Modal Cognitivism and Modal Expressivism

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Abstract

This paper aims to provide a mathematically tractable background against which to model both modal cognitivism and modal expressivism. I argue that epistemic modal algebras, endowed with a hyperintensional, topic-sensitive epistemic two-dimensional truthmaker semantics, comprise a materially adequate fragment of the language of thought. I demonstrate, then, how modal expressivism can be regimented by modal coalgebraic automata, to which the above epistemic modal algebras are categorically dual. I examine five methods for modeling the dynamics of conceptual engineering for intensions and hyperintensions. I develop a novel topic-sensitive truthmaker semantics for dynamic epistemic logic, and develop a novel dynamic epistemic two-dimensional hyperintensional semantics. I examine then the virtues unique to the modal expressivist approach here proffered in the setting of the foundations of mathematics, by contrast to competing approaches based upon both the inferentialist approach to concept-individuation and the codification of speech acts via intensional semantics.

1 Introduction

This essay endeavors to reconcile two approaches to the modal foundations of thought: modal cognitivism and modal expressivism. The novel contribution of the paper is its argument for a reconciliation between the two positions, by providing a hybrid account in which both internal cognitive architecture, on the model of epistemic possibilities, as well as modal automata, are accommodated, while retaining what is supposed to be their unique and inconsistent roles.

The notions of cognitivism and expressivism here targeted concern the role of internal – rather than external – factors in countenancing the nature of thought and information (cf. Fodor, 1975; Haugeland, 1978). Possible worlds or hyperintensional semantics is taken then to provide the most descriptively adequate

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means of countenancing the structure of the foregoing.[1] Whereas the type of modal cognitivism examined here assumes that thoughts and information take exclusively the form of internal representations, the target modal expressivist proposals assume that information states are exhaustively individuated by both linguistic behavior and conditions external to the cognitive architecture of agents.

Modal cognitivism is thus the proposal that the internal representations comprising the language of thought can be modeled via either a possible world or hyperintensional semantics. Modal expressivism has, in turn, been delineated in two ways. On the first approach, the presuppositions shared by a community of speakers have been modeled as possibilities (cf. Kratzer, 1979; Stalnaker, 1978, 1984). Speech acts have in turn been modeled as modal operators which update the common ground of possibilities, the semantic values of which are then defined relative to an array of intensional parameters (Stalnaker, op. cit.; Veltman, 1996; Yalcin, 2007). On the second approach, the content of concepts is supposed to be individuated via the ability to draw inferences. Modally expressive normative inferences are taken then to have the same subjunctive form as that belonging to the alethic modal profile of descriptive theoretical concepts (Brandom, 2014: 211-212).[2] Both the modal approach to shared information and the speech acts which serve to update the latter, and the inferential approach to concept-individuation, are consistent with mental states having semantic values or truth-conditional characterizations.

So defined, the modal cognitivist and modal expressivist approaches have been assumed to be in constitutive opposition. While the cognitivist proposal avails of modal resources in order to model the internal representations comprising an abstract language of thought, the expressivist proposal targets informational properties which extend beyond the remit of internal cognitive architecture: both the form and the parameters relevant to determining the semantic values of linguistic utterances, where the informational common ground is taken to be reducible to possibilities; and the individuation of the contents of concepts on the basis of inferential behavior.

In this paper, I provide a background mathematical theory, in order to account for the reconciliation of the cognitivist and expressivist proposals. I avail, in particular, of the duality between Boolean-valued models of epistemic modal

[1] Delineating cognitivism and expressivism by whether the positions avail of internal representations is thus orthogonal to the eponymous dispute between realists and antirealists with regard to whether mental states are truth-apt, i.e., have a representational function, rather than being non-representational and non-factive even if real (cf. Dummett, 1959; Blackburn, 1984; Price, 2013).

[2] Brandom writes, e.g.: "For modal *expressivism* tells us that modal vocabulary makes explicit normatively significant relations of subjunctively robust material consequence and incompatibility among claimable (hence propositional) contents in virtue of which ordinary empirical descriptive vocabulary *describes* and does not merely *label*, *discriminate*, or *classify*. And modal *realism* tells us that there are modal facts, concerning the subjunctively robust relations of material consequence and incompatibility in virtue of which ordinary empirical descriptive properties and facts are determinate. Together, these two claims give a definite sense to the possibility of the correspondence of modal claimings with modal facts" (op. cit.: 2012).

algebras and coalgebras; i.e., labeled transition systems defined in the setting of category theory.[3] The mappings of coalgebras permit of flexible interpretations, such that they are able to characterize both modal logics as well as discrete-state automata. I argue that the correspondence between epistemic modal algebras and modal coalgebraic automata is sufficient then for the provision of a mathematically tractable, modal foundation for thought and action, which wholly captures both the modal cognitivist and modal expressivist proposals. What will be accomplished is a model-theoretic account of the expression relation between mental states and their expression in action, via the categorical duality between coalgebras which can model automata and epistemic modal algebras which model thought.

In Section **2**, I provide details concerning the target notion of epistemic possibility at issue in this paper.

In Section **3**, I provide the background mathematical theory, in order to account for the reconciliation of the cognitivist and expressivist proposals.

In Section **4**, I provide reasons adducing in favor of modal cognitivism, and argue for the material adequacy of epistemic modal algebras as a fragment of the language of thought.

In Section **5**, I compare my approach with those advanced in the historical and contemporary literature.

In Section **6**, I provide new models for the dynamics of conceptual engineering of intensions and hyperintensions. The first method is via announcements in dynamic epistemic logic. The second method is via dynamic interpretational modalities which redefine intensions and hyperintensions which reassign topics to atomic formulas. The third method is via dynamic hyperintensional belief revision. The fourth method is via rendering epistemic two-dimensional semantics dynamic, such that updates to the epistemic space for the first parameter of a formula will determine an update to the metaphysical space for the second parameter of the formula. The fifth method models updates to two-dimensional intensions via the logic of epistemic dependency in the parameter for epistemic space which then constrains interventions to structural equation models in the parameter for metaphysical space.[4]

In Section **7**, I examine reasons adducing in favor of an expressivist natural language semantics for epistemic modals, to complement the metaphysical expressivism for epistemic modality examined in the paper.

In Section **8**, modal coalgebraic automata are argued, finally, to be preferred

[3]For an algebraic characterization of dynamic-epistemic logic, see Kurz and Palmigiano (2013). Baltag (2003) develops a coalgebraic semantics for dynamic-epistemic logic, where coalgebraic mappings are intended to record the informational dynamics of single- and multi-agent systems. The current approach differs from the foregoing by examining the duality between static epistemic modal algebras and coalgebraic automata in a single-agent system.

[4]For the origins of two-dimensional intensional semantics, see Kamp, 1967; Vlach, 1973; and Segerberg, 1973.) The distinction between epistemic and metaphysical possibilities, as they pertain to the values of mathematical formulas, is anticipated by Gödel's (1951: 11-12) distinction between mathematics in its subjective and objective senses, where the former targets all "demonstrable mathematical propositions", and the latter includes "all true mathematical propositions".

as models of modal expressivism, by contrast to the speech-act and inferentialist approaches, in virtue of the advantages accruing to the model in the philosophy of mathematics. The interest in modal coalgebraic automata consists, in particular, in the range of mathematical properties that can be recovered on the basis thereof.[5] By contrast to the above competing approaches to modal expressivism, the mappings of modal coalgebraic automata are able both to model and explain elementary embeddings in the category of sets; the intensions of mathematical terms; as well as the modal profile of Ω-logical consequence.

Section **9** provides concluding remarks.

2 The Target Conception of Epistemic Possibility

Epistemically possible worlds or scenarios can be thought of, following Chalmers, as "maximally specific ways things might be" (Chalmers, 2011: 60).

One can define epistemic possibility as a for all one knows operator following, informally, Chalmers (op. cit.)[6] and, formally, MacFarlane (2011: 164). Following MacFarlane, FAK(Φ) (read: for all I know, Φ), relative to an agent α and time τ is true at ⟨c,w,i,a⟩ iff Φ is true at ⟨c,w',i',a⟩, where c is a context, w is a possible world, i is an information state comprising a set of worlds, a is an assignment function, i' is the set of worlds not excluded by what is known by the extension of α at ⟨c,w,i,a⟩ at w and the time denoted by τ at ⟨c,w,i,a⟩, and w' is some world in i' (op. cit.). MacFarlane writes: "[A] speaker considering ⌜FAK^{I}_{now}: Φ⌝ and ⌜Might: Φ⌝ from a particular context c should hold that an occurrence of either at c would have the same truth value. This vindicates the intuition that it is correct to say "It is possible that p" just when what one knows does not exclude p" (167).

Przyjemski (2017) endorses a conception of epistemic possibility according to which it satisfies the condition of being strong, i.e. a "proposition p is epistemically possible only if it is supported by (non-overridden) evidence" (190). Strong epistemic possibility contrasts to weak epistemic possibility according to which a "proposition p is epistemically possible only if p is compatible with the relevant body of evidence" (op. cit.). Weak epistemic possibility is the limiting case of strong epistemic possibility (op. cit.).

A fourth approach to epistemic possibility defines the notion in relation to logical reasoning (Jago, 2009; Bjerring, 2012). Bjerring writes: "[W]e can now spell out deep epistemic necessity and possibility by appeal to provability in n steps of logical reasoning using the rules in R. To that end, let a proof of A in n steps of logical reasoning be a derivation of A from a set Γ of sentences – potentially the empty set – consisting of at most n applications of the rules in R. Let a disproof of A in n steps of logical reasoning be a derivation of ¬A

[5] See Wittgenstein (2001: IV, 4-6, 11, 30-31), for a prescient expressivist approach to the modal profile of mathematical formulas.

[6] "We normally say that is *epistemically possible* for a subject that p, when it might be that p for all the subject knows" (60).

from A – or from the set Γ of sentences such that A∈Γ – consisting of at most n applications of the rules in R. Similarly, let a set Γ of sentences be disprovable in n steps of logical reasoning whenever there is a derivation of A and ¬A from Γ consisting of at most n applications of the rules in R. For simplicity, I will assume that agents can rule out sets of sentence that contain {A,¬A} non-inferentially. Finally, let '$\Box_n$' and '$\Diamond_n$' be metalinguistic operators, where '$\Diamond_n$' is defined as $\neg\Box_n\neg$. Read '$\Box_n$' as 'A is provable in n steps of logical reasoning using the rules in R', and read '$\Diamond_n$'as 'A is not disprovable in n steps of logical reasoning using the rules in R'. We can then define:

(Deep-Necn) A sentence A is deeply$_n$ epistemically necessary iff $\Box_n$.

(Deep-Posn) A sentence A is deeply$_n$ epistemically possible iff $\Diamond_n$" (op. cit.).

Because I will not be concerned with the interaction between epistemic possibility and evidence, knowledge, or logical reasoning in this paper, I will define epistemic possibility in a distinct, fifth manner. This fifth way to understand epistemic possibility is via apriority, such that ϕ is epistemically possible iff ϕ is primary conceivable, where primary conceivability ($\Diamond$) is the dual of apriority ($\neg\Box\neg$, i.e. not apriori ruled out). Chalmers (2002) distinguishes between primary and secondary conceivability. Secondary conceivability is counterfactual, so rejecting the metaphysical necessity of the identity between Hesperus and Phosphorus is not secondary conceivable. Primary conceivability targets epistemically possible worlds considered as actual rather than counterfactual worlds. Chalmers also distinguishes between positive and negative conceivability and prima facie and ideal conceivability. A scenario is positively conceivable when it can be imagined with perceptual detail. A scenario is negatively conceivable when nothing rules it out apriori, as above. A scenario is prima facie conceivable when it is conceivable "on first appearances". E.g. a formula might be prima facie conceivable if it does not lead to contradiction after a finite amount of reasoning. A scenario is ideally conceivable if it is prima facie conceivable with a justification that cannot be defeated by subsequent reasoning (op. cit.).

Chalmers distinguishes between deep and strict epistemic possibilities. He writes: "[W]e might say that the notion of *strict epistemic possibility* – ways things might be, for all we know – is undergirded by a notion of *deep epistemic possibility* – ways things might be, prior to what anyone knows. Unlike strict epistemic possibility, deep epistemic possibility does not depend on a particular state of knowledge, and is not obviously relative to a subject" (62). About deep epistemic necessity, he writes: "For example, a sentence s is deeply epistemically possible when the thought that s expresses cannot be ruled out a priori / This idealized notion of apriority abstracts away from contingent limitations" (66). All references to epistemic possibility in this paper will be to Chalmers' notion of deep epistemic possibility.

Yalcin (2011; 2016) countenances the notion of epistemic possibility as strict, that is, a for all one knows operator, and adds a condition of sensitivity to subject matter. Subject matters partition logical space into the propositions which answer or resolve an interrogative concerning the subject matter (2011: Section

6).[7] The propositions which resolve the interrogative are said to be "visible" (318). Doxastic states are partial functions from interrogatives to resolutions (op. cit.). A "view" is the set of possibilities in the resolution of logical space (op. cit.). The true propositions in the resolution are said to comprise the agent's "commitments concerning the subject matter" (op. cit.). A proposition is compatible with an agent's view only if it is true at one of the worlds in the view (319). Then, "to believe that a proposition is possible, or might be, is for the proposition to be compatible with one's view, and moreover for it to be an answer to a question one is sensitive to" (320). Semantic values for epistemic modals are defined relative to worlds and parameters for "non-factual" states of information (324, 329).

