

**Abstract.** Every truth-functional three-valued propositional logic can be conservatively translated into the modal logic  $S5$ . We prove this claim constructively in two steps. First, we define a Translation Manual that converts any propositional formula of any three-valued logic into a modal formula. Second, we show that for every  $S5$ -model there is an equivalent three-valued valuation and vice versa. In general, our Translation Manual gives rise to translations that are exponentially longer than their originals. This fact raises the question whether there are three-valued logics for which there is a shorter translation into  $S5$ . The answer is affirmative: we present an elegant linear translation of the Logic of Paradox and of Strong Three-valued Logic into  $S5$ .

*Keywords:* Three-valued logic, Modal logic, Conservative translations, Expressivity.

## 1. Introduction

Translations of one logic into another logic might serve as a bridge to carry over technical results and philosophical insights. The translations of classical propositional logic into intuitionistic propositional logic that appeared in the literature around the 1930s all proved the relative consistency of classical propositional logic: if intuitionistic propositional logic is consistent, then so is classical propositional logic. The translation of intuitionistic propositional logic into the modal logic  $S4$ , based on Gödel's [9], strongly supports the provability interpretation of intuitionistic logic. Not all of these mappings of one logic into another are translations in the same sense. Some of them are only *conservative mappings*, others are *conservative translations*.<sup>1</sup> A conservative mapping  $\top$  of a logic  $L_1$  into a logic  $L_2$  is a function that preserves *valid formulas* in both ways:

$$\models_{L_1} \varphi \quad \text{iff} \quad \models_{L_2} \top(\varphi).$$

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<sup>1</sup>We take over Feitosa and D'Ottaviano's [8] terminology. They also provide a review of a variety of notions of translations of one logic into another logic and a discussion of the historical translations of the 1930s [8, pp. 211–217]. See also D'Ottaviano and Feitosa [7].

The notion of a conservative mapping is too blunt an instrument for studying translations of three-valued logics into modal logic. Let us give two examples. First, the identity function  $\top(\varphi) = \varphi$  conservatively maps the Logic of Paradox ( $LP$ ) [14] into classical propositional logic, because  $LP$  has exactly the same valid formulas as classical propositional logic. Second, the function  $\top(\varphi) = p \wedge \neg p$  (where  $p$  is some atomic formula) conservatively maps Strong Three-valued Logic ( $K_3$ ) [10, 11] into classical propositional logic, because  $K_3$  does not have any valid formulas. For our present purposes we therefore need a notion of translation that makes sharper distinctions.

The notion of a conservative translation gives us precisely what is required for our current investigations. A conservative translation  $\top$  of a logic  $L_1$  into a logic  $L_2$  is a function that preserves *valid arguments* in both ways:

$$\Pi \models_{L_1} \varphi \quad \text{iff} \quad \top(\Pi) \models_{L_2} \top(\varphi).$$

It is easy to see that the conservative mappings of  $LP$  and  $K_3$  into classical propositional logic are no conservative translations.

The set-up of our paper is as follows. First, we give a general definition of a truth-functional three-valued propositional logic. Second, we define a Translation Manual that converts any propositional formula of any three-valued logic into a modal formula. Third, we show that because for every  $S5$ -model there is a translationally equivalent three-valued valuation and vice versa, every truth-functional three-valued propositional logic can be conservatively translated into the modal logic  $S5$ . Finally, we present a linear translation that conservatively translates both  $LP$  and  $K_3$  into  $S5$ .

## 2. Three-valued Logics

A propositional language is a set of formulas built from a set  $\mathcal{P} = \{p, p', \dots\}$  of atomic formulas and a set  $\mathcal{O} = \{\otimes_0, \otimes'_0, \dots, \otimes_1, \otimes'_1, \dots\}$  of operators (the subscript indicates an operator's arity). The propositional language  $\mathcal{L}_{\mathcal{P}}^{\mathcal{O}}$  is the smallest set (in terms of set-theoretical inclusion) satisfying the following two conditions:

- (i)  $\mathcal{P} \subseteq \mathcal{L}_{\mathcal{P}}^{\mathcal{O}}$
- (ii) If  $\otimes_n \in \mathcal{O}$  and  $\varphi_1, \dots, \varphi_n \in \mathcal{L}_{\mathcal{P}}^{\mathcal{O}}$ , then  $\otimes_n(\varphi_1, \dots, \varphi_n) \in \mathcal{L}_{\mathcal{P}}^{\mathcal{O}}$ .

