Modal Ω-Logic*

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Abstract

This essay examines the philosophical significance of Ω-logic in Zermelo-Fraenkel set theory with choice (ZFC). The categorical duality between coalgebra and algebra permits Boolean-valued algebraic models of ZFC to be interpreted as coalgebras. The modal profile of Ω-logical validity can then be countenanced within a coalgebraic logic, and Ω-logical validity can be defined via deterministic automata. I argue that the philosophical significance of the foregoing is two-fold. First, because the epistemic and modal profiles of Ω-logical validity correspond to those of second-order logical consequence, Ω-logical validity is genuinely logical. Second, the foregoing provides a modal account of the interpretation of mathematical vocabulary.

1 Introduction

This essay examines the philosophical significance of the consequence relation defined in the Ω-logic for set-theoretic languages. I argue that, as with second-order logic, the modal profile of validity in Ω-Logic enables the property to be epistemically tractable. Because of the duality between coalgebras and algebras, Boolean-valued models of set theory can be interpreted as coalgebras. In Section **2**, I demonstrate how the modal profile of Ω-logical validity can be countenanced within a coalgebraic logic, and how Ω-logical validity can further be defined via automata. Finally, in Section **3**, the philosophical significance of the characterization of the modal profile of Ω-logical validity for the philosophy of mathematics is examined. I argue (i) that Ω-logical validity is genuinely logical, and (ii) that it provides a modal account of formal grasp of the concept of 'set'. Section **4** provides concluding remarks.

2 Definitions

In this section, I define the axioms of Zermelo-Fraenkel set theory with choice. I define the mathematical properties of the large cardinal axioms which can

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be adjoined to ZFC, and I provide a detailed characterization of the properties of Ω-logic for ZFC. Because coalgebras are dual to Boolean-valued algebraic models of Ω-logic, a category of coalgebraic logic is then characterized which models both modal logic and deterministic automata. Modal coalgebraic models of automata are then argued to provide a precise characterization of the modal and computational profiles of Ω-logical validity.

2.1 Axioms[1]

- Extensionality

  $\forall \mathrm{x,y}.(\forall \mathrm{z}.\mathrm{z}\in\mathrm{x} \iff \mathrm{z}\in\mathrm{y}) \rightarrow \mathrm{x} = \mathrm{y}$

- Empty Set

  $\exists \mathrm{x}.\forall \mathrm{y}.\mathrm{y}\notin\mathrm{x}$

- Pairing

  $\forall \mathrm{x,y}.\exists \mathrm{z}.\forall \mathrm{w}.\mathrm{w}\in\mathrm{z} \iff \mathrm{w} = \mathrm{x} \vee \mathrm{w} = \mathrm{y}$

- Union

  $\forall \mathrm{x}.\exists \mathrm{y}.\forall \mathrm{z}.\mathrm{z}\in\mathrm{y} \iff \exists \mathrm{w}.\mathrm{w}\in\mathrm{x} \wedge \mathrm{z}\in\mathrm{w}$

- Powerset

  $\forall \mathrm{x}.\exists \mathrm{y}.\forall \mathrm{z}.\mathrm{z}\in\mathrm{y} \iff \mathrm{z} \subseteq \mathrm{x}$

- Separation (with $\overrightarrow{x}$ a parameter)

  $\forall \overrightarrow{x},\mathrm{y}.\exists \mathrm{z}.\forall \mathrm{w}.\mathrm{w}\in\mathrm{z} \iff \mathrm{w}\in\mathrm{y} \wedge \mathrm{A}(\mathrm{w},\overrightarrow{x})$

- Infinity

  $\exists \mathrm{x}.\emptyset\in\mathrm{x} \wedge \forall \mathrm{y}.\mathrm{y}\in\mathrm{x} \rightarrow \mathrm{y} \cup \{\mathrm{y}\}\in\mathrm{x}$

- Foundation

  $\forall \mathrm{x}.(\exists \mathrm{y}.\mathrm{y}\in\mathrm{x}) \rightarrow \exists \mathrm{y}\in\mathrm{x}.\forall \mathrm{z}\in\mathrm{x}.\mathrm{z}\notin\mathrm{y}$

- Replacement

  $\forall \mathrm{x},\overrightarrow{y}.[\forall \mathrm{z}\in\mathrm{x}.\exists!\mathrm{w}.\mathrm{A}(\mathrm{z,w},\overrightarrow{y})] \rightarrow \exists \mathrm{u}.\forall \mathrm{w}.\mathrm{w}\in\mathrm{u} \iff \exists \mathrm{z}\in\mathrm{x}.\mathrm{A}(\mathrm{z,w},\overrightarrow{y})$

- Choice

  $\forall \mathrm{x}.\emptyset\notin\mathrm{x} \rightarrow \exists f\in(\mathrm{x} \rightarrow \cup\mathrm{x}).\forall \mathrm{y}\in\mathrm{x}.f(\mathrm{y})\in\mathrm{y}$

[1] For a standard presentation, see Jech (2003). The presentation here follows Avigad (2021). For detailed, historical discussion, see Maddy (1988,a).

2.2 Large Cardinals

Borel sets of reals are subsets of ω^ω or $\mathbb{R}$, closed under countable intersections and unions.[2] For all ordinals, a, such that $0 < a < \omega_1$, and $b < a$, Σ^0_a denotes the open subsets of ω^ω formed under countable unions of sets in Π^0_b, and Π^0_a denotes the closed subsets of ω^ω formed under countable intersections of Σ^0_b.

Projective sets of reals are subsets of ω^ω, formed by complementations (ω^ω – u, for u$\subseteq\omega^\omega$) and projections $[p(\mathrm{u}) = \{\langle \mathrm{x}_1, \ldots, \mathrm{x}_n\rangle \in \omega^\omega \mid \exists \mathrm{y}\langle \mathrm{x}_1, \ldots, \mathrm{x}_n, \mathrm{y}\rangle \in \mathrm{u}\}]$. For all ordinals a, such that $0 < a < \omega$, $\Pi^1{}_0$ denotes closed subsets of ω^ω; Π^1_a is formed by taking complements of the open subsets of ω^ω, Σ^1_a; and Σ^1_{a+1} is formed by taking projections of sets in Π^1_a.

The full power set operation defines the cumulative hierarchy of sets, V, such that $\mathrm{V}_0 = \varnothing$; $\mathrm{V}_{a+1} = \wp(\mathrm{V}_0)$; and $\mathrm{V}_\lambda = \bigcup_{a<\lambda}\mathrm{V}_a$.

