelberg and D. Follesdal). In J. van Heijenoort (ed.), *From Frege to Gödel*. Harvard University Press.
- Walsh, S. 2016. Fragments of Frege's *Grundgesetze* and Gödel's Constructible Universe. *Journal of Symbolic Logic*, 81:2.
- Weir, A. 2003. Neo-Fregeanism: An Embarrassment of Riches. *Notre Dame Journal of Formal Logic*, 44.
- Williamson, T. Forthcoming. Knowing by Imagining. In A. Kind (ed.), *Knowledge through Imagination*. Oxford University Press.
- Williamson, T. 2016. Absolute Provability and Safe Knowledge of Axioms. In L. Horsten and P. Welch (eds.), *Gödel's Disjunction*. Oxford University Press.
- Woodin, W.H. 2010. Strong Axioms of Infinity and the Search for V. *Proceedings of the International Congress of Mathematicians*.
- Woodin, W.H. ms. The Ω Conjecture.
- Wright, C. 2000. Neo-Fregean Foundations for Real Analysis. *Notre Dame Journal in Formal Logic*, 41:4.
- Wright, C. 2007. On Quantifying into Predicate Position. In M. Leng, A. Paseau, and M. Potter (eds.), *Mathematical Knowledge*. Oxford University Press.