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GOTTLOB FREGE
Kevin C. Klement

Introduction

Although primarily a mathematician in both training and employment, German thinker Gottlob Frege's writings are thoroughly philosophical in their outlook, style and significance. He despaired at what he saw as the average working mathematician's inattention to the fundamental questions of their discipline: what the nature of a number is, how our knowledge of them and their properties is possible, what conditions a mathematical argument must meet in order to constitute a legitimate proof, and so on. At least with regard to the proper understanding of arithmetic, he also found the views popular among philosophers of his day severely wanting. Attempting to redress these deficiencies, Frege produced works that, *in time*, profoundly changed the practices of logicians, philosophers and mathematicians, though his influence was slow in gaining ground.

Most of Frege's intellectual endeavors grew out of the attempt to get clear about the nature of arithmetical truth. Frege endorsed a position now known as *logicism*, the thesis that arithmetical truths, when properly analyzed and demonstrated, reveal themselves to be logical or analytical truths. In pursuing this project, Frege developed important and lasting criticisms of rival theories, and innovated a new approach to logic itself so profoundly different from what was prevalent at the time that it is now often described as *the* fundamental point of departure between contemporary approaches to logic and the Aristotelian tradition which had dominated for over two millennia. In thinking about the proper analysis of mathematical propositions, he also developed views about the logical segmentation of language, the distinction between the sense and reference of linguistic expressions, and the nature of truth, which have been profoundly influential in more recent (especially analytic) philosophy of language and metaphysics.

Life and works

Compared to many of his contemporaries, the details of Frege's life are relatively unknown. Friedrich Ludwig Gottlob Frege was born 8 November 1848 in Wismar, Germany. Both parents were educators at a private secondary institution for girls. Frege

studied at the *Gymnasium* at Wismar from 1864 until 1869, when he began studies in mathematics at the University of Jena. In 1871, Frege transferred to the University of Göttingen, where he primarily pursued mathematics, but also studied philosophy with the neo-Kantian philosopher and logician, Hermann Lotze. Frege completed his doctoral dissertation, on planar geometry, in late 1873 under the guidance of Ernst Schering. In 1874, on the strength of his *Habilitationsschrift*, devoted to the mathematics of complex functions, and the recommendation of a long-time supporter and benefactor in Jena, the physicist of optics, Ernst Abbe, Frege was granted a lectureship at the University of Jena, where he remained for the rest of his academic life.

At Jena, Frege was in regular contact with other neo-Kantian philosophers, including Liebmann and Windelband, and all in all, Frege seems to have been more sympathetic with their work than with the rival speculative Hegelian tradition. He took from them his anti-psychologism, his apriorism, his esteem for Leibniz, and his distrust of various forms of naturalism. He shared with them the conviction that the logical validity of a deduction is independent of its causal origins, and an understanding of logic as a science of fully general laws of truth. Nevertheless, it is largely in the ways in which his views *deviate* from those of Kant (at least) that Frege has come to be known: in particular, his contention that it is possible for purely logical sources of knowledge to acquaint us with a realm of abstract objects, and more specifically, his contention that the truths of arithmetic could be understood as analytically true.

It is not known precisely when Frege first conceived of his project of attempting to reduce the truths of arithmetic to those of logic, but it may have been as early as the time of his *Habilitation*. Frege's first major work in logic, *Begriffsschrift* (literally "Concept-script,"), published in 1879, is more or less a primer in the logical system he endorsed as an alternative to the then-dominant Boolean algebraist logics. In it, he laid out the first ever fully axiomatic deductive calculus for first- and second-order quantifier logic, and showed its power by defining within it the notion of the ancestral of a relation, used in the logical analysis of series or sequences. Strictly speaking, Frege's system is a function calculus, but the similarity between it and a predicate calculus is so strong that the *Begriffsschrift* is usually seen as the inauguration of modern predicate logic. It stopped short, however, of including the theory of classes as extensions of concepts necessary for Frege's definition of numbers.

It was certainly not an immediate success. In 1919 (p. 25), Bertrand Russell wrote, "[i]n spite of the great value of this work [Frege's *Begriffsschrift*], I was, I believe, the first person who ever read it more than twenty years after its publication." Russell's remark is an exaggeration, but a telling one. This and others of his writings did receive some attention from the leaders in the field, including Venn, Schröder, Cantor and Husserl, and over the next twenty-five years, Frege did correspond with Peano, Löwenheim, Hilbert and other top figures working on the foundations and philosophy of arithmetic. Nevertheless, overall, his works received lukewarm reviews at best. There are no doubt many contributing factors to this. While the still-widespread reputation of Frege's two-dimensional logical notation as being cumbersome or difficult to understand is almost completely undeserved, it is intimidating in its unfamiliar look, and Frege did not – in the *Begriffsschrift* anyway – do much to compare his system to

the notations of the Boolean tradition in a way that would encourage those already familiar with the alternatives to take it seriously. Frege did follow up with articles explicitly comparing his logical language to Boolean algebra, but these did not seem to gather much notice, and the harsh criticism they contain of the rival approach would likely have been more alienating than inviting. In general, Frege did not go out of his way to call attention to commonalities between his approach and interests and those of others, rarely acknowledged intellectual debts to philosophers, and even more rarely to other mathematicians. It is small wonder, then, that similar fates were in store for Frege's other major works, and it is entirely possible Frege's name was so unknown in Britain in the early years of the twentieth century that Russell was entirely justified to have leapt to the conclusion he did.

Frege's next major work, *Die Grundlagen der Arithmetik* ("The Foundations of Arithmetic"), appeared in 1884 and represents an attempt on Frege's part – occasioned by the advice of colleagues – to explain the core of his logicist program in ordinary language, in preparation for more technical works which were to follow, along with polemical replies to rival views. Although largely ignored in Frege's day, it is safe to say that this work has since become one of the most influential works in the philosophy of mathematics ever written.

Volume 1 of Frege's magnum opus, *Grundgesetze der Arithmetik* ("Basic Laws of Arithmetic") was published in 1893. Preceding it in the early 1890s, Frege published some important articles in which he outlined certain aspects of how his views on philosophical logic and the theory of meaning had changed since the writing of the *Grundlagen*, including his famous "Über Sinn und Bedeutung" ("On Sense and Reference") in 1892. The *Grundgesetze* itself was devoted largely to actual demonstrations of the most important principles of arithmetic beginning with only logical axioms and inference rules. It added to the logic of *Begriffsschrift* notation and axioms dealing with classes or extensions of concepts necessary for, among other things, defining numbers in Frege's approach. These additions were discovered to make the system of *Grundgesetze* inconsistent by Russell in 1902. Frege received a letter from Russell informing him of the contradiction just as volume 2 of *Grundgesetze* was in the initial stages of being typeset for publication. Frege was, in his own words, "thunderstruck," (PMC: 132) and hastily prepared an appendix to the volume discussing the contradiction and offering a very tentative proposal for revision.

He continued to do important work for a few more years. A series of articles entitled "Über die Grundlagen der Geometrie," ("On the Foundations of Geometry") was published between 1903 and 1906, representing Frege's side of a debate with David Hilbert over the nature of geometry and the proper understanding of axiomatic systems. However, around 1906, probably due to some combination of poor health, the early loss of his wife in 1905, frustration with his failure to find an adequate solution to Russell's paradox, and disappointment over the continued poor reception of his work, Frege seems to have lost his intellectual steam. He produced almost no published work between 1906 and his retirement in 1918, and the initially planned third volume of *Grundgesetze* never appeared.

Ironically, it was during these years that Frege's eventual legacy as an influential figure was fomented. Thanks to an appendix dedicated to Frege's work in Russell's

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Principles of Mathematics, Frege began to become known to the English-speaking world, including to Ludwig Wittgenstein, who had been an engineering student in Manchester until discovering Russell's works and, through them, Frege's, in 1910. Wittgenstein wrote to Frege in 1911 concerning the proper solution to Russell's paradox. Frege invited Wittgenstein to Jena, and engaged him in philosophical debate, and eventually recommended him to study with Russell at Cambridge. The two continued in contact for many years to follow, up through till the publication of Wittgenstein's seminal *Tractatus Logico-Philosophicus*, which lists Frege as a major influence. Also, during the years 1911–13, Rudolf Carnap attended Frege's lectures on logic and the foundations of mathematics: Frege thereby influenced Carnap's important works in logic and semantics. Carnap later described these lectures as "the most fruitful inspiration I received from university lectures" while at Jena (Reck and Awodey 2004: 18).

Upon retirement, Frege returned to the Wismar area, and also to an improved pace of writing and publishing, most notably, a series of articles including the important 1918 piece, "Der Gedanke" ("The Thought"). In 1924, having abandoned full-fledged logicism, Frege sought to describe a new approach to understanding the foundations of arithmetic which assigned Kantian pure intuitions of space and time a significant role to play. Frege did not live long enough to pursue this new tack in much detail, however. He died on 26 July 1925 at the age of seventy-six.