In the inner model program (cf. Woodin, 2001, 2010, 2011; Kanamori, 2012,a,b), the definable power set operation defines the constructible universe, L($\mathbb{R}$), in the universe of sets V, where the sets are transitive such that a$\in$C $\iff$ a$\subseteq$C; $\mathrm{L}(\mathbb{R}) = \mathrm{V}_{\omega+1}$; $\mathrm{L}_{a+1}(\mathbb{R}) = \mathrm{Def}(\mathrm{L}_a(\mathbb{R}))$; and $\mathrm{L}_\lambda(\mathbb{R}) = \bigcup_{a<\lambda}(\mathrm{L}_a(\mathbb{R}))$.

Via inner models, Gödel (1940) proves the consistency of the generalized continuum hypothesis, ${\aleph_a}^{\aleph_a} = \aleph_{a+1}$, as well as the axiom of choice, relative to the axioms of ZFC. However, for a countable transitive set of ordinals, M, in a model of ZF without choice, one can define a generic set, G, such that, for all formulas, ϕ, either ϕ or $\neg\phi$ is forced by a condition, f, in G. Let $\mathrm{M[G]} = \bigcup_{a<\kappa}\mathrm{M}_a[\mathrm{G}]$, such that $\mathrm{M}_0[\mathrm{G}] = \{\mathrm{G}\}$; with $\lambda < \kappa$, $\mathrm{M}_\lambda[\mathrm{G}] = \bigcup_{a<\lambda}\mathrm{M}_a[\mathrm{G}]$; and $\mathrm{M}_{a+1}[\mathrm{G}] = \mathrm{V}_a \cap \mathrm{M}_a[\mathrm{G}]$.[3] G is a Cohen real over M, and comprises a set-forcing extension of M. The relation of set-forcing, $\Vdash$, can then be defined in the ground model, M, such that the forcing condition, f, is a function from a finite subset of ω into {0,1}, and $f \Vdash \mathrm{u}\in\mathrm{G}$ if $f(\mathrm{u}) = 1$ and $f \Vdash \mathrm{u}\notin\mathrm{G}$ if $f(\mathrm{u}) = 0$. The cardinalities of an open dense ground model, M, and a generic extension, G, are identical, only if the countable chain condition (c.c.c.) is satisfied, such that, given a chain – i.e., a linearly ordered subset of a partially ordered (reflexive, antisymmetric, transitive) set – there is a countable, maximal antichain consisting of pairwise incompatible forcing conditions. Via set-forcing extensions, Cohen (1963, 1964) constructs a model of ZF which negates the generalized continuum hypothesis, and thus proves the independence thereof relative to the axioms of ZF.[4]

Gödel (1946/1990: 1-2) proposes that the value of Orey sentences such as the GCH might yet be decidable, if one avails of stronger theories to which new axioms of infinity – i.e., large cardinal axioms – are adjoined.[5] He writes that: 'In set theory, e.g., the successive extensions can be represented by stronger and stronger axioms of infinity. It is certainly impossible to give a combinatorial

[2]See Koellner (2013), for the presentation, and for further discussion, of the definitions in this and the subsequent paragraph.

[3]See Kanamori (2012,a: 2.1; 2012,b: 4.1), for further discussion.

[4]See Kanamori (2008), for further discussion.

[5]See Kanamori (2007), for further discussion. Kanamori (op. cit.: 154) notes that Gödel (1931/1986: fn48a) makes a similar appeal to higher-order languages, in his proofs of the incompleteness theorems. The incompleteness theorems are examined in further detail, in Section **3.2**, below.

and decidable characterization of what an axiom of infinity is; but there might exist, e.g., a characterization of the following sort: An axiom of infinity is a proposition which has a certain (decidable) formal structure and which in addition is true. Such a concept of demonstrability might have the required closure property, i.e. the following could be true: Any proof for a set-theoretic theorem in the next higher system above set theory ... is replaceable by a proof from such an axiom of infinity. It is not impossible that for such a concept of demonstrability some completeness theorem would hold which would say that every proposition expressible in set theory is decidable from present axioms plus some true assertion about the largeness of the universe of sets'.

For cardinals, x,a,C, C$\subseteq$a is closed unbounded in a, if it is closed [if x $<$ C and $\bigcup$(C$\cap$a) = a, then a$\in$C] and unbounded ($\bigcup$C = a) (Kanamori, op. cit.: 360). A cardinal, S, is stationary in a, if, for any closed unbounded C$\subseteq$a, C$\cap$S $\neq \emptyset$ (op. cit.). An ideal is a subset of a set closed under countable unions, whereas filters are subsets closed under countable intersections (361). A cardinal κ is regular if the cofinality of κ – comprised of the unions of sets with cardinality less than κ – is identical to κ. Uncountable regular limit cardinals are weakly inaccessible (op. cit.). A strongly inaccessible cardinal is regular and has a strong limit, such that if $\lambda < \kappa$, then $2^{\lambda} < \kappa$ (op. cit.).

Large cardinal axioms are defined by elementary embeddings.[6] Elementary embeddings can be defined thus. For models A,B, and conditions ϕ, j: A $\rightarrow$ B, $\phi\langle a_1, \ldots, a_n\rangle$ in A if and only if $\phi\langle j(a_1), \ldots, j(a_n)\rangle$ in B (363). A measurable cardinal is defined as the ordinal denoted by the critical point of j, crit(j) (Koellner and Woodin, 2010: 7). Measurable cardinals are inaccessible (Kanamori, op. cit.).

Let κ be a cardinal, and $\eta > \kappa$ an ordinal. κ is then η-strong, if there is a transitive class M and an elementary embedding, j: V $\rightarrow$ M, such that crit(j) = κ, j(κ) $>\eta$, and $V_\eta \subseteq$M (Koellner and Woodin, op. cit.).

κ is strong if and only if, for all η, it is η-strong (op. cit.).

If A is a class, κ is η-A-strong, if there is a j: V $\rightarrow$ M, such that κ is η-strong and j(A$\cap V_\kappa$)$\cap V_\eta$ = A$\cap V_\eta$ (op. cit.).

κ is a Woodin cardinal, if κ is strongly inaccessible, and for all A$\subseteq V_\kappa$, there is a cardinal $\kappa_A < \kappa$, such that κ_A is η-A-strong, for all η such that κ_η, $\eta < \kappa$ (Koellner and Woodin, op. cit.: 8).

