Induction and Inductive Definitions in Fragments of Second Order Arithmetic

Klaus Aehlig*

Abstract

A fragment with the same provably recursive functions as n iterated inductive definitions is obtained by restricting second order arithmetic in the following way. The underlying language allows only up to $n+1$ nested second order quantifications and those are in such a way, that no second order variable occurs free in the scope of another second order quantifier. The amount of induction on arithmetical formulae only affects the arithmetical consequences of these theories, whereas adding induction for arbitrary formulae increases the strength by one inductive definition.

1 Introduction and Related Work

The study of subsystems of second order arithmetic ("Analysis") has a long tradition in proof theory. Here we investigate a fragment that is defined by a restriction of the language. By allowing quantification of a second order variable only for formulae with at most this second order variable free, we obtain a proof theoretic weaker fragment. This fragment is motivated by a study of Altenkirch and Coquand [4] who used the non-nested case to obtain a "finitary subsystem of the polymorphic lambda calculus".

The fragment of analysis studied in this article is particularly suited as a target for the embedding of theories of iterated inductive definitions [1]. Systems with Π^1_1-comprehension have been studied by various authors [15, 9, 8]. An overview over proof theoretical aspects of inductive definitions can be found in the lecture notes volume by Buchholz, Feferman, Pohlers and Sieg [7].

The main emphasis of this article is the study of the influence of induction on the natural numbers on the provably recursive functions of these systems. Whereas induction for only arithmetical formulae only influences the arithmetical consequences, induction on arbitrary formulae increases the strength by an additional inductive definition. A technically similar observation has been made by Arai [5] when comparing the fast and the slow growing hierarchy.

The rest of this article, which is based on the author's doctoral thesis [2], is organised as follows. In Section 2 we define the formal systems under consideration. Sections 3 and 4 successively embed the systems of iterated inductive

*Mathematisches Institut, Universität München, Theresienstr. 39, 80333 München, aehlig@matheamtik.uni-muenchen.de. The author was supported by DFG Graduiertenkolleg 301 "Logik in der Informatik" when the research reported here has been carried out.

definitions in fragments of second order arithmetic first with, and then, at the price of an additional quantifier, without induction. Finally in Section 5 a proof theoretical analysis of the fragments of second order arithmetic is provided, that can be locally formalised in systems of iterated inductive definitions.

The meta theory for this article is Primitive Recursive Arithmetic, PRA for short. That is, whenever we claim a theorem to hold, we claim it to hold provably in PRA. Note that it therefore amounts to a stronger statement to say that a statement "holds", rather than just saying it is provable in, say, ID_1^c.

2 Definition of the Systems

The language of arithmetic, denoted by $\mathcal{L}_0$, consists of a single relation symbol $=$ for equality and function symbols for all primitive recursive functions. These are built from function symbols for the constant function zero of every arity (where we denote the nullary zero by 0), the unary function symbol $\mathcal{S}$ for the successor function and n-ary function symbols for the i'th projection for every $0 \leq i \leq n-1$, by arity-respecting composition and primitive recursion. The intended meaning of these function symbols is formalized in their axiomatization in Definition 2.2.

For systems based on a language extending that of arithmetic, we use the usual notation $\underline{n}$ for the n'th numeral, that is, if n is a natural number then $\underline{n}$ is short for $\underbrace{\mathcal{S}(\mathcal{S}(\ldots(\mathcal{S}}_{n}0)))$.

As logical connectives we use those of first order logic: conjunction, implication, universal quantification and absurdity. Moreover we use the other connectives as abreviations in the usual way, so $\neg A \equiv A \to \bot$, $A \vee B \equiv \neg(\neg A \wedge \neg B)$, and $\exists x A \equiv \neg \forall x \neg A$.

We use A, B, C as notations for formulae. As usual, finite lists of notations differing only in successive indices are abbreviated by putting an arrow over the notation for these entities. So, for example, $\vec{t}$ is short for $t_1, \ldots, t_n$. This may as well include the empty list, if $n = 0$. When displaying variables of a formula as in $A(\vec{x})$ we want to distinguish some of the variables that might occur in A; after having done so, we use $A(\vec{t})$ as a shorthand for the substitution $A[\vec{t}/\vec{x}\,]$. All our substitutions are assumed to be capture free in the usual sense, which is, up to α-equivalence, a well-defined notion. We hereby adopt the convention that we identify α-equal formulae. We also use $\mathcal{A}$, $\mathcal{B}$, $\mathfrak{A}$, and $\mathfrak{B}$ as notations for formulae, tacitly assuming them to have been displayed as $\mathcal{A}(x)$, $\mathcal{B}(x)$, $\mathfrak{A}(\mathfrak{X}, x)$, and $\mathfrak{B}(\mathfrak{X}, x)$ for some $\mathfrak{X}$ and x. We use a centred dot $\cdot$ to denote a distinguished first order variable.

Definition 2.1 ($\mathcal{L}_i$, Pos_i, Neg_i). Starting from the language $\mathcal{L}_0$ of arithmetic we define languages $\mathcal{L}_i$ and sets $\mathrm{Pos}_i, \mathrm{Neg}_i \subset \mathcal{L}_i[\mathfrak{X}]$ of positive and negative operator forms by induction on i. To do so, we set

$$\mathcal{L}_{i+1} = \mathcal{L}_i \cup \{\mathcal{P}_{i+1}^{\mathfrak{A}} \mid \mathfrak{A}(\mathfrak{X}, x) \in \mathrm{Pos}_i, \mathrm{FV}(\mathfrak{A}) \subset \{x, \mathfrak{X}\}\}$$

where the $\mathcal{P}_{i+1}^{\mathfrak{A}}$ are new predicate symbols (to be understood as the least fixed point of the operator $\mathfrak{A}$) and Pos_i and Neg_i are those sets of formulae that contain $\mathfrak{X}$ at most positively and at most negatively, respectively.

We use the abbreviation $\mathcal{A} \subset \mathcal{B} \equiv \forall x(\mathcal{A}(x) \to \mathcal{B}(x))$.

Definition 2.2 (basic axioms). The *basic axioms* of arithmetical theories are the following.

- The equality axioms $t = t$, $s = t \to t = s$, $s = t \to t = r \to s = r$ and $\vec{t} = \vec{s} \to \mathfrak{f}\vec{t} = \mathfrak{f}\vec{s}$, for arbitrary function symbols $\mathfrak{f}$.

- $\mathcal{S}t = 0 \to \bot$

- The defining equations for the primitive recursive function symbols. We have $\mathfrak{f}\vec{t} = 0$, if $\mathfrak{f}$ is the function symbol for the n-ary zero, $\mathfrak{f}\vec{t} = t_{i+1}$, if $\mathfrak{f}$ is the function symbol for the i'th projection, $\mathfrak{f}0\vec{t} = \mathfrak{g}\vec{t}$ and $\mathfrak{f}(\mathcal{S}x)\vec{t} = \mathfrak{h}x(\mathfrak{f}x\vec{t})\vec{t}$ if $\mathfrak{f}$ is the function symbol for the function built by primitive recursion from $\mathfrak{g}$ and $\mathfrak{h}$, and $\mathfrak{f}\vec{s} = \mathfrak{g}(\mathfrak{h}_1\vec{s})\ldots(\mathfrak{h}_n\vec{s})$, if $\mathfrak{f}$ is the function symbol for the composition of $\mathfrak{g}$ and $\vec{\mathfrak{h}}$.

Remark 2.3 (Injectivity of the successor). Let $\mathfrak{h}$ be the function symbol for binary first projection and Pred the function symbol for the function built by primitive recursion from $\mathfrak{h}$ and 0. We have the axioms $\mathrm{Pred}(\mathcal{S}x) = \mathfrak{h}x(\mathrm{Pred}x)$ and $\mathfrak{h}x(\mathrm{Pred}x) = x$. Hence, by transitivity of equality we have $\mathrm{Pred}(\mathcal{S}x) = x$. In particular the assumption $\mathcal{S}t = \mathcal{S}s$ implies $\mathrm{Pred}(\mathcal{S}t) = \mathrm{Pred}(\mathcal{S}s)$ which, implies $t = s$. So injectivity of the successor is derivable. Note that all these proofs are based on minimal logic and do not use induction.

Definition 2.4 (ID_n^c). The system ID_n^c is an extension of Peano Arithmetic in the language $\mathcal{L}_n$. That is, it is based on classical predicate logic, contains the basic axioms (equality, $\underline{1} \neq 0$, defining axioms for the primitive recursive functions) and induction on the full language $\mathcal{L}_n$.

Moreover, for arbitrary formulae $\mathcal{F}$ of the language and $0 < i \leq n$ the following axioms are added, which define $\mathcal{P}_i^{\mathfrak{A}}$ as the least fixed point of the operator $\mathfrak{A}$.

- $\mathfrak{A}(\mathcal{P}_i^{\mathfrak{A}}, \cdot) \subset \mathcal{P}_i^{\mathfrak{A}}$

- $\mathfrak{A}(\mathcal{F}, \cdot) \subset \mathcal{F} \to \mathcal{P}_i^{\mathfrak{A}} \subset \mathcal{F}$

Definition 2.5 (The negative fragment ID_n^-). The system ID_n^- is literally the same as ID_n^c, but based on minimal logic; in particular not even ex-falso-quodlibet is available.

ID_n^- is called the "negative fragment", because all connectives (conjunction, implication, universal quantification and absurdity) are negative. Recall that absurdity has no special meaning in minimal logic, but is just an unspecified nullary junctor. We still use $\exists$ and $\vee$ as abbreviations, but keep in mind that they do not behave as positive connectives.

Definition 2.6 (ID_n^*). We define ID_n^* to be the fragment of the system ID_n^- where all the operators $\mathfrak{A}$ of the fixed points $\mathcal{P}_i^{\mathfrak{A}}$ are strictly positive in the second order variable.

Proposition 2.7.

$$\mathfrak{A} \in \mathrm{Pos}_i \quad \Rightarrow \quad \mathrm{ID}_n^- \vdash \mathcal{A} \subset \mathcal{B} \to \mathfrak{A}(\mathcal{A}, \cdot) \subset \mathfrak{A}(\mathcal{B}, \cdot).$$
$$\mathfrak{A} \in \mathrm{Neg}_i \quad \Rightarrow \quad \mathrm{ID}_n^- \vdash \mathcal{A} \subset \mathcal{B} \to \mathfrak{A}(\mathcal{B}, \cdot) \subset \mathfrak{A}(\mathcal{A}, \cdot).$$

Proof. Induction on $\mathfrak{A}$. For example, the second statement in the case of "$\mathfrak{A} \to \mathfrak{B}$": $\mathfrak{A} \in \mathrm{Pos}_i$, $\mathfrak{B} \in \mathrm{Neg}_i$, hence (under the assumption $\mathcal{A} \subset \mathcal{B}$) by the induction hypotheses for arbitrary x it holds that $\mathfrak{A}(\mathcal{A}, x) \to \mathfrak{A}(\mathcal{B}, x)$ and $\mathfrak{B}(\mathcal{B}, x) \to \mathfrak{B}(\mathcal{A}, x)$. Assume $\mathfrak{A}(\mathcal{B}, x) \to \mathfrak{B}(\mathcal{B}, x)$ and $\mathfrak{A}(\mathcal{A}, x)$; hence $\mathfrak{B}(\mathcal{A}, x)$ by three applications of modus ponens.

It should be noted that the proof does not use any axioms. Hence the statement is even valid in minimal predicate logic. This will be used later (in Remark 3.12). ∎

Proposition 2.8. $\mathrm{ID}_n^- \vdash \forall x(\mathfrak{A}(\mathcal{P}_i^{\mathfrak{A}}, x) \leftrightarrow \mathcal{P}_i^{\mathfrak{A}} x)$.

Proof. We argue within ID_n^- and have to show $\forall x(\mathcal{P}_i^{\mathfrak{A}} x \to \mathfrak{A}(\mathcal{P}_i^{\mathfrak{A}}, x))$. To this end, we use the second fixed point axiom; hence we have to show $\mathfrak{A}(\mathfrak{A}(\mathcal{P}_i^{\mathfrak{A}}, \cdot), x) \to \mathfrak{A}(\mathcal{P}_i^{\mathfrak{A}}, x)$ for arbitrary x. Since $\mathfrak{A}(\mathcal{P}_i^{\mathcal{A}}, \cdot) \subset \mathcal{P}_i^{\mathfrak{A}}$ by the first fixed point axiom, we are done by monotonicity of $\mathfrak{A}$ (Proposition 2.7). ∎

Remark 2.9. As obviously $\mathrm{ID}_n^- \subset \mathrm{ID}_n^c$, Propositions 2.7 and 2.8 also hold for the system ID_n^c.

Moreover, as the proofs show, they also hold for ID_n^*, provided the formulae under consideration are in the language of ID_n^*.