It was not until the next generation of philosophical logicians, some of whom had been directly inspired by Frege himself, including Wittgenstein, Carnap, Church, Geach, Quine, Ramsey, etc., recognized and acknowledged the innovation and importance of Frege's logical and philosophical works that his influence became more fully felt. In bequeathing his unpublished work to his son, Alfred, Frege wrote prophetically, "I believe there are things here which will one day be prized much more highly than they are now. Take care that nothing gets lost." Alfred later gave the papers to Heinrich Scholz of the University of Münster for safe-keeping. Unfortunately, however, they were destroyed in a bombing raid in the Second World War. Although Scholz had made copies of some of the more important pieces, a good portion of Frege's unpublished works was lost.

Although a fierce and often satirical polemicist, Frege was a quiet, reserved man. He was right-wing in his political views, distrustful of foreigners and rather anti-Semitic. Himself Lutheran, Frege seems to have wanted all Jews expelled from Germany, or at least deprived of certain political rights. These facts of Frege's personality have gravely disappointed some of Frege's intellectual progeny, the present author included.

Contributions to logic

Frege's logical project

Although Frege agreed with Kant that the truths of geometry are synthetic *a priori*, for most of his career, Frege believed that the truths of arithmetic are analytic. In this, he agreed with Leibniz against both the Kantians, who held that the truths of arithmetic

were grounded in "pure intuition," and more empiricist thinkers such as Mill, who thought arithmetic was grounded in observation. Frege defined an analytic truth as one whose ultimate proof relies only on "general logical laws and definitions" (GL: §3). Hence, Frege set about to establish his logicist position by actually providing the requisite proofs in a deductive system whose only axioms and inference rules were purely logical. Frege situated himself within a broader movement towards greater rigor in definitions and deductions in mathematics, no doubt having in mind both Cauchy's, Weierstrass's and Dedekind's work on clarifying the notions of limits and continuity, as well as the push for more systematic proofs in geometry following the discovery of non-Euclidian interpretations (Gauss, Riemann and others).

Again following Leibniz, Frege held that natural language was unsuited to the task, so he sought to create a language that could combine the properties of what Leibniz had called a "*lingua characterica*" (a logically perspicuous language in which the meaning of each symbol is made fully precise) and a "*calculus ratiocinator*" (a language in which precise rules can be set forth to determine what does, and what does not, count as a legitimate inference).

Although there had been attempts, most notably those of Boole and his successors, to fashion logical notations that had some of these features, Frege was not sufficiently pleased with them. For example, under Boole's approach, categorical judgments in forms such as *All A are B* and *Some A are B* were captured by means of operations on classes. Boole interpreted the "logical multiplication" *AB*, of classes *A* and *B* as their intersection, i.e. the class of the members they have in common. He used the numerals "1" and "0" to represent the universal and empty classes, respectively. (The universal class is the class with everything as a member, and the empty class is the class with no members.) Then, the categorical forms, *all A are B* and *some A are B* could be represented, respectively as:

$$AB = A \quad AB \neq 0$$

The former says that the intersection of *A* and *B* has precisely the same members as *A* itself, which can be true (if and) only if all members of *A* are also members of *B*. The latter says that the intersection of *A* and *B* is not empty, i.e. that they have members in common. The same notations could be reinterpreted to deal with propositional inferences; e.g. if logical multiplication is reinterpreted as conjunction, "1" as representing truth, and "0" as falsehood, the above could be understood as the claims that *A* (materially) implies *B*, and that *A* and *B* are both true, respectively.

Frege found this approach unsuitable for a variety of reasons: (i) it reused mathematical notation in ways incompatible with their usual mathematical meanings, thereby posing an obstacle for using it to represent mathematical proofs without ambiguity, (ii) with its dual interpretations, it bifurcated its propositional and categorical/quantificational elements, thereby making it impossible to represent deductions involving both kinds of steps, and (iii) without further supplementation, it could not deal with statements of multiple generality (e.g. "*every* person loves *some* city"). It is for these reasons that Frege developed his own logical language, which he dubbed "Begriffsschrift."

Function and argument

Frege's book *Begriffsschrift* is subtitled "a formal language for pure thought, modeled upon that of arithmetic," and in particular it took over the function/argument analysis of mathematical formulæ and utilized it as a replacement for the subject/predicate analysis found in traditional logic. Thus, for example, in the complex expression "5 + 7," "+" represents a function, "5" and "7" represent its two arguments, and the whole expression "5 + 7" represents the value of that function for those arguments – it is another name for 12. According to Frege, a function expression can be understood as "incomplete" in the sense that it affords a place or places for its arguments to be written; hence, it might be better to think of the function expression as "( ) + [ ]" or "$\xi + \zeta$," where "ξ" and "ζ" (or "( )" and "[ ]") are understood as placemarkers for where the names of the arguments would be written. In this regard, function signs differ from proper names of objects, and Frege thought there was a corresponding difference between functions and objects themselves; whereas objects are "complete" functions he said, were "unsaturated" (*ungesättigt*). In a mathematical equation, such as:

$$f(x) = x^2 + 1$$

the function $f(\ )$ is said to take its argument and yield as value the result of squaring the argument and adding 1. The import of the letter "x" here used as a variable, is, according to Frege, to lend generality to the statement, so that this is said to hold for *every* possible argument to this function.

In order to make his logical language applicable more generally, Frege expanded the function expressions it contained to represent functions with arguments and values other than numbers. In his mature views of *Grundgesetze*, this included two special objects, truth-values, "the True" and "the False." A function of one argument, whose value, for every argument, is either the True or the False, Frege called a "concept." Thus, for example, if $H(\ )$ is the function such that:

$$H(x) = \begin{cases} \text{the True, if } x \text{ is human, or} \\ \text{the False, otherwise.} \end{cases}$$

then we can regard $H(\ )$ as the concept *being human*. Hence, if "a" stands for Aristotle, $H(a)$ is the True, but if "b" stands for Boston, then $H(b)$ is the False. An object whose value is the True when taken as argument to a given concept is said to "fall under" that concept. A function with two (or more) arguments whose value is always either the True or the False Frege called a "relation." Thus one way to regard the relation sign "$\xi \leq \zeta$" is as standing for a function whose value is the True in case its first argument is less than or equal to its second, and whose value is the False, otherwise.

Syntax and axiomatization of the core logic

The logical constants in Frege's system were also taken to represent functions. One such constant, called "the horizontal" or "content stroke," is used mainly to transform any term into the name of a truth-value:

$$— x = \begin{cases} \text{the True, if } x \text{ is the True, or} \\ \text{the False, if } x \text{ is anything other than the True.} \end{cases}$$

Notice that if the argument is a truth-value, the value is the same truth-value; otherwise, this yields the False. The reverse of this function is the negation stroke, which is the nearest equivalent in Frege's system to the modern "$\sim p$" or "$\neg\, p$," except that, for Frege, this is a function sign that is applied to a term to form a term, rather than a statement operator:

$$\neg\, x = \begin{cases} \text{the False, if } x \text{ is the True, or} \\ \text{the True, if } x \text{ is anything other than the True.} \end{cases}$$

Frege's near-equivalent to the material conditional sign "$p \rightarrow q$" or "$p \supset q$" involves a two-dimensional branched array with the antecedent term written below the consequent.

$$\begin{array}{c} y \\ x \end{array} = \begin{cases} \text{the False, if } x \text{ is the True and } y \text{ is other than the True, or} \\ \text{the True, otherwise.} \end{cases}$$

The three examples above are all first-level functions, meaning that they take objects as argument. The first two are first-level concepts, the last, a first-level relation.

As in algebra, Frege utilized variables to make quantified or general statements. When Roman letters are used, the generality applies to the entire proposition (as demarcated by the judgment stroke, discussed below). If Frege wished the scope of the generality to apply to only part of the proposition, he utilized Gothic (Fraktur) letters, along with a concavity as a variable-binding quantifier and scope marker, akin to the contemporary "$(x)(\ldots x \ldots)$" or "$\forall x(\ldots x \ldots)$."

$$\underset{\mathfrak{a}}{—}\, \phi(\mathfrak{a}) = \begin{cases} \text{the True, if function } \phi() \text{ has the True as value} \\ \qquad \text{for all arguments, or} \\ \text{the False, otherwise.} \end{cases}$$

According to Frege, $\underset{\mathfrak{a}}{—} \ldots \mathfrak{a} \ldots$ should be understood as a *second-level concept*, i.e. a concept that takes a first-level function $\phi()$ as *argument* and returns its own value depending on the values of the function it takes as argument. As a second-order

calculus, Frege's system also contained apparatus for quantification over first-level functions, written "_f_ … f … ."