κ is superstrong, if j: V $\rightarrow$ M, such that crit(j) = κ and $V_{j(\kappa)} \subseteq$M, which entails that there are arbitrarily large Woodin cardinals below κ (op. cit.).

Large cardinal axioms can then be defined as follows.

$\exists$xΦ is a large cardinal axiom, because:

(i) Φx is a Σ_2-formula, where 'a sentence ϕ is a Σ_2-sentence if it is of the form: There exists an ordinal α such that $V_\alpha \Vdash \psi$, for some sentence ψ' (Woodin, 2019);

(ii) if κ is a cardinal, such that V $\models \Phi(\kappa)$, then κ is strongly inaccessible; and

[6] The definitions in the remainder of this subsection follow the presentations in Koellner and Woodin (2010) and Woodin (2010, 2011).

(iii) for all generic partial orders $\mathbb{P}\in V_\kappa$, $V^{\mathbb{P}} \models \Phi(\kappa)$; I_{NS} is a non-stationary ideal; A^G is the canonical representation of reals in L($\mathbb{R}$), i.e. the interpretation of A in M[G]; H(κ) is comprised of all of the sets whose transitive closure is < κ (cf. Woodin, 2001: 569); and $L(\mathbb{R})^{\mathbb{P}max} \models \langle H(\omega_2), \in, I_{NS}, A^G\rangle \models$ 'ϕ'. $\mathbb{P}$ is a homogeneous partial order in L($\mathbb{R}$), such that the generic extension of $L(\mathbb{R})^{\mathbb{P}}$ inherits the generic invariance, i.e., the absoluteness, of L($\mathbb{R}$). Thus, $L(\mathbb{R})^{\mathbb{P}max}$ is (i) effectively complete, i.e. invariant under set-forcing extensions; and (ii) maximal, i.e. satisfies all Π_2-sentences and is thus consistent by set-forcing over ground models (Woodin, ms: 28).

Assume ZFC and that there is a proper class of Woodin cardinals; $A\in\mathbb{P}(\mathbb{R}) \cap L(\mathbb{R})$; ϕ is a Π_2-sentence; and V(G), s.t. $\langle H(\omega_2), \in, I_{NS}, A^G\rangle \models$ 'ϕ': Then, it can be proven that $L(\mathbb{R})^{\mathbb{P}max} \models \langle H(\omega_2), \in, I_{NS}, A^G\rangle \models$ 'ϕ', where 'ϕ' := $\exists A\in\Gamma^\infty\langle H(\omega_1), \in, A\rangle \models \psi$.

The axiom of determinacy (AD) states that every set of reals, $a\subseteq\omega^\omega$ is determined.

Woodin's (1999) Axiom (*) can be thus countenanced:

$AD^{L(\mathbb{R})}$ and $L[(\mathbb{P}\omega_1)]$ is a $\mathbb{P}$max-generic extension of L($\mathbb{R}$),

from which it can be derived that $2^{\aleph_0} = \aleph_2$. Thus, ¬CH; and so CH is absolutely decidable.

In more recent work, Woodin (2019) provides evidence that CH might, by contrast, be true. The truth of CH would follow from the truth of Woodin's Ultimate-L conjecture. The following definitions are from Woodin (op. cit.): 'A transitive class is an inner model if[, for the class of ordinals Ord, - HK] Ord $\subset$ M, and M $\Vdash$ ZFC'. L, the constructible reals, and HOD, the hereditarily ordinal definable sets, are inner models. 'Suppose N is an inner model and that $[a]$ is an uncountable (regular) cardinal of V. N has the $[a]$-cover property if for all $\sigma \subset$ N, if $|\sigma| < [a]$ then there exists $\tau\in$N such that: $\sigma \subset \tau$ and $|\tau| < [a]$. N has the $[a]$-approximation property if for all sets X $\subset$ N, the following are equivalent: (i) X$\in$N and (ii) For all $\sigma\in$N, if $|\sigma| < [a]$, then $\sigma \cap$ X$\in$N. Suppose N is an inner model and that $\sigma \subset$ N. Then N[σ] denotes the smallest inner model M such that N $\subseteq$ M and $\sigma\in$M. Suppose that N is an inner model and $[a]$ is strongly inaccessible. Then N has the $[a]$-genericity property if for all $\sigma \subseteq [a]$, if $|\sigma| < [a]$then N[σ] $\cap$ V_a is a Cohen extension of N $\cap$ V_a. The axiom for V = Ultimate-L states then that '(i) There is a proper class of Woodin cardinals, and (ii) For each Σ_2-sentence ϕ, if ϕ holds in V then there is a universally Baire set A $\subseteq \mathbb{R}$ such that $HOD^{L(A,\mathbb{R})} \Vdash \phi$, where a set is universally Baire if for all topological spaces Ω and for all continuous functions $\pi : \Omega \rightarrow \mathbb{R}^n$, the preimage of A by π has the property of Baire in the space Ω'. The property of Baire holds if, for a subset of a topological space A $\subseteq$ X, there is an open set U $\subset$ X such that A Ξ U is a meagre subset, where Ξ is the symmetric difference, i.e. the union of relative complements, and a subset of a topological space is meagre if it is a countable union of nowhere dense sets, where nowhere dense subsets of the topology hold if their union with an open set is not dense.[7] The Ultimate-L

[7]https://en.wikipedia.org/wiki/PropertyofBaire, https://en.wikipedia.org/wiki/Symmetricdifference, https://en.wikipedia.org/wiki/Meagreset.

Conjecture is then as follows: 'Suppose that [a] is an extendible cardinal. [a] is an extendible cardinal if for each $\lambda >$ [a] there exists an elementary embedding $j : V_{\lambda+1} \rightarrow V_{j(\lambda)+1}$ such that CRT(j) = [a] and j([a]) $> \lambda$. Then provably there is an inner model N such that: 1. N has the [a]-cover and [a]-approximation properties. 2. N has the [a]-genericity property. 3. N $\Vdash$ 'V = Ultimate-L" (Woodin, op. cit.).