Definition 2.10 (HA^2). The system HA^2 of second order arithmetic is based on second order minimal predicate logic. Its language is that of arithmetic, extended by set variables and universal quantification over them. The axioms are the basic axioms of arithmetical theories (Definition 2.2). The rules of the system are introduction and elimination of second order quantifiers in the following form

$$\frac{\Gamma \vdash A(\mathfrak{X}) \quad \mathfrak{X} \text{ not free in } \Gamma}{\Gamma \vdash \forall \mathfrak{X} A(\mathfrak{X})} \qquad\qquad \frac{\Gamma \vdash \forall \mathfrak{X} A(\mathfrak{X})}{\Gamma \vdash A(\mathcal{A})}$$

and the rules of first order minimal logic.

Since no induction is available, we should think of our universe as also containing objects which are not natural numbers. So we will often need the property of being a natural number, that is, the property of being an object for which the principle of induction holds. This predicate $\mathbb{N}\, x$ is defined in the usual way as

$$\mathbb{N}\, x \equiv \forall \mathfrak{X}. \forall y(\mathfrak{X} y \to \mathfrak{X}(\mathcal{S}(y))) \to \mathfrak{X} 0 \to \mathfrak{X} x$$

When working in fragments of second order arithmetic we use the abbreviation $\mathcal{A} \subset \mathcal{B}$ for $\forall x. \mathbb{N}\, x \to \mathcal{A}(x) \to \mathcal{B}(x)$. This differs from the use of this abbreviation in the systems of iterated inductive definitions. But as these are different systems, which even have a different language, there is no danger of confusion. Moreover, as noted in Remark 4.2, up to our canonical embeddings of ID_n^- into HA^2 these notions coincide.

Next we will formalise the fragments of HA^2 under consideration. We will define a family of subsystems of second order arithmetic. The restriction will be a restriction of the language. In HA_n^2 we will allow n nested, but not "interleaved" (in the sense of Matthes [13]), second order quantifiers, so that no second order variable is allowed to occur free in the scope of another second order quantifier.

We will also consider the amount of induction for arithmetical formulae as a second parameter. This parameter, which allows enough coding, will turn out to be independent of the proof theoretic strength (Theorem 5.27). In fact, it only affects the arithmetical consequences of these systems (Corollary 5.20).

For technical reasons, it does not suffice to count the nesting depths of second order variables but we have to use second order variables of different levels (see especially the proof of Lemma 5.8). However, universal formulae $\forall \mathfrak{X}_k.A(\mathfrak{X}_k)$ may be instantiated with arbitrary formulae of the language, as long as a legal formula is obtained (confer Remark 2.16). Allowing instantiation to the full language is essential, as in a "predicative version" of this system only the functions of Grzegorczyk's [10] class $\mathfrak{E}_4$ would be obtained as provably recursive functions [12], and a predicative version restricted to levels 0 and 1 would only yield the Kalmár [11] elementary functions [3].

From now on, we assume a fixed assignment of levels $0, 1, \ldots$ to all second-order variables except for a distinguished one, $\widehat{\mathfrak{X}}$. Let $\mathfrak{X}_i$ range over second order variables of level i. We assume that there are infinitely many second order variables of every level. The predicate $\mathbb{N}\,x$ uses a second order variable of level 0, that is, we have $\mathbb{N}\,x \equiv \forall \mathfrak{X}_0.\forall x(\mathfrak{X}_0 x \to \mathfrak{X}_0(\mathcal{S}(x))) \to \mathfrak{X}_0 0 \to \mathfrak{X}_0 x$.

Definition 2.11 ($\mathcal{I}_n^2$, $\mathcal{I}_n^*$). By induction on n we define sets $\mathcal{I}_n^2[\widehat{\mathfrak{X}}]$ of formulae of second order arithmetic as follows.

- $\mathcal{I}_0^2[\widehat{\mathfrak{X}}]$ is the set of all first order $\mathcal{L}_0[\widehat{\mathfrak{X}}]$-formulae.
- $\mathcal{I}_{n+1}^2[\widehat{\mathfrak{X}}]$ is the first order closure (conjunction, implication, first order universal quantification) of $\mathcal{L}_0[\widehat{\mathfrak{X}}] \cup \{\forall \mathfrak{X}_n A\,[\mathfrak{X}_n/\widehat{\mathfrak{X}}\,] \mid A \in \mathcal{I}_n^2[\widehat{\mathfrak{X}}]\}$.

We write $\mathcal{I}_n^2$ for the set of all $\mathcal{I}_n^2[\widehat{\mathfrak{X}}]$-formulae *without* free second order variables and $\mathcal{I}_n^2[\mathfrak{X}_n]$ for $\{A\,[\mathfrak{X}_n/\widehat{\mathfrak{X}}\,] \mid A \in \mathcal{I}_n^2[\widehat{\mathfrak{X}}]\}$. Moreover, we define the sets $\mathcal{I}_n^* = \bigcup\limits_{k \leq n} \mathcal{I}_k^2[\mathfrak{X}_k]$ with the reading that each of the $\mathfrak{X}_k$ in the union should range over *all* the second order variables of level k.

Obviously we have $\mathcal{I}_n^* \subset \mathcal{I}_{n+1}^*$. It should be noted that $\mathfrak{X}_0 0 \wedge \mathfrak{X}_1 0$ is *not* a legal formula; nor is $\mathfrak{X}_0 0 \wedge \forall \mathfrak{X}_0.(\mathfrak{X}_0 0 \to \mathfrak{X}_0 0)$.

Definition 2.12 (HA_n^2). The system HA_n^2 is defined to be the fragment of HA^2, where all occurring formulae are in $\mathcal{I}_n^*$.

Remark 2.13. It should be noted that the fragments of analysis presented here are somewhat non-standard in that elimination of second order quantification is allowed for arbitrary formulae, but the language of the system itself is restricted.

To compare the approach of this article with more conventional presentations consider the system, where the $\forall\mathfrak{X}$-elimination is restricted to variables

$$\frac{\Gamma \vdash \forall \mathfrak{X} A(\mathfrak{X})}{\Gamma \vdash A(\mathfrak{Y})}$$

and comprehension axioms of the form

$$\exists \mathfrak{X} \forall x.\mathfrak{X} x \leftrightarrow \mathcal{A}(x)$$

are present for all $\mathcal{A} \in \mathcal{I}_n^*$, maybe with other variables than x free. Then a partial cut-elimination shows that all proofs of $\mathcal{I}_n^*$-formulae in that system can be transformed into proofs in HA^2_n. In other words, our systems HA^2_n can be thought of as canonical proofs for $\mathcal{I}^*_n$-comprehension.

It may, however, be interesting to note that when showing the main Lemma 5.9 towards the admissibility of $\forall\mathfrak{X}$-elimination we actually use that we have a good overview of normal (semi-formal) proofs of $\forall\mathfrak{X} A(\mathfrak{X})$.

Remark 2.14. We note that, up to (level-*ignoring*) α-equality all formulae in $\mathcal{I}^2_n$ are built from a single second order variable. This can be proved by induction on the Definition 2.11 of $\mathcal{I}^2_n$.

Our sets $\mathcal{I}^2_n[\mathfrak{X}_n]$ are closed under substitution. More precisely, by simple induction on A one shows

Lemma 2.15. *If* $A(\mathfrak{X}_n), \mathcal{A} \in \mathcal{I}^2_n[\mathfrak{X}_n]$ *then* $A(\mathcal{A}) \in \mathcal{I}^2_n[\mathfrak{X}_n]$.

Remark 2.16. Note that in Lemma 2.15 it was crucial that the free second order variable of $\mathcal{A}$ is of level n. Substitution in free variables of too low level might lead to illegal formulae. Consider for example $A(\mathfrak{X}_1) \equiv \mathfrak{X}_1 0 \wedge \forall\mathfrak{X}_0.\mathfrak{X}_0 0 \to \mathfrak{X}_0 0$ and $\mathcal{A} \equiv \mathfrak{X}_0$. Then $A(\mathcal{A})$ is the non well-formed expression $\mathfrak{X}_0 0 \wedge \forall\mathfrak{X}_0.\mathfrak{X}_0 0 \to \mathfrak{X}_0 0$.

This will not be a problem when embedding ID^-_n into HA^2_n as all formulae we deal with in this embedding will contain no free second order variables.

In our systems HA^2_n we do not have any induction. One can think of a universe containing other first order objects than just the natural numbers. However, everything built up from zero and sucessor *is* a natural number. The following two lemmata can be shown by a simple argument within HA^2_1.

Lemma 2.17. $\mathrm{HA}^2_1 \vdash \mathbb{N}\,0$.

Lemma 2.18. $\mathrm{HA}^2_1 \vdash \forall x(\mathbb{N}\,x \to \mathbb{N}\,(\mathcal{S}x))$.

Definition 2.19 ($\mathrm{HA}^2_{\mathrm{n,(k)}}$)**.** The system $\mathrm{HA}^2_{\mathrm{n,(k)}}$ is defined to be HA^2_n extended by the axiom scheme

$$\forall x(\mathcal{A}(x) \to \mathcal{A}(\mathcal{S}x)) \to \mathcal{A}(0) \to \forall x \mathcal{A}(x)$$

for every Π^0_k-formula $\mathcal{A}$.

Remark 2.20 (Coding). In particular, $\mathrm{HA}^2_{\mathrm{n,(0)}}$ is HA^2_n plus induction for all Δ^0_0-formulae. This contains all the equations of Primitive Recursive Arithmetic and in particular coding is available. Hence arguments can be formalized in the usual way, even if coding is needed.

Obviously "HA^2_n + Arithmetical Induction" is just the union of the systems $\mathrm{HA}^2_{\mathrm{n,(k)}}$ for all k. We note that $\mathrm{HA}^2_{\mathrm{0,(k)}}$ is just an intuitionistic variant of the system $\Pi_k - \mathrm{IA}$ of Peano Arithmetic restricted to the induction axiom for Π_k formulae only.

Definition 2.21 ($\mathrm{HA}^2_{\mathrm{n,(+)}}$)**.** The system $\mathrm{HA}^2_{\mathrm{n,(+)}}$ is defined to be "HA^2_n + Full Induction", that is HA^2_n with the axiom scheme

$$\forall x(\mathcal{A}(x) \to \mathcal{A}(\mathcal{S}x)) \to \mathcal{A}(0) \to \forall x \mathcal{A}(x)$$

for *arbitrary* formulae $\mathcal{A}$ of the language.

3 Embedding of ID_n^c into $\mathrm{HA}^2_{\mathrm{n},(+)}$

First we show that the fragment $\mathrm{HA}^2_{\mathrm{n},(+)}$ is strong enough to host the obvious embedding of least fixed points as Π^1_1 sets. Since $\mathrm{HA}^2_{\mathrm{n},(+)}$ is based on minimal logic, we have to start with a double-negation translation. To do so, we use the one provided by Bucholz [6].

Definition 3.1 (A'). By induction on $A \in \mathcal{L}_i[\mathfrak{X}]$ we define its double negation translation A' and simultaneously showing $\mathfrak{A} \in \mathrm{Pos}_i$ (or $\mathfrak{A} \in \mathrm{Neg}_i$) implies $\mathfrak{A}' \in \mathrm{Pos}_i$ (or $\mathfrak{A}' \in \mathrm{Neg}_i$ respectively).

- $(\mathfrak{X}t)' \equiv \neg\neg\mathfrak{X}t$ and $(\mathcal{P}_i^{\mathfrak{A}}t)' \equiv \neg\neg\mathcal{P}_i^{\mathfrak{A}'}t$.
- $\bot' \equiv \bot$ and $A' \equiv \neg\neg A$ for $A \in \mathcal{L}_0$ atomic and different from $\bot$.
- The translation is homomorphic with respect to the logical connectives, that is $(A \wedge B)' \equiv A' \wedge B'$, $(A \to B)' \equiv A' \to B'$, and $(\forall x A)' \equiv \forall x A'$.

Since (doubly) negated formulae are stable, a simple induction on A shows

Lemma 3.2. $\mathrm{ID}_n^- \vdash \neg\neg A' \leftrightarrow A'$.

In the target of our embedding the only inductive predicates are of the form $\mathcal{P}_i^{\mathfrak{A}'}$, that is, only fixed points of properly negated forms are built. We can now show that these predicates are stable.

Lemma 3.3. $\mathrm{ID}_i^- \vdash \neg\neg\mathcal{P}_i^{\mathfrak{A}'}t \leftrightarrow \mathcal{P}_i^{\mathfrak{A}'}t$.

