Propensities are Probabilities

Jason Konek

September 22, 2014

Abstract

If chances are propensities, what reason do we have to expect them to be probabilities? I will offer a new answer to this question. It comes in two parts. First, I will defend an *accuracy-centred* account of what it is for a causal system to have precise propensities in the first place. Second, I will prove that, given some pretty weak assumptions about the nature of comparative causal dispositions, and some fairly standard assumptions about reasonable measures of inaccuracy, propensities *must* be probabilities.

Keywords: Propensity · Chance · Probability · Accuracy

1 Introduction

Suppose you toss a die. Your arm, hand, the die, the table that the die lands on, the air and surroundings — they all make up a causal system. Your toss causes the die to fly into the air and rotate. How it moves depends on the initial states of various parts of the system: the angle of your arm, how hard you flick your wrist, how the die's mass is distributed, etc. As the die rotates, chemical bonds between its faces and the air molecules break, which alters how it moves. When the die hits the table, the state of the table's surface — how hard it is, how flat it is, etc. — affects how the die tumbles and falls.

The initial state of this causal system, together with the dynamical laws governing it, ground a certain 'disposition' or 'tendency' to produce a particular final state: a '1', '2', '3', '4', '5' or '6' state. The die's *propensity* to land '1', or '2', or '3', etc., given these initial conditions, is a measure of the *strength* of this causal disposition or tendency.[1]

Propensity theorists make two important claims. First, they say that *chances* — the chance of a die landing '6', the chance of a polonium atom decaying within 138.4 days, the chance of rain tomorrow evening — are best thought of as propensities (*cf.* Popper (1959, 1983, 1990), Giere (1973), Fetzer (1982, 1983), and Miller (1994, 1996)).[2] Second, they afford propensities a certain degree of explanatory fundamentality. Propensities are not reducible to, or otherwise grounded in frequencies (actual finite frequencies or limiting frequencies of hypothetical sequences) — at least, not in a way that makes causal dispositions dispensable. Rather, those dispositions and their attendant propensities *explain* why the system produces outcomes with stable relative frequencies in long sequences of trials (*cf.*, Giere (1973), Fetzer (1981), and Hitchcock (2002)).

Of course, propensity theories come in all sorts of shapes and sizes. For example, Gillies (2000) follows Popper (1957) in defending what he calls *long-run propensity theory*. According to long-run theories, propensities are the frequencies that causal systems have the *strongest tendency to produce* in long sequences of trials. Causal systems, on this view, *do* have dispositions of various strengths to produce various different final states.

[1]If you doubt that die tosses, coin flips, etc., are genuinely chancy processes, consider "that air resistance depends partly on the chance making and breaking of chemical bonds between the coin and the air molecules it encounters" (Lewis, 1980, p. 266).

[2]More carefully, certain conditional chances are best thought of as propensities. In particular, the chance of a target variable taking a particular value, conditional on its 'parents' (the variables that exert direct causal influence on it) taking certain values, is best thought of as a propensity. The remaining chances are the probabilities that arise from propensities via the Causal Markov Condition. See Hitchcock (2012).

But propensities do *not* measure these strengths directly. Rather, they reflect *one* property (relative frequency) of *one* privileged sequence: the sequence that the system is most strongly disposed to produce. In contrast, Giere (1973), Fetzer (1982, 1983) and Miller (1994, 1996) defend *single-case propensities theories*. Single-case theories say just the opposite: propensities are straightforward measures of how strongly a system is disposed to produce one state or other on a particular occasion. As Hajek (2012) puts it, long-run theorists maintain that "a fair die has a propensity — an *extremely strong* tendency — to land '3' with long-run relative frequency 1/6. The small value of 1/6 does *not* measure this tendency. Single-case theorists maintain that "the die has a *weak* tendency to land '3'. The value of 1/6 *does* measure this tendency." We will continue to understand propensities as single-case propensities.[3]

Problem: the two big claims of propensity theory make propensities rather perplexing. On the one hand, if chances are propensities, then propensities must be probabilities. After all, chances must be probabilities to play the theoretical role that they do. For example, the Principal Principle implores us to treat chance as an *epistemic expert*. When we learn that chance's probability for X at t is x, we should straightaway adopt x as our new credence for X (unless we have information that chance lacks at t). But for chances to be worthy of such epistemic deference, they must be probabilities.[4] So if chances are propensities, then propensities must be probabilities too. On the other hand, it is *prima facie* unclear why propensities *should* be probabilities. *Frequencies* are probabilities, of course. But on the (single-case) propensity theorist's view, propensities lack any straightforward connection to frequencies that would allow them to inherit the latter's mathematical structure. On the face of it, it is no more obvious that propensities

[3]A further wrinkle: Miller (1994, pp. 18-56), and Popper (1990) take propensities to be properties not of local causal systems, but rather, of the *entire universe* at a given time. We will continue, however, to understand propensities as properties of local causal systems, or as Fetzer puts it, "a complete set of (nominally and/or causally) relevant conditions" (Fetzer, 1982, p. 195).

[4]See Joyce (1998, 2009) and Predd et al. (2009).

— measures of the strength of causal dispositions — are probabilities than it is that the numbers measuring the strength of Winnie the Pooh's desires, or Scooby-Doo's fears are probabilities.

This mystery is what Hitchcock (2012) calls *the problem of mathematical structure*. Our aim is to clear it up.[5] Our solution comes in two parts. Firstly, we will defend an *accuracy-centred* account of what it is for a causal system to have *precise* propensities at all. Secondly, we will show that, if this account is right, then given some pretty weak assumptions about the nature of causal dispositions or tendencies, and some fairly standard assumptions about reasonable measures of inaccuracy, propensities *must* be probabilities.

Cards on the table: our accuracy-centred account will be *non-reductive*. We will assume that various causal systems have *comparative* causal dispositions or tendencies: they are more strongly disposed to produce certain final states, given certain initial conditions, than others. So we will say nothing to allay the concerns of those who find causal dispositions altogether too spooky to be taken seriously. What we *will* do is this: explain what it is for those dispositions to have *precise* strengths, and show that, given some weak assumptions, the numbers measuring those precise strengths (when they exist) must be probabilities.

2 An Accuracy-Centred Account of Propensities

Propensities are, on the face of it, just one kind of *dispositional quantity*: quantities that summarise (somehow or other) which values other quantities are *disposed* to take under certain conditions. For example, the *elastic modulus* of a material measures the extent to which it is disposed to (or tends to) deform elastically (temporarily) when a certain force is applied to it. Rubber has a low elastic modulus (≈ 0.1 GPa), which reflects the fact that

[5] See Hitchcock (2012) for an overview of extant answers to the problem of mathematical structure.

stressing it a bit tends to produce a lot of strain, or deformation. (Elastic modulus is the ratio of stress to strain. So holding stress fixed, the higher the strain/deformation, the lower the modulus.) Steel, on the other hand, has a high elastic modulus (≈ 200 GPa), which reflects the fact that stressing it a bit tends to produce very little strain.

Or take another example: *molar absorptivity*. Molar absorptivity measures the extent to which a chemical compound is disposed to (or tends to) absorb light of a particular wavelength. Blue dye, for example, has a high molar absorptivity for red wavelengths (wavelengths around 625 nm). This reflects the fact that it has a strong disposition to absorb red light (the ratio of red light in to red light out tends to be very high: $\approx 98,000$ $M^{-1}cm^{-1}$).

