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THE CATEGORY OF INNER MODELS

1. INTRODUCTION

Set Theory is the mathematics of infinity. The results and arguments of set theory are characterized by enormous differences in size, as well as interactions between entities of different sizes. In this article we shall study a category of class-sized objects and its impact on small sets of real numbers.

The dialectical tension between the finite and the infinite is present in the very foundations of set theory and mathematics. Properties of the infinite are formulated as finite mathematical expressions, reasoning about the infinite is conducted by finite mathematical proofs. Gödel's incompleteness theorems prove mathematically that one cannot completely describe the infinite by finitary means: there are (number theoretic) statements involving quantifications over infinite domains which are not decided by the standard axioms of ZFC set theory; moreover, the undecidability phenomenon cannot be avoided by extensions of the axiomatic system (Gödel 1931).

Research in axiomatic set theory has discovered the independence of many principles of infinitary combinatorics. Cantor's continuum hypothesis $2^{\aleph_0} = \aleph_1$, which proposes an answer to the first nontrivial question about infinitary cardinal exponentiation was shown to be independent by Gödel (1938) and Cohen (1963). There is a wealth of set theoretical axiom systems which extend the usual axioms and which are (presumably) consistent and realizable in first order structures. This is well-documented in the standard textbooks (Jech 1978; Kanamori 1994), which we recommend as a general reference for this article.

Most research in axiomatic set theory consists in constructing and examining transitive models of the axioms of ZFC. Since the consistency results of axiomatic set theory are relative consistencies, new models of set theory have to be constructed from given ones, usually as extensions or submodels where the smaller model is a class-sized transitive submodel of the bigger one. Class-sized transitive models of ZFC are called inner

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models of set theory. Gödel's model **L** of constructible sets is the paradigm of an inner model (Gödel 1938) (see also Devlin (1984)).

Set theory has made good use of situations in which there are many inner models available. To assume the existence of many inner models seems natural if one accepts the profound inaccessibility of the infinite. Nevertheless there is the desire to classify the spectrum of "possible worlds" mathematically. It was noted many years ago and has become part of the set theoretical folklore that some natural operations and properties of inner models are related to large cardinal notions. Motivated by techniques from large cardinal theory we shall explore some aspects of the family of inner models from a category-theoretical perspective.

In Section 3 we shall consider the *category of inner models* with elementary embeddings as morphisms. In the presence of large cardinals this category is nontrivial, i.e., there are morphisms which are not the identity. The first ordinal moved by a nontrivial morphism is a large cardinal in an inner model. We investigate situations in which this category exhibits some structural richness (Section 2 and 4). Sets of real numbers may allow a representation by a commutative subsystem (diagram) of the category of inner models (Section 4). The existence of such normal forms (*embedding normal forms with witnesses*) for a set $A \subseteq \mathbb{R}$ implies the determinacy of the infinitary game with winning set A and other regularity properties (Section 5). In Section 6, we consider an operation by which an embedding of an inner model can act on another model to yield an "induced" embedding of that model. This operation is useful for coding information into diagrams of inner models. In Section 7 we give an indication how embedding normal forms for projective sets of reals can be built, which can be used to prove the famous theorem of Martin and Steel on projective determinacy (Martin and Steel 1989). The preconditions for such constructions are given by measurable cardinals and Woodin cardinals. We conclude this article with an Appendix containing more details on the construction of the embedding normal forms.

The principal message of this article is that notions of transcendental size (in the present situation: parametrized families of proper classes) are related to familiar mathematical structures and that there are natural methods of transformation between the realms of inner models and of descriptive set theory. It could be interesting and fruitful to relate technical results on the structure of the family of inner models to philosophical questions about "possible worlds". Can the spectrum of possible worlds be employed to gain information about the "actual" world as is being suggested by the determinacy results of Section 5? A programmatic answer which corresponds well to the present situation in the foundations of mathematics may be read from the concluding words of Felix Hausdorff's work *Das Chaos in kosmischer Auslese* (Hausdorff 1998, p. 209) which he published under his pseudonym *Paul Mongré*:

Werden wir also den *kosmocentrischen* Aberglauben los wie früher den geocentrischen und anthropocentrischen; erkennen wir, dass in das Chaos eine unzählbare Menge kosmischer Welten eingesponnen ist, deren jede ihren Inhabern als einzige und ausschließlich reale Welt erscheint und sie verleiten möchte, ihre qualitativen Merkmale und Besonderheiten dem transcendenten Weltkern beizulegen. Aber dieser Weltkern entzieht sich jeder noch so losen Fessel und wahrt sich die Freiheit, auf unendlich vielfache Weise zur kosmischen Erscheinung eingeschränkt zu werden; er gestattet das Nebeneinander aller dieser Erscheinungen, die als specielle Möglichkeiten, als begrifflich irgendwie abgegrenzte Theilmengen in seiner Universalität enthalten sind – ja er ist nichts anderes als eben dieses Nebeneinander und darum transcendent für die einzelne Erscheinung, die in sich selbst ihr eigenes abgeschlossenes Immanenzgebiet hat.[1]

2. INNER MODELS OF ZERMELO FRAENKEL SET THEORY

A model of set theory is a rich structure in which the usual mathematical arguments can be formulated. Such a structure can be thought of as a relational algebraic structure equipped with an elementhood relation $\in$ and operations of set formation which satisfy certain laws. These laws are expressed by the axioms of Zermelo–Fraenkel set theory including the axiom of choice (ZFC). We shall concentrate our attention on *inner models*, i.e., transitive class-sized models of set theory which can be regarded as standard models of the system ZFC. The detailed development and analysis of the theory ZFC is quite involved. The following is a brief sketch how the basic mathematical notions can be formalized set-theoretically. It mainly serves to fix some notation.

Set theory studies the informal notion of a set as described by Cantor. The *class term* $\{x \mid \varphi\}$ denotes the collection of all objects x such that φ holds, i.e., $z \in \{x \mid \varphi\} :\leftrightarrow \varphi(\frac{z}{x})$. Those z are called the *elements* of $\{x \mid \varphi\}$.

Basic operations on sets and classes can be defined with the help of class terms:

- $\emptyset = \{x \mid x \neq x\}$ is the *empty set*;
- $\{x, y\} = \{z \mid z = x \vee z = y\}$ is the *unordered pair* of x and y;
- $(x, y) = \{\{x, x\}, \{x, y\}\}$ is the *ordered pair* of x and y.

The theory of relations and functions can be built upon the notion of ordered pair as usual.

- $\bigcup x = \{z \mid \exists y \in x\ z \in y\}$ is the *union* of (the elements in) x;
- $\mathfrak{P}(x) = \{y \mid y \subseteq x\}$ is the *powerset* of x.

The principal tools to study the infinite are induction and recursion along the (transfinite) *ordinal numbers*, which were formalized by von Neumann as follows (von Neumann 1923).

- A class A is *transitive* if it is an initial segment of the $\in$-relation: $\forall x \in A\, \forall y \in x\; y \in A$;
- A is *well-ordered* by the $\in$-relation if (a) $(A, \in)$ is linearly ordered, i.e., $(A, \in)$ is transitive, non-reflexive and connected, and (b) $(A, \in)$ is well-founded, i.e., $\forall x \subseteq A(x \neq \emptyset \to \exists u \in x\, \forall v \in x\; v \notin u)$.
- an *ordinal* is a set α which is transitive and well-ordered by the $\in$-relation.

The class Ord of all ordinals is itself transitive and well-ordered by the $\in$-relation. Each ordinal α has an immediate *successor* $\alpha + 1 = \alpha \cup \{\alpha\}$. Natural numbers are those ordinals which can be reached from $0 = \emptyset$ by the $+1$-operation: n is a *natural number*, if

$$n = \emptyset \vee ((\exists m(n = m + 1)) \wedge \forall m \in n(m = \emptyset \vee \exists \ell(m = \ell + 1))).$$

The collection of all natural numbers will be denoted by ω.

A central question in set theory is which class terms $\{x \mid \varphi\}$ can be considered to be mathematical objects in the full sense, i.e., sets. Russell's antinomy shows that *not all* classes can be permitted to be sets. The axiomatization of set theory by Zermelo and Fraenkel postulates that many classes defined above are admissible as sets (Zermelo 1930). The Zermelo–Fraenkel system is sufficiently rich to carry out the development of all mathematical notions. It is based on the intuitive notion of a set and has not lead to any contradictions so far.

We give a short list of the axioms using the notation of class terms. Axioms (2.1) to (2.6) express that the $\in$-relation is *extensional* (a set is exactly determined by its elements) and well-founded and that set theoretical universes are closed relative to the basic operations introduced above.

Set Theoretical Axioms

(2.1) Extensionality: $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \to x = y)$.

(2.2) Foundation: $\forall x(x \neq \emptyset \to \exists y \in x \forall z \in x(z \notin y))$.

(2.3) Pairing: $\forall x \forall y \exists z(z = \{x, y\})$.

(2.4) Union: $\forall x \exists z(z = \bigcup x)$.

(2.5) Powerset: $\forall x \exists z(z = \mathfrak{P}(x))$.

(2.6) Infinity: $\exists z(z = \omega)$.

The remaining two schemata of axioms express the closure of set theoretical universes under certain definitions by first-order formulae. The *separation schema* could be deduced from the more powerful *replacement schema* but we include separation as it is closest to the original comprehension principle of naive set theory.

(2.7) Separation: for each first-order formula $\varphi(u, \vec{w})$ postulate: $\forall \vec{w} \forall x \exists z\, z = \{u \in x \mid \varphi(u, \vec{w})\}$.

(2.8) Replacement: for each first-order formula $\varphi(u, v, \vec{w})$ postulate: $\forall \vec{w}((\forall u, v, v'(\varphi(u, v, \vec{w}) \wedge \varphi(u, v', \vec{w}) \rightarrow v = v') \rightarrow \forall x \exists z\, z = \{v \mid \exists u \in x\, \varphi(u, v, \vec{w})\})$.