Proof. We have $\mathrm{ID}_i^- \vdash \mathfrak{A}'(\mathcal{P}_i^{\mathfrak{A}'}, t) \leftrightarrow \mathcal{P}_i^{\mathfrak{A}'}t$ by Proposition 2.8. Moreover we have $\mathrm{ID}_i^- \vdash \neg\neg\mathfrak{A}'(\mathcal{P}_i^{\mathfrak{A}'}, t) \leftrightarrow \mathfrak{A}'(\mathcal{P}_i^{\mathfrak{A}'}, t)$ by Lemma 3.2 which shows that $\mathfrak{A}'$ is stable, independently of the stability of $\mathfrak{X}$. So the claim follows. ∎

By a simple induction on $\mathfrak{A}$ we show

Proposition 3.4. $(\mathfrak{A}(\mathcal{P}_i^{\mathfrak{A}}, t))' \equiv \mathfrak{A}'(\mathcal{P}_i^{\mathfrak{A}'}, t)$.

In general, it is not the case, that $(\mathfrak{A}(\mathcal{F}, x))' \equiv \mathfrak{A}'(\mathcal{F}', x)$. Consider for example $\mathfrak{A}(\mathfrak{X},x) \equiv \mathfrak{X}x$ and $\mathcal{F}(z) \equiv \forall x.x = x$. Then $(\mathfrak{A}(\mathcal{F}, x))' \equiv \mathcal{F}'(x) \equiv (\forall x.x = x)' \equiv \forall x.\neg\neg x = x$ and $\mathfrak{A}'(\mathcal{F}', x) \equiv \neg\neg\mathcal{F}'(x) \equiv \neg\neg\forall x.\neg\neg x = x$. But at least the formulae are provably equivalent.

Proposition 3.5. $\mathrm{ID}_n^- \vdash (\mathfrak{A}(\mathcal{F}, x))' \leftrightarrow \mathfrak{A}'(\mathcal{F}', x)$.

Proof. Induction on $\mathfrak{A}$, using Lemma 3.2 for the only non-trivial case $\mathfrak{A} \equiv \mathfrak{X}t$. ∎

To complete our embedding we have to show that the translation of the axioms are provable.

Lemma 3.6.
$\mathrm{ID}_n^- \vdash (\forall x(\mathfrak{A}(\mathcal{P}_i^{\mathfrak{A}}, x) \to \mathcal{P}_i^{\mathfrak{A}}x))'$.
$\mathrm{ID}_n^- \vdash (\forall x(\mathfrak{A}(\mathcal{F}, x) \to \mathcal{F}(x)) \to \forall x(\mathcal{P}_i^{\mathfrak{A}}x \to \mathcal{F}(x)))'$.

Proof. Using the syntactical equality shown in Proposition 3.4 we calculate $(\forall x(\mathfrak{A}(\mathcal{P}_i^{\mathfrak{A}}, x) \to \mathcal{P}_i^{\mathfrak{A}} x))' \equiv \forall x(\mathfrak{A}'(\mathcal{P}_i^{\mathfrak{A}'}, x) \to \neg\neg\mathcal{P}_i^{\mathfrak{A}'} x)$. This formula is provable from the corresponding induction axiom, since trivially $\mathcal{P}_i^{\mathfrak{A}'} t \to \neg\neg\mathcal{P}_i^{\mathfrak{A}'} t$.

Again we calculate $(\forall x(\mathfrak{A}(\mathcal{F}, x) \to \mathcal{F}(x)) \to \forall x(\mathcal{P}_i^{\mathfrak{A}} x \to \mathcal{F}(x)))' \equiv$ $\forall x((\mathfrak{A}(\mathcal{F}, x))' \to \mathcal{F}'(x)) \to \forall x(\neg\neg\mathcal{P}_i^{\mathfrak{A}'} x \to \mathcal{F}'(x))$. By Lemmata 3.3 and 3.5 this formula is provably equivalent to an instance of the corresponding induction axiom. ∎

Corollary 3.7. *If* $\mathrm{ID}_n^c \vdash A$ *then* $\mathrm{ID}_n^- \vdash A'$.

Proof. By Lemma 3.6 the translations of the induction axioms are provable. The translations of the non-induction axioms (Definition 2.2) follow immediately from the corresponding axiom.

Moreover, classical logic is admissible since the translations of all formulae are stable by Lemma 3.2. ∎

Since minimal logic proves $\neg\neg\neg R(x, y) \leftrightarrow \neg R(x, y)$ we get conservativity for Π_2^0-statements.

Corollary 3.8. *For atomic* $R(x, y) \in \mathcal{L}_0$ *it holds that, if* $\mathrm{ID}_n^c \vdash \forall x \exists y R(x, y)$ *then* $\mathrm{ID}_n^- \vdash \forall x \neg\forall y \neg R(x, y)$.

As second order arithmetic is an impredicative system, the set theoretical definition of the least fixed point can be formalized easily.

Definition 3.9. For $A \in \mathcal{L}_n[\mathfrak{X}_n]$ we define a formula A^* of second order arithmetic inductively as follows.

- $(\mathcal{P}_i^{\mathfrak{A}} t)^* \equiv \forall \mathfrak{X}_i . \forall y((\mathfrak{A}(\mathfrak{X}_i, y))^* \to \mathfrak{X}_i y) \to \mathfrak{X}_i t$ and $(\mathfrak{X}_i t)^* \equiv \mathfrak{X}_i t$.
- The embedding is homomorphic for the other connectives, that is, $A^* \equiv A$ for atomic $A \in \mathcal{L}_0$, $(\forall x A)^* \equiv \forall x A^*$, $(A \wedge B)^* \equiv A^* \wedge B^*$, and $(A \to B)^* \equiv A^* \to B^*$.

A simple induction shows that the range of the embedding is the fragment we had in mind.

Lemma 3.10. *If* $A \in \mathcal{L}_n$ *then* $A^* \in \mathcal{I}_n^2$. *In particular* A^* *has* no *free second order variables.*

Lemma 3.11. $\mathfrak{A}^*((\mathcal{P}_i^{\mathfrak{A}})^*, x) \in \mathcal{I}_n^2$.

Proof. Immediate from Lemma 3.10, noting that for $\mathfrak{A} \in \mathrm{Pos}_i$ the predicate $\mathfrak{X}$ does not occur under any second-order quantifier (since Pos_i only contains first-order formulae in $\mathfrak{X}$). ∎

Remark 3.12. As in the proof of proposition 2.7 one shows that the translations of positive formulae are monotone.

Lemma 3.13.
$\mathrm{HA}^2_{\mathrm{i},(+)} \vdash (\mathfrak{A}(\mathcal{P}_i^{\mathfrak{A}}, \cdot) \subset \mathcal{P}_i^{\mathfrak{A}})^*$.
$\mathrm{HA}^2_{\mathrm{n},(+)} \vdash (\mathfrak{A}(\mathcal{F}, \cdot) \subset \mathcal{F} \to \mathcal{P}_i^{\mathfrak{A}} \subset \mathcal{F})^*$ *for* $\mathcal{F} \in \mathcal{L}_n$ *and* $i \leq n$.

Proof. By definition we have

$$(\mathfrak{A}(\mathcal{P}_i^{\mathfrak{A}},\cdot) \subset \mathcal{P}_i^{\mathfrak{A}})^* \equiv \forall x.\mathfrak{A}^*((\mathcal{P}_i^{\mathfrak{A}})^*, x) \to \forall \mathfrak{X}_i.\forall y(\mathfrak{A}^*(\mathfrak{X}_i, y) \to \mathfrak{X}_i y) \to \mathfrak{X}_i x$$

where $(\mathcal{P}_i^{\mathfrak{A}})^*(y) \equiv \forall \mathfrak{X}_i.\forall x(\mathfrak{A}^*(\mathfrak{X}_i, x)) \to \mathfrak{X}_i x) \to \mathfrak{X}_i y$.

Arguing informally in $\mathrm{HA}^2_{\mathrm{n},(+)}$, the translation can be proved as follows. Let x be arbitrary and assume $\mathfrak{A}^*(\mathcal{P}_i^{\mathfrak{A}^*}, x)$. Let $\mathfrak{X}_i$ be arbitrary and assume $\forall y.\mathfrak{A}^*(\mathfrak{X}_i, y) \to \mathfrak{X}_i y$. Using that last assumption we get $\forall y (\mathcal{P}_i^{\mathfrak{A}})^*(y) \to \mathfrak{X}_i y$. Hence, by monotonicity (remark 3.12) of $\mathfrak{A}^*$ we get $\forall y \mathfrak{A}^*((\mathcal{P}_i^{\mathfrak{A}})^*, y) \to \mathfrak{A}^*(\mathfrak{X}_i, y)$, so we can conclude $\mathfrak{X}_i x$.

By definition we have $(\mathfrak{A}(\mathcal{F},\cdot) \subset \mathcal{F} \to \mathcal{P}_i^{\mathfrak{A}} \subset \mathcal{F})^* \equiv$

$$\forall y(\mathfrak{A}^*(\mathcal{F}^*, y) \to \mathcal{F}^*(y)) \to \forall x.(\forall \mathfrak{X}_i.\forall y(\mathfrak{A}^*(\mathfrak{X}_i, y) \to \mathfrak{X}_i y) \to \mathfrak{X}_i x) \to \mathcal{F}^*(x)$$

Arguing informally in $\mathrm{HA}^2_{\mathrm{n},(+)}$ this can be proved as follows. Assume $\forall y.\mathfrak{A}^*(\mathcal{F}^*, y) \to \mathcal{F}^*(y)$. Let x be arbitrary and moreover assume $\forall \mathfrak{X}_i.\forall y(\mathfrak{A}^*(\mathfrak{X}_i, y) \to \mathfrak{X}_i y) \to \mathfrak{X}_i x$. Specialising the last assumption to $\mathcal{F}^*$ we get $\mathcal{F}^*(x)$ from our first assumption. ∎

Remark 3.14 (Monotone inductive definitions). Note that in the proof of Lemma 3.13 only monotonicity of $\mathfrak{A}^*$ was used. The only point in restricting $\mathcal{P}_i^{\mathfrak{A}}$ to positive $\mathfrak{A}$ is to have a canonical proof that $\mathfrak{A}^*$ is monotone. So one might be tempted to allow $\mathcal{P}_i^{\mathfrak{A}}$ for arbitrary $\mathfrak{A}$ and relativize the first axiom to $\mathfrak{A}$ being monotone. However, monotonicity is a second order concept and does not fit well into the arithmetical framework of systems of inductive definitions. Note that, due to the restriction to one second-order variable, some mild coding is necessary to express monotonicity: $\forall \mathfrak{X}.\mathfrak{X}\langle 0,\cdot\rangle \subset \mathfrak{X}\langle 1,\cdot\rangle \to \ldots$.

As all the axioms not concerned with the $\mathcal{P}_i^{\mathfrak{A}}$ and all the logical rules of ID_n^- are also present in $\mathrm{HA}^2_{\mathrm{n},(+)}$ we obtain

Corollary 3.15. *If* $\mathrm{ID}_n^- \vdash A$ *then* $\mathrm{HA}^2_{\mathrm{n},(+)} \vdash A^*$.

In particular, for Π^0_2-statements this corollary reads as

Corollary 3.16. *For atomic* $R(x,y) \in \mathcal{L}_0$*, if* $\mathrm{ID}_n^- \vdash \forall x \neg \forall y \neg R(x,y)$ *then* $\mathrm{HA}^2_{\mathrm{n},(+)} \vdash \forall x.\neg\forall y \neg R(x,y)$.

4 Embedding of $\mathrm{HA}^2_{\mathrm{n},(+)}$ into HA^2_{n+1}

Induction can be eliminated by relativizing all first order quantifiers to the property of being a natural number.

Definition 4.1. For $A \in \mathcal{I}^2_n[\mathfrak{X}_n]$ we define $A^\sharp \in \mathcal{I}^2_{n+1}[\mathfrak{X}_{n+1}]$ by induction on the formula A as follows.

- $(\forall x A)^\sharp \equiv \forall x.\mathbb{N}\, x \to A^\sharp$, $(\mathfrak{X}_n t)^\sharp \equiv \mathfrak{X}_{n+1} t$ and $(\forall \mathfrak{X}_n A)^\sharp \equiv \forall \mathfrak{X}_{n+1} A^\sharp$
- The other connectives are translated homomorphically, that is $(R\vec{t})^\sharp \equiv R\vec{t}$, $\perp^\sharp \equiv \perp$, $(A \wedge B)^\sharp \equiv A^\sharp \wedge B^\sharp$ and $(A \to B)^\sharp = A^\sharp \to B^\sharp$.

Remark 4.2. When introducing second order arithmetic we also fixed a different use of the symbol $\subset$. However, via the our embeddings these two notions are coherent. More precisely $(\mathcal{A} \subset \mathcal{B})^{*\sharp} \equiv \mathcal{A}^{*\sharp} \subset \mathcal{B}^{*\sharp}$.