But what does having a *precise* elastic modulus, *e.g.*, 0.1, or a *precise* molar absorptivity, *e.g.*, 98,000, amount to exactly? In virtue of what does 0.1 — that very specific value — "sum up" how much strain, or deformation a bit of stress on a piece of rubber tends to produce. (Why does 0.1, rather than 0.2, count as summing up the value that the ratio of stress to strain tends to take?) In virtue of what does 98,000 sum up how much red light a particular dye tends to absorb? (Why does 98,000, rather than 99,0000, count as summing up the value that the ratio of light in to light out tends to take?)

In part, this is a matter of convention. Why does 0.1, rather than 0.2, or 0.3, best summarise how much strain a bit of stress on a piece of rubber tends to produce? Partly because of a conventional choice to measure stress by gigapascals (GPa), rather than pounds per square inch (psi), or some other unit. But this is only *part* of the answer. The question remains: even once we have fixed our units of measurement, why 0.1, rather than 0.2, or 0.3 GPa? Why 98,000, rather than 99,0000, or 100,000 $M^{-1}cm^{-1}$? In virtue of what do these *very specific* numbers best summarise the values that their respective quantities (stress-to-strain ratio, red-light-in-to-red-light-out ratio) are *disposed* to take under certain conditions?

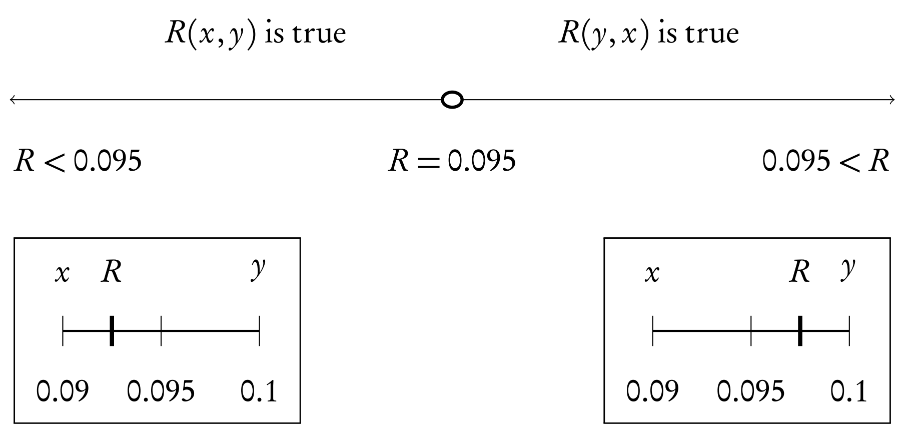
Here is one simple answer:

ACCURACY-CENTRED ACCOUNT (UNOFFICIAL): If a unique real number x best summarises which value a quantity Q tends to take in conditions C, then it does so in virtue of the fact that:

($\bigstar$) x tends to be the most ***accurate*** estimate of Q in C.

To make this more precise, in particular by swapping out ($\bigstar$) for something a bit more careful, let's introduce some notation. Let $Q(x,y)$ be the proposition that is true exactly when the causal system S in question produces a state in which Q takes a value z that is 'closer' to x than to y, *i.e.*, exactly when x is a more *accurate* estimate of Q than y. Shorter: $Q(x,y)$ says that x is closer to Q's true value — and so more accurate — than y.

For example, suppose that you have a rubber balloon. You blow into it. This produces a stress of 0.05 GPa. Now consider two estimates of the resulting stress-to-strain ratio R: $x = 0.09$ and $y = 0.1$. If applying 0.05 GPa of stress produces a strain, or deformation greater than $0.05/0.095 \approx 0.526$ on this occasion, then R is less than 0.095. So R's true value is closer to x than to y, in which case x is a more *accurate* estimate of R than y: $R(x,y)$ is true. If it produces a strain *less* than 0.526, then R is *greater* than 0.095. So R's true value is closer to y than to x, in which case y is a more accurate estimate of R than x: $R(y,x)$ is true. In diagram form:



A bit more precisely, then:

ACCURACY-CENTRED ACCOUNT (OFFICIAL): If a unique real number x best summarises which value a quantity Q tends to take in conditions C, then it does so in virtue of the fact that x tends to be the most accurate estimate of Q in C, in the following sense:

> (★★) For any $y \neq x$, S has a stronger tendency in C to make $Q(x, y)$ true than it does to make $Q(y, x)$ true.

In virtue of what, then, does 0.1 sum up how much strain, or deformation a bit of stress tends to produce? In virtue of the fact that 0.1 tends to be the most *accurate* estimate of the material's stress-to-strain ratio, according to the accuracy-centred account. Stressing the material in question (rubber in this case) tends to produce a stress-to-strain ratio R that is closer to 0.1 than 0.2, 0.3, or any other estimate. That is, for any other estimate y, the causal system S in question (the rubber, the instrument used to apply the force, etc.) has a stronger tendency to make $R(0.1, y)$ true (0.1 is closer to R's true value — and so more accurate — than y) than it does to make $R(y, 0.1)$ true. In just this sense, 0.1 tends to be the *most* accurate estimate of R. And this is just what is required for 0.1 to *best summarise* which value R is disposed to take, on the accuracy-centred view.

No doubt there are other ways one might try to explain what having a precise elastic modulus, or a precise molar absorptivity amounts to. But the explanation that the accuracy-centred account furnishes is elegant and unifying. So we will explore what else the account might do for us.

What, then, does the accuracy-centred account say about propensities? As it stands, not much. But suppose we add two little bits. First, suppose that we understand the accuracy-centred account broadly, as applying not just to single real-valued estimates x of single quantities Q, but also to *assignments* of estimates $p : \mathbb{Q} \to \mathbb{R}$ to *sets* of quantities $\mathbb{Q}$.

ACCURACY-CENTRED ACCOUNT (GENERAL): If a unique assignment of real numbers $p : \mathbb{Q} \to \mathbb{R}$ to a set of quantities $\mathbb{Q}$ best summarises which values those quantities tend to take in conditions C, then it does so in virtue of the fact that:

(⧫) p tends to be the most ***accurate*** assignment of estimates to $\mathbb{Q}$ in C.

There is good reason to understand the account broadly. Just as a single estimate x of a single quantity Q can be closer or further from the true value of that quantity — *i.e.*, more or less *accurate* — so can an assignment of estimates p to a set of quantities $\mathbb{Q}$ be more or less accurate. Indeed, *epistemic scoring rules* or *inaccuracy scores* give us the means to measure such accuracy, as we will see in §3 (see the appendix for formal details). And just as x can *tend* to be the most accurate estimate Q, so can p *tend* to be the most the most accurate assignment of estimates to quantities in $\mathbb{Q}$ (see §3-4). So we *can* understand the accuracy-centred account broadly. And we should: doing so allows us to provide a unified explanation of a more diverse range of phenomena.