So the image of a set under a definable function is again a set. The only other principle widely assumed in mathematical practice is the *axiom of choice*. In set theory this is usually employed in the equivalent form of Zermelo's *well-ordering principle* (Zermelo 1904):

(2.9) Choice: $\forall x \exists \alpha \exists f(\alpha \text{ is an ordinal } \wedge f : x \leftrightarrow \alpha)$.

The axiomatic system consisting of (2.1) to (2.8) is abbreviated as ZF, and the full system (2.1) to (2.9) as ZFC (Zermelo–Fraenkel set theory with choice). A *model of set theory* is a pair (M, E), where E is a binary relation on the domain M which satisfies the axioms ZFC. Models of the form $(M, \in)$ where $\in$ denotes the $\in$-relation restricted to M are of particular interest and are often obtained by the

MOSTOWSKI COLLAPSING LEMMA 2.1. Let (M, E) be a strongly well-founded relation, i.e., (M, E) satisfies the extensionality axiom (2.1), is well-founded, and $\{x \mid xEm\}$ is a set for all $m \in M$. Then (M, E) is isomorphic to a unique structure $(N, \in)$ where N is a transitive class.

By Gödel's incompleteness theorems we should not be able to construct models of set theory from ordinary mathematical objects. We can only expect to construct such models out of given models. This motivates the

DEFINITION 2.2. A class M is called an *inner model of set theory* if $(M, \in)$ it is a definable transitive model of the system ZFC which contains all the ordinals.

Gödel has defined the inner model $\mathbf{L}$ of *constructible sets* (Gödel 1938, see Devlin 1938). This model is the $\subseteq$-smallest inner model since its definition is *absolute* for any other inner model of set theory: If M is an inner model, then the constructible universe $\mathbf{L}^M$ constructed inside the model M is the same as the original constructible universe: $\mathbf{L}^M = \mathbf{L}$. This implies $\subseteq$-minimality: $\mathbf{L} = \mathbf{L}^M \subseteq M$.

There is always the trivial inner model $\mathbf{V} = \{x \mid x = x\}$, which is the *universe* of all sets. Gödel's *axiom of constructibility* asserts that every set in $\mathbf{V}$ belongs to the constructible sets: $\mathbf{V} = \mathbf{L}$. In this case, the family of inner models is trivial and consists only of the model $\mathbf{V}$ itself. To assume that the family of inner models is non-trivial corresponds to the idea that the set theoretical universe should be rich and allow many possibilities like the existence of non-constructible sets.

3. ELEMENTARY EMBEDDINGS AND THE CATEGORY OF INNER MODELS

A universal theme in modern mathematics is the study of structure preserving maps (homomorphisms, embeddings, isomorphisms, etc.) between structures of the same type. The appropriate framework for this is the language of categories.

In the context of models of set theory, a natural requirement for structure preserving maps is that they preserve the operations of set formation as described in the Zermelo–Fraenkel axioms. So we consider definable elementary embeddings $\pi : M \to N$ between inner models where for every first-order $\in$-formula $\varphi(\vec{u})$ and all $\vec{x} \in M$:

$$(M, \in) \models \varphi[\vec{x}] \text{ if and only if } (N, \in) \models \varphi[\pi(\vec{x})].$$

The map π can be extended to subclasses A of M: let A be definable in M from parameters $\vec{p}$ by a formula φ. Then one can define $\pi(A) = \bigcup_{x \in \mathbf{V}} \pi(A \cap x)$; the class $\pi(A)$ is definable in N from parameters $\pi(\vec{p})$ by the formula φ.

Intuitively, the collection of all inner models with elementary maps between them can be seen as a category. Because of the subtle difficulties concerning the definability of such a category in ZFC we restrict the complexity of the models and embeddings.

DEFINITION 3.1. Fix a sufficiently large natural number $n < \omega$. The *Category of Inner Models* consists of the following: *objects* are all inner models which are Σ_n-definable from parameters; *morphisms* are all elementary maps between the objects which are Σ_n-definable from parameters.

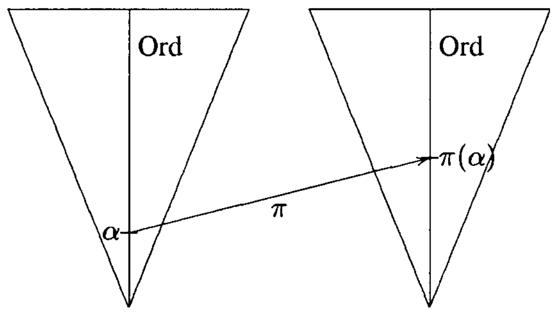

Figure 1. An elementary embedding.

Note that the family of classes which are Σ_n-definable from parameters can be described by a single Σ_{n+1}-formula. Inner models can be characterized as classes which are transitive, closed with respect to the finite set of Gödel functions and *almost universal* (see Jech 1978, Chapter 2). An embedding $\pi : M \to N$ is Σ_1-elementary iff it is fully elementary (see Kanamori 1994, p. 45). This indicates that the category of inner models can be uniformly represented within the system ZFC by a concrete but complicated formula which we shall not state explicitly.

The constant n is assumed to be big enough for the intended applications. The case $n = 1$ will cover most interesting situations and in particular models which are naturally obtainable from $\mathbf{V}$ by iteration trees. So we shall assume that $n = 1$, and for the remainder of this article "inner model" and "elementary embedding of inner models" are to be understood as objects and morphisms of the above category.[2]

The category of inner models has been applied before. Most notable is the work on connections with left-distributive algebras which was initiated by Laver (1992, 1997) and Dougherty (1997).

The formula defining the category of inner models can be evaluated within every model M of set theory. The category of inner models within M will in general be different from the category of inner models within $\mathbf{V}$. We say that an elementary embedding $\pi : M \to N$ is *internal* (in M) if π is a morphism of the category of inner models as defined in M. We shall encounter a subtle interplay between internal and non-internal embeddings in the construction of iteration trees as described in the Appendix.

Let us introduce relations for comparing inner models. We define the usual von Neumann-hierarchy (von Neumann (1925):

$$
\begin{aligned}
\mathbf{V}_0 &= \emptyset, \\
\mathbf{V}_{\alpha+1} &= \mathfrak{P}(\mathbf{V}_\alpha), \\
\mathbf{V}_\lambda &= \bigcup_{\alpha<\lambda} \mathbf{V}_\alpha, \text{ for limit ordinals } \lambda.
\end{aligned}
$$

The hierarchy exhausts the set theoretical universe: $\mathbf{V} = \bigcup_{\alpha \in \mathrm{Ord}} \mathbf{V}_\alpha$. It possesses certain absoluteness properties: if M is an inner model then for all $\alpha \in \mathrm{Ord}$: $(\mathbf{V}_\alpha)^M = \mathbf{V}_\alpha \cap M$. Here $(\mathbf{V}_\alpha)^M$ denotes the term $\mathbf{V}_\alpha$ as defined in M; relativized notations like this will also be used in connection with other class terms.

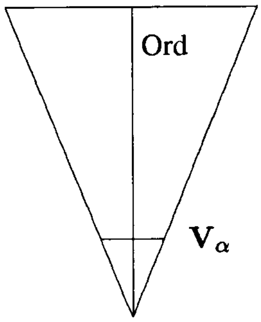

Figure 2. The von Neumann-Hierarchy.

For each ordinal α and inner models M and N we define the equivalence relation of *agreement below* α: $M \sim_\alpha N \leftrightarrow \mathbf{V}_\alpha \cap M = \mathbf{V}_\alpha \cap N$. In many set theoretical arguments, parameters have to be considered and we extend the relation of agreement to include finite sequences of parameters: Let $(M, \in, \vec{p})$ be an inner model with a finite parameter sequence $\vec{p} \in M$ and let $\alpha \in \mathrm{Ord}$, $\alpha \geq \omega$. Consider an appropriate language for the structure $(M, \in, \vec{p}, (z \mid z \in \mathbf{V}_\alpha \cap M))$ in which the sets $z \in \mathbf{V}_\alpha \cap M$ are taken as constants. We may assume that the language is absolutely coded as a subset of $\mathbf{V}_\alpha$ and that for $\beta \geq \alpha$ the theories cohere nicely:

$$\mathrm{Th}(M, \in, \vec{p}, (z \mid z \in \mathbf{V}_\alpha \cap M)) = \mathrm{Th}(M, \in, \vec{p}, (z \mid z \in \mathbf{V}_\beta \cap M)) \cap \mathbf{V}_\alpha.$$

For each infinite ordinal α and pointed inner models $(M, \vec{p})$ and $(N, \vec{q})$ of the same type define:

$$(M, \vec{p}) \sim_\alpha (N, \vec{q})$$

$$:\leftrightarrow \mathrm{Th}(M, \in, \vec{p}, (z \mid z \in \mathbf{V}_\alpha \cap M)) = \mathrm{Th}(N, \in, \vec{q}, (z \mid z \in \mathbf{V}_\alpha \cap N)).$$

Of course, $(M, \vec{p}) \sim_\alpha (N, \vec{q})$ implies that $M \sim_\alpha N$.

By Tarski's theorem on the undefinability of truth (Tarski 1935) the definition of $\sim_\alpha$ for pointed inner models can not be carried out within ZFC. Without further details we can resolve this by restricting the "Th-operator" to statements of limited quantifier-complexity. If models M and N agree below some ordinal α then methods from M can be applied to N as we shall see in Section 6.

4. LIMIT CONSTRUCTIONS IN THE CATEGORY OF INNER MODELS

If $\pi : M \to N$ is an elementary map between inner models, there are in general constants available in N which are not in the image of M. In this sense, N is an *extension* of M by constants. We shall construct iterated extensions of this kind. Sometimes an infinite sequence of such extensions can be embedded into another inner model which is equivalent to the well-foundedness of the direct limit. This criterion will be used to decide whether a real number belongs to a given set A of reals.