Lemma 4.3. *For every formula $\mathcal{A} \in \mathcal{I}_n^*$*

$$\mathrm{HA}^2_{n+1} \vdash (\forall x(\mathcal{A}(x) \to \mathcal{A}(\mathcal{S}x)) \to \mathcal{A}(0) \to \forall x\mathcal{A}(x))^\sharp.$$

Proof. Unfolding the definition yields $(\forall x(\mathcal{A}(x) \to \mathcal{A}(\mathcal{S}x)) \to \mathcal{A}(0) \to \forall x\mathcal{A}(x))^\sharp$ $\equiv (\forall x.\mathbb{N}\,x \to \mathcal{A}^\sharp(x) \to \mathcal{A}^\sharp(\mathcal{S}x)) \to \mathcal{A}^\sharp(0) \to \forall x.\mathbb{N}\,x \to \mathcal{A}^\sharp(x)$.

Arguing informally within HA^2_{n+1} we assume $\forall x.\mathbb{N}\,x \to \mathcal{A}^\sharp(x) \to \mathcal{A}^\sharp(\mathcal{S}x)$ and $\mathcal{A}^\sharp(0)$. Moreover let x be arbitrary and assume $\mathbb{N}\,x$. We have to show $\mathcal{A}^\sharp(x)$.

Instantiate $\mathbb{N}\,x$ to $\mathcal{A}^\sharp(\cdot) \wedge \mathbb{N}\,(\cdot)$ to obtain $\forall y((\mathcal{A}^\sharp(y) \wedge \mathbb{N}\,y) \to (\mathcal{A}^\sharp(\mathcal{S}(y)) \wedge \mathbb{N}\,(\mathcal{S}(y)))) \to (\mathcal{A}^\sharp(0) \wedge \mathbb{N}\,0) \to (\mathcal{A}^\sharp(x) \wedge \mathbb{N}\,x)$. Since we have $\mathcal{A}^\sharp(0)$ and (by Lemma 2.17) also $\mathbb{N}\,0$, it suffices to show $\forall y.(\mathcal{A}^\sharp(y) \wedge \mathbb{N}\,y) \to (\mathcal{A}^\sharp(\mathcal{S}(y)) \wedge \mathbb{N}\,(\mathcal{S}(y)))$. So, for arbitrary y assume $\mathcal{A}^\sharp(y) \wedge \mathbb{N}\,y$. From our first assumption we get $\mathcal{A}^\sharp(\mathcal{S}y)$ and from Lemma 2.18 we get $\mathbb{N}\,(\mathcal{S}y)$. ∎

A simple induction on the buildup of terms shows

Lemma 4.4. *For every term t we have a proof of t being total. More precisely, if the free variables of t are among $\vec{x}$ then $\mathrm{HA}^2_1 \vdash \mathbb{N}\,\vec{x} \to \mathbb{N}\,t$.*

Proof. Immediate from Lemmata 2.17, 2.18, and 4.3. ∎

Lemma 4.5. *If $\mathrm{HA}^2_{\mathrm{n},(+)}$ proves the sequent $\Gamma \vdash A$ then HA^2_{n+1} proves the sequent $\mathbb{N}\,\vec{x}, \Gamma^\sharp \vdash A^\sharp$ where $\vec{x}$ are the free variables of Γ, A.*

Proof. Induction on the derivation. All proof rules translate identical, except for the induction principle, where we use Lemma 4.3, and $\forall x$-elimination, where we use Lemma 4.4. It should be noted that at the only position where the free variables of Γ, A decrease, (when an $\forall x$-introduction is used) we can also get rid of the assumption $\mathbb{N}\,x$ via $\to$-introduction. ∎

For the special case of $\Gamma = \emptyset$ and A a sentence we obtain

Corollary 4.6. *If $\mathrm{HA}^2_{\mathrm{n},(+)} \vdash A$ for some closed formula A then $\mathrm{HA}^2_{n+1} \vdash A^\sharp$.*

In particular, for Π^0_2-statements this reads as

Corollary 4.7. *For atomic $R(x,y) \in \mathcal{L}_0$, if $\mathrm{HA}^2_{\mathrm{n},(+)} \vdash \forall x.\neg\forall y \neg R(x,y)$ then $\mathrm{HA}^2_{n+1} \vdash \forall x.\mathbb{N}\,x \to \neg\forall y(\mathbb{N}\,y \to \neg R(x,y))$.*

5 Proof Theoretic Reduction of $\mathrm{HA}^2_{\mathrm{n+1,(k)}}$ to ID^*_n

From a proof theoretic point of view, the most difficult rule of second order arithmetic is the elimination of a second order quantifier, as this rule lacks the subformula property. In fact, when stepping from $\forall\mathfrak{X} A(\mathfrak{X})$ to $A(\mathcal{A})$ the formula might get arbitrary complex.

We shall therefore develop a semi-formal notion of "normal proofs", that is, of proofs with the subformula property, in such a way that $\forall\mathfrak{X}$-elimination becomes admissible. The main obstacle to achieve this are proofs *by assumption*.

The Ω-rule, introduced by Buchholz [6] and used in the context of second order arithmetic by Buchholz and Schütte [8], solves this problem by anticipating cut elimination. If finally we consider a proof with no open assumptions, then in every subproof the open assumptions will sooner or later be replaced by proofs. So it suffices to have a proof of $A(\mathcal{A})$ for every possible proof of $\forall\mathfrak{X}_n.A(\mathfrak{X}_n)$ that does not have assumptions containing $\forall\mathfrak{X}_n$-formulae positively. But such proofs are proofs *by introduction* and hence a substitution argument works.

The overall strategy of our proof theoretic reduction is as follows. We first define a notion $d \vdash_{n,(K)} \Gamma \sqsupset A$ of semi-formal proofs (Definition 5.2). This notion of proof features the base case (atomic and $\forall\mathfrak{X}$-formulae) of partial truth predicates $f \Vdash_{(K)} \Gamma \sqsupset C$ for $\mathcal{I}^*_N$-formulae (Definition 5.11).

Since elimination of second order quantification of maximal level (Lemma 5.9) and the Ω_{k+1}-rule are admissible (Lemma 5.15), our partial truth predicates have all the needed properties of a truth predicate, including $\forall\mathfrak{X}$-elimination (Lemma 5.17). So a simple induction on HA^2_N shows that everything provable is "true" (Theorem 5.19). Since our notion of "truth" is built on the notion of provability, we obtain (Lemma 5.14) a semi-formal proof and a collapsing property (Lemma 5.10) finally gets us back to the more usual notion of proof (and hence "real truth").

Definition 5.1 (Neg^2_n**,** Pos^2_n**).** We inductively define sets Neg^2_n and Pos^2_n of second order formulae as follows.

- If $A \in \mathcal{I}^2_k[\mathfrak{X}_k]$ for some $k \leq n$ then $A \in \mathrm{Neg}^2_n \cap \mathrm{Pos}^2_n$.
- If $A \in \mathcal{I}^2_n[\mathfrak{X}_n]$ then $\forall\mathfrak{X}_n A \in \mathrm{Pos}^2_n$.
- If $A \in \mathrm{Neg}^2_n$ then $\forall x A \in \mathrm{Neg}^2_n$; if $A \in \mathrm{Pos}^2_n$ then $\forall x A \in \mathrm{Pos}^2_n$.
- If $A \in \mathrm{Neg}^2_n$ and $B \in \mathrm{Pos}^2_n$ then $A \to B \in \mathrm{Pos}^2_n$; if $A \in \mathrm{Pos}^2_n$ and $B \in \mathrm{Neg}^2_n$ then $A \to B \in \mathrm{Neg}^2_n$.
- If $A, B \in \mathrm{Neg}^2_n$ then $A \wedge B \in \mathrm{Neg}^2_n$; if $A, B \in \mathrm{Pos}^2_n$ then $A \wedge B \in \mathrm{Pos}^2_n$.

We note that $\mathrm{Neg}^2_n \subset \mathrm{Neg}^2_{n+1}$ and $\mathrm{Pos}^2_n \subset \mathrm{Pos}^2_{n+1}$ and moreover $\mathbb{N}\, x \in \mathrm{Pos}^2_0$.

We will make use of various notations of recursion theory, most prominently, the coding $\langle k_1, \ldots, k_n\rangle$ of lists of natural numbers and the i'th projection $(n)_i$ of a natural number n, thought of as the code of a list. That is, $(\langle k_1, \ldots, k_n\rangle)_i = k_{i+1}$. We assume a primitive recursive such coding with $\langle k_1, \ldots, k_n\rangle > k_i$. Moreover, by $\{e\}(k)$ we denote the "Kleene bracket": the value, if it exists, of the e'th partial recursive function at argument k. The notation $\{e\}(k) = \ell$ presupposes that $\{e\}(k)$ is defined. We note that $\{e\}(k) = \ell$ can be expressed by a Σ_1-formula in the language of arithmetic. So every theory extending PRA, hence in particular ID^*_N, provides enough induction on the natural numbers to show all the needed properties of our coding functions.

We also presuppose some canonical arithmetization ("Gödel numbering") of all our syntactical entities which we denote by $\ulcorner\urcorner$. We assume this Gödel numbering to be such that all the usual operations on syntactical entities are primitive recursive on the codes. In accordance with our convention that we identify α-equal formulae we assume that the codes are invariant under α-equivalence. We use dotted variables $\dot{x}$ within the Gödel brackets $\ulcorner\urcorner$ to signify that the object coded by x, rather than the code of x itself should be inserted at this position. We assume that all our codings are standard, that is, they have the usual monotonicity properties. In particular, we presuppose that the Gödel number of every formula is bigger than that of every proper subformula.

We use σ as a notation for first order substitutions; that is, when using σ, it is tacitly understood, that σ ranges over all first order substitutions (and only those). Application of substitution σ to a formula A is denoted by postfixing its notation, as in $A\sigma$. If $\Gamma = \{A_1, \dots, A_n\}$ is a set of formulae, we write $\Gamma\sigma$ for the set $\{A_1\sigma, \dots, A_n\sigma\}$.

Informally, normal proofs are those where no elimination follows an introduction. Technically this can be expressed by allowing elimination rules only if the major premise is an elimination, assumption or axiom.

Let $N \geq 1$ and $K \geq 0$ be a natural numbers kept fixed for the rest of this section. We will develop a notion of semi-formal proofs for $\mathrm{HA}^2_{\mathrm{N,(K)}}$ definable and properties thereof provable in ID^*_{N-1}. Recall that $\mathrm{HA}^2_{\mathrm{N,(K)}}$ is HA^2 restricted to N levels of nested (but not interleaved) quantifications, but equipped with induction for Π^0_K-formulae.

Nevertheless we will continue to state all our theorems as theorems of our meta theory PRA.

Definition 5.2 (Normal derivations $d \vdash_{n,(K)} \Gamma \sqsupset A$). We define $\mathsf{intro}(d)$ to be a shorthand for $(d)_0 > 4$.

By iterated inductive definitions we define the relations $d \vdash_{n,(K)} \Gamma \sqsupset A$ for $n < N$ and d a natural number, $A \in \mathcal{I}^*_{n+1}$ a formula and $\Gamma \subset \mathcal{I}^*_{n+1}$ a finite set of formulae.

- (Axiom) $\langle 0 \rangle \vdash_{n,(K)} \Gamma \sqsupset A$ if $A \equiv B\sigma$, where σ is an arbitrary first order substitution and B is an axiom in the sense of Definition 2.2 or of the form $\forall x(\mathcal{A}(x) \to \mathcal{A}(\mathcal{S}x)) \to \mathcal{A}(0) \to \forall x \mathcal{A}(x)$ for some Π^0_K-formula $\mathcal{A}$.
- (Assumption) $\langle 1 \rangle \vdash_{n,(K)} \Gamma \sqsupset A$ if $A \in \Gamma$.
- ($\to$-elim) If $d \vdash_{n,(K)} \Gamma \sqsupset A \to B$ with $\neg\mathsf{intro}(d)$ and $e \vdash_{n,(K)} \Gamma \sqsupset A$ then $\left\langle 2, d, e, \ulcorner A \urcorner \right\rangle \vdash_{n,(K)} \Gamma \sqsupset B$.
- ($\wedge$-elim) If $d \vdash_{n,(K)} \Gamma \sqsupset A_0 \wedge A_1$ and $\neg\mathsf{intro}(d)$ then $\left\langle 3, d, i, \ulcorner A_{1-i} \urcorner \right\rangle \vdash_{n,(K)} \Gamma \sqsupset A_i$.
- ($\forall x$-elim) If $d \vdash_{n,(K)} \Gamma \sqsupset \forall x A(x)$ with $\neg\mathsf{intro}(d)$ then $\left\langle 4, d, \ulcorner \forall x A(x) \urcorner \right\rangle \vdash_{n,(K)} \Gamma \sqsupset A(t)$.
- ($\to$-intro) If $d \vdash_{n,(K)} \Gamma, A \sqsupset B$ then $\langle 5, d \rangle \vdash_{n,(K)} \Gamma \sqsupset A \to B$.
- ($\forall x$-intro) If $d \vdash_{0,(K)} \Gamma' \sqsupset A(x)$ then $\left\langle 6, d, \ulcorner x \urcorner \right\rangle \vdash_{0,(K)} \Gamma \sqsupset \forall z A(z)$ where Γ' is the subset of those formulae of Γ not containing x free.
- (ω) If $n \geq 1$ and for all terms t it holds that $\{e\}\left(\ulcorner t \urcorner\right) \vdash_{n,(K)} \Gamma \sqsupset A(t)$ then $\langle 6, e \rangle \vdash_{n,(K)} \Gamma \sqsupset \forall x A(x)$.
- ($\wedge$-intro) If $d_0 \vdash_{n,(K)} \Gamma \sqsupset A_0$ and $d_1 \vdash_{n,(K)} \Gamma \sqsupset A_1$ then $\langle 7, d_0, d_1 \rangle \vdash_{n,(K)} \Gamma \sqsupset A_0 \wedge A_1$.