Suppose also that we take on board the following auxiliary assumption: if some assignment of real numbers $p : \Omega \to [0, 1]$ to propositions X in an algebra Ω captures the propensities of those propositions — the precise strength of a causal system S's disposition to produce a state in which those propositions X are true — then $p(X)$ *best summarises*, in some appropriate sense, what truth-value X is disposed to take ('1' for true, '0' for false). If this is right, then propensities, elastic moduli, and molar absorptivity are all of a kind: they summarise which values other quantities are *disposed* to take. And in that case, the (general) accuracy-centred account says:

ACA-PROPENSITIES (UNOFFICIAL): If a unique assignment of real numbers $p : \Omega \to [0, 1]$ to propositions in an algebra Ω captures the propensities of those propositions in conditions C, and so *best summarises* what truth-values those propositions are disposed to take in C, then it does so in virtue of the fact that:

(♣) p tends to be the most ***accurate*** assignment of truth-value estimates to Ω in C.

To make this more precise, by swapping out (♣) for something a bit more careful, let's introduce some notation. Let Ω be a Boolean algebra of propositions (closed under negation and disjunction) whose truth-values are determined by the final state of the causal system S under consideration. And let $\mathcal{T}(p,q)$ be the proposition that is true exactly when S produces a state that determines an assignment of truth-values — 1 to all truths and 0 to all falsehoods — which is 'closer' to p than to q. That is, let $\mathcal{T}(p,q)$ say that p is closer to the truth — *i.e.*, more *accurate* — than q.

For example, suppose that you have a coin. You flip it. Let H be the proposition that it comes up heads, and $\neg H$ be the proposition that it comes up tails. Now consider two assignments of truth-value estimates to these propositions, $p : \{H, \neg H\} \to [0,1]$ and $q : \{H, \neg H\} \to [0,1]$, defined by figure 1. If the coin comes up heads on this particular toss, then the actual assignment of truth-values $\mathcal{T}$ — which assigns 1 to all truths and 0 to all falsehoods — is given by ω_H (figure 2). And ω_H is 'closer' to p than to q on any reasonable way of thinking about their respective proximities. (The reason: the truth-value estimates encoded by p are *uniformly* closer to the the actual truth-values, given by ω_H, than those encoded by q). Put differently, according to any reasonable measure of accuracy, p is more *accurate* than q.[6] So $\mathcal{T}(p,q)$ is true. If the coin comes up tails, on the other hand, then the actual assignment of truth-values $\mathcal{T}$ is given by $\omega_{\neg H}$. And $\omega_{\neg H}$ is closer to q than to p. That is, q is closer to the truth — more *accurate* — than p. So $\mathcal{T}(q,p)$ is true.

	H	$\neg H$
p	2/3	1/3
q	1/3	2/3

Figure 1: p,q.

	H	$\neg H$
ω_H	1	0
$\omega_{\neg H}$	0	1

Figure 2: ω_H,$\omega_{\neg H}$.

[6]More carefully, according to any *truth-directed* inaccuracy score, p is more accurate than q relative to ω_H. See appendix for details.

We can now state the accuracy-centred account of propensities more precisely:

ACA-PROPENSITIES (OFFICIAL): If a unique assignment of real numbers $p : \Omega \rightarrow [0, 1]$ to propositions in Ω captures the propensities of those propositions in conditions C, and so *best summarises* what truth-values those propositions are disposed to take in C, then it does so in virtue of the fact that p tends to be the most accurate assignment of truth-value estimates to Ω in C, in the following sense:

> (♣♣) For any $q \neq p$, S has a stronger tendency in C to make $\mathcal{T}(p, q)$ true than it does to make $\mathcal{T}(q, p)$ true.

In virtue of what, then, does a fair die have a propensity of exactly 1/6 to land '1', or '2', or '3', etc.? In virtue of the fact that for some algebra of propositions Ω, describing the outcome of some series of tosses, there is a *unique* assignment of truth-value estimates $p : \Omega \rightarrow [0, 1]$ with $p(X_1)$=...=$p(X_6)$=1/6 that tends to be *more accurate* than any other assignment (read X_i as: the die lands on side i on the x^{th} toss). Tossing a fair die has a stronger tendency to make $\mathcal{T}(p, q)$ true than it does to make $\mathcal{T}(q, p)$ true, for any other q. In words: when you toss a fair die, you have a stronger tendency to produce an outcome in which p is more accurate than q than you do to produce one in which q is more accurate than p. And this is just what is required for 1/6, and the rest of the truth-value estimates encoded by p, to *best summarise* the truth-values that propositions in Ω are disposed to take, on the accuracy-centred view, and hence to capture their *propensities*.

The accuracy-centred account explains, in an elegant and unifying way, what having a *precise* propensity amounts to in the first place. (Though, of course, it does *not* explain what having a comparative causal disposition amounts to, *e.g.*, a stronger disposition to make $\mathcal{T}(p, q)$ true than to make $\mathcal{T}(q, p)$ true. Our accuracy-centred account is *non-reductive*.) And importantly, it delivers *genuine single-case propensities* (unlike long-run accounts, *e.g.*, Gillies (2000)) — the sort of propensities that are fit to figure into causal explanations of singular events, and all the rest (*cf.* Fetzer (1981)). According to the accuracy-centred account, the propensity of a die to land '1', or '2', or '3', etc., on a particular toss, genuinely measures the strength of the die's disposition to produce *that* particular state on *that* particular occasion. The propensity-determining truth-value estimate attaches to the proposition that the die will land '1', or '2', or '3', etc., on *that very toss*. It does not attach to some proposition describing the outcome of a *sequence* of tosses. (See §4 for additional discussion of the role of sequences of trials on the accuracy-centred account.)

Also worth emphasising: the accuracy-centred account, all on its own, genuinely leaves open the question of whether propensities are probabilities. So we have not simply assumed what we hope to explain. For all the accuracy-centred account says, the unique assignment of truth-value estimates that captures a causal system's precise propensities *could* end up being probabilistically incoherent.

Our aim now is to show that this possibility is never realised. We will show that if the accuracy-centred account account is right, then given some pretty weak assumptions about the nature of comparative causal dispositions or tendencies, and some fairly standard assumptions about reasonable measures of inaccuracy, propensities *must* be probabilities.

3 Probabilistic Structure

The short story of why propensities must be probabilities is this: propensities are the truth-value estimates that tend to be most accurate. That is what the accuracy-centred account tells us. But probabilistically incoherent truth-value estimates could not possibly tend to be most accurate. The reason: they are *accuracy-dominated* by coherent truth-value estimates (*cf.*, Joyce (1998, 2009) and Predd et al. (2009)). For every incoherent assignment of truth-value estimates $p : \Omega \to [0,1]$ to propositions in Ω, there is some coherent assignment $q : \Omega \to [0,1]$ that is more accurate come what may. And any assignment that is *necessarily* more accurate surely *tends* to be more accurate.

In fact, we can say much more than our short story does about the nature of propensities — not just that they are probabilities. Let S be an arbitrary causal system with final states $F_1, F_2, \ldots$ in $\mathscr{F}$. Let Ω be a Boolean algebra of propositions (closed under negation and disjunction) whose truth-values are determined by the final state of S. And for each consistent assignment of truth-values $\omega : \Omega \to \{0,1\}$ to propositions in Ω, let w_ω be 'the' possible world that agrees with ω: X is true at w_ω iff $\omega(X) = 1$, for all $X \in \Omega$.[7] Let $\mathcal{W}$ be the set of all such worlds.