Let us assume that there is a non-trivial elementary map $\pi : \mathbf{V} \to M$ in our category, i.e., $\pi \neq \mathrm{id}$. This assumption is equivalent to the existence of a *measurable cardinal* (see Definition 7.1): let α be the *critical point* of π, i.e., α is the minimal ordinal such that $\pi(\alpha) > \alpha$. Then $U = \{x \subseteq \alpha \mid \alpha \in \pi(x)\}$ is a non-trivial α-complete normal ultrafilter on $\mathfrak{P}(\alpha)$, hence α is a measurable cardinal as defined by St. Ulam (1930). Conversely, if there is a non-trivial α-complete normal ultrafilter U on $\mathfrak{P}(\alpha)$ then the Scott-ultrapower of $\mathbf{V}$ by U is a non-trivial elementary map π as above (Scott 1961). There are many motivations to assume the existence of a measurable cardinal. In the context of algebraic categories, "big" structures usually can be embedded into proper substructures, and it seems reasonable to assume that the universe of sets is big in a similar way.

We shall see that we can derive more maps and commutative diagrams from π, and this will lead to combinatorial consequences. We assume that π is internal in $\mathbf{V}$, so we can use π to transport the definitions of π and of M up to M: $M \models \pi(\pi) : M \to \pi(M)$. Since the notion of an elementary map of inner models is sufficiently absolute, we obtain an elementary map $\pi(\pi) : M \to \pi(M)$ into the inner model $\pi(M)$. This process can be iterated transfinitely.

4.1. *A Well-founded Direct Limit*

Define commutative systems $(M_i)_{i<\theta}$, $(\pi_{ij})_{i\leq j<\theta}$ by recursion on the length θ. Set $M_0 = \mathbf{V}$, $\pi_{00} = \mathrm{id}$, $M_1 = M$, $\pi_{01} = \pi$ and $\pi_{11} = \mathrm{id}\restriction M_1$. If θ is a limit ordinal, we simply take the union of the uniquely defined systems of smaller lengths.

For the successor step, assume that the system $(M_i)_{i<\theta}$, $(\pi_{ij})_{i\leq j<\theta}$ is defined and we have to construct $(M_i)_{i\leq\theta}$, $(\pi_{ij})_{i\leq j\leq\theta}$.

Case 1 $\theta = \bar\theta + 1$ is a successor ordinal. Then we continue by mapping π up to $M_{\bar\theta}$: Set $M_\theta = \pi_{0\bar\theta}(M_1)$, $\pi_{\bar\theta\theta} = \pi_{0\bar\theta}(\pi_{01})$, $\pi_{i\theta} = \pi_{\bar\theta\theta} \circ \pi_{i\bar\theta}$, for $i < \bar\theta$, and $\pi_{\theta\theta} = \mathrm{id}\restriction M_\theta$.

Case 2 θ is a limit ordinal. In this case, the system $(M_i)_{i\leq\theta}, (\pi_{ij})_{i\leq j\leq\theta}$ is the direct limit of the system $(M_i)_{i<\theta}, (\pi_{ij})_{i\leq j<\theta}$. The limit can be formed by a universal construction. We claim that it exists in the category of inner models. By the Mostowski Collapsing Lemma 2.1 it suffices to see that the direct limit is strongly well-founded.

So assume for a contradiction that the direct limit is ill-founded. Note that for $i_0 < \theta$ the final segment $(M_i)_{i_0\leq i<\theta}, (\pi_{ij})_{i_0\leq i\leq j<\theta}$ of the directed system is definable in M_{i_0} in a uniform way as a maximal iteration whose direct limit is ill-founded. Choose $\xi_0 \in \mathrm{Ord}$ minimal such that ξ is the first member of an ill-founded chain, i.e., there are $0 = j_0 < j_1 < \ldots \theta$ and $\xi_1, \xi_2, \ldots \in \mathrm{Ord}$ such that $\forall n < \omega\,(\pi_{j_n j_{n+1}}(\xi_n) > \xi_{n+1})$. Then the system $(M_i)_{j_1\leq i<\theta}, (\pi_{ij})_{j_1\leq i\leq j<\theta}$ has an ill-founded chain starting with ξ_1. This chain is an element of $\mathbf{V}$ and an argument using the absoluteness of well-foundedness shows that a possibly different ill-founded chain starting with ξ_1 is an element of M_{j_1}. The elementarity of π_{0j_1} yields that M_{j_1} thinks $\pi_{0j_1}(\xi_0)$ is the minimal first member of an ill-founded chain. But $\pi_{0j_1}(\xi_0) > \xi_1$, contradiction.

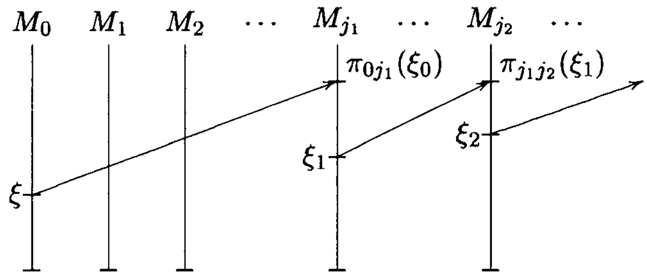

Figure 3. The well-foundedness argument for a linear iteration.

4.2. *An Ill-founded Direct Limit*

The embedding $\pi : \mathbf{V} \to M$ can be continued differently by *repetition*: define a commutative system $(\widetilde{M}_i)_{i<\omega}$ with maps $(\widetilde{\pi}_{ij})_{i\leq j<\omega}$ by recursion: Set $\widetilde{M}_0 = \mathbf{V}$, $\widetilde{\pi}_{00} = \mathrm{id}$, $\widetilde{M}_1 = M$, and $\widetilde{\pi}_{01} = \pi$. If $\widetilde{M}_n$ is defined, then set $\widetilde{\pi}_{n,n+1} = \pi\restriction\widetilde{M}_n$ and $\widetilde{M}_{n+1} = \bigcup(\widetilde{\pi}_{n,n+1})''\widetilde{M}_n$. The other maps of the system are determined by commutativity.

If α is the critical point of π then for each $n < \omega$: $\widetilde{\pi}_{n,n+1}(\alpha) > \alpha$. This implies that the direct limit of the system $(\widetilde{M}_i)_{i<\omega}, (\widetilde{\pi}_{ij})_{i\leq j<\omega}$ is ill-founded.

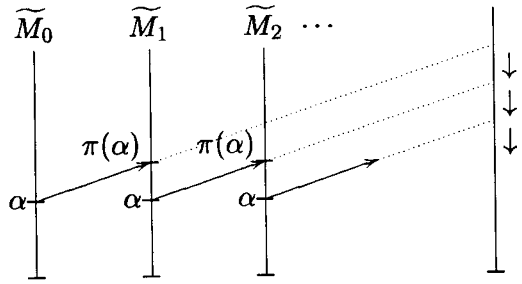

Figure 4. An ill-founded iteration.

4.3. *Embedding Normal Forms*

Combining the two iteration techniques described above one can build treelike systems of inner models so that some branches through the tree have well-founded limits whereas others have ill-founded limits. The systems will be indexed by the tree $\omega^{<\omega}$ of finite sequences of natural numbers. Since branches through $\omega^{<\omega}$ can be identified with real numbers, i.e., elements of $\mathbb{R} = \omega^{\omega}$, we can associate with every real number a well-founded or ill-founded limit through the tree of models.

DEFINITION 4.1. A commuting system $(M_s, \pi_{st})_{s \subseteq t \in \omega^{<\omega}}$ of inner models M_n and elementary maps $\pi_{st} : M_s \to M_t$ is called an *embedding normal form* (ENF) for a set $A \subseteq \mathbb{R}$ of reals if for every $p \in \mathbb{R}$:

(4.1) We have $p \in A$ if and only if the direct limit $M_p = \lim_{m \leq n < \omega}(M_{p \restriction m}, \pi_{p \restriction m, p \restriction n})$ is well-founded, and hence a transitive inner model.

This connection between inner models and reals appears attractive but it is not strong enough for the intended applications. One can prove under the assumption of a measurable cardinal that every set of reals has an embedding normal form (see Koepke (1998)). We strengthen the notion of ENF by requiring that the ill-foundedness of branches is already witnessed locally.

DEFINITION 4.2. A commuting system $(M_s, \pi_{st})_{s \subseteq t \in \omega^{<\omega}}$ together with a system $(w_s)_{s \in \omega^{<\omega}}$, $w_s : \mathbb{R} \to \mathrm{Ord}$ is called an *embedding normal form with witnesses* (ENFW) for a set $A \subseteq \mathbb{R}$ of reals if

(4.2) $(M_s, \pi_{st})_{s \subseteq t \in \omega^{<\omega}}$ is an embedding normal form for A,

(4.3) for every $s \in \omega^{<\omega}$: $w_s \in M_s$ and $\mathbb{R} \subseteq M_s$, (4.4) for every $s \subseteq t \in \omega^{<\omega}$, $s \neq t$, and $p \in \mathbb{R} \setminus A$, $p \supseteq t$ we have $\pi_{st}(w_s)(p) > w_t(p)$.

Witnesses exist if the models are sufficiently closed; a class X is κ-closed if $X^\kappa \subseteq X$.

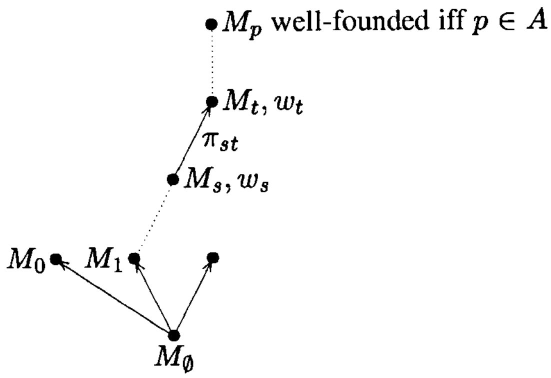

Figure 5. An ENFW tree.