2.3 Ω-Logic

For partial orders, $\mathbb{P}$, let $V^{\mathbb{P}} = V^{\mathbb{B}}$, where $\mathbb{B}$ is the regular open completion of ($\mathbb{P}$).[8] $M_a = (V_a)^M$ and $M^{\mathbb{B}}_a = (V^{\mathbb{B}}_a)^M = (V^{M^{\mathbb{B}}}_a)$. *Sent* denotes a set of sentences in a first-order language of set theory. T$\cup\{\phi\}$ is a set of sentences extending ZFC. *c.t.m* abbreviates the notion of a countable transitive $\in$-model. *c.B.a.* abbreviates the notion of a complete Boolean algebra.

Define a *c.B.a.* in V, such that $V^{\mathbb{B}}$. Let $V^{\mathbb{B}}_0 = \emptyset$; $V^{\mathbb{B}}_\lambda = \bigcup_{b<\lambda} V^{\mathbb{B}}_b$, with λ a limit ordinal; $V^{\mathbb{B}}_{a+1} = \{f\colon X \rightarrow \mathbb{B} \mid X \subseteq V^{\mathbb{B}}_a\}$; and $V^{\mathbb{B}} = \bigcup_{a \in On} V^{\mathbb{B}}_a$.

ϕ is true in $V^{\mathbb{B}}$, if its Boolean-value is $1^{\mathbb{B}}$, if and only if

$V^{\mathbb{B}} \models \phi$ iff $[\![\phi]\!]^{\mathbb{B}} = 1^{\mathbb{B}}$.

Thus, for all ordinals, a, and every *c.B.a.* $\mathbb{B}$, $V^{\mathbb{B}}_a \equiv (V_a)^{V^{\mathbb{B}}}$ iff for all x$\in V^{\mathbb{B}}$, $\exists$y$\in V^{\mathbb{B}} [\![x = y]\!]^{\mathbb{B}} = 1^{\mathbb{B}}$ iff $[\![x \in V^{\mathbb{B}}]\!]^{\mathbb{B}} = 1^{\mathbb{B}}$.

Then, ${V^{\mathbb{B}}}_a \models \phi$ iff $V^{\mathbb{B}} \models$ '$V_a \models \phi$'.

Ω-logical validity can then be defined as follows:

For T$\cup\{\phi\}\subseteq$*Sent*,

T $\models_\Omega \phi$, if for all ordinals, a, and *c.B.a.* $\mathbb{B}$, if $V^{\mathbb{B}}_a \models$ T, then $V^{\mathbb{B}}_a \models \phi$.

Supposing that there exists a proper class of Woodin cardinals and if T$\cup\{\phi\}\subseteq$*Sent*, then for all set-forcing conditions, $\mathbb{P}$:

T $\models_\Omega \phi$ iff $V^T \models$ 'T $\models_\Omega \phi$',

where T $\models_\Omega \phi \equiv \emptyset \models$ 'T $\models_\Omega \phi$'.

The Ω-Conjecture states that V $\models_\Omega \phi$ iff $V^{\mathbb{B}} \models_\Omega \phi$ (Woodin, ms). Thus, Ω-logical validity is invariant in all set-forcing extensions of ground models in the set-theoretic universe.

The soundness of Ω-Logic is defined by universally Baire sets of reals. For a cardinal, e, let a set A be e-universally Baire, if for all partial orders $\mathbb{P}$ of cardinality e, there exist trees, S and T on ω X λ, such that A = p[T] and if G$\subseteq\mathbb{P}$ is generic, then $p[T]^G = \mathbb{R}^G - p[S]^G$ (Koellner, 2013). A is universally Baire, if it is e-universally Baire for all e (op. cit.).

Ω-Logic is sound, such that V $\vdash_\Omega \phi \rightarrow$ V $\models_\Omega \phi$. However, the completeness of Ω-Logic has yet to be resolved.

Finally, in category theory, a category C is comprised of a class Ob(C) of objects a family of arrows for each pair of objects C(A,B) (Venema, 2007: 421). A functor from a category C to a category D, **E**: C $\rightarrow$ D, is an operation mapping objects and arrows of C to objects and arrows of D (422). An endofunctor on C is a functor, **E**: C $\rightarrow$ C (op. cit.).

[8]The definitions in this section follow the presentation in Bagaria et al. (2006).

A $\mathbf{E}$-coalgebra is a pair $\mathbb{A} = (A, \mu)$, with A an object of C referred to as the carrier of $\mathbb{A}$, and μ: A $\rightarrow$ $\mathbf{E}$(A) is an arrow in C, referred to as the transition map of $\mathbb{A}$ (390).

$\mathbb{A} = \langle A, \mu\text{: } A \rightarrow \mathbf{E}(A)\rangle$ is dual to the category of algebras over the functor μ (417-418). If μ is a functor on categories of sets, then coalgebraic models are dual to Boolean-algebraic models of Ω-logical validity.

The significance of the foregoing is that coalgebraic models may themselves be availed of in order to define modal logic and automata. Coalgebras provide therefore a setting in which the Boolean-valued models of set theory, the modal profile of Ω-logical validity, and automata can be interdefined. In what follows, $\mathbb{A}$ will comprise the coalgebraic model – dual to the complete Boolean-valued algebras defined in the Ω-Logic of ZFC – in which modal similarity types and automata are definable. As a coalgebraic model of modal logic, $\mathbb{A}$ can be defined as follows (407):

For a set of formulas, Φ, let $\nabla\Phi := \Box \bigvee \Phi \wedge \bigwedge \diamond\Phi$, where $\diamond\Phi$ denotes the set $\{\diamond\phi \mid \phi\in\Phi$ (op. cit.). Then,

$\diamond\phi \equiv \nabla\{\phi, T\}$,

$\Box\phi \equiv \nabla\varnothing \vee \nabla\phi$ (op. cit.)

$[\![\nabla\Phi]\!] = \{w\in W \mid R[w] \subseteq \bigcup \{[\![\phi]\!] \mid \phi\in\Phi\}$ and $\forall\phi\in\Phi$, $[\![\phi]\!] \cap R[w] \neq \emptyset\}$ (Fontaine, 2010: 17).