Yablo (2014) countenances "thick" or "directed" propositions, which combine a possible worlds semantics with subject matters (p. 21ftn.28; 49). "The subject matter of [a sentence,] S = the [similarity, i.e. reflexive and symmetric] relation m such that worlds are m-dissimilar iff S is differently true at them" (36, 41). Berto (2022: ch. 2) refers to directed propositions as two-component (2C) propositional contents and develops a semantics for 2C contents which combines possible worlds with atomic topics.

My multi-hyperintensional semantics includes atomic topics, topic-sensitive truthmakers from epistemic and metaphysical state spaces, where the truthmakers are hyperintensional parts of whole possible worlds (Fine, 2017a,b), subject matter similarity relations between topic-sensitive truthmakers,[8] and structured hyperintensions for subsentential expressions of propositions, where structured hyperintensions are functions from subsentential expressions verified by topic-sensitive truthmakers to extensions.[9] I do not countenance semantic clauses for subject similarity subject matters, in this paper.

Rossi and Özgün (2023) countenance epistemic possibility hyperintensionally, by defining it as a strict epistemic possibility operator, i.e. a for all one knows operator, interpreted analogously to positive instead of negative conceivability (2, 6-7). Hyperintensionality is secured via topic-sensitivity, and supposed to entail a condition on agent non-ideality which they refer to as not satisfying the property of "epistemic reach". They write: "the boundaries of S's epistemic reach are determined by their cognitive, computational, or conceptual limitations" (6). The significance of the hyperintensionality condition is, similarly to this paper, that it is supposed to circumvent the problem of omniscience based on intensionalism about propositions. Hyperintensional knowledge is defined as being satisfied by a "model-theoretic condition" (3), MOD, truth in all worlds , and a "hyperintensionality condition", HYPE, i.e. "grasping ϕ's topic", i.e. "a total function defined from the object language of the underlying logic

[7]The view that subject matters, broadly construed, have the form of an interrogative update on a set of worlds is anticipated by Hamblin (1958, 1973); Lewis (1988/1998); and further defended by Yalcin (2008, 2016) and Yablo (2014).

[8]Yablo (2014: 36, 41) countenances subject matters as reflexive, symmetric similarity relations between whole possible worlds.

[9]Chalmers (2006: 3.5) countenances structured intensions, i.e. functions from subsentential expressions in worlds to extensions. See Stanley (2014) and Chalmers (2014) for further discussion.

to the set {0, 1}" (4). Thus,

KNOW(ϕ) = 1 iff MOD(ϕ) =1 and HYPE(ϕ) = 1 (4).

Hyperintensional positive epistemic possibility is defined thus,

POSS(ϕ) = 1 iff MOD($\neg\phi$) = 0 and HYPE(ϕ) = 1 (8).

Rossi and Özgün apply the foregoing to Stalnaker (2006)'s conception of strong or full belief as subjective certainty, where "believing implies believing that one knows": B$\phi \rightarrow$ BKϕ; equivalently, B$\phi \iff \langle$K$\rangle$Kϕ and B$\phi \iff \neg$K$\neg$Kϕ (Rossi and Özgün, op. cit.: 10; Stalnaker, 2006: 179). $\neg$K$\neg$ and $\langle$K$\rangle$ are not, however, equivalent, because $\neg$K$\neg$ is interpreted as negative epistemic possibility and $\langle$K$\rangle$ is interpreted as positive epistemic possibility (Rossi and Özgün, op. cit.: 8). B$\phi \iff \neg$K$\neg$Kϕ entails: BEL$_{Stal}$(ϕ) = 1 iff MOD($\neg$Kϕ) = 0 and HYPE($\neg$Kϕ) = 0 (11). Construed as a positive operator because a condition on the operator is grasp of topic (8, 17), when B$\phi \iff \langle$K$\rangle$Kϕ: BEL*$_{Stal}$(ϕ) = 1 [i.e., POSS(Kϕ) = 1] iff MOD($\neg$Kϕ) = 0 and HYPE(Kϕ) = 1(Rossi and Özgün, op. cit.: 11-12). "B$\overline{\phi}$, [...] K$\overline{\phi}$ and $\langle$K$\rangle\overline{\phi}$ express that 'the agent has grasped the topic of ϕ'"(15).$*\in${K, B} (16). $\Box$ is an analyticity or apriority modality (14). Strong negative introspection, $\vdash \neg$B$\phi \rightarrow$ K$\neg$Bϕ, is invalid in Rossi and Özgün's logic.

An axiom of Rossi and Özgün's logic is a restricted closure axiom, $[\Box(\phi \rightarrow \psi) \wedge *\phi \wedge *\overline{\psi}] \rightarrow *\psi$, interpreted as "the agent knows/believes a priori consequences of what they know/believe as long as they grasp the topics of these consequences" (op. cit.). One issue with Rossi and Özgün's restricted closure axiom is that grasping the topic of the antecedent might, too, be necessary, for grasping the topic of the consequent, instead of only grasp of the topic of the consequent, $\overline{\psi}$. One might want one's account of logic and semantics to satisfy what Berto (2022: 25) refers to as "Yablo's Thesis" or "Parry Implication". (See Parry, 1968, 1989; Yablo, 2014). Yablo's Thesis states that "B is part of A iff the inference from A to B is (i) truth-preserving – A implies B (ii) aboutness-preserving – A's subject matter includes that of B" (Yablo, 2014: 15). Intuitively: "Content-inclusion is implication plus subject-matter inclusion" (op. cit.). The left-to-right direction of Yablo's Thesis is referred to as Weak Yablo's Thesis (op. cit.). The biconditional is referred to as Full Yablo's Thesis, and "gives an account of same-saying as two-way containment: ϕ and ψ say the same ... just in case they are both mutually entailing and topic-equivalent" (25-26). Yablo refers to a type of closure according to which "[s]ome conclusions are such that you should *already* know them, to know the premise" as "immanent closure" or "topical closure" (116-117). "Transeunt" closure occurs when "you are assured of knowing the conclusion only if you engage in some reasoning" (116) and "knowledge of conclusions [is] drawn from premises that do not contain them" (127). Immanent closure states that: "If S knows that P, and Q is part of P, then S knows that Q" (117).

Berto addresses the problem of possible mathematical and logical omniscience entailed by Full Yablo's Thesis and immanent closure by separating topic-inclusion from its metalinguistic and epistemic dimensions. Thus, the topic of (i) '16 +16 = 32', {16, +, =, 32}, is part of the topic of (ii) '16 + 32 = 48', {16, +, 32, =, 48}, yet Full Yablo's Thesis and immanent closure would entail that knowing (i) entails knowing (ii), which is false (55). Berto argues that the maneuver does not entail, however, that all mathematical and logical knowledge is metalinguistic knowledge (56).

Another maneuver would be to reject immanent closure, because of an objection from Alexandru Baltag (56): from $\phi \rightarrow \psi$, one can infer $[\phi \wedge (\psi \vee \neg\psi)] \rightarrow \psi$, yet, taking ϕ to be a mathematical theory and ψ to be a theorem of the theory, knowledge that a theorem in the consequent is true and grasping its topic ought not to be entailed by the disjunction in the antecedent with regard to whether the theorem is true (57).

Another maneuver might be to add logically impossible worlds to one's ontology rather than taking topic-inclusion to have the property of semantic necessity. A truthmaker for ϕ is exact when "it can necessitate the sentence while being wholly relevant to its truth" (Fine and Jago, 2019: sect. 1). According to Berto, "ϕ expresses a *semantic necessity* just in case it is true at unrestrictedly every circumstance. Correspondingly, say that ϕ and ψ express semantic co-necessities if they coincide in truth value at unrestrictedly every circumstance" (42). A set of circumstances can be interpreted as a set of possible worlds or hyperintensional states i.e. truthmakers. Berto suggests that denying that the axioms of a mathematical theory entail a theorem in the theory might require incorporating logically impossible worlds into one's ontology, "where entailment laws are violated" (58).

One issue with Rossi and Özgün's restricted closure axiom is that it satisfies Full Yablo's Thesis and immanent closure, without addressing the separability of the metalinguistic and epistemic dimensions from the topic-inclusion in Full Yablo's Thesis and immanent closure, or Baltag's objection and the possible addition of logically impossible worlds to one's ontology in order to countenance the invalidity of a disjunct being entailed by a disjunction without an instance of either disjunct.

In order for grasp of the topic of the antecedent to entail grasp of the topic of the consequent, a third type of topic-sensitive closure, distinct from immanent closure and transeunt closure, might too be required.

Chalmers defines epistemic possibility as (i) not being apriori ruled out (2011: 63, 66),[10] i.e. as the dual of epistemic necessity i.e. apriority (65),[11] and as (ii) being true at an epistemic scenario i.e. epistemically possible world (62, 64). He accepts a Plenitude principle according to which: "A thought T is epistemically possible iff there exists a scenario S such that S verifies T" (64). Chalmers advances both epistemic and metaphysical constructions of epistemic scenarios. In the metaphysical construction of epistemic scenarios, epistemic scenarios are centered metaphysically possible worlds (69). Canonical descriptions of epistemically possible worlds on the metaphysical construction are required to be speci-

[10]"One might also adopt a conception on which every proposition that is not logically contradictory is deeply epistemically possible, or on which every proposition that is not ruled out a priori is deeply epistemically possible. In this paper, I will mainly work with the latter understanding" (63).

[11]"We can say that s is deeply epistemically necessary when s is a priori: that is when s expresses actual or potential a priori knowledge" (65).

fied using only "semantically neutral" vocabulary, which is "non-twin-earthable" by having the same extensions when worlds are considered as actual or counterfactual (Chalmers, 2006: §3.5). In the epistemic construction of epistemic scenarios, an epistemic scenario consists in a set of sentence types comprising an infinitary ideal language, M, with vocabulary restricted to epistemically invariant expressions (Chalmers, 2011: 75). He defines epistemically invariant expressions thus: "[W]hen s is epistemically invariant, then if some possible competent utterance of s is epistemically necessary, all possible competent utterances of s are epistemically necessary" (op. cit.). The sentence types in the infinitary language must also be epistemically complete. A sentence s is epistemically complete if s is epistemically possible and there is no distinct sentence t such that both s ∧ t and s ∧ ¬t are epistemically possible (76). The epistemic construction of epistemic scenarios transforms the Plenitude principle into an Epistemic Plenitude principle according to which: "For all sentence tokens s, if s is epistemically possible, then some epistemically complete sentence of [M] implies s" (op. cit.). The value of a sentence in epistemically constructed epistemic scenarios determines the value of the sentence in metaphysically possible worlds when a super-rigidity condition is satisfied (Chalmers, 2012: 239, 468, 474). Chalmers writes: 'I accept Apriority/Necessity and Super-Rigid Scrutability. (Relatives of these theses play crucial roles in "The Two-Dimensional Argument against Materialism"' (241). The Apriority/Necessity Thesis is defined as the "thesis that if a sentence S contains only super-rigid expressions, s is a priori iff S is [metaphysically] necessary" (468), and Super-Rigid Scrutability is defined as the "thesis that all truths are scrutable from super-rigid truths and indexical truths" (474).

One issue with Chalmers' Epistemic Plenitude principle is that epistemic possibility figures twice in the definition of an epistemically complete sentence, and epistemic possibilities are defined as sets of epistemically complete sentences. Thus, the definition is circular. I will thus assume the metaphysical construction of epistemic scenarios in this paper. I concur, as well, that epistemic possibility is the dual of epistemic necessity i.e. apriority, but argue for an epistemic two-dimensional truthmaker semantics which avails of hyperintensional epistemic states, i.e. epistemic truthmakers or verifiers for a proposition, which comprise a state space (Fine 2017a,b; Hawke and Özgün, 2023). Epistemic states are parts of epistemically possible worlds, rather than whole worlds themselves. Apriority is thus redefined in the hyperintensional semantics. The semantics developed in this paper makes propositions multi-hyperintensional, by combining topic-, i.e. subject matter, sensitivity, with a second aspect of their subject matter being captured via epistemic and metaphysical truthmakers (see Elohim, 2024. For an outline of truthmaker semantics, see Fine, 2017a,b).

My topic-sensitive epistemic two-dimensional truthmaker semantics differs from intuitionism in logic and mathematics by being governed by a classical logic and being committed to the reality of the classical continuum. Unlike intuitionism, which reduces existence to constructions or proofs, there are epistemic, non-maximally objective, and maximally objective i.e. metaphysical verifiers for propositions. Epistemic states which serve as verifiers for the propositions concern the conceivability thereof, rather than constructive provability as in intuitionism, or ideal knowability as in epistemic arithmetic (Shapiro, 1985). Similarly to epistemic arithmetic, however, epistemic two-dimensional truthmaker semantics can capture the phenomenon of partial constructivity, e.g. a conditional mathematical claim which can be formalized neither in Heyting Arithmetic nor Peano Arithmetic, because the antecedent of the conditional concerns a property which can be effectively found, and the consequent concerns a property which cannot be effectively found (see e.g. Horsten, 1998: 7). Note as well that the the notion of conceivability and apriority here is tied to the notion of states of information which are independent of particular subjects, in agreement with the proposal in Edgington (2004: 6) according to which "a priori knowledge is independent of the state of information of the subject". While being states of information, epistemic states are yet parts of deeply epistemically possible worlds, because they are not relativized to the contingent knowledge bases of particular epistemic agents.