THEOREM 4.3. If $(M_s, \pi_{st})_{s\subseteq t\in\omega^{<\omega}}$ is an ENF for a set $A \subseteq \mathbb{R}$ in which every model M_s is $2^{\aleph_0}$-closed then there is a system $(w_s)_{s\in\omega^{<\omega}}$ of witnesses for $(M_s, \pi_{st})_{s\subseteq t\in\omega^{<\omega}}$.

Proof. For every $p \in \mathbb{R} \setminus A$ one can find a sequence $(\gamma_s^p \mid s \in b)$ of ordinals such that $s \subseteq t \in p \to \pi_{st}(\gamma_s^p) > \gamma_t^p$. Then define for $s \in T$ functions $w_s : \mathbb{R} \to \mathrm{Ord}$ by: $w_s(p) = \gamma_s^p$, if $s \in p \in \mathbb{R} \setminus A$, and $w_s(p) = 0$, else.

If $s \subseteq t \in p$ and $p \in \mathbb{R} \setminus A$ then $(\pi_{st}(w_s))(p) = \pi_{st}(w_s(p)) = \pi_{st}(\gamma_s^p) > \gamma_t^p = w_t(p)$. Q.E.D.

5. DETERMINACY FROM EMBEDDING NORMAL FORMS

Descriptive set theory studies sets arising from ordinary mathematical practice: from a logical perspective these are *pointsets*, i.e., subsets of the real numbers and of similar spaces, which are simply definable. In order of increasing definition complexity one considers the following *pointclasses*: open and closed subsets of Euclidean spaces; Borel sets; analytic sets, which are the continuous images of Borel sets; co-analytic sets, which are the complements of analytic sets; projective sets, which are obtainable from Borel sets by taking continuous images and complements finitely often. The principal aim of these studies is to extend results about the *regularity properties* of simple sets to more complex pointclasses.

A key notion in modern descriptive set theory is that of determinacy. The theory of infinite games considers games whose positions are finite sequences partially ordered by inclusion. Two players called player I and player II alternately try to lengthen a position by one move. Thereby, they determine a maximal path through the tree of positions. Player I's aim is to get this path into a previously fixed winning set while player II tries to prevent this. The winning set is *determined* if there is a *winning strategy* for one of the players.

By a classical result of Gale and Stewart (1953), topologically simple winning sets are determined. We shall show that winning sets representable by an ENFW are also determined. This is done by introducing an auxiliary game G^* which is an extension of the original game G by "side moves". One can view the original game as the auxiliary game with "hidden" side moves. The game G^* is determined due to its simple topological nature. The ENFW is then used to construct a winning strategy in G from a winning strategy in G^*. In the crucial case of the construction one moves to different models of the ENFW and employs the witnesses of the ENFW as optimal side moves for player I in G^*. If player II can win against the optimal moves, player II can also win the original game G where these moves are "hidden".

DEFINITION 5.1. A *tree* is a nonempty set of finite sequences, $T \subseteq \mathbf{V}^{<\omega}$, closed under the formation of initial segments. For $t \in \mathbf{V}^{<\omega}$ let $|t|$ denote the *length* of t. T is partially ordered by $\subseteq$. A *path* through T is a sequence p of length $\leq \omega$ such that $\forall n < \omega(p \restriction n \in T)$; p is *maximal* if there is no path through T properly extending p. A maximal path through T is also called a *play* on T. A play $p = (a_0, a_1, a_2, a_3, \ldots)$ is sometimes represented in the form

$$\begin{array}{lllllll} \text{I} & a_0 & & a_2 & & \ldots & \\ \text{II} & & a_1 & & a_3 & & \ldots \end{array}$$

to indicate that player I makes the move a_0, then player II answers a_1, player I makes the move a_2, etc. Let $[T]$ denote the set of plays of T. A *game* $G(T, A)$ on T is given by a set $A \subseteq [T]$ of *winning plays* for player I. We say that player I *wins* the play p in the game $G(T, A)$ if $p \in A$; player II *wins* if $p \in [T] \setminus A$.

The obvious question is whether one of the players possesses a winning strategy in this game. A *strategy* on T is a function $\sigma : T \to \mathbf{V}$ such that

$$\forall t \in T(t \text{ is not maximal in } (T, \subseteq) \to t^\frown\sigma(t) \in T).$$

A strategy $\sigma : T \to \mathbf{V}$ is a *winning strategy for* player I in the game $G(T, A)$ if

$$\forall p \in [T]\,((\forall 2n < |p|\,(p(2n) = \sigma(p \restriction 2n))) \to p \in A).$$

Similarly, σ is a *winning strategy for* player II if

$$\forall p \in [T]\,((\forall 2n+1 < |p|\,(p(2n+1) = \sigma(p \restriction 2n+1))) \to p \in [T] \setminus A).$$

Player I and player II cannot both have winning strategies in $G(T, A)$. $G(T, A)$ is *determined* if one of the players has a winning strategy in $G(T, A)$.

We are mainly interested in games on the real numbers. Here, T is the tree $\omega^{<\omega}$ of finite sequences of natural numbers. We identify $[T]$ with the set $\mathbb{R} = \omega^\omega$ of reals. A set $A \subseteq \mathbb{R}$ is called *determined* if $G(A) = G(\omega^\omega, A)$ is determined. *Analytic* (*projective*) *determinacy* is the statement that every analytic (projective) set of reals is determined. The determinacy of a pointclass has profound implications for its descriptive set theory (see Moschovakis (1980)).

Consider a set $A \subseteq \mathbb{R}$ which has an ENFW. We modify the game $G(A) = G(\omega^{<\omega}, A)$ to an auxiliary game $G^*(A)$ by adding side moves for player I and a system of rules such that if player I satisfies all the rules then player I has also produced a winning play for the original game $G(A)$. Let T^* consist of all finite sequences of the form

$$((a_0, f_0), a_1, (a_2, f_2), a_3, \dots, (a_{2n}, f_{2n})), \text{ or}$$

$$((a_0, f_0), a_1, (a_2, f_2), a_3, \dots, (a_{2n}, f_{2n}), a_{2n+1})$$

such that the following three conditions hold:

(5.1) $a_j \in \omega$, for $j < 2n + 2$,

(5.2) $f_{2j} : \mathbb{R} \to \theta$, for $j \leq n$,

for some fixed sufficiently large ordinal θ (we shall give an adequate lower bound for θ in (5.5)), and:

(5.3) $\forall x \in \mathbb{R} \setminus A (x \supseteq (a_0, \dots, a_{2i+2}) \to f_{2i}(x) > f_{2i+2}(x))$, for all $i < n$.

A play on T^* may be represented as

I	a_0, f_0		a_2, f_2		$\dots$	
II		a_1		a_3		$\dots$

Since there is no infinite descent in the ordinals, the functions $f_0, f_2, \dots, f_{2n}$ serve to push away the sequence $(a_0, a_1, \dots)$ from $\mathbb{R} \setminus A$

and into A. Player I wins the game $G^*(A)$ if player I is able to satisfy the rules (4.1) to (4.3) in an *infinite* play. So we define the winning set for player I by:

$$\begin{aligned} A^* &= \{p \in [T^*] \mid p \text{ is infinite}\}, \\ G^*(A) &= G(T^*, A^*). \end{aligned}$$

LEMMA 5.2. $G^*(A)$ is determined.

Proof. Call a position a *winning position* for player II if player II can force a finite play starting from that position. Now assume that player II has no winning strategy in $G^*(A)$. Then the initial position $\emptyset$ is not a winning position for player II. Whenever $t \in T^*$ is of even length $2n$ and is not a winning position for player II then there must be an extension $t^\frown\sigma(t)$ of t which is not a winning position for player II. This function σ is basically a strategy for player I and if player I follows σ in a play p in $G^*(A)$, then p is infinite. Hence player I has a winning strategy in $G^*(A)$. Note that the above is basically the Gale–Stewart argument for the determinacy of the closed game $G^*(A)$ where A^* is closed in the natural topology on $[T^*]$. Q.E.D.

Assume that player I has a winning strategy σ^* for the game $G^*(A)$. Player I is able to turn σ^* into a winning strategy for $G(A)$ by "hiding" the side-moves $f_0, f_2, \ldots$. "Internally" he reacts to the moves $a_1, a_3, \ldots$ of player II by playing $a_0, f_0, a_2, f_2, \ldots$ as given by σ^*. Officially he only plays the numbers $a_0, a_2, \ldots$ without the side moves $f_0, f_2, \ldots$. Then the play $p = (a_0, a_1, a_2, a_3, \ldots)$ is a win for player I; if not then $p \in \mathbb{R} \setminus A$ and rule (4.3) implies $f_0(p) > f_2(p) > f_4(p) > \ldots$, contradicting the well-foundedness of the ordinals. Therefore hiding the functions produced by σ^* yields a winning strategy for player I in $G(A)$.

If player I does not have a winning strategy in $G^*(A)$ then by the determinacy of the auxiliary game, player II must have a winning strategy in $G^*(A)$, call it σ^*. We shall turn this into a winning strategy for player II in the game $G(A)$. The problem is that σ^* expects to see side moves $f_0, f_2, f_4, \ldots$ to calculate his response. To apply σ^*, player II has to guess or simulate these moves, and obviously he has to simulate them in an optimal way. These simulations will be provided by the witnesses $(w_s)_{s\in\omega^{<\omega}}$ of an ENFW $(M_s, \pi_{st})_{s\subseteq t\in\omega^{<\omega}}$ for A.

