Modality and the structure of assertion

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Abstract: A solid foundation of modal logic requires a clear conception of the notion of modality. Modern modal logic treats modality as a propositional operator. I shall present an alternative according to which modality applies primarily to illocutionary force, that is, to the force, or mood, of a speech act. By a first step of internalization, modality applied at this level is pushed to the level of speech-act content. By a second step of internalization, we reach a propositional operator validating the modal logic S4. After a brief discussion of problematic modality and possibility, the article concludes with an extension of the account that identifies modality with illocutionary force. Throughout, close attention is paid to the intended interpretation of the formalism. All of the rules stipulated will be justified on the basis of this interpretation.

1   Introduction

In modern modal logic, modal operators operate on propositions, or on their formal simulacra, well-formed formulas: a modal operator (let us assume that it is unary) takes a proposition, or well-formed formula, and yields another proposition, or well-formed formula. The logico-grammatical category of such an operator is therefore that of a unary propositional connective, another standard example of which is negation. One might well ask whether this treatment of modality is conceptually the most illuminating that logic could offer. English has phrases, such as "that it is necessary", "that it is known", and "that it is possible", that transforms a that-clause into a new that-clause. Taking propositions to be expressed by that-clauses, such phrases could be

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taken to express modal operators. There is, however, another conception of modality which, following Kant, we may call the modality of a judgement. The modality of a judgement, as Kant conceived it, does not concern the content of the judgement, but rather the attitude of the judging subject to this content.

In this article I offer an account of modality and the conceptual genealogy of modal logic that may help to clarify the relation between these two conceptions of modality. The account owes much to the so-called judgemental reconstruction of modal logic of Pfenning and Davies (2001) and to some remarks on the syntax of modal logic by Sundholm (2003). In concentrating on the structure of assertion I am following Martin-Löf. The first way in which the account of modality offered here expands on Pfenning and Davies's is in its use of a finer analysis of assertion. (I take it for granted that this analysis is also an analysis of judgement, since I assume judgement and assertion to share the same logical structure. Although the term "assertion" dominates in this article, it can in most places be replaced by "judgement".) The second way is in its offering somewhat different meaning explanations. We are interested in logical systems as meaningful formalisms, to be used for reasoning with, rather than purely mathematical structures that we can only reason about. Meaning in such systems is instituted by certain stipulations that we call meaning explanations. Once basic notions such as assertion and assertoric content have been explained, such stipulations often take the form of meaning-determining rules.

2   The structure of assertion

I shall not attempt a definition of assertion, nor of the more general notion of a speech act. Instead I shall take it for granted that there is a speech act whose primary and designated role is the communication of knowledge, and I shall call that speech act assertion. A clear example is the speech act a mathematician makes when communicating a theorem, be it on the blackboard, in a paper, or over a coffee. An assertion, being an act, takes place in time, yet it has a structure that is not bound to time and that may be the object of theorizing. That at least is a presupposition of speech act theory—and of phenomenology, which studies not only speech acts, but intentionality more broadly. I will follow their lead here.

The fundamental structure of a speech act is the force/content structure. This structure can best be illustrated by series of examples where one element
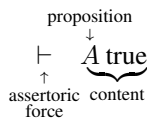
stays constant and the other varies. In normal utterances of the following three sentences,

1. He will leave the room.
2. Will he leave the room?
3. Leave the room!

the content remains the same, but the force varies. That the force varies means simply that the kind of speech act varies, namely from assertion (1) to question (2) to command (3). A series of speech acts of the same kind—with the same force—but with varying content is provided by the series of theorems, major or minor, contained in any mathematical research paper.