Finally, assume that S comes equipped with some minimal dispositional structure. In particular, assume that S determines a *comparative propensity ordering* over Ω, $\succeq$, that captures which propositions S is more or less strongly disposed to make true:

- For any $X, Y \in \Omega$, $X \succeq Y$ iff S has at least as strong of a disposition to produce a final state $F \in \mathscr{F}$ (or a collection of states in a sequence of trials) that makes X true and Y false as it does to produce one that makes Y true and X false.

- For any $X, Y \in \Omega$, $X \succ Y$ (shorthand for $X \succeq Y$ and $Y \not\succeq X$) iff S has a strictly stronger disposition to produce a final state (or a collection of states) that makes X true and Y false than it does to produce one that makes Y true and X false.

At the outset, we assume nothing further about $\succeq$. Our aim: identify some pretty weak assumptions about $\succeq$ — about the nature of comparative causal dispositions — which, in conjunction with some fairly standard assumptions about reasonable measures of inaccuracy, guarantee that propensities are probabilities.

[7]The differences between worlds that agree on the truth-values of all propositions in Ω do not matter, for our purposes. So we ignore them. And we drop the subscripts henceforth.

The real key, for our purposes — for sorting out when certain assignments of truth-value estimates *tend* to be more accurate than others — is to identify some plausible assumptions about how $\succeq$ must *extend* to the space $\mathcal{Q}$ of arbitrary real-valued quantities $Q : \mathcal{W} \to \mathbb{R}$, given how it orders propositions in Ω. We will examine three assumptions about $\succeq$'s extendability, and then put those assumptions to work for us.

First assumption:

Dominance: If quantity Q takes a higher value than quantity $Q^{\star}$ in every possible world:

$$Q(w) > Q^{\star}(w) \text{ for all } w \in \mathcal{W}$$

then $Q \succ Q^{\star}$, which says: S has a strictly stronger disposition to produce a final state in which Q take a higher value than $Q^{\star}$ than it does to produce one in which $Q^{\star}$ takes a higher value than Q.

Suppose that a benevolent gambler offers you a bet that pays £10 if some coin comes up heads on the next flip, and £20 if it comes up tails. She offers your friend a bet that is bound to pay out a little bit more: £11 if heads, £21 if tails. Then whatever else is true of our gambler's dispositions, she must be more strongly disposed to produce more winnings for your friend than for you than the other way around. This is the thought behind Dominance.

The next two assumptions are more controversial, but not by much.

Swaps: If for some $k > 0$, quantities Q and $Q^{\star}$ satisfy:

- Q takes a value that is greater than $Q^{\star}$'s by k in any world w that makes X true and Y false;
- $Q^{\star}$ takes a value that is greater than Q's by k in any world w' that makes X false and Y true;

- Q and $Q^{\star}$ take the same value in every other world;

and $X \succ Y$, then $Q \succ Q^{\star}$.

Suppose that you are bound to get precisely the same grades on your physics and history exams, except for in two scenarios: if you hire a physics tutor but not a history tutor (*Physics Tutor*), or if you hire a history tutor but not a physics tutor (*History Tutor*). You will get one grade higher in physics than in history if *Physics Tutor* is true, *e.g.*, an A and a B, respectively. Likewise, you will get one grade higher in history than in physics if *History Tutor* is true. (So the two scenarios are just the same, but with the difference in grades *swapped*.) Finally, suppose that you are more strongly disposed to make *Physics Tutor* true than to make *History Tutor* true. Then we can conclude: you are more strongly disposed to produce a higher grade in physics than in history than the other way around. This is the thought behind Swaps.

Finally:

SWEETENINGS: If for some $j > k > 0$, quantities Q and $Q^{\star}$ satisfy:

- Q takes a value that is greater than $Q^{\star}$'s by j in any world w that makes X true and Y false;
- $Q^{\star}$ takes a value that is greater than Q's by k in any world w' that makes X false and Y true;
- Q takes at least as great a value as $Q^{\star}$ in every other world;

and $X \approx Y$, then $Q \succ Q^{\star}$.

Imagine a setup much like Swaps. You will get one grade higher in physics than in history if *Physics Tutor* is true (call this difference in grades 'k'). But, unlike Swaps, you will get *two* grades higher in history than in physics if *History Tutor* is true (call this difference in grades 'j'). And you will get at least as high a grade in history as in physics in any other scenario. Finally, suppose that you are equally strongly disposed to make *Physics Tutor* and *History Tutor* true. Then in virtue of the fact that hiring a history tutor is a slightly *sweeter deal* (*i.e.*, $j > k$) we can conclude: you are more strongly disposed to produce a higher score in history than in physics than the other way around. This is the thought behind Sweetenings.

Here is the kicker: from these three fairly weak assumptions about comparative causal dispositions, and some plausible assumptions about reasonable measures of inaccuracy, it *just falls out* that the truth-value estimates $p : \Omega \to [0, 1]$ that our causal system S tends to make most accurate must have *two* properties. Not only must they be probabilities, but they must also *weakly represent* S's comparative propensity ordering: $X \succeq Y$ only if $p(X) \geq p(Y)$.

More carefully, suppose that we measure inaccuracy by an *epistemic scoring rule* or *inaccuracy score*. An inaccuracy score is a function $\mathcal{I}$, which maps assignments of truth-value estimates p and worlds w to non-negative real numbers, $\mathcal{I}(p, w)$. $\mathcal{I}(p, w)$ measures how inaccurate p is if w is actual. If $\mathcal{I}(p, w)$ equals zero, then p is minimally inaccurate at w (its estimates are maximally close to the truth). Inaccuracy increases as $\mathcal{I}(p, w)$ grows larger.

For notational convenience, let $\mathcal{I}(p)(\cdot)$ be shorthand for $\mathcal{I}(p, \cdot)$. So $\mathcal{I}(p) : \mathcal{W} \to \mathbb{R}$ is just a random variable: a function that maps worlds w to real numbers $\mathcal{I}(p, w)$. Then $\mathcal{I}(p) \succ \mathcal{I}(q)$ says: S has a strictly stronger disposition to produce a final state in which p is more inaccurate than q than it does to produce one in which q is more inaccurate than p. Or, to put it slightly differently, $\mathcal{I}(p) \succ \mathcal{I}(q)$ is equivalent to $\mathcal{T}(p, q) \prec \mathcal{T}(q, p)$. It says: S has a stronger tendency to make $\mathcal{T}(q, p)$ true (p is less accurate than q) than it does to make $\mathcal{T}(p, q)$ true (q is less accurate than p).

Then we can prove the following (see appendix):

Main Theorem: Suppose $\succeq$ satisfies Dominance, Swaps and Sweetenings. Suppose further that inaccuracy is measured by a Truth-Directed, Continuous, Strictly Proper and Convex scoring rule $\mathcal{I}$.[8] Then an assignment of truth-value estimates $p : \Omega \rightarrow [0,1]$ tends to be maximally accurate, in the sense that no other $q : \Omega \rightarrow [0,1]$ is such that $\mathcal{I}(p) \succ \mathcal{I}(q)$, only if p is a probability function, and p weakly represents $\succeq$, *i.e.*, $X \succeq Y$ only if $p(X) \geq p(Y)$, for all $X, Y \in \Omega$.