We refer to elements of this propositional language as *propositional formulas*. (If  $\mathcal{O} = \{\neg_1, \vee_2, \wedge_2, \rightarrow_2\}$ , then  $\mathcal{L}_{\mathcal{P}}^{\mathcal{O}}$  is just the familiar language of propositional logic.) In a three-valued logic propositional formulas are interpreted

by way of three-valued valuations. A three-valued valuation  $v$  on  $\mathcal{P}$  is a function  $v : \mathcal{P} \rightarrow \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ , where  $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$  is a set of three distinct truth-values. A valuation  $v$  therefore assigns to each atomic formula  $p$  in  $\mathcal{P}$  exactly one of the three truth-values  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ . A valuation  $v$  can be extended to a valuation  $v^*$  that assigns to each propositional formula  $\varphi$  in  $\mathcal{L}_{\mathcal{P}}^{\mathcal{O}}$  exactly one of the three truth-values, using truth-tables for all operators that occur in  $\varphi$ . A truth-table for an operator  $\otimes_n$  is nothing but an  $n$ -placed function  $f_{\otimes_n} : \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}^n \rightarrow \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$  that yields the truth-value of a complex formula  $\otimes_n(\varphi_1, \dots, \varphi_n)$  on the basis of the truth-values of its constituent formulas  $\varphi_1, \dots, \varphi_n$ .

DEFINITION 2.1. Let  $v$  be a three-valued valuation on  $\mathcal{P}$ . Then  $v$  can be extended to a three-valued valuation  $v^*$  on  $\mathcal{L}_{\mathcal{P}}^{\mathcal{O}}$  as follows:

$$\begin{aligned} v^*(p) &= v(p) \\ v^*(\otimes_n(\varphi_1, \dots, \varphi_n)) &= f_{\otimes_n}(\langle v^*(\varphi_1), \dots, v^*(\varphi_n) \rangle). \end{aligned}$$

Extended valuations do not suffice to define three-valued validity. We need to stipulate which truth-values are the *designated* truth-values for the premises and which ones are *designated* for the conclusion.<sup>2</sup> Accordingly, the concept  $\models_3^{\mathfrak{X}\mathfrak{Y}}$  of three-valued validity is relative to a set  $\mathfrak{X} \subseteq \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$  of designated truth-values for the premises and a set  $\mathfrak{Y} \subseteq \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$  of designated truth-values for the conclusion. An argument from a set  $\Pi$  of premises to a conclusion  $\varphi$  is  $\mathfrak{X}\mathfrak{Y}$ -valid (notation:  $\Pi \models_3^{\mathfrak{X}\mathfrak{Y}} \varphi$ ) if and only if for each valuation  $v$  it holds that if  $v^*(\psi) \in \mathfrak{X}$  for all  $\psi \in \Pi$ , then  $v^*(\varphi) \in \mathfrak{Y}$ .

### 3. A Translation Manual for Three-valued Logics

We now present a Translation Manual that translates any three-valued logic thus defined into  $S5$ . For clarity we first briefly discuss  $S5$ . The modal logic  $S5$  interprets formulas from the modal language  $\mathcal{ML}$  that is built from the set  $\mathcal{P} = \{p, p', \dots\}$  of atomic formulas and the falsum ( $\perp$ ) using negation ( $\neg$ ), disjunction ( $\vee$ ), conjunction ( $\wedge$ ), possibility ( $\diamond$ ), and necessity ( $\square$ ). We refer to elements of this modal language as *modal formulas*. The semantics of  $S5$  is as follows:

DEFINITION 3.1. An  $S5$ -model  $M (= \langle W, V \rangle)$  consists of a nonempty set  $W$  of possible worlds and a valuation function  $V$  that assigns to each atomic formula  $p$  in  $\mathcal{P}$  a subset  $V(p)$  of  $W$ . Let  $w \in W$ , let  $p \in \mathcal{P}$ , and let  $\varphi, \psi \in \mathcal{ML}$ .

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<sup>2</sup>This distinction is also made by Cobreros et al. [6, Section 3].