Schroeder (2008) provides a protracted examination of variations on the expression relation. Schroeder argues that expressivists ought to opt for an assertability account of the expression relation, such that the propositions expressed by sentences are governed by assertability conditions for the sentences rather than their truth conditions, and the expression thus doesn't concern the conveyance of information but rather norms on correct assertion of the sentence. He writes: "Every sentence in the language is associated with conditions in which it is semantically correct to use that sentence assertorically ... Assertability conditions, so conceived, are a device of the semantic theorist. They are not a kind of information that speakers intend to convey. So there is no sense in which a community of speakers could get by, managing to communicate information to each other about the world, by means of assertability conditions alone. It is only because some assertability conditions mention beliefs, and beliefs have contents about the world, that speakers can manage to convey information about the world" (op. cit.: 108, 110). The present account is not committed to Schroeder's proposed assertability expressivism. However, I note in Section 7 that Hawke and Steinert-Threlkeld (2021)'s assertability semantics for epistemic modals is consistent with the model-theoretic account of expressivism here advanced. The present account might also converge with a view which Schroeder attributes to Gibbard (1990, 2003), which he refers to as indicator expressivism, according to which mental states do not express propositional contents, but rather express ur-contents owing to an agent's intentions (§4.1). Ur-contents differ from propositional contents, by the differences in their roles in expressing normative and non-normative contents. Schroeder objects to the appeal to ur-contents, arguing that they play a role too similar to that of propositional contents because they convey descriptive information, while Gibbard simultaneously rejects the similarity (107). I think that because ur-contents express normative contents rather than non-normative ones, they are sufficiently distinct from propositional contents, and that it is innocuous for them to be descriptive in part. The present model-theoretic account of expressivism might thus be thought to be consistent with indicator expressivism.

In the following section, I provide models of epistemic modal algebras, coalgebras, and their duality, along with models for a novel topic-sensitive two-dimensional truthmaker semantics and the properties for an abstraction principle for (hyper-)intensions. In the sections that follow, I discuss the material adequacy of the approach, precedents to the approach in the literature, a novel account of conceptually engineering hyperintensions via dynamic epistemic logic and a novel dynamic epistemic two-dimensional hyperintensional semantics, and I close by discussing the limits of competing approaches.

3 Models

3.1 Epistemic Modal Algebra

An epistemic modal algebra is defined as $U = \langle A, 0, 1, \neg, \cap, \cup, \mathbf{l}, \mathbf{m}\rangle$, with A a set containing 0 and 1 (Bull and Segerberg, 2001: 28).[12]

$\mathbf{l}1 = 1$,

$\mathbf{l}(a \cap b) = \mathbf{l}a \cap \mathbf{l}b$

$\mathbf{m}a = \neg\mathbf{l}\neg a$,

$\mathbf{m}0 = 0$,

$\mathbf{m}(a \cup b) = \mathbf{m}a \cup \mathbf{m}b$, and

$\mathbf{l}a = \neg\mathbf{m}\neg a$ (op. cit.).

A valuation v on U is a function from propositional formulas to elements of the algebra, which satisfies the following conditions:

$v(\neg A) = \neg v(A)$,

$v(A \wedge B) = v(A) \cap v(B)$,

$v(A \vee B) = v(A) \cup v(B)$,

$v(\Box A) = \mathbf{l}v(A)$, and

$v(\Diamond A) = \mathbf{m}v(A)$ (op. cit.).

A frame $F = \langle W,R\rangle$ consists of a set W and a binary relation R on W (op. cit.). R[w] denotes the set $\{v \in W \mid (w,v) \in R\}$. A valuation V on F is a function such that $V(A,x) \in \{1,0\}$ for each propositional formula A and $x \in W$, satisfying the following conditions:

$V(\neg A,x) = 1$ iff $V(A,x) = 0$,

$V(A \wedge B,x) = 1$ iff $V(A,x) = 1$ and $V(B,x) = 1$,

$V(A \vee B,x) = 1$ iff $V(A,x) = 1$ or $V(B,x) = 1$ (op. cit.).

3.2 Topic-Sensitive Two-Dimensional Truthmaker Semantics

We will define a topic-sensitive truthmaker semantics over the foregoing epistemic modal algebra. According to truthmaker semantics for epistemic logic, a modalized state space model is a tuple $\langle S, P, \leq, v\rangle$, where S is a non-empty set of states, i.e. parts of the elements in A in the foregoing epistemic modal algebra U, P is the subspace of possible states where states s and t comprise a

[12] Boolean algebras with operators were introduced by Jónsson and Tarski (1951, 1952).

fusion when s $\sqcup$ t$\in$P, $\leq$ is a partial order, and v: Prop $\rightarrow$ (2^S x 2^S) assigns a bilateral proposition $\langle p^+, p^- \rangle$ to each atom p$\in$Prop with p^+ and p^- incompatible (Hawke and Özgün, 2023). Exact verification ($\vdash$) and exact falsification ($\dashv$) are recursively defined as follows (Fine, 2017a: 19; Hawke and Özgün, 2023):

s $\vdash$ p if s$\in[\![p]\!]^+$

(s verifies p, if s is a truthmaker for p i.e. if s is in p's extension);

s $\dashv$ p if s$\in[\![p]\!]^-$

(s falsifies p, if s is a falsifier for p i.e. if s is in p's anti-extension);

s $\vdash$ ¬p if s $\dashv$ p

(s verifies not p, if s falsifies p);

s $\dashv$ ¬p if s $\vdash$ p

(s falsifies not p, if s verifies p);

s $\vdash$ p $\wedge$ q if $\exists$v,u, v $\vdash$ p, u $\vdash$ q, and s = v $\sqcup$ u

(s verifies p and q, if s is the fusion of states, v and u, v verifies p, and u verifies q);

s $\dashv$ p $\wedge$ q if s $\dashv$ p or s $\dashv$ q

(s falsifies p and q, if s falsifies p or s falsifies q);

s $\vdash$ p $\vee$ q if s $\vdash$ p or s $\vdash$ q

(s verifies p or q, if s verifies p or s verifies q);

s $\dashv$ p $\vee$ q if $\exists$v,u, v $\dashv$ p, u $\dashv$ q, and s = v $\sqcup$ u

(s falsifies p or q, if s is the fusion of the states v and u, v falsifies p, and u falsifies q);

s $\vdash$ $\forall$xϕ(x) if $\exists s_1, \ldots, s_n$, with $s_1 \vdash \phi(a_1), \ldots, s_n \vdash \phi(a_n)$, and $s = s_1 \sqcup \ldots \sqcup s_n$

[s verifies $\forall$xϕ(x) "if it is the fusion of verifiers of its instances $\phi(a_1), \ldots, \phi(a_n)$" (Fine, 2017c)];

s $\dashv$ $\forall$xϕ(x) if s $\dashv$ ϕ(a) for some individual a in a domain of individuals (op. cit.)

[s falsifies $\forall$xϕ(x) "if it falsifies one of its instances" (op. cit.)];

s $\vdash$ $\exists$xϕ(x) if s $\vdash$ ϕ(a) for some individual a in a domain of individuals (op. cit.)

[s verifies $\exists$xϕ(x) "if it verifies one of its instances $\phi(a_1), \ldots, \phi(a_n)$" (op. cit.)];

s $\dashv$ $\exists$xϕ(x) if $\exists s_1, \ldots, s_n$, with $s_1 \dashv \phi(a_1), \ldots, s_n \dashv \phi(a_n)$, and $s = s_1 \sqcup \ldots \sqcup s_n$ (op. cit.)

[s falsifies $\exists$xϕ(x) "if it is the fusion of falsifiers of its instances" (op. cit.)];

s exactly verifies p if and only if s $\vdash$ p if s$\in[\![p]\!]$;

s inexactly verifies p if and only if s $\triangleright$ p if $\exists$s'$\leq$S, s' $\vdash$ p; and

s loosely verifies p if and only if, $\forall$v, s $\sqcup$ v $\vdash$ p (35-36);

s $\vdash$ Aϕ if and only if for all u$\in$P there is a u'$\in$P such that u' $\sqcup$ u$\in$P and u' $\vdash$ ϕ, where Aϕ denotes the apriority of ϕ[13]; and

[13]In epistemic two-dimensional semantics, epistemic possibility is defined as the dual of apriority or epistemic necessity, i.e. as not being ruled-out apriori (¬□¬), and follows Chalmers (2011: 66). Apriority receives, however, different operators depending on whether it is defined in truthmaker semantics or possible worlds semantics. Both operators are admissible, and the definition in terms of truthmakers is here taken to be more fundamental. The definition

$s \dashv A\phi$ if and only if there is a v∈P such that for all u∈P either $v \sqcup u \notin P$ or $u \dashv \phi$[14];

$s \vdash A(A\phi)$ if and only if for all u∈P there is a u'∈P such that u' ⊔ u ∈P and u' ⊢ ϕ and there is a u"∈P such that u' ⊔ u"∈P and u" ⊢ ϕ;

$s \vdash A(\forall x\phi(x))$ if and only if for all u∈P there is a u'∈P such that u ⊢ [u' ⊢ $\exists s_1, \ldots, s_n$, with $s_1 \vdash \phi(a_1), \ldots, s_n \vdash \phi(a_n)$, and u' = $s_1 \sqcup \ldots \sqcup s_n$];

$s \vdash A(\exists x\phi(x))$ if and only if or all u∈P there is a u'∈P such that u ⊢ [u' ⊢ ϕ(a)] for some individual a in a domain of individuals (op. cit.).

In order to account for two-dimensional indexing, we augment the model, M, with a second state space, S*, on which we define both a new parthood relation, ≤*, and partial function, V*, which serves to map propositions in a domain, D, to pairs of subsets of S*, {1,0}, i.e. the verifier and falsifier of p, such that $[\![p]\!]^+ = 1$ and $[\![p]\!]^- = 0$. Thus, M = ⟨S, S*, D, ≤, ≤*, V, V*⟩. The two-dimensional hyperintensional profile of propositions may then be recorded by defining the value of p relative to two parameters, c,i: c ranges over subsets of S, and i ranges over subsets of S*.

(*) M,s∈S,s*∈S* ⊢ p iff:
(i) $\exists c_s [\![p]\!]^{c,c} = 1$ if $s \in [\![p]\!]^+$; and
(ii) $\exists i_{s*} [\![p]\!]^{c,i} = 1$ if $s^* \in [\![p]\!]^+$

(Distinct states, s,s*, from distinct state spaces, S,S*, provide a multi-dimensional verification for a proposition, p, if the value of p is provided a truthmaker by s. The value of p as verified by s determines the value of p as verified by s*).

We say that p is hyper-rigid iff:

(**) M,s∈S,s*∈S* ⊢ p iff:
(i) $\forall c'_s [\![p]\!]^{c,c'} = 1$ if $s \in [\![p]\!]^+$; and
(ii) $\forall i_{s*} [\![p]\!]^{c,i} = 1$ if $s^* \in [\![p]\!]^+$

of apriority here differs from that of DeRose (1991: 593-594) – who defines the epistemic possibility of P as being true iff "(1) no member of the relevant community knows that P is false and (2) there is no relevant way by which members of the relevant community can come to know that P is false" – by defining epistemic possibility in terms of apriority rather than knowledge. It differs from that of Huemer (2007: 129) – who defines the epistemic possibility of P as it not being the case that P is epistemically impossible, where P is epistemically impossible iff P is false, the subject has justification for ¬P "adequate for dismissing P", and the justification is "Gettier-proof" – by not availing of impossibilities, and rather availing of the duality between apriority as epistemic necessity and epistemic possibility.

[14] A more natural clause for apriority in truthmaker semantics might perhaps be thought to be 's ⊢ A(ϕ) iff there is a t∈P such that for all t'∈P t'∈P and t' ⊢ ϕ', because the latter echoes the clause for the necessity operator according to which necessity is truth at all accessible worlds, 'M,w ⊩ □(ϕ) iff ∀w'[If R(w,w'), then M,w' ⊩ ϕ]'. However, appealing to a single state that comprises a fusion with all possible states and is a necessary verifier is arguably preferable to the claim that necessity be recorded by there being all states comprising a fusion with a first state serving to verify a proposition, p, because the latter claim is silent about whether the corresponding verifier of p in the fusion of all of those states is necessary. Thanks here to Peter Hawke.