This little theorem contains an important moral. It tells us that, for any causal system with a little bit of dispositional structure — *viz.*, a comparative propensity ordering, $\succeq$ — the truth-value estimates that ***tend*** to be most accurate (if any such estimates exist) absolutely must have two properties: (*1*) they must be *probabilities* (this follows from Dominance); (*2*) they must line up with, or *represent* $\succeq$ (this follows from Swaps and Sweetenings). And propensities *just are* the truth-value estimates that tend to be most accurate, on the accuracy-centred account. So propensities must have these properties too. Shorter: propensities must be *probabilistic representers*.

The upshot: the problem of mathematical structure is really no problem at all. Once we appreciate the deep relationship between propensities and accuracy, the mathematical structure of propensities falls out naturally. The fact that propensities are probabilities (or stronger: probabilistic representers) follows straightforwardly from some fairly standard assumptions about the nature of accuracy, and few weak assumptions about the nature of comparative causal dispositions or tendencies.

4 Existence and Uniqueness

Nothing we have said so far tells us when — if ever! — a causal system S has sufficiently rich and specific comparative dispositions or tendencies, $\succeq$, to end up with precise

[8]For discussion of these properties, see Joyce (2009).

propensities, on the accuracy-centred account. All we know so far is that *if* our system S determines precise propensities for propositions in some algebra Ω — if some unique assignment of truth-value estimates $p : \Omega \to [0, 1]$ tends to be most accurate — *then* those propensities will be probabilistic representers.

It would be nice to know if there are conditions that guarantee $\succeq$'s probabilistic representability. It would also be nice to know when $\succeq$ is representable by a *unique* probability function. If $\succeq$ is probabilistically representable, but not uniquely so — if there are multiple assignments of truth-value estimates, $p_1, p_2, \ldots$ that tend to be most accurate (in the sense of our Main Theorem, *viz.*, that no other assignment tends to be *more* accurate) — then our causal system S determines *imprecise* propensities for propositions in Ω. How strongly it is disposed to make propositions X in Ω true is characterised by a *set* of probabilities functions. The facts about how strongly S is disposed to make X true are just the facts that are *invariant* across all of the members of that set. If $\succeq$ is representable by a unique probability function, on the other hand, then it determines *precise* propensities for those propositions.

Luckily, the answers to our questions are not too hard to come by. Scott (1964), for example, proves the following:

Scott's Theorem. There is a probability function $p : \Omega \to [0, 1]$ which satisfies:

$$X \succeq Y \text{ only if } p(X) \geq p(Y)$$

just in case $\succeq$ satisfies the following conditions:

1. $\top \succ \bot$
2. $X \succeq \bot$
3. If $X_1 + \ldots + X_n = Y_1 + \ldots + Y_n$ and $X_i \succeq Y_i$ for all i, then $X_i \preceq Y_i$ for all i as well.

Axiom 1 says that the causal system S in question is more strongly disposed to make tautologies true than it is to make contradictions true. Axioms 2 says that S is at least as strongly disposed to make tautologies true as it is to make any other proposition X true. Finally, axiom 3 — sometimes called *Scott's axiom* — says that for any two sequences of propositions, $\langle X_1, ..., X_n \rangle$ and $\langle Y_1, ..., Y_n \rangle$, which contain the same number of truths as a matter of logic (*i.e.* $X_1 + ... + X_n = Y_1 + ... + Y_n$), if S is at least as strongly disposed to make X_i true as it is to make Y_i true, for each i, then it must be equally strongly disposed to make the X_i and Y_i true; it *cannot* be more strongly disposed to make X_j true than Y_j for some j.

An example will help illustrate Scott's axiom. Suppose that you have two coins, 1 and 2. Let H_i be the proposition that coin i comes up heads. Then the two sequences $\langle H_1, \neg H_1 \rangle$ and $\langle H_2, \neg H_2 \rangle$ contain the same number of truths as a matter of logic (*viz.*, 1 truth). Scott's axioms says that you cannot have both $H_1 \succeq H_2$ and $\neg H_1 \succ \neg H_2$, for example. Coin 1 cannot both be at least as strongly disposed to come up heads as coin 2, and *more* strongly disposed to come up tails.

Clearly, any system with causal dispositions to speak of determines a comparative propensity ordering $\succeq$ which satisfies axioms 1-3. No comparative propensity ordering could possibly violate these axioms. What Scott's theorem tells us, then, is that in light of this fact, any comparative propensity ordering $\succeq$ is probabilistically representable. For any such ordering $\succeq$, there is a probabilistically coherent assignment of truth-value estimates $p : \Omega \rightarrow [0, 1]$ that weakly represents it. So any causal system at the very least comes equipped with imprecise propensities, on the accuracy-centred account. Further, if $\succeq$ is *total*, so that $X \succeq Y$ or $Y \succeq X$ for all $X, Y \in \Omega$, and $\succeq$ is *non-atomic*, so that for any $X \in \Omega$ such that $X \succ \bot$, there is a $Y \in \Omega$ such that $X \& Y \succ \bot$ and $X \& \neg Y \succ \bot$, then $\succeq$ is (weakly) representable by a *unique* probability function. In that case, the system comes equipped with *precise* propensities.

What's more, these conditions are not terribly recherché. For example, if the propositions in Ω describe the potential outcomes of an infinite sequence of experiments involving S, where any combination of outcomes for any given trial is possible, then $\succeq$ will plausibly be non-atomic. So any causal system that determines a *total* comparative propensity ordering in this sort of (hypothetical) experimental context plausibly determines *precise* propensities. Or if the propositions in Ω describe the potential outcomes of single experiment involving S, but each outcome can be described in limitless detail (so that for each proposition X describing an outcome, there is another proposition Y describing an additional feature of that outcome, such that X can be divided into $X\&Y$ and $X\&\neg Y$, respectively), then $\succeq$ will plausibly be non-atomic. So any causal system that determines a total comparative propensity ordering in *this* sort of context plausibly determines *precise* propensities.

Of course, it *could* be that very few causal systems determine total comparative propensity orderings in such contexts. The accuracy-centred account *per se* takes no stand on this issue. If so, then chances, by and large, are plausibly *imprecise*.

5 Conclusion

If chances are propensities, it is a bit perplexing why we ought to expect them to be probabilities. This paper aims to clear up the mystery. It does so in two parts. First, it motivates and details an accuracy-centred account of what it is for a causal system to have precise propensities in the first place. Then, it proves that, given some fairly standard assumptions about the nature of accuracy, and some weak assumptions about the nature of comparative causal dispositions or tendencies, propensities *must* be probabilities.

6 Appendix

Choose a binary relation $\succeq$ on Ω that satisfies Dominance, Swaps and Sweetenings.

Choose an inaccuracy measure:

$$\mathcal{I}(p,w) = \sum_{X \in \Omega} s(p(X), w(X))$$

that satisfies Truth-Directedness, Continuity, Strict-Propriety, and Convexity, where $s : [0,1] \times \{0,1\} \to [0,\infty]$ is what Joyce (2009) calls a *component function*, which measures the inaccuracy of the truth-value estimate $p(X)$ when X's truth-value is $w(X)$.