Let an $\mathbf{E}$-coalgebraic modal model, $\mathbb{A} = \langle S,\lambda,R[.]\rangle$, where λ(s) is 'the collection of proposition letters true at s in S, and R[s] is the successor set of s in S', such that $\mathbb{S}$,s $\Vdash \nabla\Phi$ if and only if, for all (some) successors σ of s$\in$S, $[\Phi,\sigma(s)\in\mathbf{E}(\Vdash_{\mathbb{A}})]$ (Venema, 2007: 399, 407), with $\mathbf{E}(\Vdash_{\mathbb{A}})$ a relation lifting of the satisfaction relation $\Vdash_{\mathbb{A}} \subseteq$ S x Φ. Let a functor, $\mathbf{K}$, be such that there is a relation $\overline{\mathbf{K}} \subseteq \mathbf{K}$(A) x $\mathbf{K}$(A') (Venema, 2012: 17)). Let Z be a binary relation s.t. Z $\subseteq$ A x A' and $\wp\overline{Z} \subseteq \wp$(A) x $\wp$(A'), with

$\wp\overline{Z}$:= {(X,X') | $\forall$x$\in$X$\exists$x'$\in$X' with (x,x')$\in$Z $\wedge$ $\forall$x'$\in$X'$\exists$x$\in$X with (x,x')$\in$Z} (op. cit.). Then, we can define the relation lifting, **K!**, as follows:

$\overline{\mathbf{K}}$:= {[(π,X), (π',X')] | π = π' and (X,X')$\in\wp\overline{Z}$} (Venema, 2012: 17).[9]

A coalgebraic model of deterministic automata can be thus defined (Venema, 2007: 391). An automaton is a tuple, $\mathbb{A} = \langle A, a_I, C, \Rightarrow, F\rangle$, such that A is the state space of the automaton $\mathbb{A}$; $a_I\in A$ is the automaton's initial state; C is the coding for the automaton's alphabet, mapping numerals to properties of the natural numbers; $\Rightarrow$: A X C $\rightarrow$ A is a transition function, and F$\subseteq$ A is the collection of admissible states, where F maps A to {1,0}, such that F: A $\rightarrow$ 1 if a$\in$F and A $\rightarrow$ 0 if a$\notin$F (op. cit.). The determinacy of coalgebraic automata, the category of which is dual to the Set category satisfying Ω-logical consequence, is secured by the existence of Woodin cardinals: Assuming ZFC, that λ is a limit of Woodin cardinals, that there is a generic, set-forcing extension G $\subseteq$ the collapse of $\omega < \lambda$, and that $\mathbb{R}^* = \bigcup\{\mathbb{R}^G[a] \mid a < \lambda\}$, then $\mathbb{R}^* \models$ the axiom of determinacy (AD) (Koellner and Woodin, op. cit.: 10).

[9] The projections of a relation R, with R a relation between two sets X and Y such that R $\subseteq$ X x Y, are
X $\longleftarrow(\pi_1)$ R $(\pi_2)\longrightarrow$ Y such that π_1((x,y)) = x, and π_2((x,y)) = y. See Rutten (2019: 240).

Modal automata are defined over a modal one-step language (Venema, 2020: 7.2). With A being a set of propositional variables the set, $\texttt{Latt}(X)$, of lattice terms over X has the following grammar:

$$\pi ::= \bot \mid \top \mid x \mid \pi \wedge \pi \mid \pi \vee \pi,$$

with x∈X and $\pi \in \texttt{Latt}(A)$ (op. cit.).

The set, $\texttt{1ML}(A)$, of modal one-step formulas over A has the following grammar:

$$\alpha \in A ::= \bot \mid \top \mid \diamond\pi \mid \Box\pi \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \text{ (op. cit.).}$$

A modal P-automaton $\mathbb{A}$ is a triple, (A, Θ, a_I), with A a non-empty finite set of states, $a_I \in A$ an initial state, and the transition map

Θ: A x $\wp$P $\rightarrow$ $\texttt{1ML}(A)$

maps states to modal one-step formulas, with $\wp$P the powerset of the set of proposition letters, P (op. cit.: 7.3).

Finally, $\mathbb{A} = \langle A, \alpha{:}A \rightarrow \mathbf{E}(A)\rangle$ is dual to the category of algebras over the functor α (417-418). For a category C, object A, and endofunctor **E**, define a new arrow, α, s.t. $\alpha{:}\mathbf{E}A \rightarrow A$. A homomorphism, f, can further be defined between algebras $\langle A, \alpha\rangle$, and $\langle B, \beta\rangle$. Then, for the category of algebras, the following commutative square can be defined: (i) $\mathbf{E}A \rightarrow \mathbf{E}B$ ($\mathbf{E}f$); (ii) $\mathbf{E}A \rightarrow A$ (α); (iii) $\mathbf{E}B \rightarrow B$ (β); and (iv) $A \rightarrow B$ (f) (cf. Hughes, 2001: 7-8). The same commutative square holds for the category of coalgebras, such that the latter are defined by inverting the direction of the morphisms in both (ii) [$A \rightarrow \mathbf{E}A$ (α)], and (iii) [$B \rightarrow \mathbf{E}B$ (β)] (op. cit.).

Thus, $\mathbb{A}$ is the coalgebraic category for modal, deterministic automata, dual to the complete Boolean-valued algebraic models of Ω-logical validity, as defined in the category of sets.

Leach-Krouse (ms) defines the modal logic of Ω-consequence as satisfying the following axioms:

For a theory **T** and with $\Box\phi := \mathbf{T}^{\mathbb{B}}\alpha \Vdash \text{ZFC} \Rightarrow \mathbf{T}^{\mathbb{B}}\alpha \Vdash \phi$,

$\text{ZFC} \vdash \phi \Rightarrow \text{ZFC} \vdash \Box\phi$

$\text{ZFC} \vdash \Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$

$\text{ZFC} \vdash \Box\phi \rightarrow \phi \Rightarrow \text{ZFC} \vdash \phi$

$\text{ZFC} \vdash \Box\phi \rightarrow \Box\Box\phi$

$\text{ZFC} \vdash \Box(\Box\phi \rightarrow \phi) \rightarrow \Box\phi$

$\Box(\Box\phi \rightarrow \psi) \vee \Box(\Box\psi \wedge \psi \rightarrow \phi)$, where this clause added to GL is the logic of 'true in all V_κ for all κ strongly inaccessible' in ZFC.

3 Discussion

This section examines the philosophical significance of modal coalgebraic automata and the Boolean-valued models of set-theoretic languages to which they are dual. I argue that, similarly to second-order logical consequence, (i) the 'mathematical entanglement' of Ω-logical validity does not undermine its status as a relation of pure logic; and (ii) both the modal profile and model-theoretic characterization of Ω-logical consequence provide a guide to its epistemic tractability.[10] I argue, then, that there are several considerations adducing in favor of the claim that the interpretation of the concept of set constitutively involves modal notions. The role of the category of modal coalegebraic deterministic automata in (i) characterizing the modal profile of Ω-logical consequence, and (ii) being constitutive of the formal understanding-conditions for the concept of set, provides, then, support for a realist conception of the cumulative hierarchy.