By condition (4.4) the witnesses are descending along the ENF and this gives arbitrarily long sequences of functions satisfying the rule (5.3) in the definition of $G^*(A)$. Player II will use the witnesses to simulate optimal moves of player I. We set:

$$\begin{aligned}(5.4)\qquad \sigma(a_0) &= \pi_{\emptyset,a_0}(\sigma^*)(a_0, w_{a_0}),\\ \sigma(a_0a_1a_2) &= \pi_{\emptyset,a_0a_1a_2}(\sigma^*)(a_0, \pi_{a_0,a_0a_1a_2}(w_{a_0}), a_1, a_2, w_{a_0a_1a_2}),\\ \sigma(s) &= \pi_{\emptyset,s}(\sigma^*)(s, \pi_{s\restriction 1,s}(w_{s\restriction 1}), \pi_{s\restriction 3,s}(w_{s\restriction 3}), \ldots, w_s),\end{aligned}$$

for $|s|$ odd.

Note that the sequence $(\pi_{s\restriction 1,s}(w_{s\restriction 1}), \pi_{s\restriction 3,s}(w_{s\restriction 3}), \ldots, w_s)$ is a sequence of descending functions which lives in M_s. To view them as legal side moves the constant θ in (5.2) must have been chosen sufficiently large, e.g.,

$$(5.5)\qquad \theta > \text{supremum of the range of } w_s \text{ for every } s \in \omega^{<\omega}.$$

It is then possible to apply the mapped strategy $\pi_{\emptyset,s}(\sigma^*)$ inside M_s.

CLAIM 5.3. σ is a winning strategy for player II in $G(A)$.

Proof. Let $p = (a_0, a_1, a_2, \ldots) \in \mathbb{R}$ be a play in $G(A)$ where player II follows σ. Assume for a contradiction that $p \in A$. Then the direct limit

$$(M_p, \pi_{p\restriction m,p})_{m<\omega} = \lim_{m\le n<\omega}(M_{p\restriction m}, \pi_{p\restriction m,p\restriction n})$$

is transitive by (4.1). $p = (a_0, a_1, a_2, \ldots)$ satisfies the equations (5.4). Applying the maps $\pi_{p\restriction m,p}$ to the equations yields:

$$\begin{aligned}(5.6)\qquad a_1 &= \pi_{\emptyset,p}(\sigma^*)(a_0, \pi_{a_0,p}(w_{a_0}))\\ a_3 &= \pi_{\emptyset,p}(\sigma^*)(a_0, \pi_{a_0,p}(w_{a_0}), a_1, a_2, \pi_{a_0a_1a_2,p}(w_{a_0a_1a_2})),\\ a_{2n+1} &= \pi_{\emptyset,p}(\sigma^*)(p\restriction 2n+1, \pi_{p\restriction 1,p}(w_{p\restriction 1}), \ldots, \pi_{p\restriction 2n+1,p}(w_{p\restriction 2n+1})),\end{aligned}$$

for $n < \omega$. The sequence of functions on the right-hand side satisfies the rule (5.3): if $x \in \mathbb{R} \setminus A$ and $p\restriction 2n+3 \subseteq x$ then

$$\begin{aligned}\pi_{p\restriction 2n+1,p}(w_{p\restriction 2n+1})(x) &= \pi_{p\restriction 2n+3,p}(\pi_{p\restriction 2n+1,p\restriction 2n+3}(w_{p\restriction 2n+1})(x))\\ &> \pi_{p\restriction 2n+3,p}(w_{p\restriction 2n+3}(x))\\ &= \pi_{p\restriction 2n+3,p}(w_{p\restriction 2n+3})(x).\end{aligned}$$

Therefore,

$$(5.7)\qquad \begin{array}{llllll} \text{I} & a_0, \pi_{p\restriction 1,p}(w_{p\restriction 1}) & & a_2, \pi_{p\restriction 3,p}(w_{p\restriction 3}) & & \ldots \\ \text{II} & & a_1 & & a_3 & \ldots \end{array}$$

is a play in $\pi_{\emptyset,p}(G^*(A))$ in which player II follows the strategy $\pi_{\emptyset,p}(\sigma^*)$ and in which the rule (5.3) is observed.

An absoluteness argument shows that a similar play must actually exist inside the model M_p: Consider, in M_p, the set $\mathcal{P}$ of all finite sequences of moves in $\pi_{\emptyset,p}(G^*(A))$ in which player II follows the strategy $\pi_{\emptyset,p}(\sigma^*)$. $(\mathcal{P}, \supseteq)$ is a partial order under reverse inclusion. $(\mathcal{P}, \supseteq)$ is ill-founded in $\mathbf{V}$ as witnessed by the play (5.7). By the absoluteness of well-foundedness between $\mathbf{V}$ and the transitive model M_p, $(\mathcal{P}, \supseteq)$ is ill-founded in M_p. Hence, in M_p, there is an infinite play in $\pi_{\emptyset,p}(G^*(A))$ in which player II follows the strategy $\pi_{\emptyset,p}(\sigma^*)$.

Since $\pi_{\emptyset,p}\colon \mathbf{V} \to M_p$ is elementary, there is, in $\mathbf{V}$, an infinite play in $G^*(A)$ in which player II follows the strategy σ^*. But then σ^* is not a winning strategy for player II since player II's aim is to make plays in $G^*(A)$ finite. Contradiction. Q.E.D.

6. INDUCED EMBEDDINGS

We have seen in the preceding chapter that a set of reals having an embedding normal form with witnesses possesses strong regularity properties like determinacy. This motivates the construction of such normal forms for as many sets of reals as possible. The existence of embedding normal forms can be viewed as a certain richness of the category of inner models with elementary embeddings.

An essential technique for building iteration trees and normal forms in the proof of the Martin–Steel theorem is given by the following construction in which a given elementary embedding $\pi : M \to N$ induces an elementary embedding $\pi^* : M^* \to N^*$ of *another* inner model M^* which is in sufficient agreement with M.

Fix a non-trivial elementary embedding $\pi : M \to N$ of inner models with α being the smallest ordinal moved by π. Let M^* be an inner model such that $M \sim_{\alpha+1} M^*$. We want to define $\pi^* : M^* \to N^*$ using π. This can be done by applying the *extender* derived from π to the model M^*. In the present presentation however we shall carry out the construction as a category theoretic limit without explicit mention of extenders. We represent M^* as a direct limit of a system whose components are elements of $M^* \cap \mathbf{V}_{\alpha+1}$. The system can then be lifted by applying π to each of its components.

Take $I = \{i \in M^* \mid i : \alpha \to M^* \text{ is injective}\}$ as a class of indices for a directed system. When we want to refer to the *map* i as opposed to the *index* i we write σ_i instead of i. I is partially ordered by the relation $i \leq j$ if and only if $\text{range}(\sigma_i) \subseteq \text{range}(\sigma_j)$. For $i \in I$ let E_i be the unique binary relation on α such that $\sigma_i : (\alpha, E_i) \to (\text{range}(\sigma_i), \in)$ is an isomorphism. For $i \leq j$ define $\sigma_{ij} = \sigma_j^{-1} \circ \sigma_i$; the map $\sigma_{ij} : (\alpha, E_i) \to (\alpha, E_j)$ is a structural embedding, and $(M^*, \in), (\sigma_i)_{i \in I}$ is the transitive direct limit of the system $\mathcal{S} = (\alpha, E_i, \sigma_{ij})_{i \leq j \in I}$.

Note that for $i \leq j \in I$ the components E_i and σ_{ij} are elements of $\mathbf{V}_{\alpha+1} \cap M^* = \mathbf{V}_{\alpha+1} \cap M$. We can thus apply π to the components: define $\alpha^* = \pi(\alpha)$, $E_i^* = \pi(E_i)$ and $\sigma_{ij}^* = \pi(\sigma_{ij})$. By the elementarity of π, the lifted system $\mathcal{S}^* = (\alpha^*, E_i^*, \sigma_{ij}^*)_{i \leq j \in I}$ is a directed commutative system; let $(N^*, E^*), (\sigma_i^*)_{i \in I}$ be a direct limit of the system.

The systems $\mathcal{S}$ and $\mathcal{S}^*$ are connected by the identity map, i.e., for $i \leq j \in I$: $\text{id}\restriction\alpha \circ \sigma_{ij} = \sigma_{ij}^* \circ \text{id}\restriction\alpha$. Hence the direct limits can be connected uniquely by a map $\pi^* : M^* \to N^*$ such that for each $i \in I$ and $\xi \in \alpha$: $\pi^*(\sigma_i(\xi)) = \sigma_i^*(\xi)$. Since all morphisms considered are injective and structure preserving, we have $\pi^* : (M^*, \in) \to (N^*, E^*)$ injectively. We shall show that π^* is indeed an elementary map, and we shall later discuss conditions under which the image (N^*, E^*) is well-founded and can be taken as a transitive $\in$-model of the form $(N^*, E^*) = (N^*, \in)$.

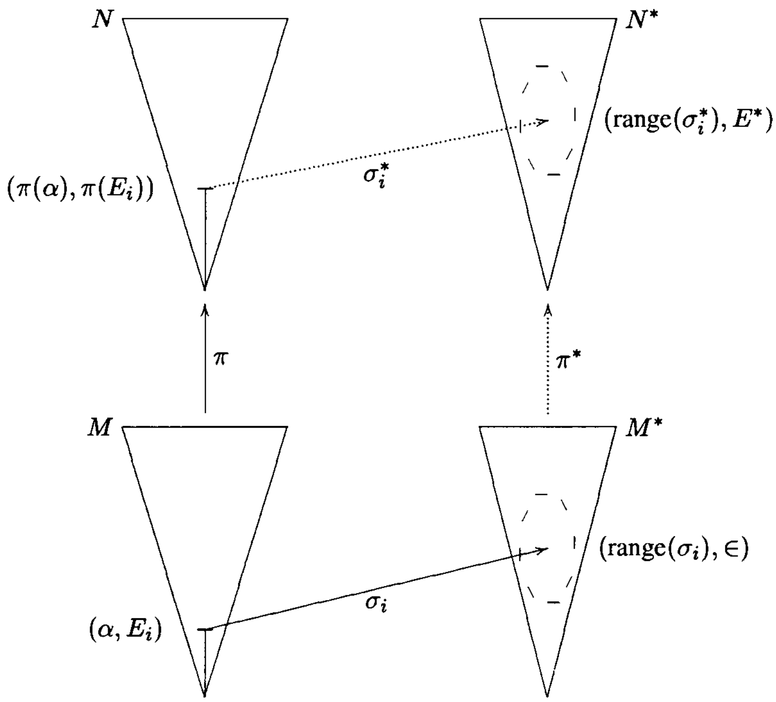

Figure 6. Inducing π^* from π.