- ($\forall\mathfrak{X}$-intro) If $d \vdash_{n,(K)} \Gamma' \sqsupset A(\mathfrak{X})$ then $\langle 8, d, \ulcorner\mathfrak{X}\urcorner\rangle \vdash_{n,(K)} \Gamma \sqsupset \forall\mathfrak{X}A(\mathfrak{X})$ where Γ' is the subset of those formulae of Γ not containing $\mathfrak{X}$ free.
- (Ω_{k+1}) If $k < n$ and $d \vdash_{n,(K)} \Gamma \sqsupset \forall\mathfrak{X}_k A(\mathfrak{X}_k)$ and (for all $\Delta \subset \mathrm{Neg}^2_k$ it holds that if $d' \vdash_{k,(K)} \Delta \sqsupset \forall\mathfrak{X}_k A(\mathfrak{X}_k)$ then $\{e\}\left(\langle d', \ulcorner\Delta\urcorner\rangle\right) \vdash_{n,(K)} \Delta, \Gamma \sqsupset B$) then $\langle 9+k, \ulcorner\forall\mathfrak{X}_k A(\mathfrak{X}_k)\urcorner, d, e\rangle \vdash_{n,(K)} \Gamma \sqsupset B$.

Remark 5.3. The reader is invited to verify that the above Definition 5.2 of the relations $\cdot \vdash_{n,(K)} \cdot \sqsupset \cdot$ for $n = 0, \ldots, N-1$ can be formalized in ID^*_{N-1}. In this formalization we would use appropriate Gödel codes instead of Γ and A. Recalling that we identify α-equal formulae also in our meta theory, the definition fits with our convention that α-equal formulae have identical Gödel codes.

So, officially, for $1 \leq n < N$ we have a formula $\mathfrak{A}_{K,n} \in \mathrm{Pos}_{n-1}$, formalizing the underlying operator and a predicate $\mathcal{P}_n^{\mathfrak{A}_{K,n}}$ for the fixed point. We write $d \vdash_{n,(K)} \Gamma \sqsupset A$ for the official $\mathcal{P}_n^{\mathfrak{A}_{K,n}} \langle d, \ulcorner\Gamma\urcorner, \ulcorner A\urcorner\rangle$.

The relation $\cdot \vdash_{0,(K)} \cdot \sqsupset \cdot$ is in fact a primitive recursive relation, and formalized as such. It should be noted, that the definition of $\cdot \vdash_{0,(K)} \cdot \sqsupset \cdot$ is finitely branching so that it is reasonable to speak of "the number of inferences". Moreover, this number can be read off the code of the derivation in a primitive recursive way.

We have "saved" an inductive definition by inspecting only a particular branch at the first "ω-branching" level and so could obtain an arithmetical relation. This technique has a strong similarity to the one used in Arai's "slow growing analogue to Buchholz' proof" [5]. There "pointwise transfinite induction" for the ordinal notation system for the ordinal of ID^c_p was shown within ID^c_{p-1}. The crucial observation was that the lowest inductive definition becomes arithmetical when for an ordinal term of type ω only the n'th element of the fundamental sequence is considered, for a fixed but arbitrary n given from the outside. In our case, to show a property for all terms we are happy with a proof just for a single variable — provided it is sufficiently new.

The reason why we nevertheless need the (ω) rule is that later we will (in Definition 5.11) define (partial) truth predicates and then will have to show (in Lemma 5.16) that everything derivable in $\cdot \vdash_{n,(K)} \cdot \sqsupset \cdot$ is true. In particular, we have to show that from a proof of a $\forall x A$ statement we get the truth of $A(t)$ for *all* terms t. In the case of a universal statement introduced by ($\forall x$-intro), soundness is ensured by admissibility of first order substitution. But such a proof would fail at an (Ω)-rule. To conclude $\ldots \vdash_{n,(K)} \Gamma\sigma \sqsupset B\sigma$ by an (Ω)-rule again we would have to provide (in a uniform way) a proof of $\ldots \vdash_{n,(K)} \Delta, \Gamma\sigma \sqsupset B\sigma$. However, the induction hypothesis would only give a proof of $\ldots \vdash_{n,(K)} \Delta\sigma, \Gamma\sigma \sqsupset B\sigma$. This should also be compared to the careful formulation of Lemma 5.8.

Remark 5.4. Note that in Definition 5.2 the "witness" d is constructed in such a way, that for given Γ and A such that $d \vdash_{n,(K)} \Gamma \sqsupset A$ holds, one can uniquely reconstruct all the formulae Γ', A' and relations $d' \vdash_{k,(K)} \Gamma' \sqsupset A'$ that led to the relation $d \vdash_{n,(K)} \Gamma \sqsupset A$ in the inductive definition. Moreover, these reconstruction is recursive as a theorem of ID^*_N. This will be used tacitly in the sequel.

Remark 5.5 (Weakening). As a theorem of $\mathrm{ID}^*_{n'}$, if $d \vdash_{n,(k)} \Gamma \sqsupset A$, $n \leq n'$, $k \leq k'$ and $\Gamma \subset \Gamma'$ then $d \vdash_{n',(k')} \Gamma' \sqsupset A$.

Lemma 5.6 (First order substitution). *There is a primitive recursive* $\mathfrak{f}$ *such that if* $d \vdash_{0,(K)} \Gamma \sqsupset A$ *then* $\mathfrak{f}(\ulcorner A \urcorner, \ulcorner \Gamma \urcorner, d, \ulcorner \sigma \urcorner) \vdash_{0,(K)} \Gamma\sigma \sqsupset A\sigma$ *and the number of inferences is* not *changed. Moreover, if* $\neg\mathsf{intro}(d)$ *then* $\neg\mathsf{intro}(\mathfrak{f}(\dots, d, \dots))$*.*

Proof. Induction on $d \vdash_{0,(K)} \Gamma \sqsupset A$. Note that (in Definition 5.2) we have constructed our set of axioms to be closed under substitution. ∎

Corollary 5.7. *If* $d \vdash_{n,(K)} \Gamma \sqsupset \forall x A(x)$ *and* d *ends in (*$\forall x$*-intro), that is, if* d *is of the form* $d = \langle 6, d', \ulcorner x \urcorner \rangle$*, then* $n = 0$ *and there is a natural number* e *that* $\{e\}\left(\ulcorner t \urcorner\right) \vdash_{0,(K)} \Gamma \sqsupset A(t)$ *and* $\{e\}\left(\ulcorner t \urcorner\right)$ *has less inferences than* d*.*

Proof. First we note that the rule ($\forall x$-intro) is only present at level 0. Then we apply Lemma 5.6 to the premise of the derivation, that is, we apply Lemma 5.6 to the derivation d'. ∎

Lemma 5.8 (Substitution for second order variables). *There is a primitive recursive* $\mathfrak{f}$ *such that the following is a theorem of* ID_n^**.*

If $d \vdash_{n,(K)} \Gamma \sqsupset A$ *then* $\mathfrak{f}(\ulcorner \Gamma \urcorner, \ulcorner A \urcorner, \ulcorner \theta \urcorner, \ulcorner \theta' \urcorner, d) \vdash_{n,(K)} \Gamma\theta'\theta \sqsupset A\theta'\theta$ *where* θ *is a substitution of second order variables with only variables of level* n *in its domain such that* $\Gamma\theta$ *and* $A\theta$ *are well-formed, and* θ' *is a permutation of second order variables of level less than* n*.*

Proof. Induction along the inductive Definition 5.2 of $d \vdash_{n,(K)} \Gamma \sqsupset A$.

For the case (Axiom) we note that our axioms do not contain second order variables.

For the cases of an elimination rule we simultaniously proof that the transformed proof is also by elimnation.

For ($\forall\mathfrak{X}$-intro) with the abstracted variable $\mathfrak{X}$ of level less than n let $\mathfrak{X}'$ be a new variable of the same level as $\mathfrak{X}$. Let $\theta'' = [\mathfrak{X}', \mathfrak{X}/\mathfrak{X}, \mathfrak{X}']\theta'$ be the composition of permutations and apply the induction hypothesis with substitutions θ and θ''. For ($\forall\mathfrak{X}$-intro) with the abstracted variable $\mathfrak{X}$ of level n let $\mathfrak{X}'$ be a new second order variable of level n. Apply the induction hypothesis for θ extended by $\mathfrak{X} \mapsto \mathfrak{X}'$. In both cases conclude by ($\forall\mathfrak{X}$-intro) again, with $\mathfrak{X}'$ as abstracted variable.

For the case (Ω_{k+1}) we have by induction hypothesis $\ldots \vdash_{n,(K)} \Gamma\theta'\theta \sqsupset \forall\mathfrak{X}_k A$. For all $\Delta \subset \mathrm{Neg}_k^2$ and $d' \vdash_{k,(K)} \Delta \sqsupset \forall\mathfrak{X}_k A$ we have by induction hypothesis $\ldots \vdash_{k,(K)} \Delta\bar{\theta}' \sqsupset \forall\mathfrak{X}_k A$ where $\bar{\theta}'$ is the inverse of θ'. Hence, by the premise of the (Ω_{k+1})-rule we have $\ldots \vdash_{n,(K)} \Delta\bar{\theta}', \Gamma \sqsupset B$. Again by induction hypothesis we obtain $\ldots \vdash_{n,(K)} \Delta, \Gamma\theta'\theta \sqsupset B\theta'\theta$, noting that $\Delta\theta = \Delta$. Therefore an application of the (Ω_{k+1})-rule again completes the derivation.

All the remaining cases are immediate by induction hypothesis, where in the case of ($\forall x$-intro) the needed renaming is provided by Lemma 5.6. ∎

Note that the proof of the following lemma does not introduce new (Ω)-inferences. The inferences are instead only reconstructed where they occur in the given semi-formal derivation. However, the lemma only considers quantification at the *topmost* level and this is important, for otherwise the restriction $\Gamma \subset \mathrm{Neg}_n^2$ would not suffice. Nevertheless, in applications of the (Ω)-rule in Lemma 5.17 the contexts are of sufficiently small level. So we will use the lemma for various values n strictly smaller than N.

Lemma 5.9 (Admissibility of $\forall\mathfrak{X}_n$-elimination). *There is a primitive recursive $\mathfrak{f}$ such that the following is a theorem of ID_n^*.*

If $d \vdash_{n,(K)} \Gamma \sqsupset \forall\mathfrak{X}_n A(\mathfrak{X}_n)$ with $\Gamma \subset \mathrm{Neg}_n^2$ and $\mathcal{A} \in \mathcal{I}_n^*$ such that $A(\mathcal{A})$ is a well-formed formula, then $\mathfrak{f}(\ulcorner\forall\mathfrak{X}_n A(\mathfrak{X}_n)\urcorner, \ulcorner\mathcal{A}\urcorner, \ulcorner\Gamma\urcorner, d) \vdash_{n,(K)} \Gamma \sqsupset A(\mathcal{A})$.

Proof. Induction along the inductive Definition 5.2 of $d \vdash_{n,(K)} \Gamma \sqsupset \forall\mathfrak{X}_n A(\mathfrak{X}_n)$.

An elimination rule would require an axiom or assumption containing $\forall\mathfrak{X}_n A(\mathfrak{X}_n)$ strictly positive, which cannot be as our axioms are first order and $\Gamma \subset \mathrm{Neg}_n^2$.

If the last rule was ($\forall\mathfrak{X}_n$-intro) then Lemma 5.8 applies.