Notice that the horizontal parts of these last three signs can themselves be regarded as content strokes, since the output is unchanged if arguments or values of the above are fed through this function. In general, making the value of one function into the argument of another is just a matter of attaching the left horizontal line of the term to be made into an argument to the right horizontal line leading to the argument place of the function into which it's fed. For example, the truth-value of its being the case that *if Socrates is human, then if not everything is human, then Boston is not human,* or, $Hs \rightarrow (\neg \forall x Hx \rightarrow \neg Hb)$, could be represented thusly:
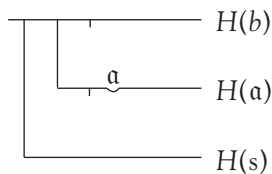
The above, however, remains simply a name of a truth-value, and does not actually assert anything. Frege therefore added a special sign, called *the judgment stroke*, a terminal vertical line which could only appear at the beginning of the expression. Unlike the others, it was not a function sign, and was instead used to *assert* that what followed the sign was a name of the True. In the above case, this would yield:
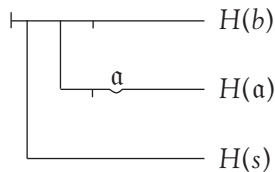
The result is a proposition with assertoric force, rather than just the name of a truth-value.

In what follows, for the convenience of the reader, we shall replace Frege's conditional and negation signs with their near equivalents, "$\rightarrow$" and "$\neg$," and utilize the more familiar notation "$\forall x(\ldots x \ldots)$" for quantifiers, but retain the unique Fregean syntax and semantics, where there is no distinction between terms and formulæ, so that, for example, "$H(\neg \mathfrak{a})$" and "$2 \rightarrow 3$" are not ill-formed. The former, for example, represents the truth-value of a truth-value's being human.

Frege did not introduce signs especially for conjunction, disjunction or the existential quantifier, and simply made use of equivalent forms using his primitive signs, e.g. "$\neg(p \rightarrow \neg q)$" for the conjunction of p and q, or "$\neg \forall x \neg \phi(x)$" in place of "$\exists x \phi(x)$." Indeed, Frege was most likely the first to discover these equivalences.

Frege's logical notation also used a primitive sign for the identity relation:

$$(x = y) = \begin{cases} \text{the True, if } x \text{ is the same object as } y, \text{ or} \\ \\ \text{the False, otherwise.} \end{cases}$$

Obviously, this, and the others above, must not be regarded as object-language definitions of Frege's logical signs, but just informal explications of their use.

Frege's *Begriffsschrift* outlined the first modern axiomatic calculus for logic, and indeed, Frege was the first even to suggest that inference or transformation rules ought to be explicitly formulated and distinguished from axioms. The axioms of his 1879 system were:

(1) $\vdash a \rightarrow (b \rightarrow a)$
(2) $\vdash [a \rightarrow (b \rightarrow c)] \rightarrow [(a \rightarrow b) \rightarrow (a \rightarrow c)]$
(3) $\vdash [a \rightarrow (b \rightarrow c)] \rightarrow [b \rightarrow (a \rightarrow c)]$
(4) $\vdash (a \rightarrow b) \rightarrow (\neg b \rightarrow \neg a)$
(5) $\vdash \neg\neg a \rightarrow a$
(6) $\vdash a \rightarrow \neg\neg a$
(7) $\vdash (a = b) \rightarrow (F(a) \rightarrow F(b))$
(8) $\vdash a = a$
(9) $\vdash \forall x F(x) \rightarrow F(a)$; along with similar principles for function variables.

Given the functional nature of Frege's language, ordinary language glosses of the above must be somewhat awkwardly worded, and even then, won't be entirely faithful to his intent. However, the following readings might be suggested.

(1) If A, then B (materially) implies A.
(2) If A implies that B implies C, then if A implies B then A implies C.
(3) If A implies that B implies C, then B implies that A implies C.
(4) If A implies B, then not-B implies not-A. (The principle of transposition.)
(5) If not-not-A, then A. (The eliminability of double negation.)
(6) If A, then not-not-A. (The introducibility of double negation.)
(7) If *a* is *b* then any concept that holds of *a* holds of *b*.
(8) *a* is *a* itself.
(9) Any concept that holds of all objects holds of any particular object.

To these Frege added inference rules of detachment (*modus ponens*) and universal generalization (applicable either to an entire judgment, or within a consequent of the main conditional assuming the variable replaced is not found in the antecedent), and implicitly utilized a rule of replacement (which became explicit in *Grundgesetze*). Together (1)–(6) provide what amounts to a complete axiomatization of propositional logic, in the sense that all truth-functionally tautologous forms can be derived from them, although (3) is redundant (provable from the others). With (7), (8) and the explicitly given form of (9), they constitute a complete axiomatization of first-order quantified logic with identity, and the system as a whole represents a consistent and

Henkin-complete axiomatization of second-order logic. (Henkin completeness is weaker than standard completeness; a standardly complete axiomatization of second-order logic beginning with a finite axiom set is now known to be impossible as a corollary of Gödel's incompleteness theorems.) The development of this system is undeniably one of the watershed events in the history of logic.

Given the extent to which they are taken for granted today, it can be difficult to appreciate the advantages Frege's system offered over its predecessors. Frege's new analysis in terms of scoped variable-binding quantifiers applied to truth-functional forms provided new insights even into the four basic forms of categorical logic (i.e. the square of opposition). By understanding "all A are B" as "$\vdash \forall x(A(x) \rightarrow B(x))$," Frege first made it possible to capture the logical connections between such statement pairs as "either all students are hard-working or all students are intelligent" and "all students are either hard-working or intelligent" (for example, that the first implies the second). Frege's system was not only adequately able to render statements involving multiple generality, but was even able to distinguish adequately the different meanings of such statements as "every person loves some city" in terms of quantifier scope distinctions:

$$\vdash \forall x[Person(x) \rightarrow \exists y(City(y) \wedge Loves(x,y))]$$

means that every person loves some city *or other*, but

$$\vdash \exists y[City(y) \wedge \forall x(Person(x) \rightarrow Loves(x,y))]$$

means that some one city in particular is loved by everyone. When compared to the baroque explanations and theories that other logicians of the time, or even after Frege, invoked to explain the same phenomena (compare even Russell 1903: Ch. 5), Frege's treatment stands out as a model of eloquence and simplicity. And while work was being done from within the Boolean tradition, by such logicians as Peirce, Schröder and Peano, that moved it closer to matching the expressive power and exactness of Frege's approach, including, for example, independent (re)discovery of the universal and existential quantifiers by Peirce around the same time, these efforts were, relatively speaking, fragmented and piecemeal by comparison.

Value-ranges, extensions and Russell's paradox

Frege offered a slightly different axiomatization of his core logic in the 1893 *Grundgesetze* system, which included fewer axioms but additional inference rules. He also added two additional primitives to his logical notation, most importantly, a sign consisting of a Greek vowel written with a *spiritus lenis* accent acting as a variable binding second-level function sign: "$\grave{\alpha}(\dots\alpha\dots)$." Frege described the value of this second-level function for a given argument first-level function as the "*Werthverlauf*" of the first-level function – a word translated into English sometimes as "value-range," "course-of-values" or "graph." The word was one of several used widely by German mathematicians at the time for the representation of the arguments and values of a

mathematical function, such as on a Cartesian plane (e.g. a wave for the sine or cosine functions or a parabola for a quadratic function). Frege's value-ranges can be understood similarly, but broadened, once again, to include functions whose arguments and values were not numbers. Obviously, Frege did not have in mind a "graph" considered as a physical object, or anything else intuitable, but rather something more like the complete argument-value pairings generated by a function thought of as a single abstract object. Frege identified the value-range of a concept with its *extension*, and sometimes also used the word "class" to describe the sorts of things expressions of the form "$\grave{\alpha}\phi(\alpha)$" name when $\phi()$ is a concept. The label is appropriate, since if $F(\ )$ and $G(\ )$ are concepts that pair precisely the same truth-values with the same objects, they would have the same "extension" i.e. have precisely the same things falling under them.

The other added sign was a first-level function sign to be used in conjunction with the value-range notation:

$$\setminus x = \begin{cases} \text{the sole object falling under the concept whose extension is } x, \text{ assuming } x \\ \qquad\qquad\qquad\qquad \text{is the extension of such a concept, or} \\ x \text{ itself, otherwise.} \end{cases}$$

Frege called this the "substitute for the definite article," since where $\phi()$ is a concept under which only one thing falls, "$\setminus \grave{\alpha}\phi(\alpha)$" might be read "the ϕ."