A standard assumption of speech act theory is that the content of a speech act is to be identified with a proposition in the sense of modern logic. There are, of course, various ways of understanding the notion of proposition in logic, but on any understanding, it is an object of some kind: a truth value, a set of possible worlds, a type of proof objects, etc. A moment of reflection shows that a proposition understood in any of these ways cannot play the role of the content of a speech act. In particular, it cannot play the role of the content of an assertion: one cannot attach assertoric force to a truth value, a set, or a type. What one can attach assertoric force to is the content that a truth value is equal to the truth value True, that the actual world is an element of a set of possible worlds, or that a type of proof objects is non-empty.

Martin-Löf (2003) concluded that we must distinguish the notion of the content of a speech act from the notion of proposition. In the first instance, this gives rise to a three-levelled analysis of assertion:

$$\overset{\text{proposition}}{\underset{\underset{\substack{\uparrow \\ \text{assertoric} \\ \text{force}}}{}}{\vdash} \underbrace{\overset{\downarrow}{A}\text{ true}}_{\text{content}}}$$

The word "true" here is not a truth predicate, since "A true" does not stand for a proposition, but for a content. I shall call it the truth particle. It stands for an operation that transforms a proposition into a content. The precise specification of this operation will depend on the underlying notion of proposition. For instance, if A is a truth value, then A true is the content that A is equal to the True. If A is a type of proof objects, as in Martin-Löf's constructive meaning theory, then A true is the content that A is non-empty.

Hare (1970, 1989) observed that Frege's assertion sign—which in Frege's writings is a vertical stroke, but here, in a tradition following the *Principia*

Mathematica, is the turnstile—serves a double role: it is both a sign of assertoric force and a sign of what Hare called subscription, that is, a sign to indicate that the assertion has in fact been made. This latter indication is no part of the assertion sign as we shall use it here. For us, the assertion sign will be purely a sign of assertoric force.

The displayed structure is the structure of a categorical assertion. Deductive reasoning consists not only of categorical assertions, but also of hypothetical assertions. A hypothetical assertion is an assertion, hence its force is the assertoric force, but its content has hypothetical form:

$$\vdash \underbrace{A_1 \text{ true}, \ldots, A_n \text{ true} \Rightarrow A \text{ true}}_{\text{content}}$$

The arrow here is not a propositional connective, such as implication, but a content-forming operation, namely an operation that forms hypothetical content. We call contents to the left of the arrow hypotheses and the content to the right the succedent. If the set Γ of hypotheses is empty, then the content $\Gamma \Rightarrow A$ true is just the content A true. If the set of hypotheses is non-empty, the arrow is explained by means of a modus-ponens-like rule, the precise form of which depends on the form of the hypotheses. In the present case, the rule is as follows:

$$\frac{\vdash \Gamma, A \text{ true} \Rightarrow C \qquad \vdash A \text{ true}}{\vdash \Gamma \Rightarrow C}$$

Here and in what follows, C is any content that may serve as succedent. A cut rule can now be justified:

$$\frac{\vdash \Gamma, A \text{ true} \Rightarrow C \qquad \vdash \Gamma' \Rightarrow A \text{ true}}{\vdash \Gamma, \Gamma' \Rightarrow C} \qquad \text{(true-cut)}$$

The justification proceeds by induction on the size of Γ'.

Cut rules such as (true-cut) above and (apod-cut) below are introduced in this article only to facilitate the justification of certain further rules that will be used, in Section 5, in the validation of S4. Whether a cut rule is a primitive rule in the formal system or is merely admissible matters nothing to our purposes. What does matter is that the cut rules introduced are justified by the meaning of the arrow.

We assume that formation rules for propositions are included in the formalism, as they are in Martin-Löf's type theory. There is thus a form of content, A prop, expressing that A is a proposition. (To explain this form of

content is just to explain what a proposition is.) We use this form of content in the formulation of two rules that are justified by the meaning-determining rule for the arrow. The first is the rule of assumption:

$$\frac{\vdash A \text{ prop}}{\vdash A \text{ true} \Rightarrow A \text{ true}} \qquad \text{(Assm)}$$

The second is a rule of weakening:

$$\frac{\vdash \Gamma \Rightarrow A \text{ true} \qquad \vdash B \text{ prop}}{\vdash \Gamma, B \text{ true} \Rightarrow A \text{ true}} \qquad \text{(Weak)}$$

The justification of the latter proceeds by induction on the size of Γ.