The limit of a directed system is isomorphic to the limit of any cofinal subsystem. We can thin out the systems $\mathcal{S}$ and $\mathcal{S}^*$ to cofinal subsystems which are Σ_n-elementary for a given n: Let $i \in I$. By the Levy reflection principle choose $\theta \in \mathrm{Ord}$ such that $\mathrm{range}(\sigma_i) \subseteq \mathbf{V}_\theta \cap M^*$ and $\mathbf{V}_\theta \cap M^* \prec_{\Sigma_n} M^*$. By the downward Löwenheim–Skolem theorem applied in M^* there is $X \in M^*$ such that $X \prec_{\Sigma_n} \mathbf{V}_\theta \cap M^*$, $\mathrm{range}(\sigma_i) \subseteq X$, and $M^* \models \mathrm{card}(X) = \alpha$. Choose $j \in I$ such that $\mathrm{range}(\sigma_j) = X$. This shows that the class of indices j for which $\sigma_j : (\alpha, E_j) \to (M^*, \in)$ is Σ_n-elementary is cofinal in I. The identical map $\pi \restriction \alpha = \mathrm{id} \restriction \alpha : (\alpha, E_j) \to (\alpha^*, E_j^*)$ is Σ_ω-elementary by the elementarity of π. Therefore the connecting map $\pi^* : (M^*, \in) \to (N^*, E^*)$ is at least Σ_n-elementary. Since $n \in \omega$ is arbitrary, π^* is fully elementary.

Let us assume until further notice that the structure (N^*, E^*) is strongly well-founded. By the Mostowski Collapsing Lemma 2.1, the structure is isomorphic to a unique transitive $\in$-structure. We can thus assume without loss of generality that (N^*, E^*) is of the form $(N^*, \in)$ where N^* is transitive. Then the map π^* is called the embedding of M^* *induced* by π. Since N^* satisfies the axioms of ZFC, N^* is an inner model. We shall study the relations between the original map π and the map π^* induced by π.

Consider $i \in I$ such that $\mathrm{range}(\sigma_i)$ is transitive. Then the map $\sigma_i : (\alpha, E_i) \to (\mathrm{range}(\sigma_i), \in)$ is the Mostowski isomorphism of (α, E_i). By the elementarity of π, $\pi(\sigma_i) : (\alpha^*, E_i^*) \to (\mathrm{range}(\pi(\sigma_i)), \in)$ is the Mostowski isomorphism of (α^*, E_i^*). On the other hand, $\sigma_i^* : (\alpha^*, E_i^*) \to (N^*, \in)$ is the natural embedding of (α^*, E_i^*) into the direct limit. We claim that $\mathrm{range}(\sigma_i^*)$ is transitive: Let $x \in y \in \mathrm{range}(\sigma_i^*)$. Choose $\xi < \alpha^*$ such that $y = \sigma_i^*(\xi)$. Choose $j \geq i$ and $\zeta < \alpha^*$ such that $x = \sigma_j^*(\zeta)$. $x = \sigma_j^*(\zeta) \in y = \sigma_i^*(\xi)$ implies that $\zeta \; E_j^* \; \sigma_{ij}^*(\xi)$. It suffices to show the following

6.1. There is $\nu < \alpha^*$ such that $\zeta = \sigma_{ij}^*(\nu)$; then $x = \sigma_j^*(\zeta) = \sigma_j^*(\sigma_{ij}^*(\nu)) = \sigma_i^*(\nu) \in \mathrm{range}(\sigma_i^*)$.

Proof. By assumption, $\mathrm{range}(\sigma_i)$ is transitive. This implies that $\mathrm{range}(\sigma_i)$ is an $\in$-initial segment of $\mathrm{range}(\sigma_j)$. Then $\mathrm{range}(\sigma_{ij})$ is an E_j-initial segment of α. This can be expressed as: $\forall \gamma < \alpha \forall \delta < \alpha \exists \eta < \alpha$ $(\delta \; E_j \; \sigma_{ij}(\gamma) \to \delta = \sigma_{ij}(\eta))$. We apply the elementary map π to this fact: $\forall \gamma < \alpha^* \forall \delta < \alpha^* \exists \eta < \alpha^* (\delta \; E_j^* \; \sigma_{ij}^*(\gamma) \to \delta = \sigma_{ij}^*(\eta))$. Then the claim follows with $\gamma = \xi$, $\delta = \zeta$, $\eta = \nu$. Q.E.D.

Since $\mathrm{range}(\sigma_i^*)$ is transitive, σ_i^* is the Mostowski collapse of (α^*, E_i^*). Since the Mostowski collapse is uniquely determined, we have:

6.2. $\sigma_i^* = \pi(\sigma_i)$.

6.3. $\pi \restriction (H_{\leq\alpha})^{M^*} = \pi^* \restriction (H_{\leq\alpha})^{M^*}$.

Proof. Let $x \in (H_{\leq\alpha})^{M^*}$. Choose $i \in I$ such that range(σ_i) is transitive and $\sigma_i(0) = x$. Then by 6.2, we have that $\pi(x) = \pi(\sigma_i(0)) = \pi(\sigma_i)(0) = \sigma_i^*(0) = \pi^*(\sigma_i(0)) = \pi^*(x)$. Q.E.D.

6.4. $N \sim_{\alpha^*} N^*$.

Proof. The ordinal α is strongly inaccessible in M. Thus $\mathbf{V}_\alpha \cap M \in (H_{\leq\alpha})^{M^*}$. $\mathbf{V}_{\alpha^*} \cap N = \pi(\mathbf{V}_\alpha \cap M) = \pi^*(\mathbf{V}_\alpha \cap M^*) = \mathbf{V}_{\alpha^*} \cap N^*$. Q.E.D.

The agreement of inner models below some level is crucial for many constructions in the category of inner models. The large cardinal notions considered in the next chapter mainly involve postulates on agreement.

We now discuss conditions under which the image structure (N^*, E^*) is well-founded. The relation E^* is set-like:

6.5. If $z \in N^*$, then $\{x \in N^* \mid xE^*z\} \in \mathbf{V}$.

Proof. Let $xE^*z \in N^*$. Choose $i \in I$ and $\zeta < \alpha^*$ such that $z = \sigma_i^*(\zeta)$. We may assume that $0 \in$ range(σ_i). Choose $\eta \in$ Ord such that range$(\sigma_i) \subseteq \mathbf{V}_\eta$. Choose $k \in I$, $k \geq i$ and $\xi < \alpha^*$ such that $x = \sigma_k^*(\xi)$. Define $j \in I$ by $j(\gamma) = k(\gamma)$ if $k(\gamma) \in \mathbf{V}_\eta$, and $j(\gamma) = 0$ else. Then $i \leq j \leq k$. By construction, the following formula holds:

$$\forall\gamma < \alpha\forall\delta < \alpha\ (\sigma_k(\gamma) \in \sigma_i(\delta) \to \sigma_k(\gamma) = \sigma_j(\gamma)).$$

Applying the inverse of σ_k to this formula yields:

$$\forall\gamma < \alpha\forall\delta < \alpha\ (\gamma\ E_k\ \sigma_{ik}(\delta) \to \gamma = \sigma_{jk}(\gamma)).$$

We now apply π to get:

$$\forall\gamma < \alpha^*\forall\delta < \alpha^*\ (\gamma\ E_k^*\ \sigma_{ik}^*(\delta) \to \gamma = \sigma_{jk}^*(\gamma)).$$

And finally, we apply σ_k^*, and have:

$$\forall\gamma < \alpha^*\forall\delta < \alpha^*\ (\sigma_k^*(\gamma)\ E^*\ \sigma_i^*(\delta) \to \sigma_k^*(\gamma) = \sigma_j^*(\gamma)).$$

Since $x = \sigma_k^*(\xi)\ E^*\ \sigma_i^*(\zeta)$ we have $x = \sigma_k^*(\xi) = \sigma_j^*(\xi)$. Hence

$$\{x \in N^* \mid xE^*z\} \subseteq \{\sigma_j^*(\xi) \mid j \in I, \text{range}(\sigma) \subseteq \mathbf{V}_\eta, \xi < \alpha^*\},$$

which is a set. Q.E.D.

6.6. If M^* is countably closed, $(M^*)^\omega \subseteq M^*$, then (N^*, E^*) is strongly well-founded.

Proof. By the previous claim we only have to check well-foundedness. Assume E^* is ill-founded. Then there are indices $i_n \in I$ and ordinals $\xi_n < \alpha^*$ such that for $n < \omega$: $\sigma^*_{i_{n+1}}(\xi_{n+1})\ E^*\ \sigma^*_{i_n}(\xi_n)$. We may assume that $i_0 \leq i_1 \leq \ldots$. Then for $n < \omega$: $\xi_{n+1}\ E^*_{i_{n+1}}\ \sigma^*_{i_n i_{n+1}}(\xi_n)$. This means that the direct limit of the system $\mathcal{E}^* = (\alpha^*, E^*_{i_m}, \sigma^*_{i_m i_n})_{m \leq n < \omega}$ is ill-founded.

Define the system $\mathcal{E} = (\alpha, E_{i_m}, \sigma_{i_m i_n})_{m \leq n < \omega}$. This system can be embedded into $(M^*, \in)$ and hence is well-founded. Since M^* is countably closed, $\mathcal{E} \in M^*$. Since $M \sim_{\alpha+1} M^*$, we have $\mathcal{E} \in M$. $\mathcal{E}^* = \pi(\mathcal{E})$, and by the elementarity of π, $\mathcal{E}^*$ is well-founded in N and hence in $\mathbf{V}$. Contradiction. Q.E.D.