Frege added two additional axioms to his system governing these new signs (– recall that Frege had renumbered his axiom list –):

(Basic Law V) ⊢ $(\grave{\varepsilon} F(\varepsilon) = \grave{\alpha} G(\alpha)) = \forall x(F(x) = G(x))$
(Basic Law VI) ⊢ $a = \setminus \grave{\varepsilon}(a = \varepsilon)$

The second of these laws basically says that a is *the* thing identical to a, which is relatively straightforward. The first, the notorious Law V of the *Grundgesetze*, also seems straightforward at first blush. Since "=" placed between names of truth-values works more or less as a material biconditional in Frege's system, Law V states that $F(\ )$ and $G(\ )$ determine the same value-range (or have the same "graph") if and only if they have the same value for every argument, which, in the case of number-theoretic functions, seems perfectly reasonable. In the case where $F(\ )$ and $G(\ )$ are concepts, Law V amounts to the assertion that two concepts have the same extension if and only if they are coextensive. It is not difficult to see why Frege might have taken this to be an analytic truth or basic law of logic, and – although this reading is controversial and somewhat problematic – some read it as an attempted implicit definition of the notion of the value-range of a function, i.e. an attempt to explicate what sort of a thing a value-range would be by giving the identity conditions of value-ranges in terms of already-understood logical vocabulary.

Unfortunately, however, the addition of Basic Law V to the other principles of Frege's system renders it inconsistent due to a version of Russell's paradox. Extensions of concepts are objects, and hence, we may ask of each one whether or not it falls

under its defining concept. Some, such as the extension of the concept of self-identity, or the concept of being a class, do so. Others, such as the extension of the concept *being a cat*, will not (since the class of cats is not a cat). Now, however, consider the concept $W(\xi)$, definable as $\exists F[(\xi) = \dot{\alpha}F(\alpha) \wedge \neg F(\xi)]$, i.e. the concept of being a value-range/extension that does not fall under (one of) its defining concepts. This concept has its own value-range, $\dot{\varepsilon}W(\varepsilon)$. Does $\dot{\varepsilon}W(\varepsilon)$ fall under $W(\xi)$? Suppose it does. Then $\dot{\varepsilon}W(\varepsilon)$ must not fall under some concept $F(\xi)$ of which it is the extension. However, since it is also the extension of $W(\xi)$, by Basic Law V, $W(\xi)$ and $F(\xi)$ must be coextensive, and since it doesn't fall under $F(\xi)$, it also must not fall under $W(\xi)$, contradicting our hypothesis. But suppose instead that it doesn't fall under $W(\xi)$. Then it must fall under *every* concept of which it is the extension, including $W(\xi)$, so we are back at contradiction. In a classical logic such as Frege's, a single contradiction in the system explodes so that it then becomes possible to derive every single well-formed proposition in the language as *both* true and false, thereby trivializing his attempt give arithmetic a logical foundation.

After learning of this contradiction, Frege identified Basic Law V as culpable (though this is not altogether a safe assumption; for discussion, see Dummett 1991: Ch. 17; Boolos 1998: Ch. 14). Reflecting on the above argument, it is clear that Basic Law V comes in only once, in allowing us to conclude that because $F(\xi)$ and $W(\xi)$ have the same extension, they must be coextensive. Hence, Frege reached the conclusion that this inference is not always valid, and that it is possible for two concepts to have the same extension *even when they are not coextensive*, or differ with regard to what objects fall under them. In particular, he proposed that if two concepts differed by at most what their values were for their own extension(s), then they have the same extension. After Frege's death, this proposal was later discovered to lead to more complicated contradictions (Quine 1955; Landini 2006). Nevertheless, Frege's general rubric of supposing that two non-coextensive concepts may determine the same "extension" – or something appropriately renamed but playing a similar role within an overall logical theory – has been profitably explored by more recent logicians to develop consistent theories (see, e.g. Boolos 1998: Ch. 11; Shapiro 2003). Moreover, the notation and axioms governing classes can be removed wholescale from Frege's *Grundgesetze* system, leaving a perfectly functioning and (again) Henkin-complete axiomatization of second-order logic, and so Frege's achievement is not altogether marred by Russell's paradox.

The philosophy of logic

Frege understood logic to constitute the *a priori* science of the laws of truth, claiming that "true" points the way for logic "just as 'beautiful' points the way for æsthetics and 'good' for ethics" (Ged: 325). In keeping with the neo-Kantian tradition, Frege resisted psychologism about logic. Logicians need not concern themselves with the psychological processes that underlie thinking, inferring or reasoning. The laws of logic would remain true even if no one's reasoning ever accorded with them, or people in fact reasoned differently than they do. They are laws of *truth* in a descriptive sense,

but they are laws of *thinking* only in a prescriptive sense. They tell us which chains of reasoning are truth-preserving, and hence which allow for us to infer the truth of a certain conclusion *justifiably* from certain premises. Frege understood the laws of logic to be general laws, universally and equally applicable to all domains of objects and concepts.

Where Frege differed from some neo-Kantians, Kant himself, and many others since, was in rejecting the understanding of logic as comprising *purely formal* principles, devoid of content and subject matter, and as such, incapable of expanding our knowledge. We have already seen that Frege understood the logical constants of his *Begriffsschrift* as representing concepts, relations and other functions, no less real or substantial than the meaning of the predicate, "… is human." What's more is that by combining such functors to form any complete name of a truth-value, and then removing any one (or more) constituent name(s) or variable(s), one achieves the name of a possibly new concept (or relation). Frege's replacement rule for second-order variables allowed one to instantiate variables for functions to any complex concept expression so formed, and hence is equivalent to the modern (impredicative) *comprehension principle* for second-order logic, i.e. for every open sentence "…*x*…" containing "*x*" but not "*F*" free, Frege held:

$$\vdash \exists F \forall x (Fx \leftrightarrow \ldots x \ldots).$$

This represents a powerful means for the recognition of new logical concepts, and eventually (along with the recognition of logical objects, described below), the recognition of mathematical concepts as well. According to Frege, logic is *ampliative*; it can extend our knowledge even when the proofs involved start off with self-evident axioms and utilize only transformation rules that guarantee truth-preservation. As Frege put the matter, the truth of the conclusions thereby reached is in one sense "contained within" the truth of the self-evident logical principles with which one begins, but only "as plants are contained in their seeds, not as beams are contained in a house" (GL: §88).

Prior to the discovery of Russell's paradox, Frege even thought that by means of making the transition from concepts to their extensions, or more generally, from functions to their value-ranges, logic could afford us with knowledge of objects, in contradistinction to the Kantian dictum that the content for our reasonings and judgments must be provided by the faculty of sensibility (intuitions, perception, or mental imagery). Frege was explicit that, even when their members are concrete objects, he did not regard classes as aggregates or collections of those members. Instead, he claimed that "the extension of a concept is constituted in being … by the concept itself" (CP: 224–5; cf. PW: 183). This would seem a rather puzzling remark were it not for Frege's understanding of concepts as functions. Frege claimed that functions with the same value for every argument, "coincided," and hence the concepts of *having a heart* and *having a kidney*, which map the same things to the True, and the same things to the False, were the same concept. (Frege could still claim that the two expressions "… has a heart" and "… has a kidney" differ in sense, so long as

they have the same referent; see "Sense and Reference," fifth section.) In that regard, Frege's concepts were already extensional in their identity conditions, and it is not unreasonable to think of the extensions or value-ranges of concepts differing from the concepts themselves only insofar as the concepts themselves are "unsaturated" while the extensions are objects. On this understanding, it is natural to think that insofar as logic alone suffices for awareness of the concepts, it can also afford us knowledge of their extensions. Frege, however, came to have doubts about this way of thinking after the discovery of Russell's paradox (PMC: 140–1; GG: II 265).

Frege's philosophy of mathematics

Criticisms of rival views

Although best remembered in the philosophy of mathematics for his positive arguments in favor of logicism, Frege also provided powerful and influential arguments against other positions about the nature of mathematical truth and knowledge.

We have seen that Frege was a harsh critic of psychologism in logic. He thought similarly about psychologism in mathematics. Numbers cannot be equated with anyone's mental images, nor truths of mathematics with psychological truths. Mathematical truths are objective, not subjective (GL: §26; GG: xviii–xxv).

Frege was also a critic of Mill's view that arithmetical truths are empirical truths, based on observation. Frege pointed out that we can just as well count things that cannot be observed, or cannot all be observed together. It would be perfectly possible for us to realize that two distinct tastes and two distinct musical notes constitute four types of experiences altogether without experiencing them all in a unified perceptual state. Moreover, the limited range of observations we have had seem inadequate to make sense of our knowledge of very large numbers, where we are unlikely to have observed precisely that many things together – not to mention that *no one* has ever had an experience of zero things – rendering inexplicable our knowledge of those numbers' mathematical properties (GL: §§7–8, 24). Moreover, if our knowledge of numbers were to be based on observation, numbers would have to be observable properties of external things, or aggregates of things. Frege points out that the same observable conglomerate can be seen as made up of a different number of things depending on how the parts are counted. *One* deck of cards contains *fifty-two* cards, but each card consists of a multitude of atoms. There is no one uniquely determined "number" of the whole conglomeration (GL: §§22–3). He also reiterated the arguments of others: that mathematical knowledge seems apodictic and *a priori*, and hence not the result of (fallible) sense perception.