3   Introducing modality

Martin-Löf's three-levelled analysis of assertion gives us three possible operands for modality. I shall outline a view according to which modality applies primarily to the assertion as a whole, or more precisely, to assertoric force. By a process I shall call internalization, following both Pfenning and Davies (2001) and Sundholm (2003), the modality is pushed inwards, first to the level of content, modifying the truth particle, and next to the level of propositions, yielding a modal operator in the usual sense of modal logic.

Letting modality operate primarily on the force of an assertion seems to me to make good sense of Kant's claim that modality contributes nothing to content, but concerns rather the relation between this content and "thought as such" (Kant, 1781/1787, A75/B100). The force of a speech act can naturally be said to correspond to the attitude that the utterer takes to its content—assuming, of course, that the utterance is a sincere one. In the *Ideas* (1913), Husserl dedicated a long discussion (§§ 103–112) to the phenomenon of modified "thetic character", which was Husserl's term in that book for what in effect is a generalization of the notion of force to intentional acts more broadly. Searle and Vanderveken (1985, pp. 63–64) introduce various operations on the forces of speech acts which they designate by means of prefixed boxes, though they do not speak of these operations as modalities.

I shall be concerned, in the first instance, with apodeictic modality. Operating on the force of an assertion, apodeictic modality indicates that the assertion in question has been demonstrated, or proved. (The Greek noun *apodeixis* was used to mean demonstration, both in the sense of showing forth something or someone and in the sense of scientific demonstration.)

I shall not attempt to define the bounds of apodeictic knowledge, but I do take it for granted that mathematical knowledge is, in the typical case, apodeictic. Knowledge supported, not by demonstration, but by testimony, is an instance of what may be called assertoric knowledge. That Caesar crossed the Rubicon I can know only assertorically, since I have no way of going back in time to witness with my own senses that river crossing. The notion of assertoric knowledge turns out to play an essential role in the explanation of the validity of inference (Klev, 202x).

Given this gloss on apodeictic modality, it is natural to postulate the following "necessitation" rule as meaning determining for it:

$$\frac{\begin{array}{c} \mathcal{D} \\ \vdash C \end{array}}{\Vdash C} \qquad (\Vdash\text{-intro})$$

In words this says that the content C, which in general has hypothetical form, $\Gamma \Rightarrow C'$, may be apodeictically asserted provided one has demonstrated the assertion $\vdash C$. A demonstration is a sequence of valid inferences beginning from axioms. The boldface line is used to indicate that this is not a usual form of inference. If a colleague in the mathematics department makes an assertion of the form $\vdash A \wedge B$ true, I may, without further ado, infer $\vdash A$ true and $\vdash B$ true. I may, however, not infer $\Vdash A \wedge B$ true unless I have seen, or otherwise become confident that there is a demonstration of this assertion. The rule of $\Vdash$-introduction is context sensitive in a way rules of inference usually are not. To apply it, one needs to take into account, not only the premiss assertion, but the whole demonstration, $\mathcal{D}$, that precedes it.

4 First internalization

The level of assertion is the appropriate level for the application of apodeictic modality, since what one demonstrates are assertions and not assertoric contents or propositions. We can, however, make good sense of internalizing the apodeictic modality of an assertion by making it part of its content. More precisely, modality at this level can be taken to modify the truth particle. I shall write "apod", short for "apodeictically true", and introduce the form of content A apod. (Pfenning and Davies use "valid" for the same purpose.) The general form of hypothetical content is then

$$A_1 \text{ apod}, \ldots, A_n \text{ apod}; B_1 \text{ true}, \ldots, B_m \text{ true} \Rightarrow A \text{ true/apod}$$

Writing Δ for the set of apodeictic hypotheses and Γ for the set of truth hypotheses, the same can be written more compactly as follows:

$$\Delta; \Gamma \Rightarrow A \text{ true}/\text{apod}$$

The slash is used here to indicate that the form of the succedent is either A true or A apod.