Epistemic (primary), subjunctive (secondary), and 2D hyperintensions can be defined as follows, where hyperintensions are functions from states to extensions, and intensions are functions from worlds to extensions. Epistemic two-dimensional truthmaker semantics receives substantial motivation by its capacity (i) to model conceivability arguments involving hyperintensional metaphysics, and (ii) to avoid the problem of mathematical omniscience entrained by intensionalism about propositions:

- Epistemic Hyperintension:

  $\texttt{pri}(x) = \lambda s.[\![x]\!]^{s,s}$, with s a state in the state space defined over the foregoing epistemic modal algebra, U

- Subjunctive Hyperintension:

  $\texttt{sec}_{v_{@}}(x) = \lambda w.[\![x]\!]^{v@,w}$, with w a state in metaphysical state space W

In epistemic two-dimensional semantics, the value of a formula or term relative to a first parameter ranging over epistemic scenarios determines the value of the formula or term relative to a second parameter ranging over metaphysically possible worlds. The dependence is recorded by 2D-intensions. Chalmers (2006: 102) provides a conditional analysis of 2D-intensions to characterize the dependence: "Here, in effect, a term's subjunctive intension depends on which epistemic possibility turns out to be actual. / This can be seen as a mapping from scenarios to subjunctive intensions, or equivalently as a mapping from (scenario, world) pairs to extensions. We can say: the two-dimensional intension of a statement S is true at (V, W) if V verifies the claim that W satisfies S. If $[\mathrm{A}]_1$ and $[\mathrm{A}]_2$ are canonical descriptions of V and W, we say that the two-dimensional intension is true at (V, W) if $[\mathrm{A}]_1$ epistemically necessitates that $[\mathrm{A}]_2$ subjunctively necessitates S. A good heuristic here is to ask "If $[\mathrm{A}]_1$ is the case, then if $[\mathrm{A}]_2$ had been the case, would S have been the case?". Formally, we can say that the two-dimensional intension is true at (V, W) iff '$\Box_1([\mathrm{A}]_1 \rightarrow \Box_2([\mathrm{A}]_2 \rightarrow \mathrm{S}))$' is true, where '$\Box_1$' and '$\Box_2$' express epistemic and subjunctive necessity respectively".

- 2D-Hyperintension:

  $\texttt{2D}(x) = \lambda s \lambda w [\![\mathrm{x}]\!]^{s,w} = 1.$

If a formula is two-dimensional and the two parameters for the formula range over distinct spaces, then there won't be only one subject matter for the formula, because total subject matters are construed as sets of verifiers and falsifiers and there will be distinct verifiers and falsifiers relative to each space over which each parameter ranges. This is especially clear if one space is interpreted epistemically and another is interpreted metaphysically. Availing of topics, i.e. subject matters, however, and assigning the same topics to each of the states from the distinct spaces relative to which the formula gets its value is one way of ensuring that the two-dimensional formula has a single subject matter. This would answer Berto (2022)'s problem of 1C contents and "topic-diverging (co-)necessities" (§2.3.3). According to 1C semantics, "either topics reduce to truth conditions, or vice versa" (§2.3). Making truthmakers topic-sensitive would avoid the issues outlined by Berto (op. cit.) with regard to topic-divergence for truthmaker semantics.

Following the presentation of topic models in Berto (2018; 2019), Canavotto et al (2020), and Berto and Hawke (2021), atomic topics comprising a set of topics, T, record the hyperintensional intentional content of atomic propositions, i.e. what the atomic propositions are about at a hyperintensional level. Topic fusion is a binary operation, such that for all x, y, z∈T, the following properties are satisfied: idempotence (x ⊕ x = x), commutativity (x ⊕ y = y ⊕ x), and associativity [(x ⊕ y) ⊕ z = x ⊕ (y ⊕ z)] (Berto, 2018: 5). Topic parthood is a partial order, ≤, defined as ∀x,y∈T(x ≤ y $\iff$ x ⊕ y = y) (op. cit.: 5-6). Atomic topics are defined as follows: Atom(x) $\iff$ ¬∃y < x, with < a strict order. Topic parthood is thus a partial ordering such that, for all x, y, z∈T, the following properties are satisfied: reflexivity (x ≤ x), antisymmetry (x ≤ y ∧ y ≤ x → x = y), and transitivity (x ≤ y ∧ y ≤ z → x ≤ z) (6). A topic frame can then be defined as {W, R, T, ⊕, t}, with t a function assigning atomic topics to atomic formulas. For formulas, ϕ, atomic formulas, p, q, r (p_1, p_2, ...), and a set of atomic topics, Utϕ = {p_1, ... p_n}, the topic of ϕ, t(ϕ) = ⊕Utϕ = t(p_1) ⊕ ... ⊕ t(p_n) (op. cit.). Topics are hyperintensional, though not as fine-grained as syntax. Thus t(ϕ) = t($\neg\neg\phi$), tϕ = t($\neg\phi$), t($\phi \wedge \psi$) = t(ϕ) ⊕ t(ψ) = t($\phi \vee \psi$) (op. cit.).

The diamond and box operators can then be defined relative to topics:
$\langle$M,w$\rangle \Vdash \Diamond^t\phi$ iff $\langle R_{w,t}\rangle(\phi)$
$\langle$M,w$\rangle \Vdash \Box^t\phi$ iff $[R_{w,t}](\phi)$, with
$\langle R_{w,t}\rangle(\phi)$:= {w'∈Wt'∈T | $R_{w,t}$[w', t'] ∩ $\phi \neq \emptyset$ and t'(ϕ) ≤ t(ϕ)
$[R_{w,t}](\phi)$:= {w'∈Wt'∈T | $R_{w,t}$[w', t'] ⊆ ϕ and t'(ϕ) ≤ t(ϕ).

We can then combine topics with truthmakers rather than worlds, thus countenancing doubly hyperintensional semantics, i.e. topic-sensitive epistemic two-dimensional truthmaker semantics:

- Topic-Sensitive Epistemic Hyperintension:

  $\texttt{pri}_t(x) = \lambda s \lambda t.[\![x]\!]^{s\cap t, s\cap t}$, with s a truthmaker from an epistemic state space.

- Topic-Sensitive Subjunctive Hyperintension:

  $\texttt{sec}_{v_@\cap t}(x) = \lambda w \lambda t.[\![x]\!]^{v_@\cap t, w\cap t}$, with w a truthmaker from a metaphysical state space.

- Topic-Sensitive 2D-Hyperintension:

  $\texttt{2D}(x) = \lambda s \lambda w \lambda t[\![x]\!]^{s\cap t, w\cap t} = 1$.

Topic-sensitive two-dimensional truthmaker semantics can be availed of to account for the interaction between the epistemic and metaphysical profiles of abstraction principles, set-theoretic axioms including large cardinal axioms, rational intuition, and indefinite extensibility.

3.3 An Abstraction Principle for Epistemic (Hyper)intensions

In this section, I specify a homotopic abstraction principle for epistemic (hyper)intensions. Intensional isomorphism, as a jointly necessary and sufficient condition for the identity of intensions, is first proposed in Carnap (1947: §14). The isomorphism of two intensional structures is argued to consist in their logical, or L-, equivalence, where logical equivalence is co-extensive with the notions of both analyticity (§2) and synonymy (§15). Carnap writes that: '[A]n expression in S is L-equivalent to an expression in S' if and only if the semantical rules of S and S' together, without the use of any knowledge about (extra-linguistic) facts, suffice to show that the two have the same extension' (p. 56), where semantical rules specify the intended interpretation of the constants and predicates of the languages (4).[15] The current approach differs from Carnap's by basing the equivalence relation necessary for an abstraction principle for epistemic intensions on Voevodsky's (2006) Univalence Axiom, which collapses identity with isomorphism in the setting of intensional type theory.[16]

Topological Semantics

In the topological semantics for modal logic, a frame is comprised of a set of points in topological space, X, and an accessibility relation, R:
$F = \langle X,R \rangle$;
$X = (X_x)_{x \in X}$; and
$R = (Rxy)_{x,y \in X}$ iff $R_x \subseteq X_x$ x X_x, s.t. if Rxy, then $\exists o \subseteq X$, with $x \in o$ s.t. $\forall y \in o(Rxy)$,
where the set of points accessible from a privileged node in the space is said to be open.[17] A model defined over the frame is a tuple, $M = \langle F,V \rangle$, with V a valuation function from subsets of points in F to propositional variables taking the values 0 or 1. Necessity is interpreted as an interiority operator on the space:
$M,x \Vdash \Box\phi$ iff $\exists o \subseteq X$, with $x \in o$, such that $\forall y \in o$ $M,y \Vdash \phi$.

Homotopy Theory

Homotopy Theory countenances the following identity, inversion, and concatenation morphisms, which are identified as continuous paths in the

[15] For criticism of Carnap's account of intensional isomorphism, based on Carnap's (1937: 17) 'Principle of Tolerance' to the effect that pragmatic desiderata are a permissible constraint on one's choice of logic, see Church (1954: 66-67).

[16] Note further that, by contrast to Carnap's approach, epistemic intensions are here distinguished from linguistic intensions. For topological Boolean-valued models of epistemic set theory – i.e., a variant of ZF with the axioms augmented by epistemic modal operators interpreted as informal provability and having a background logic satisfying S4 – see Scedrov (1985), Flagg (1985), and Goodman (1990).

[17] In order to ensure that the Kripke semantics matches the topological semantics, X must further be Alexandrov; i.e., closed under arbitrary unions and intersections. Thanks here to Peter Milne.

topology. The formal clauses, in the remainder of this section, evince how homotopic morphisms satisfy the properties of an equivalence relation.[18]

Reflexivity

$\forall$x,y:A$\forall$p(p : x $=_A$ y) : τ(x,y,p), with A and τ designating types, 'x:A' interpreted as 'x is a token of type A', p $\bullet$ q is the concatenation of p and q, $\texttt{refl}_x$: x $=_A$ x for any x:A is a reflexivity element, and e:$\prod_{x:A}\tau$(a,a,$\texttt{refl}_\alpha$) is a dependent function[19]:
$\forall\alpha$:A$\exists$e(α) : τ(α,α,$\texttt{refl}_\alpha$);
p,q : (x $=_A$ y)
$\exists$r$\in$e : p $=_{(x=_A y)}$ q
$\exists\mu$: r $=_{(p=_{(x=_A y)}q)}$ s.

Symmetry

$\forall$A$\forall$x,y:A$\exists$H$_\Sigma$(x=y $\rightarrow$ y=x)
H$_\Sigma$:= p $\mapsto$ p^{-1}, such that
$\forall$x:A($\texttt{refl}_x \equiv \texttt{refl}_x{}^{-1}$).

Transitivity

$\forall$A$\forall$x,y:A$\exists$H$_T$(x=y $\rightarrow$ y=z $\rightarrow$ x=z)
H$_T$:= p $\mapsto$ q $\mapsto$ p $\bullet$ q, such that
$\forall$x:A[$\texttt{refl}_x \bullet \texttt{refl}_x \equiv \texttt{refl}_x$].

Homotopic Abstraction

$\prod_{x:A}$B(x) is a dependent function type. For all type families A,B, there is a homotopy:

H := [(f $\sim$ g) :$\equiv$ $\prod_{x:A}$(f(x) = g(x)], where
$\prod_{f:A\rightarrow B}$[(f $\sim$ f) $\wedge$ (f $\sim$ g $\rightarrow$ g $\sim$ f) $\wedge$ (f $\sim$ g $\rightarrow$ g $\sim$ h $\rightarrow$ f $\sim$ h)],
such that, via Voevodsky's (op. cit.) Univalence Axiom, for all type families A,B:U, there is a function:
$\texttt{idtoeqv}$: (A $=_U$ B) $\rightarrow$ (A $\simeq$ B),
which is itself an equivalence relation:
(A $=_U$ B) $\simeq$ (A $\simeq$ B).

[18] The definitions and proofs at issue can be found in the Univalent Foundations Program (op. cit.: ch. 2.0-2.1). A homotopy is a continuous mapping or path between a pair of functions.

[19] A dependent function is a function type 'whose codomain type can vary depending on the element of the domain to which the function is applied' (Univalent Foundations Program (op. cit.: §1.4).

Epistemic intensions take the form,
$\texttt{pri}(x) = \lambda c.[\![x]\!]^{c,c}$,
with c an epistemically possible world.

Abstraction principles for epistemic intensions take, then, the form of function type equivalence:

- $\exists$f,g[f(x) = g(x)] $\simeq$ [f(x) $\simeq$ g(x)].[20]

3.4 Modal Coalgebraic Automata

Modal coalgebraic automata can be thus characterized. Let a category C be comprised of a class Ob(C) of objects and a family of arrows for each pair of objects C(A,B) (Venema, 2007: 421). A functor from a category C to a category D, **E**: C $\to$ D, is an operation mapping objects and arrows of C to objects and arrows of D (422). An endofunctor on C is a functor, **E**: C $\to$ C (op. cit.).

A **E**-coalgebra is a pair $\mathbb{A}$ = (A, μ), with A an object of C referred to as the carrier of $\mathbb{A}$, and μ: A $\to$ **E**(A) is an arrow in C, referred to as the transition map of $\mathbb{A}$ (390).

As, further, a coalgebraic model of modal logic, $\mathbb{A}$ can be defined as follows (407):

For a set of formulas, Φ, let $\nabla\Phi := \Box \bigvee \Phi \wedge \bigwedge \Diamond\Phi$, where $\Diamond\Phi$ denotes the set $\{\Diamond\phi \mid \phi\in\Phi\}$ (op. cit.). Then,

$\Diamond\phi \equiv \nabla\{\phi, \mathrm{T}\}$,

$\Box\phi \equiv \nabla\varnothing \vee \nabla\phi$ (op. cit.).

$[\![\nabla\Phi]\!] = \{\mathrm{w}\in\mathrm{W} \mid \mathrm{R[w]} \subseteq \bigcup \{[\![\phi]\!] \mid \phi\in\Phi\}$ and $\forall\phi\in\Phi$, $[\![\phi]\!] \cap \mathrm{R[w]} \neq \emptyset\}$ (Fontaine, 2010: 17).