Frege also takes issue with the Kantian position that arithmetic is grounded in pure intuitions of space and time. Frege notes that the notion of "magnitude" involved in speaking of intuitions of various magnitudes is often too vague to be useful, that "pure" intuition fares little better than direct observation when it comes to zero and large numbers, and that any attempt to explain how intuitive knowledge gained about small numbers could be extended to large ones is liable to draw an arbitrary boundary

between large and small (GL: §§5–6, 12). Perhaps Frege's most forceful response to this kind of position, however, is his positive argument that e.g. the basic properties of the kind of succession involved in the series of natural numbers could be defined and proven purely logically, without appeal to a representation of the succession of time or any other form of intuition. Of course, establishing this required giving the proofs: hence the importance of carrying out the project in an artificial logical language with precisely formulated inference rules, where no appeal to intuition could go undetected.

Finally, Frege was an ardent opponent of formalism, the view that arithmetic can be understood as the study of uninterpreted formal systems. While Frege's logical language represented a kind of formal system, he insisted that it was important only because of what its signs represent and its propositions mean (CP: 112ff). Frege believed that at its worst, formalism was guilty of simply conflating the signs with what they represent, e.g. concluding, absurdly, that numbers are signs, or that "5 + 7 = 12" written in Arabic numerals, is not the same truth as the formula, "V + VII = XII" written in Roman numerals (cf. e.g. GG: §100). Even at their best, Frege believed that formalist theories in mathematics were guilty of a kind of confusion of concept and object. By simply laying out a formal system (even a consistent one) and holding its subject matter to be constituted by the fact that it is what is characterized by those axioms, one is really, according to Frege, defining a *concept* (e.g. "object satisfying the axioms") which one, many, or even *no* systems of objects might fall under (GL: §97; CP: 120–1). Without providing a specific interpretation of the vocabulary used in the theory so that it does represent a specific content, the formal system cannot even be thought of as *truth-apt*, much less *actually true*.

Frege suggests that mistaken views in the philosophy of mathematics are often the result of attempting to understand the meaning of number terms in the wrong way, for example, in attempting to understand their meaning independently of the remainder of the propositions (sentences) wherein they appear. If simply asked to consider what "two" means independently of the context of a proposition, we are likely simply to imagine the numeral "2" or perhaps some conglomeration of two things. Thus, in the *Grundlagen* (p. x), Frege espouses his famous *context principle*, "never to ask for the meaning of a word in isolation, but only in the context of a proposition." The *Grundlagen* is an earlier work, written before Frege had made the distinction between sense and reference (see "Sense and Reference," fifth section, below). It is an active matter of debate to what extent this principle coheres with Frege's later theory of meaning, and in particular, the compositionality principle of sense, but what is clear is that it plays an important role in his own philosophy of mathematics in the *Grundlagen*.

The definition of number

According to Frege, most everyday uses of number words place them within propositions that, as wholes, make claims about concepts. In particular, they are used to specify how many times a certain concept is instantiated. Consider, for example:

I have six cards in my hand.

or

There are ten members of Congress from Wisconsin.

These propositions seem to tell us how many times the concepts of *being a card in my hand* and *being a member of Congress from Wisconsin* are instantiated. Thus, Frege concludes that ascriptions of number are statements about concepts. Frege was able to show that it is possible to define what it means for a concept to be instantiated a certain number of times purely logically by making use of quantifiers and identity. To say that the concept $F(\ )$ is instantiated zero times is to say that there are no objects that fall under $F(\ )$, or, equivalently, that everything does not fall under $F(\ )$, i.e.:

$$\vdash \forall x \neg F(x).$$

To say that $F(\ )$ is instantiated exactly once is to say that something, x, falls under $F(\ )$, and that nothing *else* does:

$$\vdash \exists x [F(x) \wedge \forall y (\neg F(y) \vee y = x)].$$

To say that $F(\ )$ is instantiated twice is to say that there are *distinct* x and y which fall under $F(\ )$ and that everything either does not fall under $F(\ )$ or is one of them:

$$\vdash \exists x \, \exists y [F(x) \wedge F(y) \wedge x = y \wedge \forall z (\neg F(z) \vee z = x \vee z = y)]$$

and so on. In general, for each natural number n, we seem to be able, using logical constants only, to define a formula involving a variable "F" that will hold of a given value if and only if it is a concept instantiated by precisely n objects.

However, Frege rejects thinking of this as a complete understanding of what numbers are. The above definitions do not define the number words "1," "2" or "3" but only the second-level concept expressions, "the number of things that fall under … is 1" and "the number of things that fall under … is 2." In mathematics, and even in ordinary language, however, numerals and other expressions for numbers often occur as complete expressions or proper names and hence must be thought of as referring to objects. (See "Objects and the Hierarchy of Concepts," fifth section, below, for explanation of Frege's distinction between objects and concepts of various levels.) Nevertheless, Frege does not think it fruitful simply to ask what object a number word stands for; determining what sorts of objects numbers might be is constrained by the fact that number words typically occur in statements about concepts, and hence any account of their nature must do justice to this fact. Frege then suggests that we might get a better handle on what objects numbers are by considering statements concerning their identity or non-identity. Here Frege appeals to an insight expressed by Hume according to which the number of Fs is the same as the number of Gs if and only if

we can pair up each F with a unique G and vice-versa. It is possible, in second-order logic, to define what it means for there to exist a one-to-one correlation between two concepts, which we can call the relation of *equinumerosity*:

$$(F(x) \cong_x G(x)) \quad =_{\text{Df.}} \quad \exists R \big[\forall x [F(x) \to \exists y (G(y) \wedge \forall z (R(x,z) \leftrightarrow z = y))] \wedge \\ \forall x [G(x) \to \exists y (F(y) \wedge \forall z (R(z,x) \leftrightarrow z = y))] \big]$$

or, in other words, $F(\xi)$ and $G(\xi)$ are *equinumerous* just in case there is a relation R that correlates each F to one (and only one) G and each G to one (and only one) F. Using the notation "$(\#_x)F(x)$" for "the number of xs such that $F(x)$" (Frege's own notation was $\mathfrak{N}\dot{a}F(\alpha)$), we can then state the following:

(Hume's Principle) $\vdash ((\#_x)F(x) = (\#_y)G(y)) \leftrightarrow (F(x) \cong_x G(x))$

i.e. the number of Fs is the number of Gs if and only if F is equinumerous with G. This principle seems to be an analytic statement that just explicates what we *mean* when we speak of numbers. Notice, however, that it does not actually give a definition of the notation "$(\#_x)\phi(x)$" or provide a way of eliminating it in terms of core logical notation. Instead, it only gives us a criterion for the truth or falsity of identity statements formed using such notation. It tells us when one number is identical with another, purely logically, but not yet what a number *is*. It does not even fix the truth conditions for all statements of identity involving numbers since it does not tell us whether or not $(\#_x)F(x) = q$ when q is not given to us *as* the number belonging to some concept. To adapt an example Frege uses in a slightly different context, it does not tell us whether or not the number of Fs is Julius Cæsar. Hence, Frege is unwilling to rest content with this as a "definition" of number words either.

Nevertheless, Frege holds that Hume's Principle is another important step forward, since any account of the nature of numbers must validate it, even if it on its own cannot precisely afford us knowledge of what numbers are. At this point, Frege turns to his theory of extensions of concepts in order to identify objects that can play this role which our understanding of logic alone can make known to us (as explained in the subsection, "The Philosophy of Logic," third section). In the *Grundlagen*, Frege suggests that we can define "the number of Fs" as "the extension of the concept 'equinumerous with F'," thereby making numbers into extensions of second-level concepts. In the later *Grundgesetze*, noting the ready interchange between speaking of a concept and speaking of its extension, Frege modifies this definition so that "the number of Fs" (or "$(\#_x)F(x)$") is taken as the extension of the concept *being the extension of a concept that is equinumerous with F*, thereby making numbers into extensions of concepts applicable to extensions, and anticipating Russell's 1901 definition of numbers as classes of like-membered classes. A (cardinal) *number* generally can be defined as any object that is "the number of" some concept. Since equinumerosity is an equivalence relation, both symmetric and transitive, if we assume that F and G are equinumerous, then (the extension of) any other concept H will be included in the number of Fs if and only if it is included in the number of Gs. Conversely, if anything

that is included in the number of Fs is also included in the number of Gs, since this includes (the extension of) F itself, F must be equinumerous with G. Therefore, the proposal validates Hume's Principle (though of course it does so by invoking Frege's problematic notion of the extension of a concept, which eventually was shown to lead to inconsistency.)