That the apod particle internalizes apodeictically modified force is captured by its introduction rule:

$$\frac{\Vdash A \text{ true}}{\vdash A \text{ apod}} \qquad (\text{apod-intro})$$

The meaning of the arrow is extended in the obvious way, giving rise to a cut rule and rules of assumption and weakening. The cut rule is as follows:

$$\frac{\vdash \Delta, A \text{ apod}; \Gamma \Rightarrow C \qquad \vdash \Delta'; \Gamma' \Rightarrow A \text{ apod}}{\vdash \Delta, \Delta'; \Gamma, \Gamma' \Rightarrow C} \qquad (\text{apod-cut})$$

The peculiar nature of apodeictic hypotheses is captured by the following pair of rules:

$$\frac{\vdash \Delta, A \text{ apod}; \Gamma \Rightarrow C \qquad \Vdash \Delta; \emptyset \Rightarrow A \text{ true}}{\vdash \Delta; \Gamma \Rightarrow C} \qquad (\text{apod-true-cut})$$

$$\frac{\Vdash \Delta; \emptyset \Rightarrow A \text{ true}}{\vdash \Delta; \emptyset \Rightarrow A \text{ apod}} \qquad (\text{apod-intro*})$$

Although (apod-intro*) is in effect a generalization of (apod-intro), it cannot replace the latter as meaning determining for the apod particle, owing to the occurrence of this particle in the hypotheses, Δ.

Justification of (apod-true-cut) and (apod-intro).* We justify these rules simultaneously by induction on the size of the set Δ of apodeictic hypotheses.

For size 0, (apod-intro*) is just (apod-intro), which is valid by stipulation, so consider (apod-true-cut). Its right-hand premiss has the form $\Vdash A$ true, whence we may infer $\vdash A$ apod. Its left-hand premiss has the form $\vdash A \text{ apod}; \Gamma \Rightarrow C$, whence we may infer $\vdash \Gamma \Rightarrow C$ by (apod-cut).

For size greater than 0, let us first consider (apod-true-cut). We wish to justify inferences of the following form:

$$\frac{\vdash \Delta, B \text{ apod}, A \text{ apod}; \Gamma \Rightarrow C \qquad \Vdash \Delta, B \text{ apod}; \emptyset \Rightarrow A \text{ true}}{\vdash \Delta, B \text{ apod}; \Gamma \Rightarrow C} \qquad (\gamma)$$

Under the assumption that the two premisses are correct, we must show that the conclusion is correct. We do the latter by showing, firstly, that the following inference is correct:

$$\frac{\vdash B \text{ apod}}{\vdash \Delta; \emptyset \Rightarrow A \text{ apod}} \tag{δ}$$

Assume, therefore, that $\vdash B$ apod is correct. From the meaning of the apod particle, it follows that also $\Vdash B$ true is correct, whence there is a demonstration of $\vdash B$ true. This demonstration can be extended to a demonstration of $\vdash B$ apod, whence we may infer $\Vdash B$ apod. Using (true-cut) with the usual turnstile, $\vdash$, replaced by the reinforced turnstile, $\Vdash$, we may then infer $\Vdash \Delta; \emptyset \Rightarrow A$ true from the right-hand premiss of (γ). Now we use the induction hypothesis on (apod-intro*) to infer $\vdash \Delta; \emptyset \Rightarrow A$ apod. The justification of (δ) is thus complete. From the meaning of the arrow, it follows that $\vdash \Delta, B$ apod$; \emptyset \Rightarrow A$ apod is correct. From the left-hand premiss of (γ) we may infer $\vdash \Delta, B$ apod$; \Gamma \Rightarrow C$ by (apod-cut). This completes the induction step for (apod-true-cut).