Let an **E**-coalgebraic modal model, $\mathbb{A} = \langle \mathrm{S},\lambda,\mathrm{R}[.]\rangle$, where λ(s) is 'the collection of proposition letters true at s in S, and R[s] is the successor set of s in S', such that $\mathbb{S}$,s $\Vdash \nabla\Phi$ if and only if, for all (some) successors σ of s$\in$S,

[20] Observational type theory countenances 'structure identity principles' which are type equivalences between identification types, and the theory is said to be observational because the type formation rules satisfy structure-preserving definitional equality. Higher observational type theory holds for propositional equality. 'The idea of higher observational type theory is to make these and analogous structural characterizations of identification types be part of their definitional inference rules, thus building the structure identity principle right into the rewrite rules of the type theory' (2023: https://ncatlab.org/nlab/show/higher+observational+type+theory). Shulman (2022) argues that higher observational type theory is one way to make the Univalence Axiom computable. Wright (2012c: 120) defines Hume's Principle as a pair of inference rules, and higher observational type theory might be one way to make first-order abstraction principles defined via inference rules, although not higher-order abstraction principles, computable. The Burali-Forti paradox could be circumvented, because the target abstraction principles wouldn't be based on isomorphism like the Univalence Axiom. See Burali-Forti (1897/1967). Hodes (1984) and Hazen (1985) note that abstraction principles based on isomorphism with unrestricted comprehension entrain the paradox. I avoid the Burali-Forti paradox in my abstraction principle for two-dimensional hyperintensions because the definition is not augmented to second-order logic like in the abstractionist foundations of mathematics, is instead taken in isolation, and the definition defines functions from sets of epistemic states taken as actual to sets of metaphysical states to extensions.

[Φ,σ(s)$\in$**E**($\Vdash_{\mathbb{A}}$)] (Venema, 2007: 399, 407), with **E**($\Vdash_{\mathbb{A}}$) a relation lifting of the satisfaction relation $\Vdash_{\mathbb{A}} \subseteq$ S x Φ. Let a functor, **K**, be such that there is a relation $\overline{\mathbf{K}} \subseteq$ **K**(A) x **K**(A') (Venema, 2012: 17)). Let Z be a binary relation s.t. Z $\subseteq$ A x A' and $\wp\overline{Z} \subseteq \wp$(A) x $\wp$(A'), with

$\wp\overline{Z}$:= {(X,X') | ∀x∈X∃x'∈X' with (x,x')∈Z ∧ ∀x'∈X'∃x∈X with (x,x')∈Z} (op. cit.). Then, we can define the relation lifting, $\overline{\mathbf{K}}$, as follows:

$\overline{\mathbf{K}}$:= {[(π,X), (π',X')] | $\pi = \pi$' and (X,X')$\in\wp\overline{Z}$} (op. cit.), with π a projection mapping of $\overline{\mathbf{K}}$.[21]

The relation lifting, $\overline{\mathbf{K}}$, associated with the functor, **K**, satisfies the following properties (Enqvist et al, 2019: 586):

- $\overline{\mathbf{K}}$ extends **K**. Thus $\overline{\mathbf{K}}f = \mathbf{K}f$ for all functions f: $X_1 \rightarrow X_2$;
- $\overline{\mathbf{K}}$ preserves the diagonal. Thus $\overline{\mathbf{K}}\mathrm{Id}_X = \mathrm{Id}_{KX}$ for any set X and functor, Id, where Id_C maps a set S to the product S x C (583, 586);
- $\overline{\mathbf{K}}$ is monotone. R $\subseteq$ Q implies $\overline{\mathbf{K}}$R $\subseteq$ $\overline{\mathbf{K}}$Q for all relations R,Q $\subseteq$ X_1 x X_2;
- $\overline{\mathbf{K}}$ commutes with taking converse. $\overline{\mathbf{K}}R^\circ = (\overline{\mathbf{K}}R)^\circ$ for all relations R $\subseteq$ X_1 x X_2;
- $\overline{\mathbf{K}}$ distributes over relation composition. $\overline{\mathbf{K}}$(R ; Q) = $\overline{\mathbf{K}}$R ; $\overline{\mathbf{K}}$Q, for all relations R $\subseteq$ X_1 x X_2 and Q $\subseteq$ X_2 x X_3, provided that the functor **K** preserves weak pullbacks (op. cit.). Venema and Vosmaer (2014: §4.2.2) define a weak pullback as follows: "A weak pullback of two morphisms f : X → Z and g : Y → Z with a shared codomain Z is a pair of morphisms p_X : P → X and p_Y : P → Y with a shared domain P, such that (1) $f \circ p_X = g \circ p_Y$, and (2) for any other pair of morphisms q_X : Q → X and q_Y : Q → Y with $f \circ q_X = g \circ q_Y$, there is a morphism q : Q → P such that $p_X \circ q = q_X$ and $p_Y \circ q = q_Y$. This pullback is "weak" because we are not requiring q to be unique. Saying that [a set functor] T : **Set** → **Set** preserves weak pullbacks means that if p_X : P → X and p_Y : P → Y form a weak pullback of f : X → Z and g : Y → Z, then Tp_X : TP → TX and Tp_Y : TP → TY form a weak pullback of Tf : TX → TZ and Tg : TY → TZ".

A coalgebraic model of deterministic automata can finally be thus defined (Venema, 2007: 391). An automaton is a tuple, $\mathbb{A} = \langle$A, a_I, C, Ξ, F$\rangle$, such that A is the state space of the automaton $\mathbb{A}$; a_I∈A is the automaton's initial state; C is the coding for the automaton's alphabet, mapping numerals to the natural numbers; Ξ: A X C → A is a transition function, and F $\subseteq$ A is the collection of admissible states, where F maps A to {1,0}, such that F: A → 1 if a∈F and A → 0 if a∉F (op. cit.).

[21] The projections of a relation R, with R a relation between two sets X and Y such that R $\subseteq$ X x Y, are
X ⟵(π_1) R (π_2)⟶ Y such that π_1((x,y)) = x, and π_2((x,y)) = y. See Rutten (2019: 240).

Modal automata are defined over a modal one-step language (Venema, 2020: 7.2). With A being a set of propositional variables the set, $\mathtt{Latt}$(X), of lattice terms over X has the following grammar:

$$\phi ::= \bot \mid \top \mid x \mid \phi \wedge \phi \mid \phi \vee \phi,$$

with x∈X and $\phi \in \mathtt{Latt}$(A) (op. cit.).

The set, $\mathtt{1ML}$(A), of modal one-step formulas over A has the following grammar:

$$\alpha \in A ::= \bot \mid \top \mid \Diamond\phi \mid \Box\phi \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \text{ (op. cit.).}$$

A modal P-automaton $\mathbb{A}$ is a triple, (A, Θ, a_I), with A a non-empty finite set of states, $a_I \in A$ an initial state, and the transition map

Θ: A x ℘P → $\mathtt{1ML}$(A)

maps states to modal one-step formulas (op. cit.: 7.3).

The crux of the reconciliation between algebraic models of cognitivism and the formal foundations of modal expressivism is based on the duality between categories of algebras and coalgebras: $\mathbb{A} = \langle A, \alpha{:}A \to \mathbf{E}(A)\rangle$ is dual to the category of algebras over the functor α (417-418). For a category C, object A, and endofunctor **E**, define a new arrow, α, s.t. $\alpha{:}\mathbf{E}A \to A$. A homomorphism, f, can further be defined between algebras $\langle A, \alpha\rangle$, and $\langle B, \beta\rangle$. Then, for the category of algebras, the following commutative square can be defined: (i) $\mathbf{E}A \to \mathbf{E}B$ ($\mathbf{E}f$); (ii) $\mathbf{E}A \to A$ (α); (iii) $\mathbf{E}B \to B$ (β); and (iv) $A \to B$ (f) (cf. Hughes, 2001: 7-8). The same commutative square holds for the category of coalgebras, such that the latter are defined by inverting the direction of the morphisms in both (ii) [$A \to \mathbf{E}A$ (α)], and (iii) [$B \to \mathbf{E}B$ (β)] (op. cit.).

The significance of the foregoing is twofold. First and foremost, the above demonstrates how a formal correspondence can be effected between algebraic models of cognition and coalgebraic models which provide a natural setting for modal logics and automata. The second aspect of the philosophical significance of modal coalgebraic automata is that – as a model of modal expressivism – the proposal is able to countenance fundamental properties in the foundations of mathematics, and circumvent the issues accruing to the attempt so to do by the competing expressivist approaches.

4 Material Adequacy

The material adequacy of epistemic modal algebras as a fragment of the representational theory of mind is witnessed by the prevalence of possible worlds and hyperintensional semantics – the model theory for which is algebraic (cf. Blackburn et al., 2001: ch. 5) – in cognitive psychology and artificial intelligence.

Contemporary vision science endeavors to account for the issue of underdetermination, with regard to the transition from the receipt of retinal lightwave spectra to the perceptual representations of physical particulars. In order to account for the transition, the visual system is taken to be comprised of implicit computations that are governed by the Bayesian probability calculus, and the probability measure is interpreted as a function of likelihood (cf. Mamassian et al, 2002; Burge, 2010; Rescorla, 2013). The visual system is presented with a distribution of possibilities, concerning e.g. whether light is emanating from above or emanating from below. The set of possibilities is pointed, as the visual system calculates the likelihood that one of the possibilities is actual. The visual system's implicit calculations are a vindication of Helmholtz's conjecture that visual perception is derived by types of "unconscious inductive inference" (see Helmholtz, 1878/1977: 132, 175-176). The possibility assigned the highest likelihood of being actual is referred to as a perceptual constancy. The designated possibility places, then, a condition on the accuracy of the attribution of properties, such as boundedness and volume, to distal, physical objects.

Marcus (2001) writes that: 'A multilayer perceptron consists of a set of *input nodes*, one or more sets of *hidden nodes*, and a set of *output nodes* ... These nodes are attached to each other through *weighted connections*; the weights of these connections are generally adjusted by some sort of *learning algorithm* ... *Nodes* are *units* that have activation [real] values ... *Input* and *output nodes* also have *meanings* or *labels* that are assigned by an external programmer ... The *meanings* of nodes (their labels) play no direct role in the computation: a network's computations depend only on the activation values of nodes and not on the labels of those nodes' (7-8). Both a single and multiple nodes can serve to represent the variables for a target domain. The values of a target domain for variables are universally quantified over and the function is one-one, mapping a number of inputs to an equivalent number of outputs (35-36). Models of the above algebraic rules can be defined in both classical and weighted, connectionist systems (42-45). Temporal synchrony or dynamic variable-bindings are stored in short-term memory (56-57), while information relevant to long-term variable-bindings are stored in 'binary registers' i.e. 'bits' (41, 54-56). 'Operations are defined in parallel over these sets of binary bits. When a programmer issues a command to copy the contents of variable **x** into variable **y**, the computer copies in parallel each of the bits that represents variable **x** into the corresponding bits that represent variable **y**' (41). Examples of the foregoing algebraic rules on variable-binding include both the syntactic concatenation of morphemes and noun phrase reduplication in linguistics (37-39, 70-72), as well as learning algorithms (45-48). Conditions on variable-binding are further examined, including treating the binding relation between variables and values as tensor products – i.e., an application of a multiplicative axiom for variables and their values treated as vectors (53-54, 105-106). In order to account for recursively formed, complex representations, which he refers to as structured propositions, Marcus argues instead that the syntax and semantics of such representations can be modeled via an ordered set of registers, which he refers to as 'treelets' (108).

A strengthened version of the algebraic rules on variable-binding can be accommodated in models of epistemic modal algebras, when the latter are augmented by cylindrifications, i.e., operators on the algebra simulating the treatment of quantification, and diagonal elements.[22] By contrast to Boolean Algebras with Operators, which are propositional, cylindric algebras define first-order logics. Intuitively, valuation assignments for first-order variables are, in cylindric modal logics, treated as possible worlds of the model, while existential and universal quantifiers are replaced by, respectively, possibility and necessity operators ($\Diamond$ and $\Box$) (Venema, 2013: 249). For first-order variables, $\{v_i \mid i < \alpha\}$ with α an arbitrary, fixed ordinal, $v_i = v_j$ is replaced by a modal constant $\mathbf{a}_{i,j}$ (op. cit: 250). The following clauses are valid, then, for a model, M, of cylindric modal logic, with $E_{i,j}$ a monadic predicate and T_i for i,j < α a dyadic predicate:

M,w $\Vdash$ p $\iff$ w$\in$V(p);

M,w $\Vdash$ $\mathbf{a}_{i,j}$ $\iff$ w$\in E_{i,j}$;

M,w $\Vdash$ $\Diamond_i\psi$ $\iff$ there is a v with wT_iv and M,v $\Vdash$ ψ (252).

Cylindric frames need further to satisfy the following axioms (op. cit.: 254):

1. $p \to \Diamond_i p$
2. $p \to \Box_i\Diamond_i p$
3. $\Diamond_i\Diamond_i p \to \Diamond_i p$
4. $\Diamond_i\Diamond_j p \to \Diamond_j\Diamond_i p$
5. $\mathbf{a}_{i,i}$
6. $\Diamond_i(\mathbf{a}_{i,j} \wedge p) \to \Box_i(\mathbf{a}_{i,j} \to p)$

[Translating the diagonal element and cylindric (modal) operator into, respectively, monadic and dyadic predicates and universal quantification: $\forall xyz[(T_i xy \wedge E_{i,j} y \wedge T_i xz \wedge E_{i,j} z) \to y = z]$ (op. cit.)]