The final claim shows that a given degree of closure is preserved by the formation of induced embeddings.

6.7. Assume that in our situation M^* is η-closed and $(\alpha^*)^\eta \subseteq N$ where $\eta < \alpha$. Then N^* is η-closed.

Proof. Consider an η-sequence $(z_\delta)_{\delta<\eta} = (\sigma^*_{i_\delta}(\xi_\delta))_{\delta<\eta}$ from N^* with indices i_δ and ordinals $\xi_\delta < \alpha^*$. By the η-closure of M^* we can choose an index $j \in I$ such that $\forall \delta < \eta\ i_\delta < j$. Then $(z_\delta)_{\delta<\eta} = (\sigma^*_j(\zeta_\delta))_{\delta<\eta}$ for appropriate ordinals $\zeta_\delta < \alpha^*$. By assumption, $(\zeta_\delta)_{\delta<\eta} \in N$ and so $(\zeta_\delta)_{\delta<\eta} \in N \cap \mathbf{V}_{\alpha^*} = N^* \cap \mathbf{V}_{\alpha^*}$. Choose a surjective map $f : \alpha \to M \cap \mathbf{V}_\alpha$, $f \in M$. Since $M \sim_{\alpha+1} M^*$, $f \in M^*$. Define a map $g : \alpha \to M^*$, $g \in M^*$ by setting: $g(\nu) = (j(\mu_\delta))_{\delta<\eta}$ if $f(\nu) = (\mu_\delta)_{\delta<\eta}$, and $g(\nu) = 0$ else. By eliminating repetitions in the values of g we can turn g into an injective function k. Then k is an index in I. Choose $\nu < \alpha^*$ minimal such that $(\zeta_\delta)_{\delta<\eta} = \pi(f)(\nu)$. Then $(z_\delta)_{\delta<\eta} = (\sigma^*_j(\zeta_\delta))_{\delta<\eta} = \sigma^*_k(\nu) \in N^*$. Hence N^* is η-closed. Q.E.D.

7. LARGE CARDINALS AND THE CONSTRUCTION OF ENFS

The initial assumptions for the construction of systems of elementary embeddings of inner models are *large cardinal* axioms. It is now customary to formulate large cardinal notions like measurable cardinals and strong cardinals in terms of elementary embeddings of inner models.[3] This has proved to be a strong unifying principle in large cardinal theory (see the survey article by Kanamori and Magidor (1978)). The strength of a large cardinal assumption is expressed by the degree of agreement between the sources and the targets of the postulated embeddings. We have seen in the previous chapter that such agreement allows the construction of new embeddings out of given ones. All elementary embeddings to be considered will be taken from the family of internal maps as defined in Section 4. So the definition of this family is part of the definitions of the subsequent large cardinal notions. There are, however, combinatorial equivalences which do not depend on particular families of inner models and embeddings.

Since embeddings induced by π can move models different from the model where π is defined one is able to build complicated branching systems of models out of large cardinal assumptions. The constructions require careful control of the agreement among models. In the proof of the Martin–Steel theorem ENFWs for $\mathbf{\Pi}^1_n$-sets are constructed by recursion on n. Woodin cardinals provide the exact large cardinal strength for the successor case of the recursion. We list some relevant large cardinal notions in order of strength. Using 6.7 we require sufficient closure of the image models so that the induced embeddings possess well-founded image models (6.6) and witnesses (Theorem 4.3). One can prove that the definitions are equivalent to the same formulations without closure requirements. A measurable cardinal – the weakest notion considered here – is the obvious assumption for making the structure of the category of inner models non-trivial.

DEFINITION 7.1. A cardinal α is *measurable* if there is an elementary embedding $\pi : \mathbf{V} \to M$ with critical point α into an α-closed inner model M.

A measurable cardinal is strongly inaccessible hence the image model M in this definition is certainly $2^{\aleph_0}$-closed.

DEFINITION 7.2. A cardinal α is *strong* (Gaifman 1974) if for every $x \in \mathbf{V}$ there is an elementary embedding $\pi : \mathbf{V} \to M$ with critical point α into an α-closed inner model M such that $x \in M$.

DEFINITION 7.3. Let $\alpha < \delta$ and $\vec{p}$ be a finite sequence of parameters. Then α is *strong in $\vec{p}$ up to δ* if for all $\eta < \delta$ there is an elementary embedding $\pi : \mathbf{V} \to M$ with critical point α into an α-closed inner model M such that $(\mathbf{V}, \vec{p}) \sim_\eta (M, \pi(\vec{p}))$.

DEFINITION 7.4. A cardinal δ is a *Woodin cardinal* (Shelah and Woodin 1990) if for all finite sequences $\vec{p}$ of parameters there is $\alpha < \delta$ which is strong in $\vec{p}$ up to δ.

The growth of strength from measurable to Woodin cardinals is formidable: below each strong cardinal there are cofinally many measurable cardinals; if δ is a Woodin cardinal then $\mathbf{V}_\delta \models$ "there are cofinally many strong cardinals". Woodin cardinals imply the existence of many elementary embeddings with favourable preservation properties. We shall indicate in the following how this can be used in the construction of ENFWs for arbitrary projective sets.

The projective sets are obtained recursively from open sets in some product space $\mathbb{R}^n$ by finitely many complementations and projections. Provided there is a measurable cardinal one can construct ENFWs for closed sets by the iteration methods of chapter 3. The recursion step requires to show: if a set $A \subseteq \mathbb{R}^{n+1}$ has a sufficiently closed ENF then the *complement* $\mathbb{R}^{n+1} \setminus A$ and the *projection* $\{a \in \mathbb{R}^n \mid \exists b \in \mathbb{R}\ (a, b) \in A\}$ both have sufficiently closed ENFs.

The main idea of the proof is already present in the simpler case of complements. For simplicity we shall consider 1-dimensional sets, i.e., $n = 0$. So assume that the set $A \subseteq \mathbb{R}$ has an ENFW $\mathcal{N} = (N_s, \pi_{st})_{s\subseteq t\in\omega^{<\omega}}$ together with witnesses $(w_s)_{s\in\omega^{<\omega}}$. The aim is to construct an ENF $\mathcal{M} = (M_s, \sigma_{st})_{s\subseteq t\in\omega^{<\omega}}$ for $\mathbb{R} \setminus A$. Simultaneously one defines an auxiliary system $\mathcal{M}^* = (M^*_s, \sigma^*_{st})_{s\subseteq t\in\omega^{<\omega}}$ which reflects many properties of the given system $\mathcal{N}$ and in particular the well-foundedness of branches.

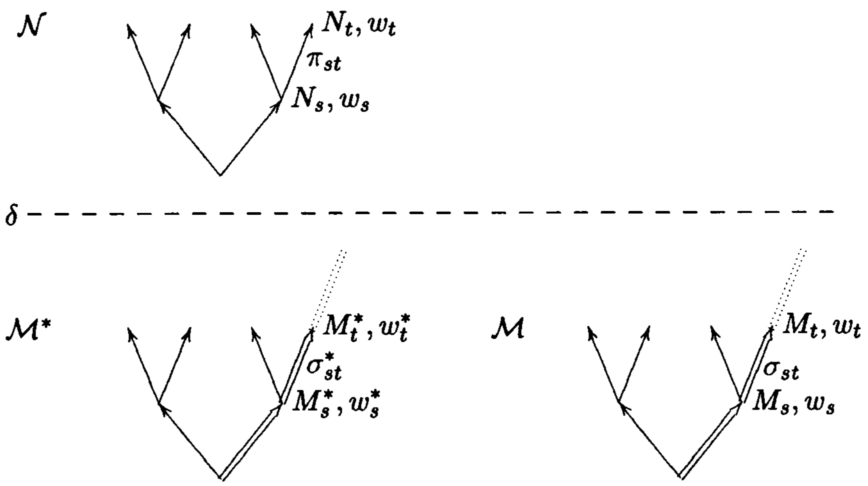

Figure 7. Constructing an ENF $\mathcal{M}$ from $\mathcal{N}$. The two highlighted branches constitute an alternating chain.

We state the crucial properties of this diagram:

FACT 7.5. For all $p \in \mathbb{R}$ we have N_p is well-founded if and only if M^*_p is well-founded, where N_p resp. M^*_p denote the direct limits of the branches corresponding to the real p in $\mathcal{N}$ resp. $\mathcal{M}^*$.

FACT 7.6. The systems $\mathcal{M}$ and $\mathcal{M}^*$ are complementary in the sense that for all $p \in \mathbb{R}$ we have M_p is well-founded if and only if M_p^* is ill-founded.

Granting these two properties we have an embedding normal form representation of the complement of A as required.

FACT 7.7. $p \in \mathbb{R} \setminus A$ if and only if N_p is ill-founded if and only if M_p^* is ill-founded if and only if M_p is well-founded.

In conclusion we see that under appropriate initial assumptions the category of inner models and elementary embeddings exhibits rich structural properties which can be used to analyse situations in descriptive set theory.

8. APPENDIX: MORE TECHNICAL DETAILS

For a reader who is keen on understanding the whole proof of the Martin–Steel theorem we shall now sketch the crucial part of the construction in terms of the category of inner models. For a complete argument along these lines but formulated in the different language of extenders we refer to Koepke (1998).

The fact 7.6 is obtained by constructing the branches $(M_{p\restriction n})_{n<\omega}$ and $(M^*_{p\restriction n})_{n<\omega}$ through $\mathcal{M}$ and $\mathcal{M}^*$ as complimentary branches of an *alternating chain*. In our particular situation this amounts to the following three requirements:

(8.1) $M_\emptyset = M^*_\emptyset = \mathbf{V}$.