The induction step for (apod-intro*) follows immediately by letting Γ be $\emptyset$ and C be A apod. □

One more rule pertaining to the apod particle is needed for the development of S4 in the following section. From the meaning-determining role of (apod-intro) it is clear that the following rule is justified:

$$\frac{\vdash \Delta; \Gamma \Rightarrow A \text{ apod}}{\vdash \Delta; \Gamma \Rightarrow A \text{ true}} \tag{apod-elim}$$

Rules governing the propositional connectives are formulated only by means of the truth particle, not by means of the apod article. For instance, the rules for implication are as follows:

$$\frac{\vdash \Gamma, A \text{ true} \Rightarrow B \text{ true}}{\vdash \Gamma \Rightarrow A \supset B \text{ true}} \qquad \frac{\vdash \Gamma \Rightarrow A \supset B \text{ true} \qquad \vdash \Gamma' \Rightarrow A \text{ true}}{\vdash \Gamma, \Gamma' \Rightarrow B \text{ true}}$$

As will be seen in an example in the next section, one can use (apod-elim) and (apod-intro) to analyze propositions under the apod particle.

In this section and the next we have relied on Pfenning and Davies (2001, § 4), though with some deviations. Three deviations are worth noticing. The three-levelled analysis of assertion led us to introduce the force operator $\Vdash$, not used by Pfenning and Davies. Unlike them, we have allowed the apod particle to occur in the succedent, not only in the hypotheses. Finally, we have offered a justification of, and not simply stipulated, the rule (apod-true-cut).

5   Second internalization

The second internalization introduces modality as a propositional operator. We stipulate, first of all, that $\Box A$ is a proposition whenever A is a proposition. That $\Box$ is indeed the internalization of the apod particle is clear from its meaning-determining introduction rule:

$$\frac{\vdash A \text{ apod}}{\vdash \Box A \text{ true}} \qquad (\Box\text{-intro})$$

From the meaning of the arrow, the corresponding rule that includes hypotheses is justified:

$$\frac{\vdash \Delta; \Gamma \Rightarrow A \text{ apod}}{\vdash \Delta; \Gamma \Rightarrow \Box A}$$

No restrictions are placed on the set Δ, Γ of hypotheses here, and it remains unchanged in the passage from premiss to conclusion. This is uncommon in natural-deduction formulations of modal logic (cf. e.g. Prawitz, 1965, ch. vi). A typical and easily formulable restriction says—translated to the present framework—that all hypotheses must have the form $\Box B$ true. Such a restriction works well formally, but places the meaning-determining role of ($\Box$-intro) in jeopardy, owing to the negative occurrences of $\Box$ in its premiss.

For the elimination rule we may use a natural deduction formulation:

$$\frac{\Box A \text{ true} \qquad \begin{array}{c} (A \text{ apod}) \\ \mathcal{D} \\ C \end{array}}{C} \qquad (\Box\text{-elim})$$

This elimination rule follows a pattern familiar, for instance, from $\vee$-elimination. It is justified through the stipulation of the following reduction:

$$\frac{\begin{array}{c}\mathcal{D}\\ A \text{ apod}\end{array}}{\dfrac{\Box A \text{ true}}{C} \quad \begin{array}{c}(A \text{ apod})\\ \mathcal{D}'\\ C\end{array}} \quad \rightsquigarrow \quad \begin{array}{c}\mathcal{D}\\ \mathcal{D}'\\ C\end{array} \qquad (\Box\text{-red})$$

The right-hand derivation here is obtained by replacing each occurrence of A apod in leaf position in $\mathcal{D}'$ by the derivation $\mathcal{D}$, whose conclusion is A apod. Such a replacement is justified by the rule (apod-cut).