- *Truth-Directedness*: For any p and q, and any world w, if p's truth-value estimates are uniformly closer to the truth than q's at w, so that:

  $|p(X) - w(X)| \leq |q(X) - w(X)|$ for all $X \in \Omega$,

  and:

  $|p(Y) - w(Y)| < |q(Y) - w(Y)|$ for some $Y \in \Omega$,

  then $\mathcal{I}(p,w) < \mathcal{I}(q,w)$.

- *Continuity*. For any $w \in \mathcal{W}$, $\mathcal{I}(p,w)$ is a continuous function of p.

- *Strict Propriety*. For any probabilistically coherent p and $q \neq p$, $\mathbf{Exp}_p(\mathcal{I}(p)) < \mathbf{Exp}_p(\mathcal{I}(q))$.

- *Convexity*. For any p and q, and any $0 < \lambda < 1$, $\mathcal{I}(\lambda \cdot p + (1-\lambda) \cdot q, w) < \lambda \cdot \mathcal{I}(p,w) + (1-\lambda) \cdot \mathcal{I}(q,w)$.

  - Corollary: for any p and q, any $0 < \lambda < 1$, and any $X \in \Omega$, $s(\lambda \cdot p(X) + (1-\lambda) \cdot q(X), w(X)) < \lambda \cdot s(p(X), w(X)) + (1-\lambda) \cdot s(q(X), w(X))$

Finally, choose an assignment of truth-value estimates $p : \Omega \to [0,1]$ to propositions in Ω.

<u>*To Show*</u>: p tends to be maximally accurate, in the sense that there is no assignment of truth-value estimates q such that $\mathcal{I}(p) \succ \mathcal{I}(q)$, if and only if:

(i) p is a probability function, and

(ii) p weakly represents $\succeq$, *i.e.*, $X \succeq Y$ only if $p(X) \geq p(Y)$.

<u>*Left-to-Right Direction*</u>: Suppose that p is not a probability function. Then, since $\mathcal{I}$ is continuous and strictly proper, there is some probability function $q : \Omega \to [0,1]$ that is more accurate come what may: $\mathcal{I}(p,w) > \mathcal{I}(q,w)$ for all $w \in \mathcal{W}$ (Predd et al., 2009, p. 4788). By Dominance, then, $\mathcal{I}(p) \succ \mathcal{I}(q)$. Hence, there is no q such that $\mathcal{I}(p) \succ \mathcal{I}(q)$ only if p is a probability function.

Now suppose that p does not weakly represent $\succeq$. So for some $X, Y \in \Omega$, $X \succeq Y$ and $p(X) < p(Y)$.

<u>*Case 1*</u>: $X \succeq Y$ and $X \not\preceq Y$. Let $q(X) = p(Y)$, $q(Y) = p(X)$, and $q(Z) = p(Z)$ for all other $Z \in \Omega$.

<u>*To Show*</u>:

(1*a*) For any $w, w' \in \mathcal{W}$ such that:

$(a)\ w(X) = 1$	$(b)\ w(Y) = 0$
$(c)\ w'(X) = 0$	$(d)\ w'(Y) = 1$

$\mathcal{I}(p,w) - \mathcal{I}(q,w) = \mathcal{I}(q,w') - \mathcal{I}(p,w') > 0.$

(1*b*) For any $w'' \in \mathcal{W}$ such that $w''(X) = w''(Y) = 0$ or $w''(X) = w''(Y) = 1, \mathcal{I}(p,w'') = \mathcal{I}(q,w'')$.

<u>*Proof of (1a)*</u>: Choose any $w, w' \in \mathcal{W}$ that satisfy (a)-(d). Now note:

$$
\begin{aligned}
(I)\ \mathcal{I}(p,w)-\mathcal{I}(q,w) &= \sum_{X\in\Omega} s(p(X),w(X))-s(q(X),w(X)) \\
&= s(p(X),w(X))-s(q(X),w(X)) \\
&\quad +s(p(Y),w(Y))-s(q(Y),w(Y))
\end{aligned}
$$

And also:

$$
\begin{aligned}
(II)\ \mathcal{I}(q,w')-\mathcal{I}(p,w') &= s(q(X),w'(X))-s(p(X),w'(X)) \\
&\quad +s(q(Y),w'(Y))-s(p(Y),w'(Y))
\end{aligned}
$$

Since $q(X)=p(Y)$ and $q(Y)=p(X)$, (*II*) gives us (*III*):

$$
\begin{aligned}
(III)\ \mathcal{I}(q,w')-\mathcal{I}(p,w') &= s(p(Y),w'(X))-s(q(Y),w'(X)) \\
&\quad +s(p(X),w'(Y))-s(q(X),w'(Y))
\end{aligned}
$$

And since $w(X)=w'(Y)$ and $w(Y)=w'(X)$, (*III*) gives us (*IV*):

$$
\begin{aligned}
(IV)\ \mathcal{I}(q,w')-\mathcal{I}(p,w') &= s(p(Y),w(Y))-s(q(Y),w(Y)) \\
&\quad +s(p(X),w(X))-s(q(X),w(X))
\end{aligned}
$$

So $\mathcal{I}(p,w)-\mathcal{I}(q,w)=\mathcal{I}(q,w')-\mathcal{I}(p,w')$, from (*I*) and (*IV*).

It only remains to show that $\mathcal{I}(p,w)-\mathcal{I}(q,w)>0$. To that end, observe that since $p(X)<q(X)$,

$$|q(X)-w(X)|=1-q(x)<1-p(X)=|p(X)-w(X)|.$$

And since $q(Y)<p(Y)$,

$$|q(Y)-w(Y)|=q(Y)<p(Y)=|p(Y)-w(Y)|.$$

Finally, since $q(Z)=p(Z)$ for all other $Z\in\Omega$,

$$|q(Z)-w(Z)|\leq|p(Z)-w(Z)|.$$

So $\mathcal{I}(p,w)>\mathcal{I}(q,w)$, by Truth-Directedness.

Therefore:

$$\mathcal{I}(p,w)-\mathcal{I}(q,w)=\mathcal{I}(q,w')-\mathcal{I}(p,w')>0.$$

Proof of (1b): Choose any $w''\in\mathcal{W}$ such that $w''(X)=w''(Y)=0$ or $w''(X)=w''(Y)=1$. Then we have:

$$\begin{aligned}&\mathcal{I}(p,w'')-\mathcal{I}(q,w'')\\ &=s(p(X),w''(X))-s(q(X),w''(X))\\ &+s(p(Y),w''(Y))-s(q(Y),w''(Y))\\ &=s(p(X),w''(X))-s(p(Y),w''(X))\\ &+s(p(Y),w''(X))-s(p(X),w''(X))=0\end{aligned}$$

Hence $\mathcal{I}(p,w'')=\mathcal{I}(q,w'')$.

Finally, given $(1a)$ and $(1b)$, and given that $X\succ Y$ ($X\succeq Y$ and $X\not\preceq Y$), we have $\mathcal{I}(p)\succ\mathcal{I}(q)$, by Swaps.

Case 2: $X\succeq Y$ and $X\preceq Y$. Let $q(X)=p(Y)$, $q(Y)=p(X)$, and $q(Z)=p(Z)$ for all other $Z\in\Omega$. Let $r=1/2\cdot p+1/2\cdot q$. So $r(X)=r(Y)=(p(X)+p(Y))/2$, and $r(Z)=p(Z)$ for all other $Z\in\Omega$.