3.1 Ω-Logical Validity is Genuinely Logical

Frege's (1884/1980; 1893/2013) proposal – that cardinal numbers can be explained by specifying a biconditional between the identity of, and an equivalence relation on, concepts, expressible in the signature of second-order logic – is the first attempt to provide a foundation for mathematics on the basis of logical axioms rather than rational or empirical intuition. In Frege (1884/1980. cit.: 68) and Wright (1983: 104-105), the number of the concept, **A**, is argued to be identical to the number of the concept, **B**, if and only if there is a one-to-one correspondence between **A** and **B**, i.e., there is a bijective mapping, R, from **A** to **B**. With Nx: a numerical term-forming operator,

- $\forall\mathbf{A}\forall\mathbf{B}[\mathrm{Nx}{:}\ \mathbf{A} = \mathrm{Nx}{:}\ \mathbf{B} \equiv \exists\mathrm{R}[\forall\mathrm{x}[\mathbf{A}\mathrm{x} \to \exists\mathrm{y}(\mathbf{B}\mathrm{y} \wedge \mathrm{Rxy} \wedge \forall\mathrm{z}(\mathbf{B}\mathrm{z} \wedge \mathrm{Rxz} \to \mathrm{y} = \mathrm{z}))] \wedge \forall\mathrm{y}[\mathbf{B}\mathrm{y} \to \exists\mathrm{x}(\mathbf{A}\mathrm{x} \wedge \mathrm{Rxy} \wedge \forall\mathrm{z}(\mathbf{A}\mathrm{z} \wedge \mathrm{Rzy} \to \mathrm{x} = \mathrm{z}))]]]$.

Frege's Theorem states that the Dedekind-Peano axioms for the language of arithmetic can be derived from the foregoing abstraction principle, as augmented to the signature of second-order logic and identity.[11] Thus, if second-order logic may be counted as pure logic, despite that domains of second-order models are definable via power set operations, then one aspect of the philosophical significance of the abstractionist program consists in its provision of a foundation for classical mathematics on the basis of pure logic as augmented with non-logical implicit definitions expressed by abstraction principles.

There are at least three reasons for which a logic defined in ZFC might not undermine the status of its consequence relation as being logical. The first reason for which the mathematical entanglement of Ω-logical validity might be innocuous is that, as Shapiro (1991: 5.1.4) notes, many mathematical properties cannot be defined within first-order logic, and instead require the expressive resources of second-order logic. For example, the notion of well-foundedness cannot be expressed in a first-order framework, as evinced by considerations of compactness. Let E be a binary relation. Let m be a well-founded model, if

[10] The phrase, 'mathematical entanglement', is owing to Koellner (2010: 2).

[11] Cf. Dedekind (1888/1963) and Peano (1889/1967). See Wright (1983: 154-169) for a proof sketch of Frege's theorem; Boolos (1987) for the formal proof thereof; and Parsons (1964) for an incipient conjecture of the theorem's validity.

there is no infinite sequence, $a_0, \ldots, a_i$, such that $Ea_0, \ldots, Ea_{i+1}$ are all true. If m is well-founded, then there are no infinite-descending E-chains. Suppose that T is a first-order theory containing m, and that, for all natural numbers, n, there is a T with $n + 1$ elements, $a_0, \ldots, a_n$, such that $\langle a_0, a_1 \rangle, \ldots, \langle a_n, a_{n-1} \rangle$ are in the extension of E. By compactness, there is an infinite sequence such that that $a_0 \ldots a_i$, s.t. $Ea_0, \ldots, Ea_{i+1}$ are all true. So, m is not well-founded.

By contrast, however, well-foundedness can be expressed in a second-order framework:

$\forall X[\exists xXx \to \exists x[Xx \wedge \forall y(Xy \to \neg Eyx)]]$, such that m is well-founded iff every non-empty subset X has an element x, s.t. nothing in X bears E to x.

One aspect of the philosophical significance of well-foundedness is that it provides a distinctively second-order constraint on when the membership relation in a given model is intended. This contrasts with Putnam's (1980) claim, that first-order models *mod* can be intended, if every set s of reals in *mod* is such that an ω-model in *mod* contains s and is constructible, such that – given the Downward Lowenheim-Skolem theorem[12] – if *mod* is non-constructible but has a submodel satisfying 's is constructible', then the model is non-well-founded and yet must be intended. The claim depends on the assumption that general understanding-conditions and conditions on intendedness must be co-extensive, to which I will return in Section **4.2**

A second reason for which Ω-logic's mathematical entanglement might not be pernicious, such that the consequence relation specified in the Ω-logic might be genuinely logical, may again be appreciated by its comparison with second-order logic. Shapiro (1998) defines the model-theoretic characterization of logical consequence as follows:

'(10) Φ is a logical consequence of [a model] Γ if Φ holds in all possibilities under every interpretation of the nonlogical terminology which holds in Γ' (148).

A condition on the foregoing is referred to as the 'isomorphism property', according to which 'if two models M, M' are isomorphic vis-a-vis the nonlogical items in a formula Φ, then M satisfies Φ if and only if M' satisfies Φ' (151).

Shapiro argues, then, that the consequence relation specified using second-order resources is logical, because of its modal and epistemic profiles. The epistemic tractability of second-order validity consists in 'typical soundness theorems, where one shows that a given deductive system is truth-preserving' (154). He writes that: '[I]f we know that a model is a good mathematical model of logical consequence (10), then we know that we won't go wrong using a sound deductive system. Also, we can know that an argument is a logical consequence . . . via a set-theoretic proof in the metatheory' (154-155).

The modal profile of second-order validity provides a second means of accounting for the property's epistemic tractability. Shapiro argues, e.g., that: 'If the isomorphism property holds, then in evaluating sentences and arguments, the only 'possibility' we need to 'vary' is the size of the universe. If enough sizes

[12]The Downward Lowenheim-Skolem theorem claims that for any first-order model M, M has a submodel M' whose domain is at most denumerably infinite, s.t. for all assignments s on, and formulas $\phi(x)$ in, M', M,$s \Vdash \phi(x) \iff M'$,$s \Vdash \phi(x)$.

are represented in the universe of models, then the modal nature of logical consequence will be registered ... [T]he only 'modality' we keep is 'possible size', which is relegated to the set-theoretic metatheory' (152).