For illustration we display, in a natural deduction format, a demonstration that the K scheme is true:

$$
\cfrac{
  \cfrac{
    \cfrac{A \supset B \text{ apod}^3}{A \supset B \text{ true}}
  }{}
}{}
$$

$$
\begin{array}{c}
\cfrac{\dfrac{A \supset B \text{ apod}^3}{A \supset B \text{ true}} \qquad \dfrac{A \text{ apod}^4}{A \text{ true}}}{\begin{array}{c}B \text{ true}\\ \overline{\phantom{B \text{ true}}}\\ B \text{ apod}\end{array}}
\end{array}
$$

$$
\cfrac{\Box(A \supset B)\text{ true}^1 \qquad \cfrac{\Box A \text{ true}^2 \qquad \cfrac{\cfrac{\cfrac{A \supset B \text{ apod}^3}{A \supset B \text{ true}} \quad \cfrac{A \text{ apod}^4}{A \text{ true}}}{\cfrac{B \text{ true}}{B \text{ apod}}}}{\cfrac{\Box B \text{ true}}{}}\, 4}{\cfrac{\Box B \text{ true}}{\cfrac{\Box A \supset \Box B \text{ true}}{}}\, 2}\, 3}{\cfrac{\Box A \supset \Box B \text{ true}}{\Box(A \supset B) \supset (\Box A \supset \Box B)\text{ true}}\, 1}
$$

The dotted line indicates two inference steps which we can make explicit by writing out the first part of this derivation as a proper demonstration:

$$
\cfrac{
  \cfrac{\vdash A \supset B \text{ apod} \Rightarrow A \supset B \text{ apod}}{\vdash A \supset B \text{ apod} \Rightarrow A \supset B \text{ true}} \qquad
  \cfrac{\vdash A \text{ apod} \Rightarrow A \text{ apod}}{\vdash A \text{ apod} \Rightarrow A \text{ true}}
}{
  \cfrac{\cfrac{\vdash A \supset B \text{ apod},\, A \text{ apod} \Rightarrow B \text{ true}}{\Vdash A \supset B \text{ apod},\, A \text{ apod} \Rightarrow B \text{ true}}}{\vdash A \supset B \text{ apod},\, A \text{ apod} \Rightarrow B \text{ apod}}
}
$$

A boldface inference line is used to indicate application of ($\Vdash$-intro) from the third to the fourth line. As already noted, application of this rule requires that we reflect on the whole of the preceding demonstration, not only the premiss.

The derivations of $\vdash \Box A \supset A$ true and $\vdash \Box A \supset \Box\Box A$ true employ the same ideas. Thus, all the defining schemes of S4 are derivable.

Our formulation of ($\Box$-intro) relies on the possibility of using the apod particle in the succedent. In Pfenning and Davies's formulation, the premiss of ($\Box$-intro) is rather $\Delta; \cdot \Rightarrow A$ true, rendering their $\Box$-introduction similar to the rule (apod-intro*) from the previous section. The present formulation ($\Box$-intro) has been preferred here for two reasons: it explicitly introduces $\Box$ as an internalization of apod, and it fits perfectly with ($\Box$-elim) as elimination rule.

The idea of distinguishing hypotheses and succedents of two kinds in sequent calculi for modal logic was explored already by Blamey and Humberstone (1991). They did not introduce any novel logico-grammatical categories—such as our contents and assertions—but operated just with formulae, corresponding to our propositions. They did, however, remark (p. 776) that "the move from truth-functional to modal logic" may best be

made, not "by adding a new primitive connective with new rules governing it, but rather by extending one's conception of the objects to be manipulated by such rules." In the present terminology, the objects manipulated by such rules are indeed contents rather than propositions.