Logicism and its prospects

With his conception of number in hand, in the *Grundlagen*, Frege began to sketch how many of the basic arithmetical truths of number theory, beginning with the theory of natural numbers, might be deduced. He gave a list of several basic principles governing them, not identical to, but equivalent with, the Peano-Dedekind axioms. Full axiomatic proofs of these principles, stated in Frege's logical notation, were given in his *Grundgesetze*. Zero is defined as the number of the concept *non-self-identical*, or $(\#_x)(x \neq x)$. For any numbers m and n, m can be said to be the *successor* of n, by definition, just in case for some concept F, $m = (\#_x)Fx$ and there is a y such that $F(y)$ and $n = (\#_x)(F(x) \wedge x \neq y)$. We can say that a concept ϕ is "successor-hereditary" if it's the case that whenever $\phi(n)$ and m is the successor of n, then it also holds that $\phi(m)$. We can say that m follows n in the successorship-series just in case m has every successor-hereditary property that n has. The *natural numbers* can be defined as zero and all those numbers that follow zero in the successorship-series. We can establish that this series never runs out with the observation that, for any member n of this series, the number of natural numbers up to and including it, will be the number m that succeeds n.

Much in keeping with Georg Cantor's theory of transfinite cardinal numbers, Frege points out that there are some concepts whose number is not a natural (finite) number, such as the concept of *natural number* itself. Infinite numbers of various sizes, still definable in terms of equinumerosity or one-to-one correspondence, are also countenanced within Frege's approach. Moreover, Frege believed that integers (including negative numbers), and real numbers, could be defined in terms of natural numbers and certain relations between them, and had been well on his way towards establishing the basic principles governing them when his project was derailed by the discovery of Russell's paradox.

In light of its inconsistency, and the fact that his proposed way out of the inconsistency also failed, it can be difficult to gauge what the lasting significance of Frege's logicist project is, and opinions differ significantly. Of course, the invention of quantifier logic, which remains intact after the inconsistent naive class theory is removed, is no small feat. Yet, there is reason to think that even the mathematical part of Frege's logicist project was in its own way a success. It sketched a methodology for logically defining numbers and proving their basic properties which could be largely retained excepting only various possible modifications to escape inconsistency. Frege's approach directly influenced the approaches to such issues by later mathematical logicians, including Whitehead and Russell, Carnap, Wittgenstein, Leśniewski, Church, Ramsey and Quine, at least *some* of whom took themselves to be offering

their own brands of logicism. More recently, it has been stressed by Crispin Wright that Frege's proofs of (what amount to) the Peano-Dedekind axioms stem exclusively from Hume's Principle and the other, consistent and widely accepted, principles of second-order logic (or can be changed to do so), and that the role Basic Law V plays in Frege's treatment of natural number theory, at least, is almost exclusively limited to obtaining Hume's Principle. Hence, Wright has urged that if Hume's Principle can itself be argued to be a logical, or at least analytic, truth, prospects remain for the pursuit of neo-Fregean forms of logicism. Whatever one thinks about this suggestion, Wright is no doubt right that the result that the whole of second-order Peano arithmetic can be derived from Hume's Principle alone is a significant discovery, and it has come to be known as "Frege's theorem."

In 1931, Kurt Gödel discovered that no deductive system that utilizes a finite or recursively specifiable collection of axioms can be both consistent and have every arithmetical truth as a theorem. This has only clouded further the evaluation of Frege's logicism. Notice that the result certainly doesn't establish, at least not in any direct fashion, that there are arithmetical truths that are not logical truths. Indeed, a corollary of Gödel's results show that no finite or recursively specifiable collection of axioms for second-order logic can be both consistent and have every standardly valid or logically true formula as a theorem; this certainly doesn't establish that there are logical truths that are not logical truths! Gödel's results do deal a significant blow to Frege's own method of trying to *establish* the analyticity of arithmetic, insofar as Frege thought this meant finding "*the* proof" of any given proposition, as if there is one preferred, all-encompassing, logical calculus in which this is to be done. Still, the problem persists only if we insist there has to be a *single* such calculus or deductive system, since Gödel's results do *not* show that there are any truths of mathematics which cannot be established in *any* deductive system of logic, only that they can't *all* be established in the *same* one. (And notice that accepting multiple logical *calculi* is not tantamount to believing in multiple "logics.")

Moreover, it is entirely unclear how and to what extent these results give a point in favor of any of the major rivals to logicism. Supposing it were true that Frege or a neo-Fregean had succeeded in capturing all of Peano arithmetic, and the core principles of real number analysis, in a purely logical or analytic deductive system, but was happy to admit that there were some arcane truths involving complex recursive number systems it didn't capture. Still, *all* of the arithmetic we learn in our primary and secondary schools would be included. It would seem odd in the extreme to insist then that there was some *other* metaphysical or epistemological basis for mathematics as a whole, which just doesn't *happen* to be necessary for *any* of the so-called "arithmetical truths," which we make use of in day-to-day calculations or normally associate with the word "arithmetic." It is also very hard to imagine a plausible case being made that some non-logical source of knowledge is precisely what fills the void left by the undecidable sentences posited by Gödel's arguments.

Language and metaphysics

Frege's work in the foundations of arithmetic and logic led him to investigate more fully both the nature of language, and certain fundamental questions about the structure of reality and our knowledge of it. Commentators on Frege disagree about precisely to what extent Frege was interested in metaphysics or the nature of language for their own sakes, and what brought him to these investigations. Nevertheless, it is clear what the primary connection is. Frege's logical language aimed to capture precisely that which was thought relevant to inference – what, early on, he called the "content" – of a premise or conclusion. This led him to investigate how precisely these contents should be understood, and in particular, (i) how they are expressed and logically articulated in ordinary language, (ii) whether the notion of content is univocal or further distinctions are necessary, and (iii) what the nature of these contents is, whether they are the products of the mind, or objective, and whether or not they are the same for all who grasp them.

Objects and the Hierarchy of Concepts

As discussed in the subsection, "Function and Argument," third section, the syntax of Frege's logical language is modeled on the function/argument analysis of complex mathematical terms. Frege believed that ordinary language, to the extent that it is articulated in a logically non-defective way, is capable of the same kind of segmentation. Thus, proper names, descriptions, or other "complete" expressions must be distinguished from "incomplete expressions," such as "the capital of …" or a predicate phrase like "… plays tennis," which bear the hallmark of function expressions generally, i.e. that they suggest or hold open a spot for the sign for an argument or arguments to be written. Recall that for the mature Frege, a concept is simply a function whose value is always a truth-value. Beginning with any complete proposition, if a constituent name is removed, the result is a concept expression, and the removed name can be considered the sign for its argument. For some propositions, it is possible to divide or decompose them into concept expression and argument in more than one way. Thus, Frege says, "Jupiter is larger than Mars," can equally well be divided into the concept expression, "Jupiter is larger than …" and the object name, "Mars," or into the concept expression, "… is larger than Mars" and the argument name, "Jupiter."

Frege believed that corresponding to the distinction between complete and incomplete expressions there was an analogous distinction between the kinds of things to which they referred. Objects, the referents of complete expressions, were regarded as "self-standing" or "saturated." Concepts and other functions, the referents of incomplete expressions, were regarded as "unsaturated" somehow. Frege regarded this lack of "saturation" as explaining the "predicative" nature of concepts; it is what makes them *suitable* to be predicated *of* something. Multiple names of objects written in succession cannot form a meaningful proposition; only if one or more of the expressions making up a proposition stands for a function, concept or relation, can a phrase be regarded as

a significant whole. (Frege thought there was also a corresponding saturated/unsaturated distinction to be found at the level of the senses of these expressions; more on this in the next section.)

Frege regarded the distinction between objects and concepts (and other functions) that take objects as arguments, i.e. the kind that might be represented by removing a name from a proposition, at the base of an hierarchy of different "orders," or, in his mature terminology, different "levels" of concepts (or other functions). Functions that take objects as argument are called "first-level"; functions that take first-level functions as argument are called "second-level," and so on. The name of a second-level concept might be obtained by removing a first-level function expression from a complete proposition. Thus, if we remove "… is human" from the proposition, "for any x, if x is human, then x is mortal," to obtain "for any x, if … x …, then x is mortal," our new incomplete expression names a concept that will map its argument concept to the True just in case all objects that fall under that argument concept also fall under the concept of mortality. Notice that a second-level function expression, like a first-level function expression, contains a gap or spot to be completed by another expression, except that the kind of expression that would fit that gap is of a different sort. If an argument expression is inserted, the two "fit together" in the appropriate way. At the level of what the expressions stand for, again Frege thinks there is something comparable: the second-level function "mutually saturates" with its argument first-level function to yield an object as value. But notice, however, that a concept expression of any level is never of the right sort to complete, or be completed by, itself or another concept expression of the same level.