To Show:

$(2a)$ For any $w,w'\in\mathcal{W}$ such that:

(a) $w(X)=1$	(b) $w(Y)=0$
(c) $w'(X)=0$	(d) $w'(Y)=1$

$\mathcal{I}(p,w)-\mathcal{I}(r,w)>\mathcal{I}(r,w')-\mathcal{I}(p,w')>0.$

$(2a)$ For any $w''\in\mathcal{W}$ such that $w''(X)=w''(Y)=0$ or $w''(X)=w''(Y)=1, \mathcal{I}(p,w'')\geq\mathcal{I}(r,w'')$.

Proof of (2a): Choose any $w,w'\in\mathcal{W}$ that satisfy (a)-(d). As noted in case 1:

$$(I^*)\ \mathcal{I}(p,w)-\mathcal{I}(r,w) = s(p(X),w(X))-s(r(X),w(X)) + s(p(Y),w(Y))-s(r(Y),w(Y))$$

And:

$$(II^*)\ \mathcal{I}(r,w')-\mathcal{I}(p,w') = s(r(X),w'(X))-s(p(X),w'(X)) + s(r(Y),w'(Y))-s(p(Y),w'(Y))$$

By Convexity:

$$\begin{aligned} & s(r(X),w(X)) \\ & = s(1/2\cdot p(X)+1/2\cdot q(X),w(X)) \\ & < 1/2\cdot s(p(X),w(X))+1/2\cdot s(q(X),w(X)). \end{aligned}$$

Likewise for $s(r(Y),w(Y))$, $s(r(X),w'(X))$ and $s(r(Y),w'(Y))$. Therefore:

$$\begin{aligned} (III^*)\quad & \mathcal{I}(p,w)-\mathcal{I}(r,w) > \\ & s(p(X),w(X))-[1/2\cdot s(p(X),w(X))+1/2\cdot s(q(X),w(X))] \\ & +s(p(Y),w(Y))-[1/2\cdot s(p(Y),w(Y))+1/2\cdot s(q(Y),w(Y))] \\ & = 1/2\cdot s(p(X),w(X))-1/2\cdot s(q(X),w(X)) \\ & +1/2\cdot s(p(Y),w(Y))-1/2\cdot s(q(Y),w(Y)) \end{aligned}$$

And:

$$\begin{aligned} (IV^*)\quad & \mathcal{I}(r,w')-\mathcal{I}(p,w') < \\ & = [1/2\cdot s(p(X),w'(X))+1/2\cdot s(q(X),w'(X))]-s(p(X),w'(X)) \\ & +[1/2\cdot s(p(Y),w'(Y))+1/2\cdot s(q(Y),w'(Y))]-s(p(Y),w'(Y)) \\ & = 1/2\cdot s(q(X),w'(X))-1/2\cdot s(p(X),w'(X)) \\ & +1/2\cdot s(q(Y),w'(Y))-1/2\cdot s(p(Y),w'(Y)) \end{aligned}$$

Lastly, since $q(X)=p(Y)$ and $q(Y)=p(X)$, (IV^*) gives us (V^*):

$$\begin{aligned} (V^*)\quad & \mathcal{I}(r,w')-\mathcal{I}(p,w') < \\ & = 1/2\cdot s(p(Y),w'(X))-1/2\cdot s(q(Y),w'(X)) \end{aligned}$$

$$+1/2 \cdot s(p(X), w'(Y)) - 1/2 \cdot s(q(X), w'(Y))$$

And since $w(X) = w'(Y)$ and $w(Y) = w'(X)$, (V^*) gives us (VI^*):

$$(VI^*) \qquad \mathcal{I}(r, w') - \mathcal{I}(p, w') <$$

$$= 1/2 \cdot s(p(Y), w(Y)) - 1/2 \cdot s(q(Y), w(Y))$$

$$+1/2 \cdot s(p(X), w(X)) - 1/2 \cdot s(q(X), w(X))$$

So $\mathcal{I}(p, w) - \mathcal{I}(r, w) > \mathcal{I}(r, w') - \mathcal{I}(p, w')$, from ($III^*$) and ($VI^*$). It only remains to show that $\mathcal{I}(r, w') - \mathcal{I}(p, w') > 0$. To that end, observe that since $p(X) < r(X)$,

$$|p(X) - w'(X)| = p(X) < r(X) = |r(X) - w'(X)|.$$

And since $r(Y) < p(Y)$,

$$|p(Y) - w'(Y)| = 1 - p(Y) < 1 - r(Y) = |r(Y) - w'(Y)|.$$

Lastly, since $r(Z) = p(Z)$ for all other $Z \in \Omega$,

$$|p(Z) - w'(Z)| \leq |r(Z) - w'(Z)|.$$

So $\mathcal{I}(p, w') < \mathcal{I}(r, w')$, by Truth-Directedness.

Therefore:

$$\mathcal{I}(p, w) - \mathcal{I}(r, w) > \mathcal{I}(r, w') - \mathcal{I}(p, w') > 0.$$

Proof of (2b): Choose any $w'' \in \mathcal{W}$ such that $w''(X) = w''(Y) = 0$ or $w''(X) = w''(Y) = 1$. Then we have:

$$\mathcal{I}(p, w'') - \mathcal{I}(r, w'') =$$

$$s(p(X), w''(X)) - s(r(X), w''(X))$$

$$+s(p(Y), w''(Y)) - s(r(Y), w''(Y))$$

By Convexity, then, we have:

$$\mathcal{I}(p, w'') - \mathcal{I}(r, w'') >$$

$$s(p(X), w''(X)) - [1/2 \cdot s(p(X), w''(X)) + 1/2 \cdot s(q(X), w''(X))]$$

$$+$$

$$s(p(Y), w''(Y)) - [1/2 \cdot s(p(Y), w''(Y)) + 1/2 \cdot s(q(Y), w''(Y))]$$

$$= 1/2 \cdot s(p(X), w''(X)) - 1/2 \cdot s(q(X), w''(X))$$

$$+1/2 \cdot s(p(Y), w''(Y)) - 1/2 \cdot s(q(Y), w''(Y))$$

$$= 1/2 \cdot s(p(X), w''(X)) - 1/2 \cdot s(p(Y), w''(X))$$

$$+1/2 \cdot s(p(Y), w''(X)) - 1/2 \cdot s(p(X), w''(X)) = 0$$

Hence $\mathcal{I}(r, w'') < \mathcal{I}(p, w'')$.

Finally, given ($2a$) and ($2b$), and given that $X \approx Y$ ($X \succeq Y$ and $X \preceq Y$), we have $\mathcal{I}(p) \succ \mathcal{I}(q)$, by Sweetenings.

Intermediate conclusion: if q fails to weakly represent $\succeq$, either per case 1 or 2, then there is some q such that $\mathcal{I}(p) \succ \mathcal{I}(q)$. So there is no q such that $\mathcal{I}(p) \succ \mathcal{I}(q)$ only if p weakly represents $\succeq$.

Right-to-Left Direction: Suppose that

(i) p is a probability function, and

(ii) p weakly represents $\succeq$, *i.e.*, $X \succeq Y$ only if $p(X) \geq p(Y)$.

To Show: p tends to be maximally accurate, in the sense that there is no assignment of truth-value estimates q such that $\mathcal{I}(p) \succ \mathcal{I}(q)$.