6   Problematic modality

Besides apodeictic and assertoric modality, the third and last modality of judgement recognized by Kant is the problematic modality. In Kant's table of categories, which he took to be derived from his table of judgements, problematic modality corresponds to possibility. From Kant's discussion of problematic modality, it appears that the conception of possibility here must be a rather weak one. An especially instructive passage is the following (A75/B101):

> The problematic proposition [*Satz*] is therefore that which only expresses logical possibility (which is not objective), i.e., a free choice to allow such a proposition to count as valid, a merely arbitrary assumption [*Aufnehmung*] of it in the understanding.

Transferred to the current setting, this passage (and others from the same section of the *Critique*) suggests to me that *any* proposition in the present sense may be judged with problematic modality to be true. Every proposition in the present sense "expresses logical possibility" in the sense that it may be assumed to be true.

It is therefore natural to stipulate the following rule as meaning determining for problematic modality:

$$\frac{\vdash A \text{ prop}}{\vdash\hspace{-0.6em}\sim A \text{ true}} \qquad (\hspace{-0.2em}\sim\hspace{-0.2em}\text{-intro})$$

Problematically modified assertoric force is thus indicated by means of a squiggly turnstile. Its applicability is restricted here to content of the form A true. Restriction on the form of content to which a given force can apply is a familiar phenomenon in speech act theory (cf. the so-called propositional content rule of Searle, 1969, p. 63.) I shall not discuss whether introduction rules for problematic modality can be given that is less restrictive as to the form of the operand. Meaning-determining rules for problematic modality are not without interest, since $\vdash\hspace{-0.6em}\sim C$ in effect asserts that C is a content.

We capture the tight relation that Kant appears to have seen between problematic modality and the notion of assumption by laying down the following elimination rule:

$$\frac{\vdash A \text{ true}}{\vdash A \text{ true} \Rightarrow A \text{ true}} \qquad (\text{⊬-elim})$$

It will be clear that a force regulated by these two rules is of little use: an assertion of the form $\vdash A$ prop already provides for what we might want to do with the problematically modified assertion $\vdash A$ true. It is sometimes asked whether there is a separate speech act of assumption. We might call $\vdash$ the force of such a speech act and gloss $\vdash A$ true as entertaining A to be true. The logical use of assumptions, however, comes out properly only through the occurrence of contents as hypotheses in assertions of hypothetical form.

The first internalization of $\vdash$ gives us just the form of content A prop, and the second internalization gives an operator π with the introduction rule

$$\frac{\vdash A \text{ prop}}{\vdash \pi A \text{ true}}$$

Such an operator is of little interest here, but it may be of interest in a formal language that allows for the formation of sentences deemed meaningless.

7 Possibility

Problematic modality, in particular its second internalization, π, is quite far removed from what standard modal logic would make one expect of a possibility modality. In particular, one would not expect in standard modal logic to find $\Diamond A$ to be true for every proposition A. Pfenning and Davies (2001, § 5) extended their account of modal logic to a more standard possibility modality by relying on possible-worlds intuitions. The account follows the same pattern as their treatment of necessity: it begins with rules for a content-forming particle, poss, which is then employed for the formulation of natural deduction rules.

The particle poss operates on a proposition A to form the content A poss. Where Γ and Δ are as before, and E is a—possibly empty—set of hypotheses of the form B poss, the general form of hypothetical content is now

$$E; \Delta; \Gamma \Rightarrow A \text{ true/apod/poss}$$

The cut rule for poss is the obvious one, and rules of assumption and weakening follow.

Pfenning and Davies lay down the following two rules for the poss particle:

$$\frac{\vdash A \text{ true}}{\vdash A \text{ poss}} \qquad \frac{\vdash E; \Delta; \Gamma \Rightarrow A \text{ poss} \qquad \vdash \Delta; A \text{ true} \Rightarrow B \text{ poss}}{\vdash E; \Delta; \Gamma \Rightarrow B \text{ poss}}$$

It is natural to regard both of these as (meaning-determining) introduction rules. The first rule gives the base case for the application of poss, and the second rule, the successor case. In the second premiss of the second rule, it is essential that A true is the only truth hypotheses and that there are no possibility hypotheses. If A poss is taken to mean that there is a world in which A is true, then the second rule may be compared to $\exists$-elimination, and the hypothesis A true may be compared to the hypothesis that a world w is given where A is true. To be allowed to use this hypothesis in an application of $\exists$-application, we cannot assume to know anything more about w than this, whence the restriction on the hypotheses in the second premiss. From the meaning of the arrow it follows that the first rule remains justified under the extension of the content by arbitrary hypotheses.