7. $\mathbf{a}_{i,j} \iff \Diamond_k(\mathbf{a}_{i,k} \wedge \mathbf{a}_{k,j})$.

Finally, a cylindric modal algebra of dimension α is an algebra, $\mathbb{A} = \langle A, +, \bullet, -, 0, 1, \Diamond_i, \mathbf{a}_{ij}\rangle_{i,j<\alpha}$, where $\Diamond_i$ is a unary operator which is normal ($\Diamond_i 0 = 0$) and additive [$\Diamond_i(x + y) = \Diamond_i x + \Diamond_i y$] (257).

The philosophical interest of cylindric modal algebras to Marcus' cognitive models of algebraic variable-binding is that the valuation assignments to variables in the Epistemic Modal Algebra are epistemically possible worlds, while universal quantification is interpreted as epistemic necessitation. The interest of translating universal generalization into operations of epistemic necessitation is, finally, that – by identifying epistemic necessity with apriority – both the algebraic rules for variable-binding and the recursive formation of structured propositions can be seen as operations, the implicit knowledge of which is apriori.

In artificial intelligence, the subfield of knowledge representation draws on epistemic logic, where belief and knowledge are interpreted as necessity operators (Meyer and van der Hoeck, 1995; Fagin et al., 1995). Possibility and necessity may receive other interpretations in mental terms, such as that of conceivability and apriority (i.e. truth in all epistemic possibilities, or inconceivability that not ϕ). The language of thought hypothesis maintains that thinking occurs in a mental language with a computational syntax and a seman-

[22]See Henkin et al (op. cit.: 162-163) for the introduction of cylindric algebras, and for the axioms governing the cylindrification operators.

tics. The philosophical significance of cognitivism about epistemic modality is that it construes epistemic intensions and hyperintensions as abstract, computational functions in the mind, and thus provides an explanation of the relation that human beings bear to epistemic possibilities. Intensions and hyperintensions are semantically imbued abstract functions comprising the computational syntax of the language of thought. The functions are semantically imbued because they are defined relative to a parameter ranging over either epistemically possible worlds or epistemic states in a state space, and extensions or semantic values are defined for the functions relative to that parameter. Cognitivism about epistemic modality argues that thoughts are composed of epistemic intensions or hyperintensions. Cognitivism about epistemic modality provides a metaphysical explanation or account of the ground of thoughts, arguing that they are grounded in epistemic possibilities and either intensions or hyperintensions which are themselves internal representations comprising the syntax and semantics for a mental language. This is consistent with belief and knowledge being countenanced in an epistemic logic for artificial intelligence, as well. Epistemic possibilities are constitutively related to thoughts, and figure furthermore in the analysis of notions such as apriority and conceivability, as well as belief and knowledge in epistemic logic for artificial intelligence.

My claim is only that epistemic intensions and hyperintensions – i.e. functions from epistemically possible worlds or epistemic states to extensions – are computable functions comprising a fragment of the language of thought, leaving it open whether the mind is more generally a Turing machine. I thus hope to avoid taking a position here on whether human cognition is generally computational in light of Gödel's (1931/1986) incompleteness theorems. Gödel's disjunction claims that either (I) the mind is a Turing machine and thus there are sentences which are undecidable, i.e. not provable, such as the Generalized Continuum Hypothesis, because (i) formal systems are recursively enumerable, i.e. formalizable by Turing machines, and (ii) the first incompleteness theorem entails that, in *consistent* formal systems, the provability via the recursive enumerability of sentences is distinct from the truth of Gödel sentences (1931/1986: 195), or (II) the mind surpasses the computability via the recursive enumerability of sentences in a Turing machine, and currently undecidable sentences are provable i.e. decidable owing to (i) mathematical intuition instead of computable mechanism, and (ii) Gödel's acceptance of rational optimism.[23]

[23]Gödel's proofs of the incompleteness theorems are as follows. The presentation follows that of Raatikainen (2022). I will quote the entire text, because the definitions and characterizations are mostly owing to Raatikainen. 'A numeral canonically denoting a natural number **n** is abbreviated as $\overline{n}$. A formalized theory F is ω-consistent if it is not the case that for some formula A(x), both $F \vdash \neg A(\overline{n})$ for all **n**, and $F \vdash \exists x A(x)$. A set S of natural numbers is strongly representable in F if there is a formula A(x) of the language of F with one free variable x such that for every natural number **n**:

'$\mathbf{n} \in S \Rightarrow F \vdash A(\overline{n})$;

'$\mathbf{n} \notin S \Rightarrow F \vdash \neg A(\overline{n})$.

The signature of Robinson Arithmetic i.e. **Q** is first-order Peano Arithmetic without the induction schema, with 0 a constant for zero, a unary function symbol s for successor, and binary function symbols + and • for addition and multiplication. The axioms of **Q** are:

5 Precedent

The proposal that possible worlds semantics comprises the model for thoughts and propositions is anticipated by Wittgenstein (1921/1974); Chalmers (2011); and Jackson (2011). Their approaches depart, however, from the one here examined in the following respects.

Wittgenstein writes: "Logical pictures can depict the world. / A picture has a logico-pictorial form in common with what it depicts. / A picture depicts reality by representing a possibility of existence and non-existence of states of affairs. / A picture represents a possible situation in logical space. / A picture contains the possibility of the situation that it represents ... A logical picture of facts is a thought. / 'A state of affairs is thinkable': what this means is that we can picture it to ourselves. / The totality of true thoughts is a picture of the world. / A thought contains the possibility of the situation of which it is the thought. What is thinkable is possible too" (op. cit.: 2.19-2.203, 3-3.02).

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1. $\forall x \neg s(x) = 0$
2. $\forall_{x,y} s(x) = s(y) \rightarrow x = y$
3. $\forall_x x = 0 \vee \exists_y x = s(y)$
4. $\forall_x x + 0 = x$
5. $\forall_{x,y} x + s(y) = s(x + y)$
6. $\forall_x X \bullet 0 = 0$
7. $\forall_{x,y} x \bullet s(y) = x \bullet y + x$

(https://ncatlab.org/nlab/show/Robinson+arithmetic).

'A set S of natural numbers is weakly representable in F if there is a formula A(x) of the language of F such that for every natural number **n**:

'$\mathbf{n} \in S \iff F \vdash A(\overline{n})$.

'The representability theorem says then that in any consistent formal system which contains Robinson Arithmetic i.e. **Q**:

'1. A set (or relation) is strongly representable if and only if it is recursive;

'2. A set (or relation) is weakly representable if and only if it is recursively enumerable.

'Suppose that there is a coding of symbols and formulas by the natural numbers. The Gödel number of a formula A is denoted as $\ulcorner A \urcorner$.

'Suppose that the diagonalization lemma holds, such that $F \vdash Q \iff A(\ulcorner Q \urcorner)$.

'For the first incompleteness theorem, the diagonalization lemma is applied to the negation of the provability predicate, $\neg \mathrm{Prov}_F(x)$, which yields the following sentence:

'(Z) $F \vdash M_F \iff \neg \mathrm{Prov}_F(\ulcorner M_F \urcorner)$.

'Assume that M_F is provable. By the weak representability of provability-in-F by $\mathrm{Prov}_F(x)$, F would also prove $\mathrm{Prov}_F(M_F)$. Because F proves Z – i.e. $F \vdash M_F \iff \neg \mathrm{Prov}_F(\ulcorner M_F \urcorner)$ – F would then prove $\neg M_F$. So F would be inconsistent. Thus, if F is consistent, then M_F is not provable in F.

'Assume that F is ω-consistent. Assume, then, that $F \vdash \neg M_F$. Then F cannot prove M_F, because it would then be ω-inconsistent. Thus no natural number **n** is the Gödel number of a proof of M_F. Because the proof relation is strongly representable, for all **n**, $F \vdash \neg \mathrm{Prf}_F(\overline{n}, \ulcorner M_F \urcorner)$. If $F \vdash \exists x \mathrm{Prf}_F(x, \ulcorner M_F \urcorner)$, F is not ω-consistent. Thus F does not prove $\exists x \mathrm{Prf}_F(x, \ulcorner M_F \urcorner)$, i.e. F does not prove $\mathrm{Prov}_F(\ulcorner M_F \urcorner)$. By the equivalence recorded in (Z), F does not prove $\neg M_F$.

'For the second incompleteness theorem: Suppose that consistency, Con(F), is defined as $\neg \mathrm{Prov}_F(\ulcorner \bot \urcorner)$, where $\bot$ expresses an inconsistent formula such as $\overline{0} = \overline{1}$. Formalizing the proof of the first incompleteness theorem in F yields $F \vdash \mathrm{Cons}(F) \rightarrow M_F$. If Cons(F) were provable in F, so would be M_F. Cons(F) is thus unprovable, given the first incompleteness theorem.'

For further discussion, see Gödel (1951); Lucas (1961); Penrose (1989; 1994); the essays in Horsten and Welch (2016); and Koellner (2018a,b). See Elohim (2024), for the convergence between modal and hyperintensional computational automata and rational intuition.

Wittgenstein (op. cit.: 1-1.1) has been interpreted as endorsing an identity theory of propositions, which does not distinguish between internal thoughts and external propositions (cf. McDowell, 1994: 27; and Hornsby, 1997: 1-3). How the identity theory of propositions is able to accommodate Wittgenstein's suggestion that a typed hierarchy of propositions can be generated – only if the class of propositions has a general form and the sense of propositions over which operations range is invariant by being individuated by the possibilities figuring as their truth and falsity conditions (cf. Wittgenstein, 1979: 21/11/16, 23/11/16, 7/11/17; and Potter, 2009: 283-285 for detailed discussion) – is an open question. Wittgenstein (1921/1974: 5.5561) writes that "Hierarchies are and must be independent of reality", although provides no account of how the independence can be effected.

Jackson (2008: 48-50) distinguishes between personal and subpersonal theories by the role of neural science in individuating representational states (cf. Shea, 2013, for further discussion), and argues in favor of a "personal-level implicit theory" for the possible worlds semantics of mental representations.

Chalmers' approach comes closest to the one here proffered, because he argues for a hybrid cognitivist-expressivist approach as well, according to which epistemic intensions – i.e. functions from epistemically possible worlds to extensions – are individuated by their inferential roles (2012a: 462-463; 2021). Chalmers endorses what he refers to as "anchored inferentialism", and in particular "acquaintance inferentialism" for intensions, according to which "there is a limited set of primitive concepts, and all other concepts are grounded in their inferential role with respect to these concepts", where "the primitive concepts are acquaintance concepts" (2012a: 463, 466) and "[a]cquaintance concepts may include phenomenal concepts and observational concepts: primitive concepts of phenomenal properties, spatiotemporal properties, and secondary qualities" (2010b: 11). According to Chalmers, "anchored inferential role determines a primary intension. The relevant role can be seen as an internal (narrow or short-armed) role, so that the content is a narrow content" (5). The inferences in question are taken to be "suppositional" inferences, from a base class of truths, PQTI – i.e. truths about physics, consciousness, and indexicality, and a that's all truth – determining canonical specifications of epistemically possible worlds, to other truths (3). With regard to how suppositional inference, i.e. "scrutability", plays a role in the definitions of intensions, Chalmers writes that "[t]he primary intension of [a sentence] S is true at a scenario [i.e. epistemically possible world] w iff [A] epistemically necessitates S, where [A] is a canonical specification of w", where "[A] epistemically necessitates S iff a conditional of the form '[A] $\rightarrow$ S' is apriori" and the apriori entailment is the relation of scrutability (2006). Chalmers (2012a: 245) is explicit about this: "The intension of a sentence S (in a context) is true at a scenario w iff S is a priori scrutable from [A] (in that context), where [A] is a canonical specification of w (that is, one of the epistemically complete sentences in the equivalence class of w) ... A Priori Scrutability entails that this sentence S is a priori scrutable (for me) from a canonical specification [A] of my actual scenario, where [A] is something along the lines of $PQTI$". "The secondary intension of S is true at a world w iff [A] metaphysically necessitates S", where "[A] metaphysically necessitates S when a subjunctive conditional of the form 'if [A] had been the case, S would have been the case' is true" (op. cit.). Thus, suppositional inference, i.e. scrutability, determines the intensions of two-dimensional semantics.

In this paper, intensions and hyperintensions are countenanced as semantically imbued functions. Intensions and hyperintensions as functions comprise the computational syntax for the language of thought, but they are semantically imbued because they are functions from epistemic possibilities to extensions.

An anticipation of this proposal is Tichy (1969), who defines intensions as Turing machines. Adriaans (2020) provides an example of intensions modeled using a Turing machine, as well.[24] The expression

$U_j\overline{T_i}(x) = y$

has the following components. "The universal Turing machine U_j is a **context** in which the computation takes place. It can be interpreted as a **possible computational world** in a modal interpretation of computational semantics. / The sequences of symbols $\overline{T_i}x$ and y are **well-formed data**. / The sequence $\overline{T_i}$ is a self-delimiting description of a program and it can be interpreted as a piece of well-formed **instructional data**. / The sequence $\overline{T_i}x$ is an **intension**. The sequence y is the corresponding **extension**. / The expression $U_j\overline{T_i}(x) = y$ states the result of the program $\overline{T_i}x$ in world U_j is y. It is a **true sentence**".