By the form of the conclusion the only other inference the derivation could end in is (Ω_{k+1}), where we can use the induction hypothesis. ∎

Lemma 5.10 (Collapsing). *There is a partially recursive function $\mathfrak{f}$ such that the following is a theorem of ID_n^*.*

If $d \vdash_{n,(K)} \Gamma \sqsupset A$ with $\Gamma \subset \mathrm{Neg}_\ell^2$ and $A \in \mathrm{Pos}_\ell^2$ for some $\ell < n$ then $\mathfrak{f}(\ell, d, \ulcorner\Gamma\urcorner, \ulcorner A\urcorner) \vdash_{\ell,(K)} \Gamma \sqsupset A$.

Proof. Induction along the inductive Definition 5.2 of $d \vdash_{n,(K)} \Gamma \sqsupset A$, showing the claim simultaneously for all $\ell < n$.

In the case of the (Ω_{k+1})-rule with $\ell \leq k$ we have $\mathfrak{f}(k, d, \dots) \vdash_{k,(K)} \Gamma \sqsupset \forall\mathfrak{X}_k A$ by induction hypothesis. Since $\ell \leq k$ and hence $\Gamma \subset \mathrm{Neg}_\ell^2 \subset \mathrm{Neg}_k^2$ by the premise of the rule we get $\{e\}\left(\langle \mathfrak{f}(k, d, \dots), \ulcorner\Gamma\urcorner\rangle\right) \vdash_{n,(K)} \Gamma \sqsupset B$. By application of the induction hypothesis we get $\mathfrak{f}\left(\ell, \{e\}(\dots), \ulcorner\Gamma\urcorner\right) \vdash_{\ell,(K)} \Gamma \sqsupset B$.

In the case of the (Ω_{k+1})-rule with $\ell > k$ we have by induction hypothesis $\mathfrak{f}(\ell, d, \dots) \vdash_{\ell,(K)} \Gamma \sqsupset \forall\mathfrak{X}_k A$ and $\mathfrak{f}(\ell, \{e\}\left(\langle d', \ulcorner\Delta\urcorner\rangle\right), \dots) \vdash_{\ell,(K)} \Delta, \Gamma \sqsupset B$ if $\Delta \subset \mathrm{Neg}_k^2$ and $d' \vdash_{k,(K)} \Delta \sqsupset \forall\mathfrak{X}_k A(\mathfrak{X}_k)$. Hence an application of the (Ω_{k+1})-rule yields the desired derivation.

If $\ell = 0$ and the last rule was the (ω) rule we use the premise of this rule for some new variable y and conclude by ($\forall x$-intro). The remaining cases are trivial. ∎

With our semi-formal notion of normal proofs we have an appropriate semantics for $\forall\mathfrak{X}$-statements. Based on this semantics we define (partial) truth predicates in the usual way, similar to Tait's computability predicates [14].

To be proof theoretically optimal, these predicates are defined on the meta-level, by a family of formulae. Of course, an additional inductive definition would suffice to define these predicates internally, requiring, however, a stronger system.

Definition 5.11 ($f \Vdash_{(K)} \Gamma \sqsupset C$). By induction on $\mathfrak{k}$ we define a family $f \Vdash_{(K),\mathfrak{k}} \Gamma \sqsupset C$ of formulae in the language of ID_{N-1}^* such that for all $C \in \mathcal{I}_N^*$ with less than $\mathfrak{k}$ logical symbols, all $f \in \mathbb{N}$ and $\Gamma \subset \mathrm{Neg}_{N-1}^2$ the following properties hold.

- $f \Vdash_{(K),\mathfrak{k}} \Gamma \sqsupset C \quad \Leftrightarrow \quad f \vdash_{N-1,(K)} \Gamma \sqsupset C$
  if C atomic or of the form $\forall\mathfrak{X} A$
- $f \Vdash_{(K),\mathfrak{k}} \Gamma \sqsupset A \to B \quad \Leftrightarrow$
  $\forall g \in \mathbb{N}, \Delta \subset \mathrm{Neg}_{N-1}^2 (g \Vdash_{(K),\mathfrak{k}-1} \Delta \sqsupset A$
  $\Rightarrow \{f\}\left(\langle g, \ulcorner\Delta\urcorner\rangle\right) \Vdash_{(K),\mathfrak{k}-1} \Gamma, \Delta \sqsupset B)$
- $f \Vdash_{(K),\mathfrak{k}} \Gamma \sqsupset \forall x A(x) \quad \Leftrightarrow \quad \forall t(\{f\}\left(\ulcorner t\urcorner\right) \Vdash_{(K),\mathfrak{k}-1} \Gamma \sqsupset A(t))$

- $f \Vdash_{(K),\mathfrak{k}-1} \Gamma \sqsupset A \wedge B$
  $\Leftrightarrow \quad (f)_0 \Vdash_{(K),\mathfrak{k}} \Gamma \sqsupset A$ and $(f)_1 \Vdash_{(K),\mathfrak{k}-1} \Gamma \sqsupset B$

Definition 5.11 could be defined by induction on C. However, we need slightly more uniformity. So, officially, we define a family $A_{K,\mathfrak{k}}(x,y,z)$ of formulae indexed by $\mathfrak{k}$, with free number variables for f, $\ulcorner\Gamma\urcorner$ and $\ulcorner C\urcorner$ in such a way that whenever the number of logical symbols in C is at most $\mathfrak{k}$ then $A_{K,\mathfrak{k}}(f,\ulcorner\Gamma\urcorner,\ulcorner C\urcorner)$ is equivalent to $f \Vdash_{(K)} \Gamma \sqsupset C$ as defined (say) by induction on C. This can easily be achieved by setting (with the $\mathfrak{A}_{K,N-1}$ of Remark 5.3)

$$
\begin{aligned}
A_{K,\mathfrak{k}}(x,y,z) :\equiv (&\text{“}z \text{ codes a formula which is atomic or of the form } \forall\mathfrak{X}B\text{”} \\
&\quad\wedge \quad \mathcal{P}_{N-1}^{\mathfrak{A}_{K,N-1}}\langle x,y,z\rangle) \\
&\vee (\text{“}x \text{ codes a formula of the form } B \to C\text{”} \\
&\quad\wedge \quad \forall g \forall \Delta \subset \mathrm{Neg}_{N-1}.\ A_{K,\mathfrak{k}-1}(g,\ulcorner\Delta\urcorner,\ulcorner B\urcorner) \\
&\qquad\qquad\qquad\qquad \to A_{K,\mathfrak{k}-1}(\{f\}\left(\langle g,\ulcorner\Delta\urcorner\rangle\right),\ulcorner\Gamma,\Delta\urcorner,\ulcorner C\urcorner)) \\
&\vee \ldots
\end{aligned}
$$

It should also be noted that for $\mathfrak{k}' \geq \mathfrak{k}$ it is provable in ID^*_{N-1}, that if z codes a formula with at most $\mathfrak{k}$ logical symbols then $A_{\mathfrak{k}'}(x,y,z)$ and $A_{\mathfrak{k}}(x,y,z)$ are equivalent. This can be shown by induction on $\mathfrak{k}$. From now on we will write $f \Vdash_{(K)} \Gamma \sqsupset C$ for $A_{K,\mathfrak{k}}(f,\ulcorner\Gamma\urcorner,\ulcorner C\urcorner)$ with appropriately chosen $\mathfrak{k}$, tacitly assuming $\mathfrak{k}$ to be big enough. It should be obvious, that for each of the following theorems we provide a primitive recursive family of proofs.

Remark 5.12. As weakening is admissible for derivations (Remark 5.5), an easy induction shows that it is also admissible for the truth relation. More precisely, provably in ID^*_{N-1}, if $f \Vdash_{(K)} \Gamma \sqsupset A$, $\Gamma \subset \Gamma'$ then $f \Vdash_{(K)} \Gamma' \sqsupset A$.

Lemma 5.13 (Renaming of second order variables). *There is a primitive recursive function* $\mathfrak{f}$*, such that for every natural number* $\mathfrak{k}$ *the following is a theorem of* ID^*_{N-1}*.*

If C *is a formula with at most* $\mathfrak{k}$ *logical symbols,* θ *a level-preserving permutation of second order variables, and* $f \Vdash_{(K)} \Gamma \sqsupset C$ *then* $\{\mathfrak{f}(\ulcorner C\urcorner,\ulcorner\theta\urcorner)\}(f) \Vdash_{(K)} \Gamma\theta \sqsupset C\theta$*.*

We write $\mathfrak{f}^C_\theta$ *for* $\{\mathfrak{f}(\ulcorner C\urcorner,\ulcorner\theta\urcorner)\}(\cdot)$*.*

Proof. By induction on the logical complexity of C.

If C is atomic or of the form $\forall\mathfrak{X}A$ then Lemma 5.8 applies.

If $C = A \to B$ and $g \Vdash_{(K)} \Delta \sqsupset A\theta$ then, by induction hypothesis $\mathfrak{f}^B_{\bar\theta}(g) \Vdash_{(K)} \Delta\bar\theta \sqsupset A$, where $\bar\theta$ is the inverse of θ. Hence, $\{f\}\left(\mathfrak{f}^B_{\bar\theta}(g)\right) \Vdash_{(K)} \Gamma, \Delta\bar\theta \sqsupset B$ and by induction hypothesis, $\mathfrak{f}^A_\theta(\{f\}\left(\mathfrak{f}^B_{\bar\theta}(g)\right)) \Vdash_{(K)} \Gamma\theta, \Delta \sqsupset B\theta$.

The cases $A \wedge B$ and $\forall xA$ are immediate by induction hypothesis. ∎

Lemma 5.14. *There are primitive recursive functions* $\mathfrak{f}$ *and* $\mathfrak{g}$*, such that for every* $\mathfrak{k}$ *the following is a theorem of* ID^*_{N-1}*.*

If C *is a formula with at most* $\mathfrak{k}$ *logical symbols, then*

- $e \vdash_{N-1,(K)} \Gamma \sqsupset C$*,* $\neg\mathsf{intro}(e)$*,* $C \in \mathrm{Neg}^2_{N-1} \quad\Rightarrow\quad \{\mathfrak{f}(\ulcorner C\urcorner,\ulcorner\Gamma\urcorner)\}(e) \Vdash_{(K)} \Gamma \sqsupset C$*,*
- $f \Vdash_{(K)} \Gamma \sqsupset C$*,* $C \in \mathrm{Pos}^2_{N-1} \quad\Rightarrow\quad \{\mathfrak{g}(\ulcorner C\urcorner,\ulcorner\Gamma\urcorner)\}(f) \vdash_{N-1,(K)} \Gamma \sqsupset C$*.*

Again we write $\mathfrak{f}^C_\Gamma$ *for* $\{\mathfrak{f}(\ulcorner C\urcorner, \ulcorner\Gamma\urcorner)\}(\cdot)$ *and* $\mathfrak{g}^C_\Gamma$ *for* $\{\mathfrak{g}(\ulcorner C\urcorner, \ulcorner\Gamma\urcorner)\}(\cdot)$.

Proof. Induction on the logical complexity of C. We only consider the non-trivial cases.

If C is of the form $A \to B$ we argue as follows.

- We have $A \in \mathrm{Pos}^2_{N-1}$ and $B \in \mathrm{Neg}^2_{N-1}$. Assume $g \Vdash_{(K)} \Delta \sqsupset A$. By induction hypothesis $\mathfrak{g}^A_\Delta(g) \vdash_{N-1,(K)} \Delta \sqsupset A$. Since $\neg\mathsf{intro}(e)$ by assumption, we get $\langle 2, e, \mathfrak{g}^A_\Delta(g), \ulcorner A\urcorner\rangle \vdash_{N-1,(K)} \Gamma, \Delta \sqsupset B$. Hence by induction hypothesis $\mathfrak{f}^B_{\Gamma,\Delta}(\langle 2, e, \mathfrak{g}^A_\Delta(g), \ulcorner A\urcorner\rangle) \Vdash_{(K)} \Gamma, \Delta \sqsupset B$.
- We have $A \in \mathrm{Neg}^2_{N-1}$ and $B \in \mathrm{Pos}^2_{N-1}$. By definition we have $\langle 1\rangle \vdash_{N-1,(K)} A \sqsupset A$. Hence by induction hypothesis $\mathfrak{f}^A_A(\langle 1\rangle) \Vdash_{(K)} A \sqsupset A$. Thus $\{f\}\left(\langle \mathfrak{f}^A_A(\langle 1\rangle), \ulcorner A\urcorner\rangle\right) \Vdash_{(K)} \Gamma, A \sqsupset B$. So $\mathfrak{g}^B_{A,\Gamma}(\{f\}(\ldots)) \vdash_{N-1,(K)} \Gamma, A \sqsupset B$ by induction hypothesis. We conclude by ($\to$-intro).

If C is of the form $\forall x A(x)$ we argue in the following way.