Having introduced the poss particle, we can use it to give natural deduction rules for $\Diamond$ rendering it the internalization of poss. The rules are precise parallels to the rules for $\Box$:

$$\frac{\vdash A \text{ poss}}{\vdash \Diamond A \text{ true}} \Diamond\text{-intro} \qquad \frac{\Diamond A \text{ true} \qquad \begin{array}{c} (A \text{ poss}) \\ \mathcal{D} \\ C \end{array}}{C} \Diamond\text{-elim}$$

The corresponding reduction rule is justified by the cut rule for the poss particle. Using these rules, one can derive $\vdash A \supset \Diamond A$ true, $\vdash \Diamond\Diamond A \supset \Diamond A$ true, and $\vdash \Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$ true.

Whereas the diamond, $\Diamond$, is thus rendered the internalization of the poss particle, it is difficult to see from the two introduction rules for poss that it could be rendered the internalization of a modification of assertoric force or indeed of any illocutionary force. The second premiss of the second rule poses a problem for such a rendering, since poss there occurs in the succedent, but is not allowed to occur in the hypotheses. Since the introduction rules for poss relied on possible-worlds intuitions rather than speech act theory, it is perhaps only to be expected that no such rendering is forthcoming.

8   Generalizing the account

Let Φ be an illocutionary force that is applicable to content of the form A true, perhaps with some restriction on the proposition A. A natural question is then whether, following our account of apodeictic and problematic modality, two steps of internalization can be carried out:

$$\Phi\ A\ \text{true} \quad \longrightarrow \quad \vdash A\ \text{phi} \quad \longrightarrow \quad \vdash \phi A\ \text{true}$$

Not every modal operator of standard modal logic can be regarded as such a ϕ. The operators of temporal logic are a clear example. The possibility modality described in the previous section, motivated by possible-worlds intuitions, may be another. These modal operators are, however, not counterexamples to the thesis that every force Φ of the kind described gives rise to a modal operator ϕ.

One might ask whether plain assertoric force, $\vdash$, is such a counterexample. It is not. There is a clear sense in which the truth particle plays a role at the level of content similar to that played by assertoric force at the level of force: each of them is the standard form with respect to modalization. Under the first internalization, modified assertoric force is replaced by unmodified assertoric force, $\vdash$, and under the the second internalization, a modified truth particle is replaced by the unmodified truth particle, true. The truth particle is thus naturally seen as the internalization of assertoric force. The internalization of the truth particle, in turn, is the truth modality, T:

$$\frac{\vdash A\ \text{true}}{\vdash T(A)\ \text{true}} \qquad (T\text{-intro})$$

Its elimination rule may be formulated simply as the inverse of this introduction rule:

$$\frac{\vdash T(A)\ \text{true}}{\vdash A\ \text{true}} \qquad (T\text{-elim})$$

As one would expect of the internalization of the truth particle, the truth modality is neutral and leaves everything as it was. The second internalization, quite generally, offers a way of expressing a modified truth particle in terms of the unmodified truth particle, but with a modified proposition. It is clear that, in order to express A true in terms of the truth particle, no changes are needed to the proposition A.

The view suggested is thus that every illocutionary force applicable to content of the form A true gives rise to a modal operator. Such a view in turn

suggests that there are deep connections between modal logic and speech act theory. Investigating those connections further could be beneficial to the philosophical study of modal logic.

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