Suppose for reductio that there is some q such that $\mathcal{I}(p) \succ \mathcal{I}(q)$.

Case 1: $\mathcal{I}(p) \succ \mathcal{I}(q)$ follows from Dominance. In that case, $\mathcal{I}(p, w) > \mathcal{I}(q, w)$ for all $w \in \mathcal{W}$. But no assignment of truth-value estimates q accuracy-dominates any probabilistically coherent p relative to any strictly proper scoring rule $\mathcal{I}$. If it did, we would have $\mathbf{Exp}_p(\mathcal{I}(p)) > \mathbf{Exp}_p(\mathcal{I}(q))$. $\Rightarrow\Leftarrow$.

Case 2: $\mathcal{I}(p) \succ \mathcal{I}(q)$ follows from Swaps. In that case, there is some $X, Y \in \Omega$ such that

$X \succ Y$, and some $k > 0$ such that

- $\mathcal{I}(p,w) - \mathcal{I}(q,w) = k$ for any $w \in \mathcal{W}$ such that $w(X) = 1$ and $w(Y) = 0$

- $\mathcal{I}(q,w') - \mathcal{I}(p,w') = k$ for any $w' \in \mathcal{W}$ such that $w'(X) = 0$ and $w'(Y) = 1$

- $\mathcal{I}(p,w'') = \mathcal{I}(q,w'')$ for any $w'' \in \mathcal{W}$ such that $w''(X) = w''(Y) = 0$ or $w''(X) = w''(Y) = 1$

But then by (ii), $p(X) \geq p(Y)$. And in that case

$$\begin{aligned}
\text{Exp}_p(\mathcal{I}(p)) - \text{Exp}_p(\mathcal{I}(q)) &= \sum_w p(w)[\mathcal{I}(p,w) - \mathcal{I}(q,w)] \\
&= k \cdot [p(X \& \neg Y) - p(\neg X \& Y)] \\
&= k \cdot [p(X \& \neg Y) - p(\neg X \& Y)] + k \cdot [p(X \& Y) - p(X \& Y)] \\
&= k \cdot [p(X) - p(Y)] \\
&\geq 0
\end{aligned}$$

But since $\mathcal{I}$ is strictly proper, $\text{Exp}_p(\mathcal{I}(p)) < \text{Exp}_p(\mathcal{I}(q))$. $\Rightarrow\Leftarrow$.

Case 3: $\mathcal{I}(p) \succ \mathcal{I}(q)$ follows from Sweetenings. In that case, there is some $X, Y \in \Omega$ such that $X \approx Y$, and some $j > k > 0$ such that

- $\mathcal{I}(p,w) - \mathcal{I}(q,w) = j$ for any $w \in \mathcal{W}$ such that $w(X) = 1$ and $w(Y) = 0$

- $\mathcal{I}(q,w') - \mathcal{I}(p,w') = k$ for any $w' \in \mathcal{W}$ such that $w'(X) = 0$ and $w'(Y) = 1$

- $\mathcal{I}(p,w'') \geq \mathcal{I}(q,w'')$ for any $w'' \in \mathcal{W}$ such that $w''(X) = w''(Y) = 0$ or $w''(X) = w''(Y) = 1$

But then by (ii), $p(X) = p(Y)$. And in that case

$$
\begin{aligned}
\text{Exp}_p(\mathcal{I}(p)) - \text{Exp}_p(\mathcal{I}(q)) &= \sum_w p(w)[\mathcal{I}(p,w) - \mathcal{I}(q,w)] \\
&\geq j \cdot p(X\&\neg Y) - k \cdot p(\neg X\&Y) \\
&> j \cdot [p(X\&\neg Y) - \cdot p(\neg X\&Y)] \\
&= j \cdot [p(X\&\neg Y) - p(\neg X\&Y)] + j \cdot [p(X\&Y) - p(X\&Y)] \\
&= j \cdot [p(X) - p(Y)] \\
&= 0
\end{aligned}
$$

But since $\mathcal{I}$ is strictly proper, $\text{Exp}_p(\mathcal{I}(p)) < \text{Exp}_p(\mathcal{I}(q))$. $\Rightarrow\Leftarrow$.

Conclusion: p tends to be maximally accurate, in the sense that there is no assignment of truth-value estimates q such that $\mathcal{I}(p) \succ \mathcal{I}(q)$, if and only if:

(i) p is a probability function, and

(ii) p weakly represents $\succeq$, *i.e.*, $X \succeq Y$ only if $p(X) \geq p(Y)$.

References

Fetzer, J. (1981). Probability and explanation. *Synthese 48*(3), 371–408.

Fetzer, J. (1982). Probabilistic explanations. In *PSA: Proceedings of the Biennial Meeting of Philosophy of Science Association*, Volume 2, pp. 194–207.

Fetzer, J. (1983). Probability and objectivity in deterministic and indeterministic situations. *Synthese 57*, 367–386.

Giere, R. (1973). Objective single-case probabilities and the foundations of statistics. In P. Suppes (Ed.), *Logic, Methodology and Philosophy of Science*, Volume IV. New York: North-Holland.

Gillies, D. (2000). Varieties of propensity. *British Journal for the Philosophy of Science 51*, 807–835.

Hajek, A. (2012). Interpretations of probability. *The Stanford Encyclopedia of Philosophy*.

Hitchcock, C. (2002). *International Encyclopedia of the Social and Behavioral Sciences*, Volume 18, Chapter Probability and Chance, pp. 12089–12095. London: Elsevier.

Hitchcock, C. (2012). Cause and chance. Ms.

Joyce, J. M. (1998). A nonpragmatic vindication of probabilism. *Philosophy of Science 65*, 575–603.

Joyce, J. M. (2009). Accuracy and coherence: Prospects for an alethic epistemology of partial belief. In F. Huber and C. Schmidt-Petri (Eds.), *Degrees of Belief*, Volume 342. Dordrecht: Springer.

Lewis, D. (1980). A subjectivist's guide to objective chance. In *Philosophical Papers 2*, pp. 83–132. New York: Oxford University Press.

Miller, D. (1994). *Critical Rationalism: A Restatement and Defence*. Chicago and Lasalle, IL: Open Court.

Miller, D. (1996). Propensities and indeterminism. In O. A (Ed.), *Karl Popper: Philosophy and Problems*, pp. 121–147. Cambridge: Cambridge University Press.

Popper, K. (1957). The propensity interpretation of the calculus of probability, and the quantum theory. In S. Korner (Ed.), *Observation and Interpretation, Proceedings of* *the Ninth Symposium of the Colston Research Society*, University of Bristol, pp. 65–70, 88–89.

Popper, K. (1959). The propensity interpretation of probability. *British Journal for the Philosophy of Science 10*, 25–42.

Popper, K. (1983). *Realism and the Aim of Science*. London: Hutchinson.

Popper, K. (1990). *A World of Propensities*. Bristol: Thoemmes.

Predd, J. B., R. Seiringer, E. H. Lieb, D. N. Osherson, H. V. Poor, and S. R. Kulkarni (2009, October). Probabilistic Coherence and Proper Scoring Rules. *IEEE Transactions on Information Theory 55*(10), 4786–4792.

Scott, D. (1964). Measurement structures and linear inequalities. *Journal of mathematical psychology 1*(2), 233–247.