I will avail, in this paper, of Adriaans (2020)'s definition of intensions as Turing machines. The variable, x, in the (hyper-)intension, $\overline{T_i}x$, ranges over epistemically possible worlds or states and metaphysically possible worlds or states, and is a function from epistemic states verifying sentences, where the epistemic states are taken as actual, to the value of the sentences verified by metaphysical states, to the sentences' extensions.

This is consistent with the inferences of scrutability playing a role in the individuation of intensions and hyperintensions, but whereas Chalmers grounds inferences in dispositions (2010: 10; 2021), I claim that the inferences drawn from the canonical specifications of epistemic possibilities to arbitrary truths are apriori computations between mental representations.

6 Conceptual Engineering of Intensions and Hyperintensions

How can intensions and hyperintensions be revised, given that they are here countenanced as computable functions comprising the syntax of the language of thought? Note that the epistemically possible worlds or hyperintensional truthmakers, and the topics to which they are sensitive, which figure as input to intensions and hyperintensions, can be externally individuated. If so, then they are susceptible to updates by external sources. One might want further to

[24] Approaches to conceiving of intensions as computable functions have been pursued, as well, by Muskens (2005), Moschovakis (2006), and Lappin (2014). The computational complexity of algorithms for intensions has been investigated by Mostowski and Wojtyniak (2004), Mostowski and Szymanik (2012), and Kalocinski and Godziszewski (2018).

engage in the project of conceptually engineering one's intensions and hyperintensions, perhaps in order to engage in an ameliorative project relevant to using more socially just concepts (see Haslanger, 2012, 2020 for further discussion). Conceptual engineering of intensions and hyperintensions can then be effected by five methods. The first is via announcements in dynamic epistemic logic. The second method is via dynamic interpretational modalities which concern the possible reassignment of topics to atomic formulas. The third method is via dynamic hyperintensional belief revision. We here propose a novel truthmaker semantics for the first and second methods.

The language of public announcement logic has the following syntax (see Baltag and Renne, 2016):

$$\phi := p \mid \phi \wedge \phi \mid \neg\phi \mid [a]\phi \mid [\phi!]\psi$$

$[a]\phi$ is interpreted as the 'the agent knows ϕ'. $[\phi!]\psi$ is an announcement formula, and is intuitively interpreted as "whenever ϕ is true, ψ is true after we eliminate all not-ϕ possibilities (and all arrows to and from these possibilities)".

Semantics for public announcement logic is as follows:

M, w $\Vdash \phi$ if and only if w$\in$V(ϕ)

M, w $\Vdash \phi \wedge \psi$ if and only if M, w $\Vdash \phi$ and M, w $\Vdash \psi$

M, w $\Vdash \neg\phi$ if and only if M, w $\nVdash \phi$

M, w $\Vdash [a]\phi$ if and only if M, w $\Vdash \phi$ for each v satisfying wR$_a$v

M, w $\Vdash [\phi!]\psi$ if and only if M, w $\nVdash \phi$ or M[ϕ!], w $\Vdash \psi$,

where M[ϕ!] = (W[ϕ!], R[ϕ!], V[ϕ!]) is defined by

W[ϕ!] := (v$\in$W | M, v $\Vdash \phi$) (intuitively, "retain only the worlds where ϕ is true") (op. cit.),

xR[ϕ!]$_a$y if and only if xR$_a$y (intuitively, "leave arrows between remaining words unchanged"), and

v$\in$V[ϕ!](p) if and only if v$\in$V(p) (intuitively, "leave the valuation the same at remaining worlds").

Fine (2006) and Uzquiano (2015) countenance interpretational modalities. Fine (2005)'s modality is postulational, dynamic, and prescriptive. The dynamic modality is interpreted so as to concern the execution of computer programs which entrain e.g. the introduction of objects into a domain which conform to a certain property. Fine (2006) advances an interpretational modality which concerns the possible reinterpretation of quantifier domains in accounting for indefinite extensibility. Uzquiano's modality is interpretational and also relevant to capturing the property of indefinite extensibility. The modality is mathematical, and concerns the possible reinterpretations of the intensions of non-logical vocabulary such as the membership relation, $\in$.

In this paper, I propose to render Fine's and Uzquiano's interpretational modalities dynamic. The dynamic interpretational modalities are interpreted as program executions which entrain reinterpretations of intensions as well as reinterpretations of hyperintensions, where the latter reassign topics to atomic formulas.

My proposal is that both announcement formulas, $[\phi!]\psi$, and Fine and Uzquiano's dynamic modalities ought to be rendered hyperintensional, such that the box operators are further interpreted as topic-sensitive necessary truthmakers. The dynamic interpretational modalities can just take the clause for A(ϕ) as above. For announcement formulas, $[\phi!]\psi$ if and only if either (i) for all s∈P there is no s'∈P such that s' ⊔ s ∈P and s' ⊢ ϕ or (ii) M[ϕ!], s ⊢ ψ,

where M[ϕ!] = ⟨S[ϕ!], ≤[ϕ!], v[ϕ!]⟩ is defined by

S[ϕ!] := s'∈S | M, s' ⊢ ϕ (intuitively, retain only states which verify ϕ),

≤[ϕ!] if and only if s≤s' (intuitively, leave relations between remaining states unchanged), and

v[ϕ!] if and only if v: Prop → (2^S x 2^S) which assigns a bilateral proposition $\langle \phi^+, \phi^- \rangle$ to ϕ∈Prop (intuitively, leave the valuation the same at remaining states). States are topic-sensitive such that s in the foregoing abbreviates s ∩ t.

This would suffice for what Chalmers (2020) refers to as conceptual re-engineering, rather than "de novo" conceptual engineering, of intensions and hyperintensions. Conceptual re-engineering concerns the refinement or replacement of extant concepts, while de novo engineering concerns the introduction of new concepts. The third method for conceptual re-engineering contents would be via Berto and Özgün (2021)'s logic for dynamic hyperintensional belief revision, which includes a topic-sensitive upgrade operator. On this method, the worlds and topics for formulas are both updated in cases of belief revision.

A fourth novel method can be countenanced, namely making epistemic two-dimensional semantics dynamic. On this approach, an epistemic action such as an announcement which updates the first, epistemic parameter for a formula would entrain an update to a second parameter ranging over metaphysically possible worlds or states in a state space. Using two-dimensional intensions, such that the value of a formula relative to a first parameter ranging over epistemic states determines the value of the formula relative to a second parameter ranging over metaphysical states, an update (announcement, epistemic action) to the epistemic space over which the first parameter of a formula ranges induces an update to the metaphysical space over which a second parameter for a formula ranges. With M* a model including a class of epistemic states, S, and a class of metaphysical states, W, two-dimensional updates have the form:

M*, w ⊩ $[\phi!]\psi$ if and only if M*, w ⊮ ϕ or M*[ϕ!], w ⊩ ψ,

where M*[ϕ!] = (S[ϕ!], $W[\phi!]^{S[\phi!]}$, R[ϕ!], V[ϕ!]). $W[\phi!]^{S[\phi!]}$ records the dynamic two-dimensional update of metaphysical states, W, conditional on the update of epistemic states, S, and the rest is defined as above.

A fifth method for modeling updates might be via the interventions of structural equation models which reassign values to exogenous variables which then determines the values of endogenous variables (see e.g. Pearl, 2009).[25] Using two-dimensional intensions, the updates to the epistemic parameter of a formula might be modeled using Baltag (2016)'s Logic of Epistemic Dependency. As Baltag writes: 'An *epistemic dependency formula* $K_a^{x_1,\ldots,x_n}y$ says that an

[25]Thanks here to Hannes Leitgeb for mentioning structural equation models with regard to a possible example of metaphysical updates.

agent knows the value of some variable y *conditional* on being given the values of the variables $x_1, \dots, x_n \dots$ if we use the abbreviation $(w(\overrightarrow{x})) = (v(\overrightarrow{x}))$ for the conjunction $(w(x_1)) = (v(x_1)) \wedge (w(x_n)) = (v(x_n))$, then we put

$w \Vdash K_a^{x_1,\dots,x_n} y$ iff $\forall v \sim_a w \ (w(\overrightarrow{x})) = (\mathrm{v}(\overrightarrow{x})) \Rightarrow v(y) = w(y)$.

In words: an agent knows y given $x_1, \dots, x_n$ if the value of y is the same in all the epistemic alternatives that agree with the actual world on the values of $x_1, \dots, x_n$. This operator has connections with Dependence Logic and allows us to "pre-encode" the dynamics of the value-announcement operator [!x]ϕ" (136).

Epistemic updates via announcements would then, via two-dimensional intensions and hyperintensions, induce an intervention in the metaphysical space in the parameter defining the second dimension of a formula, by reassigning values of exogenous variables so as to constrain the values of endogenous variables in structural equations.

My dynamic logic for conceptual engineering thus accounts for topic-preservation at the level of variables.[26]

The Epistemic Church-Turing Thesis can receive a similar two-dimensional hyperintensional formalization. Carlson (2016: 132) presents the schema for the Epistemic Church-Turing Thesis as follows:

With $\Box$ interpreted as a knowledge operator, '$\Box\forall x\exists y\Box\phi \rightarrow \exists e\Box\forall x\exists y[E(e, x, y) \wedge \phi]$,

'where e does not occur free in ϕ and E is a fixed formula of L_{PA} [i.e the language of Peano Arithmetic] with free variables v_0, v_1, v_2 such that, letting N be the standard model of arithmetic,

'$N \Vdash E(e, x, y)[e, x, y \mid a, m, n]$

'iff on input m, the a^{th} Turing machine halts and outputs n. For convenience, we will write $\{t_1\}\{t_2\} \simeq t_3$ for $E(t_1, t_2, t_3)$ when t_1, t_2, t_3 are terms'. Carlson defines $(x_1, \dots, x_n) \mid (y_1, \dots, y_1)$ as denoting the 'function which maps x_i to y_i for each i = 1, …, n' (op. cit.: 130). Hyperintensionally reformalized, the Epistemic Church-Turing Thesis is then:

$A\forall x\exists yA\phi \rightarrow \exists eA\forall x\exists y[E(e, x, y) \wedge \phi]$.

The two-dimensional hyperintensional profile of the Epistemic Church-Turing Thesis can be countenanced by adding a topic-sensitive truthmaker from a metaphysical state space and making its value dependent on the value of the epistemically necessary truthmaker $A(\phi)$, which has the same clause as truthmaker apriority above. Thus:

$A^{(w\cap t)}\forall x\exists yA^{(w\cap t)}\phi \rightarrow \exists eA^{(w\cap t)}\forall x\exists y[E(e, x, y) \wedge \phi]$.

An application of the two-dimensional Epistemic Church-Turing Thesis is to the foregoing dynamic epistemic two-dimensional semantics. Two-dimensional Turing machines can be availed of in order to provide mechanistic, constructive definitions of the epistemic actions and metaphysical interventions and their dependence in the two-dimensional semantics. Aside from defining epistemic intensions as computable functions, where the functions comprise the computable

[26] The issue of topic-preservation in conceptual engineering is discussed by Haslanger (2000, 2021), Sawyer (2018; 2020), Cappelen (2018; 2020), Prinzing (2018), Pinder (2020), Nado (2021), and Koch (2023).

syntax of the language of thought, the author records here their preference for non-mechanistic approaches to epistemic modality, such as the interpretation thereof as informal provability or as an inference package.

7 Expressivist Natural Language Semantics for Epistemic Modals

I assume a dissociation between the natural language semantics for epistemic modals and an account of mental states as epistemic possibilities or hyperintensional epistemic states. However, my expressivism about epistemic modality might be thought to adduce in favor of expressivism about epistemic modals. Let expressivism about a domain of discourse be the claim that an utterance from that domain expresses a mental state, rather than states a fact (Hawke and Steinert-Threlkeld, 2021). Hawke and Steinert-Threlkeld (op. cit., 480) distinguish between semantic expressivism and pragmatic expressivism. Expressivism about epistemic modality takes the property expressed by $\Diamond\phi$ to be $\{\mathbf{s} \subseteq W\colon \mathbf{s} \nVdash \neg p\}$, where **s** is a state of information, W is a set of possible worlds, and **s** $\Vdash \phi$ if and only if ϕ is assertable relative to **s**, if and only if the state of information is compatible with ϕ (op. cit.). Semantic expressivism incorporates a "psychologistic semantics" according to which the value of ϕ is a partial function from information states to truth-values, such that "the mental type expressed by ϕ is characterized in terms of the assertability relation $\Vdash$" and "the definition of $\Vdash$ is an essential part of that of $[\![\ ]\!]$" (481). Pragmatic expressivism rejects the psychologistic semantics condition, and "allows for a *gap* between the compositional semantic theory and $\Vdash$" (op. cit.). Hawke and Steinert-Threlkeld's semantic expressivist semantics for epistemic modals converges with the metaphysical expressivism about epistemic modality here adumbrated, although the proposal in this paper is also consistent with pragmatic expressivist accounts of epistemic modals which reject psychologistic semantics.