Frege takes this distinction to be of utmost philosophical importance. He goes so far as to call the confusion of concept and object, or concepts of different levels, "the grossest possible" (GG: xxv). As we have already seen, he argues that numbers must be considered self-standing objects, and his rationale for this conclusion is that numerals and other number expressions are syntactically complete. He also argues that the concept of existence, as represented by the existential quantifier, "$\exists x(...x...)$," must be understood as, at least at the fundamental level, a second-level concept (GL: §68; FuB: 25), and, alongside other confusions, diagnoses the ontological argument for God's existence as a failure of inattentiveness to this (GL: §53; FuB: 27). Frege admits, however, that our normal ways of speaking about concepts and objects within philosophical contexts is often imprecise, and apt to mislead. Part of this is because any attempt to talk about a concept or other function as if it were an object is doomed to failure. It is natural to think of "the color of the sky" as referring to the same thing as the adjective "blue," but Frege claims that this cannot be so, since the former but not the latter is an object (GL: §106) Notoriously, Frege argues that despite what we might initially assume, the phrase "the concept horse," since it is not itself incomplete or predicative, cannot name a concept, and hence, accepts the *truth* of the following paradoxical-seeming assertion (BuG: 184):

The concept horse is not a concept.

Benno Kerry (1887), Bertrand Russell (1903), and others since have despaired that a result like this renders Frege's position unintelligible at best and inconsistent at worst. Indeed, the issue seems to pose a rather direct problem for interpreting Frege's own language when stating his own theory. How are we to understand the words "object" and "concept" that Frege uses? If "… is an object" is a first-level concept expression, then it is true of everything whatever of which it can be meaningfully predicated, and so doesn't seem to mark off a distinction between objects and other sorts of entities. (And if it is interpreted differently, parallel problems arise.) Frege himself admitted the problem, adding that "by a kind of necessity of language, my expressions, taken literally, miss my thought" and that hence, "I was relying upon a reader who would be ready to meet me halfway – who does not begrudge a pinch of salt" (BuG: 193). This has led some commentators to the conclusion that Frege believes there are insights into metaphysical categories, founded (in his own words) "deep in the nature of things" (FuB: 31) that escape expression in language, and has led others to conclude that Frege regarded his own elucidations of his views of logic as nothing more than nonsense used as a stepping stone on the way to mastering his logical notation, but both groups of commentators find Frege's struggles with these issues to prefigure Wittgenstein's discussion in the *Tractatus* of ineffable truths, those that can be "shown" but not "said."

Sense and reference

Distinct from, and cutting across, the distinction between concept and object is the distinction Frege makes between two components of meaning expressions may possess, the distinction between, in German, "*Sinn*" (sense) and "*Bedeutung.*" The proper translation of Frege's "Bedeutung" is controversial, with "meaning," "denotation," "nominatum," "indication" and "significance" all in the running, though here we follow Dummett (1973) in substituting either "referent" or "reference" depending on whether we mean the thing referred to (the referent), or the relationship between the word or phrase and this thing (reference). The distinction was first outlined in Frege's 1891 piece, "Funktion und Begriff," and was expanded upon in his celebrated essay, "Über Sinn und Bedeutung," a year later.

During Frege's time, there was a dispute among certain mathematicians over the sign "=" routinely used in mathematical equations. Consider:

$$4 \times 2 = 11 - 3.$$

A number of Frege's contemporaries (Weierstrass among them) were wary of viewing this as a statement of identity proper, and instead posited some weaker form of "equality" whereby 4×2 and $11 - 3$ could be considered equal-in-magnitude or equal-in-number without being one and the same thing. Their rationale was rather straightforward: the two cannot in all ways be thought to be the same: the former is a product, the latter a difference, and so on. In his early work, 1879's *Begriffsschrift*, Frege apparently put enough credence into this viewpoint to think it worth using a different

sign, "≡" for "identity of content" so that "=" could be reserved for mathematical equality instead.

Before distinguishing between sense and reference, Frege merely spoke of the "content" of a word or phrase. He came to the conclusion that a judgment recorded in the form:

$$\vdash (a \equiv b)$$

must not actually be about the contents of the names "*a*" and "*b*," but rather about the names themselves, since – assuming the judgment is true – "*a*" and "*b*" typically have the same content, and hence the judgment above as a whole would otherwise not differ in content from the analytic judgment $\vdash(a \equiv a)$, whereas the former judgment may be synthetic depending on how these names determine their content.

In his mature work, Frege is an explicit opponent of the view that mathematical equality needs to be distinguished from identity proper, and is usually read as no longer holding the view that identity statements involve claiming something about signs rather than things themselves. (For an exception, see Caplan and Thau 2001.) However, Frege now differentiates two parts of the content of a well-formed expression. In the case of "4 × 2" and "11 − 3," for example, we can make a distinction between the actual number designated – the common referent of both these expressions – and the way in which that number is presented or picked out – the senses of the two expressions, which differ in the two cases. In Frege's terminology, a word or phrase is said to *express* its sense, and *refer to* (i.e. *bedeutet*) its referent. The distinction applies outside of mathematics as well. Another famous example Frege gives is of the pair, "the morning star" and "the evening star," both of which refer to the planet Venus, but in virtue of very different properties that it has. In the case of a proper name or complete expression, the referent is the object that it stands for, and the sense is what contains the "mode of presentation" or "cognitive content" used to represent that object.

Frege is explicit that he believes that the sense/reference distinction can be applied to other kinds of phrases as well as complete propositions. In the case of a concept expression, "… is an equilateral triangle," the referent is a concept, again, understood as a kind of function from objects to truth-values. Frege believes that functions "coincide" (– he reserves the label "identical" for pairs of objects –) when they always have the same value for the same arguments, or in the case of concepts, when the same objects fall under them (PW: 120–2). Because the function mapping all equilateral triangles (and nothing else) to the True is the same as the function mapping all equiangular triangles to the True, the difference in meaning between "… is an equilateral triangle," and "… is an equiangular triangle," must be one of sense rather than reference. Frege concludes that the notion of sense captures what other logicians have in mind when speaking of the "intension" of a concept, though Frege is clear that the referent of a concept phrase – the concept itself – must still be distinguished from its extension, in virtue of the fact that the latter is an object not a function (see e.g. PMC: 63–4).

Frege believes in compositionality principles at both levels. The reference of a complex expression is a function of the referents of the parts; indeed, the referent of one of the parts *is* a function that is applied to the *referent(s)* of other part(s) to yield the referent of the whole. The sense of the whole expression is also a function of the senses of the parts; indeed, Frege often suggests something stronger: that the sense of the complex expression *is* a kind of whole composed of the senses of the parts. (The incompleteness of the senses of function expressions provides the "logical glue" holding the whole together.) He uses this as an explanation for how it is we are able to understand new propositions we've never heard before: we put together the already-understood senses of the individual parts in order to grasp the sense of the whole (CP: 390). There is, however, some uncertainty among Frege commentators as to how this view coheres with the context principle of *Grundlagen*, which seems to put our understanding of the content of parts of a proposition secondary to that of the entire proposition, as well as what's sometimes called Frege's "priority thesis," according to which the parts of a complete judgment are to be found by analyzing the judgment, and not vice-versa (PW: 17, 253; PMC: 101).

Returning to the puzzle about identity, consider the two cases:

(A)   the morning star = the morning star
(B)   the morning star = the evening star.

It seems clear that in (A) the two expressions flanking the identity sign have the same sense, but in (B) the two signs flanking the identity sign have different senses, but in both cases they have the same referent. This latter fact explains why these two claims are both true. However, Frege believes that the "cognitive value" or "informativity" of a proposition is determined by its sense. Hence, we have an explanation of how it can be that (B) is informative whereas (A) is not, despite that each part of one corresponds to a part of the other with the same referent. Frege calls the sense of a complete proposition a "thought" (*Gedanke*), and believes the referent of a complete proposition is its truth-value. Indeed, Frege sometimes defines thoughts as those senses whose referents are truth-values (e.g. GG: §2). Since these are determined functionally, (A) and (B) can differ in the thought expressed, but must have the same truth-value (as they do). Thus, in his mature work, Frege's prior single notion of "content" was replaced by the dual notions of "thought" and "truth-value."

Unfortunately, Frege tells us very little about what senses are and how they are related to the referents they present. A long-standing interpretative tradition (e.g. Kripke 1972) reads him as thinking of the sense of a proper name as involving certain descriptive information, or a condition, which must hold of one individual and one individual alone in order for that individual to be the referent. There is some evidence in favor of this reading. Frege says that each sense "illuminates a single aspect of the referent, supposing it to have one," and that "comprehensive knowledge of the referent would require us to be able to say immediately whether any given sense attaches to it" (SuB: 27), suggesting that which senses present which referents is a function of the properties of those referents. Evidence also comes by way of certain examples Frege

gives, such as his suggestion that for some people the sense of the name "Aristotle" can be described in terms of "the pupil of Plato and teacher of Alexander the great" (SuB: 27n; cf. Ged: 65). Nevertheless, it is not entirely clear that this gives the right picture for all Fregean senses. With regard to senses of other kinds of expressions, Frege tells us even less (see Klement 2002: Ch. 3).