Shapiro's remarks about the considerations adducing in favor of the logicality of non-effective, second-order validity generalize to Ω-logical validity. In the previous section, the modal profile of Ω-logical validity was codified by the duality between the category, $\mathbb{A}$, of coalgebraic modal logics and complete Boolean-valued algebraic models of Ω-logic. As with Shapiro's definition of logical consequence, where Φ holds in all possibilities in the universe of models and the possibilities concern the 'possible size' in the set-theoretic metatheory, the Ω-Conjecture states that $V \models_{\Omega} \phi$ iff $V^{\mathbb{B}} \models_{\Omega} \phi$, such that Ω-logical validity is invariant in all set-forcing extensions of ground models in the set-theoretic universe.

Finally, the epistemic tractability of Ω-logical validity is secured, both – as on Shapiro's account of second-order logical consequence – by its soundness, but also by its being the dual of coalgebraic category of deterministic automata, where the determinacy thereof is again secured by the existence of Woodin cardinals.

3.2 Intensionality and the Concept of Set

In this section, I argue, finally, that the modal profile of Ω-logic can be availed of in order to account for the understanding-conditions of the concept of set.

Putnam (op. cit.: 473-474) argues that defining models of first-order theories is sufficient for both understanding and specifying an intended interpretation of the latter. Wright (1985: 124-125) argues, by contrast, that understanding-conditions for mathematical concepts cannot be exhausted by the axioms for the theories thereof, even on the intended interpretations of the theories. He suggests, e.g., that:

'[I]f there really were uncountable sets, their existence would surely have to flow from the concept of set, as intuitively satisfactorily explained. Here, there is, as it seems to me, no assumption that the content of the ZF-axioms cannot exceed what is invariant under all their classical models. [Benacerraf] writes, e.g., that: 'It is granted that they are to have their 'intended interpretation': '∈' is to mean set-membership. Even so, and conceived as encoding the intuitive concept of set, they fail to entail the existence of uncountable sets. So how can it be true that there are such sets? Benacerraf's reply is that the ZF-axioms are indeed faithful to the relevant informal notions only if, in addition to ensuring that '∈' means set-membership, we interpret them so as to observe the constraint that 'the universal quantifier has to mean all or at least all sets' (p. 103). It follows, of course, that if the concept of set does determine a background against which Cantor's theorem, under its intended interpretation, is sound, there is more to the concept of set that can be explained by communication of the intended sense of '∈' and the stipulation that the ZF-axioms are to hold. And the residue is contained, presumably, in the informal explanations to which, Benacerraf reminds us, Zermelo intended his formalization to answer. At least, this must be so if the 'intuitive concept of set' is capable of being explained at all. Yet it is notable that Benacerraf nowhere ventures to supply the missing informal explanation – the story which will pack enough into the extension of 'all sets' to yield Cantor's theorem, under its intended interpretation, as a highly non-trivial corollary' (op. cit).

In order to provide the foregoing explanation in virtue of which the concept of set can be shown to be associated with a realistic notion of the cumulative hierarchy, I will argue that there are several points in the model theory and epistemology of set-theoretic languages at which the interpretation of the concept of set constitutively involves modal notions. The intensionality at issue is consistent with realist positions with regard to both truth values and the ontology of abstracta.[13]

One point is in the coding of the signature of the theory, T, in which Gödel's incompleteness theorems are proved (cf. Halbach and Visser, 2014). Gödel's incompleteness theorems can be thus outlined.[14] 'A numeral canonically denoting a natural number **n** is abbreviated as $\overline{n}$. A formalized theory F is ω-consistent if it is not the case that for some formula A(x), both F $\vdash \neg$A($\overline{n}$) for all **n**, and F $\vdash \exists$xA(x). A set S of natural numbers is strongly representable in F if there is a formula A(x) of the language of F with one free variable x such that for every natural number **n**:

'$\mathbf{n} \in \mathrm{S} \Rightarrow \mathrm{F} \vdash \mathrm{A}(\overline{n})$;

'$\mathbf{n} \notin \mathrm{S} \Rightarrow \mathrm{F} \vdash \neg A(\overline{n})$.

'A set S of natural numbers is weakly representable in F if there is a formula A(x) of the language of F such that for every natural number **n**:

'$\mathbf{n} \in \mathrm{S} \iff \mathrm{F} \vdash \mathrm{A}(\overline{n})$.

'The representability theorem says then that in any consistent formal system which contains Robinson Arithmetic i.e. **Q**[15]:

'1. A set (or relation) is strongly representable if and only if it is recursive;

'2. A set (or relation) is weakly representable if and only if it is recursively enumerable.

'Suppose that there is a coding of symbols and formulas by the natural numbers. The Gödel number of a formula A is denoted as $\ulcorner$A$\urcorner$.

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[13] For the modal commitments of the abstractionist foundations of mathematics, i.e. necessitism, see Khudairi (ms).

[14] The presentation follows that of Raatikainen (2022). I will quote the entire text, because the definitions and characterizations are mostly owing to Raatikainen.

[15] The signature of **Q** is first-order Peano Arithmetic without the induction schema, with 0 a constant for zero, a unary function symbol s for successor, and binary function symbols + and • for addition and multiplication. The axioms of **Q** are:

1. $\forall_x \neg$s(x) = 0
2. $\forall_{x,y}$s(x) = s(y) $\rightarrow$ x = y
3. $\forall_x$x = 0 $\vee$ $\exists_y$x = s(y)
4. $\forall_x$x + 0 = x
5. $\forall_{x,y}$x + s(y) = s(x + y)
6. $\forall_x$X • 0 = 0
7. $\forall_{x,y}$x • s(y) = x • y + x

(https://ncatlab.org/nlab/show/Robinson+arithmetic).

‘Suppose that the diagonalization lemma holds, such that F $\vdash$ Q $\iff$ A($\ulcorner$Q$\urcorner$).

‘For the first incompleteness theorem, the diagonalization lemma is applied to the negation of the provability predicate, $\neg\mathrm{Prov}_F(\mathrm{x})$, which yields the following sentence:

‘(Z) F $\vdash \mathrm{M}_F \iff \neg\mathrm{Prov}_F(\ulcorner \mathrm{M}_F \urcorner)$.