- Since $\neg\mathsf{intro}(d)$, we have $\langle 4, e, \ulcorner\forall x A(x)\urcorner\rangle \vdash_{N-1,(K)} \Gamma \sqsupset A(t)$, hence $\mathfrak{f}^{A(t)}_\Gamma(\langle 4, e, \ulcorner\forall x A(x)\urcorner\rangle) \Vdash_{(K)} \Gamma \sqsupset A(t)$.
- We have for every term t that $\{f\}\left(\ulcorner t\urcorner\right) \Vdash_{(K)} \Gamma \sqsupset A(t)$, hence by induction hypothesis $\mathfrak{g}^{A(t)}_\Gamma(\{f\}\left(\ulcorner t\urcorner\right)) \vdash_{n,(K)} \Gamma \sqsupset A(t)$. The ($\omega$)-rule applies. (If $N = 1$ then $\mathfrak{g}^{A(y)}_\Gamma(\{f\}\left(\ulcorner y\urcorner\right)) \vdash_{0,(K)} \Gamma \sqsupset A(y)$ for a new variable y, and we apply ($\forall x$)-intro.) ◻

Lemma 5.15 (Admissibility of (Ω_{k+1}) for $k \le N-1$). *There is a primitive recursive function* $\mathfrak{f}$ *such that for every* $\mathfrak{k}$ *the following is a theorem of* ID^*_{N-1}.

Let C *be a formula with at most* $\mathfrak{k}$ *logical symbols and assume for* $\Gamma \subset \mathrm{Neg}^2_{N-1}$ *that* $d \Vdash_{(K)} \Gamma \sqsupset \forall\mathfrak{X}_k A$ *and* $k \le N-1$. *Assume moreover that for all* $\Delta \subset \mathrm{Neg}^2_k$ *if* $d' \vdash_{k,(K)} \Delta \sqsupset \forall\mathfrak{X}_k A$ *then* $\{e\}\left(\langle d', \ulcorner\Delta\urcorner\rangle\right) \Vdash_{(K)} \Delta, \Gamma \sqsupset C$. *Then* $\{\mathfrak{f}(\ulcorner C\urcorner, \ulcorner\Gamma\urcorner)\}(\langle d, e\rangle) \Vdash_{(K)} \Gamma \sqsupset C$.

Again we write $\mathfrak{f}^C_\Gamma$ *for* $\{\mathfrak{f}(\ulcorner C\urcorner, \ulcorner\Gamma\urcorner)\}(\cdot)$.

Proof. Induction on the logical complexity of C.

Case C is atomic or of the form $\forall\mathfrak{X} B$ and so we have to show $\ldots \vdash_{N-1,(K)} \Gamma \sqsupset C$. If $k < N-1$, we may apply an (Ω_{k+1})-inference, using $d \vdash_{N-1,(K)} \Gamma \sqsupset \forall\mathfrak{X}_k A$ and $\{e\}\left(\langle d', \ulcorner\Delta\urcorner\rangle\right) \vdash_{N-1,(K)} \Delta, \Gamma \sqsupset C$. If $k = N-1$ then $\{e\}\left(\langle d, \ulcorner\Gamma\urcorner\rangle\right) \Vdash_{(K)} \Gamma \sqsupset C$ by assumption.

Case $C = B \to B'$. Let $\Gamma' \subset \mathrm{Neg}^2_{N-1}$ and assume $g \Vdash_{(K)} \Gamma' \sqsupset B$. We have $d \Vdash_{(K)} \Gamma, \Gamma' \sqsupset \forall\mathfrak{X}_k A$. Moreover, for any $\Delta \subset \mathrm{Neg}^2_k$, $d' \vdash_{k,(K)} \Delta \sqsupset \forall\mathfrak{X}_k A$ we have $\{e\}\left(\langle d', \ulcorner\Delta\urcorner\rangle\right) \Vdash_{(K)} \Gamma, \Delta \sqsupset B \to B'$ by assumption and hence $\left\{\{e\}\left(\langle d', \ulcorner\Delta\urcorner\rangle\right)\right\}\left(\langle g, \ulcorner\Gamma'\urcorner\rangle\right) \Vdash_{(K)} \Gamma, \Delta, \Gamma' \sqsupset B'$. So by induction hypothesis (using $\Gamma, \Gamma' \subset \mathrm{Neg}^2_{N-1}$) we get $\ldots \Vdash_{(K)} \Gamma, \Gamma' \sqsupset B'$.

The remaining cases follow immediately by the induction hypotheses. ◻

Lemma 5.16. *There is a partially recursive* $\mathfrak{f}$ *such that for every natural number* $\mathfrak{k}$ *the following is a theorem of* ID^*_{N-1}.

Assume that each of the formulae $A, \vec{A}$ *has at most* $\mathfrak{k}$ *symbols. Assume moreover that* $d \vdash_{N-1,(K)} \Delta, \vec{A} \sqsupset A$ *and* $f_i \Vdash_{(K)} \Gamma \sqsupset A_i$ *for* $i = 1, \ldots, \ell$ *and some* $\Delta, \Gamma \subset \mathrm{Neg}^2_{N-1}$. *Then* $\mathfrak{f}(\langle d, \langle f_1, \ulcorner A_1\urcorner\rangle, \ldots, \langle f_n, \ulcorner A_n\urcorner\rangle\rangle) \Vdash_{(K)} \Delta, \Gamma \sqsupset A$.

Proof. Within ID^*_{N-1} we argue by the induction principle provided by the inductive Definition 5.2 of $\cdot \vdash_{n,(K)} \cdot \sqsupset \cdot$, where in the case of $\cdot \vdash_{0,(K)} \cdot \sqsupset \cdot$ we actually use induction on the number of inferences. We only consider the non-trivial cases.

Case (Axiom). By Lemma 5.14, noting that axioms are in $\mathcal{I}^2_0 \subset \mathrm{Neg}^2_0 \subset \mathrm{Neg}^2_{N-1}$.

Case (→-elim). By induction hypothesis $e \Vdash_{(K)} \Delta, \Gamma \sqsupset A \to B$ and $e' \Vdash_{(K)} \Delta, \Gamma \sqsupset A$ for appropriate e and e'. Hence $\{e'\}(e) \Vdash_{(K)} \Delta, \Gamma \sqsupset B$ follows from the definition of our partial truth predicates.

Case (→-intro). By induction hypothesis we know that $f \Vdash_{(K)} \Gamma' \sqsupset A$ implies $\mathfrak{f}(\langle d, \langle f_1, \ulcorner A_1 \urcorner\rangle, \ldots, \langle f_n, \ulcorner A_n \urcorner\rangle, \langle f, \ulcorner A \urcorner\rangle\rangle) \Vdash_{(K)} \Delta, \Gamma, \Gamma' \sqsupset B$. Hence $e \Vdash_{(K)} \Delta, \Gamma \sqsupset A \to B$ by Definition 5.11 of the truth predicates, where e is an index of the recursive function $f \mapsto \mathfrak{f}\left(\langle d, \langle f_1, \ulcorner A_1 \urcorner\rangle, \ldots, \langle f_n, \ulcorner A_n \urcorner\rangle, \langle f, \ulcorner A \urcorner\rangle\rangle\right)$.

Case (∀x-intro). By Corollary 5.7 we find an index e such that for every term t we have $\{e\}\left(\ulcorner t \urcorner\right) \vdash_{0,(K)} \Delta, \vec{A} \sqsupset A(t)$ with a derivation with no more inferences. Hence by induction hypothesis we get the desired truth relation in a recursive way for every t.

Case (∀$\mathfrak{X}$-intro). We have $d \vdash_{n,(K)} \Delta, \vec{A} \sqsupset A(\mathfrak{X})$ and $\mathfrak{X}$ not free in $\Delta, \vec{A}$. Let $\mathfrak{Y}$ be a new second order variable and $\theta = [\mathfrak{Y}, \mathfrak{X}/\mathfrak{X}, \mathfrak{Y}]$. Noting that $\mathfrak{X}$ does not occur in A_i, Lemma 5.13 yields $\mathfrak{f}^{A_i}_\theta(f_i) \Vdash_{(K)} \Gamma\theta \sqsupset A_i$. Hence, by induction hypothesis $\ldots \Vdash_{(K)} \Gamma\theta, \Delta \sqsupset A(\mathfrak{X})$, and by Lemma 5.14 we get $\ldots \vdash_{N-1,(K)} \Gamma\theta, \Delta \sqsupset A(\mathfrak{X})$. By (∀$\mathfrak{X}$-intro) we obtain $\ldots \vdash_{N-1,(K)} \Gamma\theta, \Delta \sqsupset \forall\mathfrak{X} A(\mathfrak{X})$ and hence by Lemma 5.8, since $\mathfrak{X}$ is not free in Δ and $\mathfrak{Y}$ is new, $\ldots \vdash_{N-1,(K)} \Gamma, \Delta \sqsupset \forall\mathfrak{X} A(\mathfrak{X})$. This is what we had to show.

Case (Ω_{k+1}). We have $k < N-1$ and $d \vdash_{N-1,(K)} \Delta, \vec{A} \sqsupset \forall\mathfrak{X}_k A(\mathfrak{X}_k)$. Moreover, for all $\Delta' \subset \mathrm{Neg}^2_k$ it holds that if $d' \vdash_{k,(K)} \Delta' \sqsupset \forall\mathfrak{X}_k A(\mathfrak{X}_k)$ then $\{e\}\left(\langle d', \ulcorner \Delta' \urcorner\rangle\right) \vdash_{N-1,(K)} \Delta', \Delta, \vec{A} \sqsupset B$.

By induction hypothesis we have $\ldots \Vdash_{(K)} \Delta, \Gamma \sqsupset \forall\mathfrak{X}_k A(\mathfrak{X}_k)$. Moreover, also by induction hypothesis we know for every $\Delta' \subset \mathrm{Neg}^2_k$ that if $d \vdash_{k,(K)} \Delta' \sqsupset \forall\mathfrak{X}_k A(\mathfrak{X}_k)$ then $\ldots \Vdash_{(K)} \Delta', \Delta, \Gamma \sqsupset B$. Using the assumption that $\Delta, \Gamma \subset \mathrm{Neg}^2_{N-1}$ we can apply Lemma 5.15 to obtain $\ldots \Vdash_{(K)} \Delta, \Gamma \sqsupset B$. □

Lemma 5.17 (Admissibility of (∀$\mathfrak{X}$-elimination)). *There is a partially recursive $\mathfrak{f}$ such that for all A and for every $\mathcal{A} \in \mathcal{I}^*_N$, such that $A(\mathcal{A})$ is well formed, the following is a theorem of* ID^*_{N-1}.

If $\Gamma \subset \mathrm{Neg}^2_{N-1}$ *and* $f \Vdash_{(K)} \Gamma \sqsupset \forall\mathfrak{X}_k A(\mathfrak{X}_k)$ *then* $\mathfrak{f}(f, \ulcorner\Gamma\urcorner, \ulcorner A \urcorner, \ulcorner \mathfrak{X}_k \urcorner, \ulcorner \mathcal{A} \urcorner) \Vdash_{(K)} \Gamma \sqsupset A(\mathcal{A})$.

Proof. If $k = N-1$ then we argue as follows: $f \vdash_{N-1,(K)} \Gamma \sqsupset \forall\mathfrak{X}_{N-1} A(\mathfrak{X}_{N-1})$ by assumption, hence $\ldots \vdash_{N-1,(K)} \Gamma \sqsupset A(\mathcal{A})$ by Lemma 5.9, and $\ldots \Vdash_{(K)} \Gamma \sqsupset A(\mathcal{A})$ by Lemma 5.16.

If $k < N-1$ we use the (Ω_{k+1})-rule to construct a derivation of $A(\mathcal{A})$. From this derivation we get a the truth of $A(\mathcal{A})$ by Lemma 5.16. We have $f \vdash_{N-1,(K)} \Gamma \sqsupset \forall\mathfrak{X}_k A$ by assumption. So assume $\Delta \subset \mathrm{Neg}^2_k$ and $d \vdash_{k,(K)} \Delta \sqsupset \forall\mathfrak{X}_k A$. Then by Lemma 5.9 we have $\ldots \vdash_{k,(K)} \Delta \sqsupset A(\mathcal{A})$, and therefore also $\ldots \vdash_{N-1,(K)} \Delta, \Gamma \sqsupset A(\mathcal{A})$. □

Lemma 5.18 (Admissibility of (∀$\mathfrak{X}$-introduction)). *There is a partially recursive $\mathfrak{f}$ such that for every formula A the following is a theorem of* ID^*_{N-1}.

If $f \Vdash_{(K)} \Gamma \sqsupset A(\mathfrak{X}_k)$ for some $k \leq N-1$ and $\Gamma \subset \mathrm{Neg}^2_{N-1}$, such that $\mathfrak{X}_k$ is not free in Γ, then $\mathfrak{f}(f, \ulcorner \mathfrak{X}_k \urcorner, \ulcorner A \urcorner, \ulcorner \Gamma \urcorner) \Vdash_{(K)} \Gamma \sqsupset \forall \mathfrak{X}_k A(\mathfrak{X}_k)$.