Another development which is worth mentioning is Holliday and Mandelkern (forthcoming)'s orthologic and possibility semantics for epistemic modals, which is non-classical by rejecting the laws of distributivity, disjunctive syllogism, and orthomodularity, while negation is defined as orthocomplementation rather than psuedocomplentation such that the inference from '$p \wedge \Diamond\neg p \vdash \bot$' to '$\Diamond\neg p \vdash \neg p$' does not hold. Possibility semantics rejects a primeness condition according to which a world x makes disjunction true iff it makes the disjuncts true. Rather, in possibility semantics, x makes a disjunction true just in case for every refinement x' $\sqsubseteq$ x, there is a further refinement x" $\sqsubseteq$ x' which makes one of the disjuncts true (see Holliday, 2021, for further discussion). One issue for the foregoing is that possibility semantics might be unnatural, by relying on refinements of refinements to make one of the disjuncts of a disjunction true, instead of one instance of refinements making one of the disjuncts true.

In the the remainder of the paper, I endeavor to demonstrate the advantages accruing to the present approach to countenancing modal expressivism via modal coalgebraic automata, via a comparison of the theoretical strength of the proposal when applied to characterizing the fundamental properties of the foundations of mathematics, by contrast to the competing approaches to modal expressivism and the limits of their applications thereto.

8 Modal Expressivism and the Philosophy of Mathematics

When modal expressivism is modeled via speech acts on a common ground of presuppositions, the application thereof to the foundations of mathematics is limited by the manner in which necessary propositions are characterized.

Because for example a proposition is taken, according to the proposal, to be identical to a set of possible worlds, all necessarily true mathematical formulas can only express a single proposition; namely, the set of all possible worlds (cf. Stalnaker, 1978; 2003: 51). Thus, although distinct set-forming operations will be codified by distinct axioms of a language of set theory, the axioms will be assumed to express the same proposition: The axiom of Pairing in set theory – which states that a unique set can be formed by combining an element from each of two extant sets: $\forall$x,y.$\exists$z.$\forall$w.w$\in$z $\iff$ w = x $\vee$ w = y – will be supposed to express the same proposition as the Power Set axiom – which states that a set can be formed by taking the set of all subsets of an extant set: $\forall$x.$\exists$y.$\forall$z.z$\in$y $\iff$ z $\subseteq$ x. However, that distinct operations – i.e., the formation of a set by selecting elements from two extant sets, by contrast to forming a set by collecting all of the subsets of a single extant set – are characterized by the different axioms is readily apparent. As Williamson (2016: 244) writes: "...if one follows Robert Stalnaker in treating a proposition as the set of (metaphysically) possible worlds at which it is true, then all true mathematical formulas literally express the same proposition, the set of all possible worlds, since all true mathematical formulas literally express necessary truths. It is therefore trivial that if one true mathematical proposition is absolutely provable, they all are. Indeed, if you already know one true mathematical proposition (that 2 + 2 = 4, for example), you thereby already know them all. Stalnaker suggests that what mathematicians really learn are in effect new contingent truths about which mathematical formulas we use to express the one necessary truth, but his view faces grave internal problems, and the conception of the content of mathematical knowledge as contingent and metalinguistic is in any case grossly implausible."

Thomasson (2007) argues for a version of modal expressivism which she refers to as 'modal normativism', according to which alethic modalities are to be replaced by deontic modalities taking the form of object-language, modal indicative conditionals (op. cit.: 136, 138, 141). The modal indicative conditionals serve to express constitutive rules pertaining, e.g., to ontological dependencies which state that: "Necessarily, if an entity satisfying a property exists then a distinct entity satisfying a property exists" (143-144), and generalizes to other expressions, such as analytic conditionals which state, e.g., that: "Necessarily, if an entity satisfies a property, such as being a bachelor, then the entity satisfies a distinct yet co-extensive property, such as being unmarried" (148).

A virtue of Thomasson's interpretation of modal indicative conditionals as expressing both analytic and ontological dependencies is that it would appear to converge with the 'If-thenist' proposal in the philosophy of mathematics. 'If-thenism' is an approach according to which, if an axiomatized mathematical language is consistent, then (i) one can either bear epistemic attitudes, such as fictive acceptance, toward the target system (cf. Leng, 2010: 180) or (ii) the system (possibly) exists [cf. Russell (op. cit.: §1); Hilbert (1899/1980: 39); Menger (1930/1979: 57); Putnam (1967); Shapiro (2000: 95); Chihara (2004: Ch. 10); and Awodey (2004: 60-61)].[27] However, there are at least two issues for the modal normativist approach in the setting of the philosophy of mathematics.

One general issue for the proposal is that the treatment of quantification remains unaddressed, given that there are translations from modal operators, such as figure in modal indicatives, into existential and universal quantifiers.[28]

A second issue for the normative indicative conditional approach is that Thomasson's normative modalities are unimodal. They are thus not sufficiently fine-grained to capture distinctions such as Gödel's (op. cit.) between mathematics in its subjective and objective senses.[29] Further distinctions between the types of mathematical modality can be delineated which permit epistemic types of mathematical possibility to serve as a guide as to whether a formula is metaphysically mathematically possible.[30] The convergence between epis-

[27]See Leng (2009), for further discussion. Field (1980/2016: 11-21; 1989: 54-65, 240-241) argues in favor of the stronger notion of conservativeness, according to which consistent mathematical theories must be satisfiable by internally consistent theories of physics. More generally, for a class of assertions, A, comprising a theory of fundamental physics, and a class of sentences comprising a mathematical language, M, any sentences derivable from A+M ought to be derivable from A alone. Another variation on the 'If-thenist' proposal is witnessed in Field (2001: 333-338), who argues that the existence of consistent forcing extensions of set-theoretic ground models adduces in favor of there being a set-theoretic pluriverse, and thus entrains indeterminacy in the truth-values of undecidable sentences. For a similar proposal, which emphasizes the epistemic role of examining how instances of undecidable sentences obtain and fail so to do relative to forcing extensions in the set-theoretic pluriverse, see Hamkins (2012: §7).

[28]The formal correspondence between modalities and quantifiers is anticipated by Aristotle (*De Interpretatione*, 9; *De Caelo*, I.12), who defines the metaphysical necessity of a proposition as its being true at all times. For detailed discussion of Aristotle's theory, see Waterlow (1982). For a contemporary account of the multi-modal logic for metaphysical and temporal modalities, see Dorr and Goodman (2019). For contemporary accounts of the correspondence between modal operators and quantifiers see von Wright (1952/1957); Montague (1960/1974: 75); Lewis (1975/1998; 1981/1998); Kratzer (op. cit.; 1981/2012); and Kuhn (1980). For the history of modal logic, see Goldblatt (2006).

[29]See footnote 4 for the relevant definitions.

[30]Reinhardt (1974a: §6) proposes the use of imaginary sets and classes as 'imaginary experiments' (204), in order to define imaginary projections corresponding to the universe of sets which define Reinhardt large large cardinals. An objection to the foregoing is advanced by Maddy (1988) who objects to the 'use of counterfactual situations to distinguish these new entities from sets' (754). Maddy writes: 'I think even those with strong modal intuitions will have trouble imagining how there might be more pure sets and ordinals than there are. After all, V is supposed to contain all the sets and ordinals there could possibly be' (op. cit.).

temic and metaphysical mathematical modalities can be countenanced via a two-dimensional semantics. Thus, by eschewing alethic modalities for unimodal, normative indicatives, the normative modalities are unable to account for the relation between the alethic interpretation of modality and, e.g., logical mathematical modalities treated as consistency operators on languages (cf. Field, 1989: 249-250, 257-260; Leng: 2007; 2010: 258), or for the convergence between epistemic possibilities concerning decidability and their bearing on the metaphysical modal status of undecidable sentences.

According, finally, to Brandom's (op. cit.) modal expressivist approach, terms are individuated by their rules of inference, where the rules are taken to have a modal profile translatable into the counterfactual forms taken by the transition functions of automata (cf. Brandom, 2008: 142). In order to countenance the metasemantic truth-conditions for the object-level, pragmatic abilities captured by the automata's counterfactual transition states, Brandom augments a first-order language comprised of a stock of atomic formulas with an incompatibility function (141). An incompatibility function, I, is defined as the incoherence of the union of two sentences, where incoherence is a generalization of the notion of inconsistency to nonlogical vocabulary.

$\mathrm{x} \cup \mathrm{y} \in Inc \iff \mathrm{x} \in I(\mathrm{y})$ (141-142).

Incompatibility is supposed to be a modal notion, such that the union of the two sentences is incompossible (126). A sentence, β is an incompatibility-consequence, $\Vdash_I$, of a sentence, α, iff there is no sequence of sentences, $<\gamma_1, \ldots, \gamma_n>$, such that it can be the case that $\alpha \Vdash_I <\gamma_1, \ldots, \gamma_n>$, yet not be the case that $\beta \Vdash_I <\gamma_1, \ldots, \gamma_n>$ (125). To be incompatible with a necessary formula is to be compatible with everything that does not entail the formula (129-130). Dually, to be incompatible with a possible formula is to be incompatible with everything compatible with something compatible with the formula (op. cit.).

There are at least two, general issues for the application of Brandom's modal expressivism to the foundations of mathematics.

The first issue is that the mathematical vocabulary – e.g., the set-membership relation, $\in$ – is axiomatically defined. I.e., the membership relation is defined by, inter alia, the Pairing and Power Set axioms of set-theoretic languages. Thus, mathematical terms have their extensions individuated by the axioms of the language, rather than via a set of inference rules that can be specified in the absence of the mention of truth values. Even, furthermore, if one were to avail of modal notions in order to countenance the intensions of the mathematical vocabulary at issue – i.e., functions from terms in intensional contexts to their extensions – the modal profile of the intensions is orthogonal to the properties encoded by the incompatibility function. Fine (2006) avails, e.g., of interpretational modalities in order to countenance the possibility of reinterpreting quanitifier domains, and of thus accounting for variance in the cardinality of the domains of quantifier expressions. The interpretational possibilities are specified as operational conditions on tracking increases in the size of the cardinality of the universe. Uzquiano (2015) argues, as mentioned, that it is always possible to reinterpret the intensions of non-logical vocabulary, as one augments one's language with stronger axioms of infinity and climbs thereby farther up the cumulative hierarchy of sets. The reinterpretations of, e.g., the concept of set are effected by the addition of new large cardinal axioms, which stipulate the existence of larger inaccessible cardinals. However, it is unclear how the incompatibility function – i.e., a modal operator defined via Boolean negation and a generalized condition on inconsistency – might similarly be able to model the intensions pertaining to the ontological expansion of the cumulative hierarchy.

The second issue is that Brandom's inferential expressivist semantics is not compositional (Brandom, 2008: 135-136). While the formulas of the semantics are recursively formed – because the decomposition of complex formulas into atomic formulas is decidable[31] – formulas in the language are not compositional, because they fail to satisfy the subformula property to the effect that the value of a logically complex formula is calculated as a function of the values of the component logical connectives applied to subformulas therein (op. cit.).[32]

By contrast to the limits of Brandom's approach to modal expressivism, modal coalgebraic automata can circumvent both of the issues mentioned in the foregoing. In response to the first issue, concerning the axiomatic individuation and intensional profiles of mathematical terms, mappings of modal coalgebraic automata can be interpreted in order to provide a precise delineation of the intensions of the target vocabulary. In response, finally, to the second of the above issues, the values taken by modal coalgebraic automata are both decidable and computationally feasible, while the duality of coalgebras to Boolean-valued models of modal algebras ensures that the formulas therein retain their compositionality. The decidability of coalgebraic automata can further be witnessed by the role of modal coalgebras in countenancing the modal profile of Ω-logical consequence, where – given a proper class of Woodin cardinals – the values of mathematical formulas can remain invariant throughout extensions of the ground models comprising the set-theoretic universe (cf. Woodin, 2010; Author, 2019). The individuation of large cardinals can further be characterized by the functors of modal coalgebras, when the latter are interpreted so as to countenance the elementary embeddings constitutive of large cardinal axioms in the category of sets (Author, 2023).

9 Concluding Remarks

In this essay, I have endeavored to account for a mathematically tractable background against which to model both modal cognitivism and modal expressivism. I availed, to that end, of the duality between epistemic modal algebras and modal coalgebraic automata. Epistemic modal algebras were shown to com-

[31]Let a decision problem be a propositional function which is feasibly decidable, if it is a member of the polynomial time complexity class; i.e., if it can be calculated as a polynomial function of the size of the formula's input [see Dean (2015) for further discussion].

[32]Note that Incurvati and Schlöder (2020) advance a multilateral inferential expressivist semantics for epistemic modality which satisfies the subformula property. (Thanks here to Luca Incurvati.) Incurvati and Schlöder (2021) extend the semantics to normative vocabulary, but it is an open question whether their semantics is adequate for mathematical vocabulary as well.

prise a materially adequate fragment of the language of thought, given that models thereof figure in both cognitive psychology and artificial intelligence. With regard to conceptual engineering of intensions and hyperintensions, I introduced a novel topic-sensitive truthmaker semantics for dynamic epistemic logic as well as a novel dynamic epistemic two-dimensional hyperintensional semantics. It was then shown how the approach to modal expressivism here proffered, as regimented by the modal coalgebraic automata to which the epistemic modal algebras are dual, avoids the pitfalls attending to the competing modal expressivist approaches based upon both the inferentialist approach to concept-individuation and the approach to codifying the speech acts in natural language via intensional semantics. The present modal expressivist approach was shown, e.g., to avoid the limits of the foregoing in the philosophy of mathematics, as they concerned the status of necessary propositions; the inapplicability of inferentialist-individuation to mathematical vocabulary; and failures of compositionality.

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