‘Assume that M_F is provable. By the weak representability of provability-in-F by $\mathrm{Prov}_F(\mathrm{x})$, F would also prove $\mathrm{Prov}_F(\mathrm{M}_F)$. Because F proves Z – i.e. F $\vdash \mathrm{M}_F \iff \neg\mathrm{Prov}_F(\ulcorner \mathrm{M}_F \urcorner)$ – F would then prove $\neg\mathrm{M}_F$. So F would be inconsistent. Thus, if F is consistent, then M_F is not provable in F.

‘Assume that F is ω-consistent. Assume, then, that F $\vdash \neg\mathrm{M}_F$. Then F cannot prove M_F, because it would then be ω-inconsistent. Thus no natural number **n** is the Gödel number of a proof of M_F. Because the proof relation is strongly representable, for all **n**, F $\vdash \neg\mathrm{Prf}_F(\overline{n}, \ulcorner \mathrm{M}_F \urcorner)$. If F $\vdash \exists \mathrm{x}\mathrm{Prf}_F(\mathrm{x}, \ulcorner \mathrm{M}_F \urcorner)$, F is not ω-consistent. Thus F does not prove $\exists \mathrm{x}\mathrm{Prf}_F(\mathrm{x}, \ulcorner \mathrm{M}_F \urcorner)$, i.e. F does not prove $\mathrm{Prov}_F(\ulcorner \mathrm{M}_F \urcorner)$. By the equivalence recorded in (Z), F does not prove $\neg\mathrm{M}_F$.

‘For the second incompleteness theorem: Suppose that consistency, Con(F), is defined as $\neg\mathrm{Prov}_F(\ulcorner \perp \urcorner)$, where $\perp$ expresses an inconsistent formula such as $\overline{0} = \overline{1}$. Formalizing the proof of the first incompleteness theorem in F yields F $\vdash$ Cons(F) $\rightarrow \mathrm{M}_F$. If Cons(F) were provable in F, so would be M_F. Suppose that F $\vdash \mathrm{M}_F \iff$ Cons(F). Cons(F) is thus unprovable, given the first incompleteness theorem.’

In the foregoing, the choice of coding bridges the numerals in the language with the properties of the target numbers. The choice of coding is therefore intensional, and has been marshalled in order to argue that the very notion of syntactic computability – via the equivalence class of partial recursive functions, λ-definable terms, and the transition functions of discrete-state automata such as Turing machines – is constitutively semantic (cf. Rescorla, 2015). Further points at which intensionality can be witnessed in the phenomenon of self-reference in arithmetic are introduced by Reinhardt (1986). Reinhardt (op. cit.: 470-472) argues that the provability predicate can be defined relative to the minds of particular agents – similarly to Quine’s (1968) and Lewis’ (1979) suggestion that possible worlds can be centered by defining them relative to parameters ranging over tuples of spacetime coordinates or agents and locations – and that a theoretical identity statement can be established for the concept of the foregoing minds and the concept of a computable system.

A second point at which understanding-conditions may be shown to be constitutively modal can be witnessed by the conditions on the epistemic entitlement to assume that the language in which Gödel’s second incompleteness theorem is proved is consistent (cf. Dummett, 1963/1978; Wright, 1985). Wright (op. cit.: 91, fn.9) suggests that ‘[T]o treat [a] proof as establishing consistency is implicitly to exclude any doubt . . . about the consistency of first-order number theory’. Wright’s elaboration of the notion of epistemic entitlement, appeals to a notion of rational ‘trust’, which he argues is recorded by the calculation of ‘expected epistemic utility’ in the setting of decision theory (2004; 2014: 226, 241). Wright notes that the rational trust subserving epistemic entitlement will be pragmatic, and makes the intriguing point that 'pragmatic reasons are not a special genre of reason, to be contrasted with e.g. epistemic, prudential, and moral reasons' (2012: 484). Crucially, however, the very idea of expected epistemic utility in the setting of decision theory makes implicit appeal to the notion of possible worlds, where the latter can again be determined by the coalgebraic logic for modal automata.

A third consideration adducing in favor of the thought that grasp of the concept of set might constitutively possess a modal profile is that the concept can be defined as an intension – i.e., a function from possible worlds to extensions. The modal similarity types in the coalgebraic modal logic may then be interpreted as dynamic-interpretational modalities, where the dynamic-interpretational modal operator has been argued to entrain the possible reinterpretations both of the domains of the theory's quantifiers (cf. Fine, 2005, 2006), as well as of the intensions of non-logical concepts, such as the membership relation (cf. Uzquiano, 2015).[16]

The fourth consideration avails directly of the modal profile of Ω-logical consequence. While the above dynamic-interpretational modality will suffice for possible reinterpretations of mathematical terms, the absoluteness of the consequence relation is such that, if the Ω-conjecture is true, then Ω-logical validity is invariant in all possible set-forcing extensions of ground models in the set-theoretic universe. The truth of the Ω-conjecture would thereby place an indefeasible necessary condition on a formal understanding of the intension for the concept of set.

4 Concluding Remarks

In this essay I have examined the philosophical significance of the duality between modal coalgebraic models of automata and Boolean-valued algebraic models of modal Ω-logic. I argued that – as with the property of validity in second-order logic – Ω-logical validity is genuinely logical. I argued, then, that modal coalegebraic deterministic automata, which characterize the modal profile of Ω-logical consequence, are constitutive of the interpretation of mathematical concepts such as the membership relation.

[16]For an examination of the philosophical significance of modal coalgebraic automata beyond the philosophy of mathematics, see Baltag (2003). Baltag (op. cit.) proffers a colagebraic semantics for dynamic-epistemic logic, where coalgebraic functors are intended to record the informational dynamics of single- and multi-agent systems. For an algebraic characterization of dynamic-epistemic logic, see Kurz and Palmigiano (2013). For further discussion, see Khudairi (ms). The latter proceeds by examining undecidable sentences via the epistemic interpretation of two-dimensional semantics. See Reinhardt (1974), for a similar epistemic interpretation of set-theoretic languages, in order to examine the reduction of the incompleteness of undecidable sentences on the counterfactual supposition that the language is augmented by stronger axioms of infinity; and Maddy (1988,b), for critical discussion. Chihara (2004) argues, as well, that conceptual possibilities can be treated as imaginary situations with regard to the construction of open-sentence tokens, where the latter can then be availed of in order to define nominalistically adequate arithmetic properties.

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