One nice thing about the interpretative tradition is that, if correct, it sheds light on something Frege certainly does believe: that it is possible for there to exist expressions that have a sense but no referent. Frege gives such examples as "Odysseus" and "the least rapidly converging series." Given that Frege believes that the reference of a complex expression is a function of the referents of its parts, and that the referent of a complete proposition is its truth-value, he concludes that when a phrase such as these occur within a proposition, while the proposition may be considered to express a thought as sense, as a whole it is devoid of reference, and hence is neither true nor false. If the interpretative tradition is right, the explanation for this is surely that the sense fails to present a reference because no one individual uniquely satisfies the requisite condition. (The thought expressed by the whole doesn't actually assert that there is one such individual; it merely presupposes it – see SuB: 39–40.) Frege regarded the existence of referent-less phrases as a defect of ordinary language to be avoided within a logical language, where stating precise rules of truth-preservation is the goal.

Lastly, Frege thinks that in certain contexts, the reference of words "shift" so that they have as their referents what would ordinarily be their senses. Consider the following two propositions:

(C)   Ptolemy believes that the morning star is a planet.
(D)   Ptolemy believes that the evening star is a planet.

It seems perfectly natural to suppose that one of these might be true, and the other false, assuming Ptolemy to be unaware that the morning star and evening star are the same. However, if the truth or falsity of (C) and (D) is determined functionally by the referents of their parts, it seems that Frege would be committed to holding that (C) and (D) must have the same truth-value. Worse, he would seem committed to holding that anyone who believes one truth believes all of them. Frege responds to such puzzles with his theory of "*oratio obliqua*" or "indirect speech," according to which, in belief reports and similar contexts, words have as referents what would ordinarily be their senses. Hence, rather than referring to the same truth-value, the dependent clauses in (C) and (D) refer to distinct thoughts, and so Frege is free to regard one of these as true and the other as false without abandoning the principle that the reference of the whole is determined by the referents of the parts. Frege's deliberation on this issue has sparked a lively, and still ongoing debate (over a century later), about whether or not words within such contexts really should be taken as meaning something other than what they normally mean, and whether, if we answer yes to that question, this requires us to posit the existence of a potentially infinite number of distinct meanings (*senses* or *referents*) the same words might have given the possibility of embedding one such indirect speech report within another.

Thoughts and truth

As we have seen, Frege believed that the referent of a complete proposition, when it has one, is its truth-value, either the True or the False. This view comports with, and indeed, is probably the source of, his understanding of concepts and relations as functions whose values, when given arguments, are truth-values. Frege gives two broad reasons in favor of this view. First, he claims (rather cavalierly) that, in general, the reference of expressions becomes important to us precisely when we become interested in the truth or falsity of what we are saying. If simply telling a story for amusement, then it may not matter to us at all whether or not the name "Odysseus" refers to a person, but if we become interested in the historical accuracy of the story, i.e. its truth or falsity, then it will matter (SuB: 32–3). His other reason stems from his conviction that the reference of the whole must be determined by the referents of the parts, adding, rhetorically (SuB: 35), "what feature except the truth-value can be found that belongs to such propositions quite generally and remains unchanged by substitutions" in which "a part of the proposition is replaced by an expression with the same referent"? Frege doesn't give examples, but it seems likely that he meant such transformations as between sentences (A) and (B) in the previous section, although it is not obvious in that case that *nothing* else remains in common between them other than their truth-value. Years later, followers of Frege on this point gave more complicated examples. Assuming that both substitutions of coreferential descriptions preserve reference, and that logically equivalent propositions have the same referent as well, Alonzo Church (1956: 25) argued that we must accept that the following all have the same referent:

(E)   Sir Walter Scott is the author of *Waverly*.
(F)   Sir Walter Scott is the writer of twenty nine *Waverly* novels.
(G)   The number of *Waverly* novels written by Sir Walter Scott is twenty nine.
(H)   The number of counties in Utah is twenty nine.

Since the first and the last seem to have nothing interesting in common except their truth-value, we are lead to Frege's conclusion that truth-values are the referents of propositions. Arguments of this form have come to be called "the Frege-Church slingshot."

Thoughts, the senses of complete propositions, can thus be understood as senses that present truth-values. It is their truth or falsity that is at issue in any inquiry. Frege is clear that the relationship between thoughts and their truth-values must be understood as the relationship of senses to referents, not the relationship of subject and predicate. Frege notes that the words "it is true that …" adds nothing to the sense of the proposition that would fill the ellipsis; they therefore cannot represent a genuine property. Moreover, Frege does not think that it is possible to give a non-circular account or definition of what truth, or "the True," is. Truth cannot consist of correspondence of a thought with something else, according to Frege, since nothing *completely* corresponds with anything distinct from itself, and if truth relied on

correspondence with something in this-or-that respect, we would inevitably be led to the question as to whether it was *true* that there was just such a correspondence, and we would be stuck in a vicious regress. Frege thinks similar remarks apply to any other property we might suggest as our characterization of what truth is, since recognition of the possession of a property is always *ipso facto* the recognition of a thought as being true. Frege concludes that truth is "*sui generis* and indefinable" (Ged: 327). (I note in passing that one might attempt to define a thought as true when the object it presents as referent is $—\alpha= \alpha$, keeping in mind that " $—\alpha = \alpha$" is, in Frege's notation, considered a name of the True. Although extensionally correct, this definition would be circular, since the logical function expressions involved were characterized in terms of the True.) Rather, Frege thinks that logical activity in a way indirectly spells out what truth is, particularly the activity of forming judgments and making inferences and assertions (over and above merely grasping thoughts and considering them). "The meaning of the word *true* is spelled out in the laws of truth" (Ged: 59; cf. PW: 251–2).

According to Frege, thoughts (and other senses) exist in a "third realm," distinct from both the concrete physical world, and the psychological world of ideas, feelings and sensations. Frege rejects the supposition that thoughts can be thought of as concrete objects or built up out of them, adding that "Mont Blanc with its masses of snow and ice is no part of the thought that Mont Blanc is over 4000m high" (PMC: 187), while noting that thoughts need to be finer grained than the objects they're about. Secondly, Frege strenuously rejects the suggestion that thoughts can be considered as ideas, citing as his reasons that unlike ideas, which are private to the individual who has them and are created by our mental activities and destroyed by their cessation, thoughts are timeless, can be shared by multiple individuals, and exist (and can be true or false) regardless of having been grasped or not by any thinkers. Frege insists that we must not confuse the thought with the act of thinking. If each person's thought were different from another's, like each person's ideas and perceptions differ, then it would be impossible to communicate or disagree with others in a true sense. (Interestingly, however, Frege does acknowledge a certain kind of exception to the communicability of senses and thoughts, by recognizing a special way in which each of us is presented to him or herself which no one else is capable of grasping: the notion of "I" when used in our own cogitations. This has generated significant interest, in part, because it seems out of sorts with the remainder of Frege's theory, and in part because it seems as if it might be an early recognition of the distinctive character of *de se* belief and knowledge, as discussed, for example, in John Perry's work.)

Frege's legacy

Frege is a philosopher whose work has dramatically changed how it is we think about nearly all areas of abstract study: abstract truths, abstract objects and abstract knowledge. In some cases, he did this directly, such as in his groundbreaking work in quantificational logic and the philosophy of mathematics, where his innovations were many and his arguments cogent. In other cases, he did this indirectly, by failing in such a drastic, but such an understandable, fashion, as with his inconsistent theory of the extensions of concepts, that investigating where and how Frege went wrong has been every bit as

instructive to ongoing researches as work not so flawed. With his theories of meaning and truth, perhaps, the jury is still out regarding in which category they belong, though I think there is increasing evidence these may go the way of Basic Law V. Either way, his influence, although limited during his own lifespan, is now pervasive. Even 125 years after its publication, Frege's *Grundlagen* is likely still the single most widely read work in the philosophy of mathematics. We find working philosophers of language write about Frege's views in the same way one might write about an esteemed contemporary whose work must be considered as viable as anyone's (e.g. Salmon 1986; Kripke 2008). It is perhaps in the area of logic, however, that the effects of his work are most ubiquitous – indeed, perhaps so thoroughly so that they're almost invisible. First-order predicate logic, extracted out of Frege's function calculus, is now the default standard logical calculus for conducting any technical proof. More complicated logical calculi are almost unthinkable except as building upon, or *intentionally* and systematically deviating from, the Fregean (or "classical") approach. Moreover, while, in practice, most working mathematicians don't construct proofs using the full rigor of a *Begriffsschrift*-style derivation, most do, I think, attempt to fashion their proofs in such a way as to provide enough details that a proof of this level of rigor *could* be constructed out of it, and would regard it as not really a proof at all if this were not possible. To be sure, however, these were not changes that had no hope of being brought about without Frege (given the parallel work of Peirce, Schröder, Peano, etc.), nor are they changes Frege brought about *alone*, or indeed, could have helped bring about at all without the championing of later intellectual heirs who eventually did come to prize his work much more highly than it had been at first.

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Further reading

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