Proof. We have $A(\mathfrak{X}_k) \in \mathrm{Pos}^2_{N-1}$. Hence by Lemma 5.14 we have $\ldots \vdash_{N-1,(K)} \Gamma \sqsupset A(\mathfrak{X}_k)$, so by an application of ($\forall\mathfrak{X}$-intro) we get a derivation of $\forall \mathfrak{X}_k A(\mathfrak{X}_k)$, so the truth predicate holds for that formula. ∎

Theorem 5.19 (Cut elimination). *For every proof in HA^2_N of A from assumptions $\vec{A}$ there is a partially recursive $\mathfrak{f}$ such that for every Γ the following is a theorem of ID^*_{N-1}.*

For every σ, if $f_i \Vdash_{(K)} \Gamma\sigma \sqsupset A_i\sigma$ for all i then $\mathfrak{f}(\ulcorner \sigma \urcorner, \vec{f}) \Vdash_{(K)} \Gamma\sigma \sqsupset A\sigma$.

Proof. Induction on the HA^2_N derivation. For the axioms we use Lemma 5.14 using that our axioms are closed under substitution. The cases of introduction and elimination of second order quantifiers are covered by Lemmata 5.18 and 5.17. The remaining cases are trivial using the nature of $\cdot \Vdash_{(K)} \cdot \sqsupset \cdot$ as partial truth predicate. ∎

Corollary 5.20 (Conservativity for first order formulae). *If $\mathrm{HA}^2_{\mathrm{N,(K)}} \vdash A$ for some first order formula A then $\mathrm{ID}^*_{N-1} \vdash (\mathrm{HA}^2_{\mathrm{0,(K)}} \vdash A)$.*

As $\mathrm{HA}^2_{\mathrm{0,(K)}}$ is a subsystem of Π_K-IA the last statement in particular implies $\mathrm{ID}^*_{N-1} \vdash (\Pi_K\text{-IA} \vdash A)$.

Proof. From the proof of A we obtain by Theorem 5.19 that $\mathrm{ID}^*_{N-1} \vdash \exists f (f \Vdash_{(K)} \emptyset \sqsupset A)$. By Proposition 5.14 we get $\mathrm{ID}^*_{N-1} \vdash \exists g (g \vdash_{N-1,(K)} \emptyset \sqsupset A)$ and by Proposition 5.10 we obtain $\mathrm{ID}^*_{N-1} \vdash \exists g' (g' \vdash_{0,(K)} \emptyset \sqsupset A)$. Inspection of Definition 5.2 shows that all the rules of $\cdot \vdash_{0,(K)} \cdot \sqsupset \cdot$ are also proof rules of $\mathrm{HA}^2_{\mathrm{0,(K)}}$, hence ID^*_{N-1} can read g' as a proof in $\mathrm{HA}^2_{\mathrm{0,(K)}}$. ∎

Remark 5.21. Since the truth of $\mathcal{L}_{N-1}$-formulae can be defined by one additional inductive definition on top of the $N-1$ inductive definitions of ID^*_{N-1}, an induction (within ID^*_N) on ID^*_{N-1}-proofs shows for $\mathcal{L}_{N-1}$-formulae B that

$$\mathrm{ID}^*_N \vdash \quad \forall \vec{x}.\ (\mathrm{ID}^*_{N-1} \vdash B(\vec{x})) \to \mathcal{P}^{\mathrm{True}}_N \ulcorner B(\dot{\vec{x}}) \urcorner$$

Finally, by (meta)induction on the $\mathcal{L}_{N-1}$-formula B we show that

$$\mathrm{ID}^*_N \vdash \forall x.\, \mathcal{P}^{\mathrm{True}}_N \ulcorner B(\dot{\vec{x}}) \urcorner \leftrightarrow B(\vec{x})$$

so that we obtain that ID^*_N reflects ID^*_{N-1} in the sense that

$$\mathrm{ID}^*_N \vdash \quad \forall \vec{x}.\ (\mathrm{ID}^*_{N-1} \vdash B(\vec{x})) \to B(\vec{x})$$

and we therefore get (by taking $B(\ulcorner A \urcorner) \equiv (\Pi_K\text{-IA} \vdash A)$) as an immediate consequence of Corollary 5.20 that the following is a theorem of ID^*_N.

All arithmetical consequences of $\mathrm{HA}^2_{\mathrm{N,(K)}}$ are already consequences of Π_K-IA.

Remark 5.22. Believing in ID^*_N, Remark 5.21 might look disturbing.

The systems $\mathrm{HA}^2_{\mathrm{n,(K)}}$ increase in consistency strength over PRA with growing n, irrespectively of the amount K of arithmetical induction allowed. On the other hand, the arithmetical consequences are only determined by the amount of arithmetical induction.

But what about the consistency statement for $\mathrm{HA}^2_{\mathrm{n},(\mathrm{K})}$, which is provable in HA^2_{n+1}?

The answer is, that in the absence of full induction the correct notion of the consistency statement is no longer the Π^0_1-statement "For all proofs it is not the case…", but the (classically) Σ^1_1-statement "For all *natural numbers* it is not the case that they code a proof…" which has the shape $\forall x.\mathbb{N}\,x \to \dots$ and is no longer an arithmetical formula.

Remark 5.23. Arguing within set theory, it is possible to obtain the conservativity result for arithmetical formulae (Remark 5.21) by the following model theoretic argument.

Assume $\mathrm{HA}^2_{\mathrm{N},(\mathrm{K})} \vdash A$ for some first order formula A. We want to show $\mathrm{HA}^2_{0,(\mathrm{K})} \vdash A$. By soundness we have $\mathrm{HA}^2_{\mathrm{N},(\mathrm{K})} \models A$ and by completeness it suffices to show $\mathrm{HA}^2_{0,(\mathrm{K})} \models A$. So assume $\mathfrak{M} \models \mathrm{HA}^2_{0,(\mathrm{K})}$ for some first order structure $\mathfrak{M}$, recalling that $\mathrm{HA}^2_{0,(\mathrm{K})}$ is a first order theory. We have to show $\mathfrak{M} \models A$. Let $\tilde{\mathfrak{M}}$ be the second order extension of $\mathfrak{M}$ obtained by taking the full power set(!) as second order structure. Then $\tilde{\mathfrak{M}} \models \mathrm{HA}^2_{\mathrm{N},(\mathrm{K})}$. In fact, $\tilde{\mathfrak{M}}$ even is a model of full comprehension (but not necessarily of full induction). Hence $\tilde{\mathfrak{M}} \models A$. But for first order formulae A we have $\tilde{\mathfrak{M}} \models A$ if and only if $\mathfrak{M} \models A$.

However, it should be noted that this simple argument does *not* render Corollary 5.20 useless, as it provides also the precise proof theoretical strength of this conservativity statement. Note that the conservativity statement in particular implies the consistency of $\mathrm{HA}^2_{\mathrm{N},(\mathrm{K})}$ over Peano Arithmetic. So Corollary 5.20 is optimal, as $\mathrm{HA}^2_{\mathrm{N},(\mathrm{K})}$ and ID^*_{N-1} are equiconsistent by Corollary 5.28.

Lemma 5.24. *There is a primitive recursive* $\mathfrak{f}$ *with* $\mathrm{ID}^*_0 \vdash (\forall n.\mathfrak{f}n \Vdash_{(K)} \emptyset \sqsupset \mathbb{N}\,\underline{n})$.

Proof. Argue informally in ID^*_0. For every n we have to construct a derivation (in the sense of $\cdot \vdash_0 \cdot \sqsupset \cdot$) of $\forall \mathfrak{X}_0.\mathfrak{X}_0 0 \to \forall x(\mathfrak{X}_0 x \to \mathfrak{X}_0(\mathcal{S}(x))) \to \mathfrak{X}_0\underline{n}$. To do so, it suffices to construct a derivation of $\mathfrak{X}_0\underline{n}$ of $\mathfrak{X}_0 0$ and $\forall x(\mathfrak{X}_0 x \to \mathfrak{X}_0(\mathcal{S}(x)))$. We do this in the obvious way and verify correctness of this construction by induction(!) on n. ∎

Lemma 5.25. *If* $\mathrm{HA}^2_{\mathrm{N},(\mathrm{K})} \vdash \forall x.\mathbb{N}\,x \to \neg\forall y \neg R(x,y)$ *then there is a partially recursive* $\mathfrak{f}$ *such that* $\mathrm{ID}^*_{N-1} \vdash \forall x.R(x,\mathfrak{f}x)$.

Proof. Assume $\mathrm{HA}^2_{\mathrm{N},(\mathrm{K})} \vdash \forall x.\mathbb{N}\,x \to \neg\forall y \neg R(x,y)$. By Theorem 5.19 we obtain $\mathrm{ID}^*_{N-1} \vdash \exists f(f \Vdash_{(K)} \emptyset \sqsupset \forall x.\mathbb{N}\,x \to \neg\forall y\neg R(x,y))$. By Lemma 5.24 we get $\mathrm{ID}^*_{N-1} \vdash \forall n(\big\{\{f\}\,(\ulcorner\dot{\underline{n}}\urcorner)\big\}\,(\mathfrak{f}(n)) \Vdash_{(K)} \emptyset \sqsupset \neg\forall y\neg R(\underline{n},y))$.

Now argue informally in ID^*_{N-1}. By Proposition 5.14 we have a partially recursive $\mathfrak{g}$ such that $\forall n(\mathfrak{g}(n) \vdash_{N-1,(K)} \emptyset \sqsupset \neg\forall y\neg R(\underline{n},y))$. Hence by Proposition 5.10 we get another partially recursive $\mathfrak{g}'$ such that $\forall n(\mathfrak{g}'(n) \vdash_{0,(K)} \emptyset \sqsupset \neg\forall y\neg R(\underline{n},y))$. This is a derivation in the system Π_K-IA. *Since every formula provably in* Π_K-IA *is true,* there really is a y such that $R(n,y)$ holds. Since R is decidable, this y can be found in a recursive way. ∎

Remark 5.26. In the proof of Lemma 5.25 we used the well-known property that arithmetic reflects the system Π_K-IA which reads as follows.

$$\mathrm{PA} \vdash \forall n.(\Pi_K\text{-IA} \vdash \exists y R(\underline{n},y)) \to \exists y R(n,y)$$

However, we only needed a simpler form thereof, where we only have to deal with proofs that are already partially cut-eliminated.

Summing up, we get the main result of this article.

Theorem 5.27. *Let n and k be a natural number and $R(\cdot,\cdot)$ a primitive recursive atom. Then the following are equivalent.*

(α) $\mathrm{ID}^c_n \vdash \forall x \exists y R(x,y)$

(β) $\mathrm{ID}^-_n \vdash \forall x \neg\forall y \neg R(x,y)$

(γ) $\mathrm{HA}^2_{\mathrm{n},(+)} \vdash \forall x \neg\forall y \neg R(x,y)$

(δ) $\mathrm{HA}^2_{n+1} \vdash \forall x.\mathbb{N}\, x \to \neg\forall y(\mathbb{N}\, y \to \neg R(x,y))$

(δ') $\mathrm{HA}^2_{\mathrm{n+1,(k)}} \vdash \forall x.\mathbb{N}\, x \to \neg\forall y(\mathbb{N}\, y \to \neg R(x,y))$

(ε) $\mathrm{HA}^2_{n+1} \vdash \forall x.\mathbb{N}\, x \to \neg\forall y \neg R(x,y)$

(ε') $\mathrm{HA}^2_{\mathrm{n+1,(k)}} \vdash \forall x.\mathbb{N}\, x \to \neg\forall y \neg R(x,y)$

(ζ) *For some natural number e it is the case that $\mathrm{ID}^*_n \vdash \forall x R(x, \{\underline{e}\}(x))$ and this statement is to be read that in particular $\mathrm{ID}^*_n \vdash \forall x \exists y.\, \{\underline{e}\}(x) = y$.*

Proof. (α)⇒(β) is Corollary 3.8; (β)⇒(γ) is Corollary 3.16; (γ)⇒(δ) is Corollary 4.7; (δ)⇒(ε), (ε)⇒(ε'), (δ)⇒(δ') and (δ')⇒(ε') are trivial; (ε')⇒(ζ) is Lemma 5.25 and (ζ)⇒(α) holds since ID^*_n is a subsystem of ID^c_n. ∎

Corollary 5.28. *Taking for R in Theorem 5.27 an inconsistent formula, say $1 = 0$, we get equiconsistency of ID^c_n, ID^-_n, $\mathrm{HA}^2_{\mathrm{n},(+)}$, HA^2_{n+1}, $\mathrm{HA}^2_{\mathrm{n+1,(k)}}$, and ID^*_n.*

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