(8.2) For every $n < \omega$, the embedding $\sigma_{p\restriction n, p\restriction n+1} : M_{p\restriction n} \to M_{p\restriction n+1}$ is induced by a map which is internal in $M^*_{p\restriction n}$.

(8.3) For every $n < \omega$, the embedding $\sigma^*_{p\restriction n, p\restriction n+1} : M^*_{p\restriction n} \to M^*_{p\restriction n+1}$ is induced by a map which is internal in $M_{p\restriction n+1}$.

Schematically the two branches can be represented as in Figure 8 where the broken arrows indicate that an internal map induces an elementary embedding of another model.

An alternating chain can also be seen as linear construction

$$M_\emptyset, M_{p\restriction 1}, M^*_{p\restriction 1}, M_{p\restriction 2}, M^*_{p\restriction 2}, \ldots.$$

At each stage of the construction an internal map is chosen and used to induce an elementary map of an earlier model in the sequence; the resulting

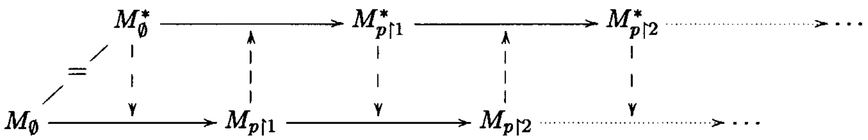

Figure 8. An alternating chain.

image is put as the next model on the sequence. This is a generalization of the iteration process studied in Section 4. The generalization retains some traces of the construction of a well-founded direct limit, and by a crucial lemma of Martin and Steel (1989), the alternating chain has at least one well-founded branch.

If we can find functions $w^*_{p\restriction 0}$, $w^*_{p\restriction 1}$, $\ldots$ in $M^*_{p\restriction 0}$, $M^*_{p\restriction 1}$, $\ldots$ which behave like the witnesses $w_{p\restriction 0}, w_{p\restriction 1}, \ldots$ of the ENF for A we have:

If N_p is ill-founded as witnessed by $w_{p\restriction 0}, w_{p\restriction 1}, \ldots$, then M^*_p is illfounded as witnessed by $w^*_{p\restriction 0}, w^*_{p\restriction 1}, \ldots$ and then M_p is well-founded by the above-mentioned lemma of Martin and Steel.

The case when N_p is well-founded is treated by a different argument.

The construction of the alternating chain requires a high degree of agreement between structures on both branches since one uses the method of induced embeddings. Let us state some properties needed in the recursive construction of the alternating chain:

The construction takes place under the assumption of a sufficiently big Woodin cardinal δ. In the course of the construction δ yields a sequence $\kappa^*_0 < \kappa_0 < \kappa^*_1 < \kappa_1 < \ldots < \delta$ of large cardinals such that:

(8.4) $N_{p\restriction n} \models$ "$M_{p\restriction n} \models \kappa^*_n$ is strong in $\pi_{p\restriction 0, p\restriction n}(w_{p\restriction 0}), \ldots, \pi_{p\restriction n, p\restriction n}(w_{p\restriction n})$ up to δ",

(8.5) $N_{p\restriction n} \models$ "$M_{p\restriction n} \sim_{\kappa^*_n+1} M^*_{p\restriction n}$", and

(8.6) $N_{p\restriction n} \models$ "$(M_{p\restriction n}, \pi_{p\restriction 0, p\restriction n}(w_{p\restriction 0}), \ldots, \pi_{p\restriction n, p\restriction n}(w_{p\restriction n})) \sim_{\kappa^*_n} (M^*_{p\restriction n}, \sigma^*_{p\restriction 0, p\restriction n}(w^*_{p\restriction 0}), \ldots, \sigma^*_{p\restriction n, p\restriction n}(w^*_{p\restriction n}))$".

Note that the facts about the alternating chain are stated in $N_{p\restriction n}$ where the witnessing parameters $\pi_{p\restriction 0, p\restriction n}(w_{p\restriction 0}), \ldots, \pi_{p\restriction n, p\restriction n}(w_{p\restriction n})$ are living. To continue we have to incorporate the witness $w_{p\restriction n+1}$ into the construction. So we lift properties (8.4) to (8.6) to $N_{p\restriction n+1}$ by the elementary embedding $\pi_{p\restriction n, p\restriction n+1}$, and get:

(8.7) $N_{p\restriction n+1} \models$ "$M_{p\restriction n} \models \kappa_n^*$ is strong in $\pi_{p\restriction 0,p\restriction n+1}(w_{p\restriction 0}), \ldots, \pi_{p\restriction n,p\restriction n+1}(w_{p\restriction n})$ up to δ",

(8.8) $N_{p\restriction n+1} \models$ "$M_{p\restriction n} \sim_{\kappa_n^*+1} M^*_{p\restriction n}$", and

(8.9) $N_{p\restriction n+1} \models$ "$(M_{p\restriction n}, \pi_{p\restriction 0,p\restriction n+1}(w_{p\restriction 0}),\ldots,\pi_{p\restriction n,p\restriction n+1}(w_{p\restriction n})) \sim_{\kappa_n^*} (M^*_{p\restriction n}, \sigma^*_{p\restriction 0,p\restriction n}(w^*_{p\restriction 0}), \ldots, \sigma^*_{p\restriction n,p\restriction n}(w^*_{p\restriction n}))$".

We consider the initial part of the alternating chain to be given by a *term* which can be interpreted in $N_{p\restriction n}$ and in $N_{p\restriction n+1}$.

Within $N_{p\restriction n+1}$ there is a map internal in $M_{p\restriction n}$ with critical point κ_n^* which can be applied to $M^*_{p\restriction n}$ to yield $\pi^*_{p\restriction n,p\restriction n+1} : M^*_{p\restriction n} \to M^*_{p\restriction n+1}$. The internal map can be chosen strong enough so that there are $\kappa_n > \kappa_n^*$ and $w^*_{p\restriction n+1}$ with the following properties:

(8.10) $N_{p\restriction n+1} \models$ "$M^*_{p\restriction n+1} \models \kappa_n$ is strong in $\sigma^*_{p\restriction 0,p\restriction n+1}(w^*_{p\restriction 0}), \ldots, \sigma^*_{p\restriction n,p\restriction n+1}(w^*_{p\restriction n})), w^*_{p\restriction n+1}$ up to δ",

(8.11) $N_{p\restriction n+1} \models$ "$M^*_{p\restriction n+1} \sim_{\kappa_n+1} M_{p\restriction n}$", and

(8.12) $N_{p\restriction n+1} \models$ "$(M^*_{p\restriction n+1}, \sigma^*_{p\restriction 0,p\restriction n+1}(w^*_{p\restriction 0}), \ldots, \sigma^*_{p\restriction n,p\restriction n+1}(w^*_{p\restriction n})), w^*_{p\restriction n+1})$
$\sim_{\kappa_n} (M_{p\restriction n}, \pi_{p\restriction 0,p\restriction n+1}(w_{p\restriction 0}), \ldots, \pi_{p\restriction n,p\restriction n+1}(w_{p\restriction n}), w_{p\restriction n+1})$".

As above one can find an internal map in $M^*_{p\restriction n+1}$ which induces a map $\pi_{p\restriction n,p\restriction n+1} : M_{p\restriction n} \to M_{p\restriction n+1}$ and a $\kappa^*_{n+1} > \kappa_n$ such that:

(8.13) $N_{p\restriction n+1} \models$ "$M_{p\restriction n+1} \models \kappa^*_{n+1}$ is strong in $\pi_{p\restriction 0,p\restriction n+1}(w_{p\restriction 0}),\ldots, \pi_{p\restriction n,p\restriction n+1}(w_{p\restriction n}), w_{p\restriction n+1}$ up to δ",

(8.14) $N_{p\restriction n+1} \models$ "$M_{p\restriction n+1} \sim_{\kappa^*_{n+1}+1} M^*_{p\restriction n+1}$", and

(8.15) $N_{p\restriction n+1} \models$ "$(M_{p\restriction n+1}, \pi_{p\restriction 0,p\restriction n+1}(w_{p\restriction 0}),\ldots, \pi_{p\restriction n,p\restriction n+1}(w_{p\restriction n}), w_{p\restriction n+1})$
$\sim_{\kappa^*_{n+1}} (M^*_{p\restriction n+1}, \sigma^*_{p\restriction 0,p\restriction n+1}(w^*_{p\restriction 0}),\ldots, \sigma^*_{p\restriction n,p\restriction n+1}(w^*_{p\restriction n}), w^*_{p\restriction n+1})$".

This corresponds to the initial situation (8.4) to (8.6) and shows that the construction can be continued. There are many subtle points to be arranged to make this construction work of which we mention two: the constrution has to be local, i.e., the definition of $M_{p\restriction n}$ and of $M^*_{p\restriction n}$ should only depend on $p\restriction n$ and not on all of p; all structures should be sufficiently closed so that the questions of well-foundedness of induced images and of existence of witnesses is resolved.

NOTES

[1] So let us cast aside the *cosmocentric* superstition just like we cast aside the geocentric and anthropocentric superstition before; let us realize that there are myriads of cosmic worlds spun into the chaos – each of them appearing to its inhabitants as the sole and only real world and misleading them to assign its qualitative and particular characteristics to the transcendental core of the world. But this core escapes every bond however loose and keeps its freedom to be restricted to a cosmic appearance in infinitely varied ways. It allows the coexistence of all these appearances which are contained in its universality as particular possibilities and as conceptually defined subsets. Indeed it is nothing else than just this coexistence and thus transcendent for a particular appearance which has a closed realm of immanence in itself. (Translation P.K.)

[2] Alternatively one could also work in a class theoretic system and study classes which are inner models.

[3] The notion of a "flipping property" allows to characterize not only embedding cardinals but also combinatorial large cardinal notions. See Abramson et al. (1977), Barnabel (1989), Di Prisco and Marek (1985) and Di Prisco and Zwicker (1980).

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