Then

$M, w \models_{S5} p$	iff	$w \in V(p)$
$M, w \not\models_{S5} \perp$		
$M, w \models_{S5} \neg\varphi$	iff	$M, w \not\models_{S5} \varphi$
$M, w \models_{S5} \varphi \vee \psi$	iff	$M, w \models_{S5} \varphi$ or $M, w \models_{S5} \psi$
$M, w \models_{S5} \varphi \wedge \psi$	iff	$M, w \models_{S5} \varphi$ and $M, w \models_{S5} \psi$
$M, w \models_{S5} \diamond\varphi$	iff	there is a $w'$ in $W$ such that $M, w' \models_{S5} \varphi$
$M, w \models_{S5} \Box\varphi$	iff	for all $w'$ in $W$ it holds that $M, w' \models_{S5} \varphi$ .

We write  $M \models_{S5} \varphi$ , if  $M, w \models_{S5} \varphi$  for all  $w$  in  $W$ . An argument from a set  $\Pi$  of premises to a conclusion  $\varphi$  is  $S5$ -valid (notation:  $\Pi \models_{S5} \varphi$ ) if and only if for each  $S5$ -model  $M (= \langle W, V \rangle)$  and each  $w$  in  $W$  it holds that if  $M, w \models_{S5} \psi$  for all  $\psi$  in  $\Pi$ , then  $M, w \models_{S5} \varphi$ .

In an  $S5$ -model there are three mutually exclusive and jointly exhaustive possibilities for each atomic formula  $p$  : either  $p$  is true in all possible worlds, or  $p$  is true in some possible worlds and false in others, or  $p$  is false in all possible worlds. Our Translation Manual first maps the three possible truth-values of any atomic formula  $p$  to these three possibilities, and then maps the three possible truth-values of any complex formula  $\otimes_n(\varphi_1, \dots, \varphi_n)$  to truth-functional combinations of the mapped truth-values of its constituent formulas according to the strictures of  $\otimes_n$ 's truth-table:

DEFINITION 3.2. (*Translation Manual*) Let  $p \in \mathcal{P}$  and let  $\varphi_1, \dots, \varphi_n \in \mathcal{L}_{\mathcal{P}}^{\mathcal{O}}$ . Then

$$\begin{aligned} \mathbf{a}(p) &= \Box p \\ \mathbf{b}(p) &= \Diamond p \wedge \neg \Box p \\ \mathbf{c}(p) &= \neg \Diamond p \end{aligned}$$

$$\begin{aligned} \mathbf{a}(\otimes_n(\varphi_1, \dots, \varphi_n)) &= \bigvee_{\langle \mathbf{r}_1, \dots, \mathbf{r}_n \rangle \in \mathbf{a}(f_{\otimes_n})} (\mathbf{r}_1(\varphi_1) \wedge \dots \wedge \mathbf{r}_n(\varphi_n)) \\ \mathbf{b}(\otimes_n(\varphi_1, \dots, \varphi_n)) &= \bigvee_{\langle \mathbf{r}_1, \dots, \mathbf{r}_n \rangle \in \mathbf{b}(f_{\otimes_n})} (\mathbf{r}_1(\varphi_1) \wedge \dots \wedge \mathbf{r}_n(\varphi_n)) \\ \mathbf{c}(\otimes_n(\varphi_1, \dots, \varphi_n)) &= \bigvee_{\langle \mathbf{r}_1, \dots, \mathbf{r}_n \rangle \in \mathbf{c}(f_{\otimes_n})} (\mathbf{r}_1(\varphi_1) \wedge \dots \wedge \mathbf{r}_n(\varphi_n)), \end{aligned}$$

where  $\mathbf{r}(f_{\otimes_n}) = \{ \langle \mathbf{r}_1, \dots, \mathbf{r}_n \rangle \in \{ \mathbf{a}, \mathbf{b}, \mathbf{c} \}^n : f_{\otimes_n}(\langle \mathbf{r}_1, \dots, \mathbf{r}_n \rangle) = \mathbf{r} \}$ .

Note that  $\mathbf{r}(\otimes_n(\varphi_1, \dots, \varphi_n)) = \perp$ , if  $\mathbf{r}(f_{\otimes_n}) = \emptyset$ . Given a formula  $\varphi$  and a set  $\mathfrak{X} \subseteq \{ \mathbf{a}, \mathbf{b}, \mathbf{c} \}$  of truth-values, we write  $\mathfrak{X}(\varphi)$  for  $\bigvee_{\mathbf{r} \in \mathfrak{X}} \mathbf{r}(\varphi)$ . (Again, note

that  $\mathfrak{X}(\varphi) = \perp$ , if  $\mathfrak{X} = \emptyset$ .) Similarly, given a set  $\Pi$  of formulas and a set  $\mathfrak{X} \subseteq \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$  of truth-values, we write  $\mathfrak{X}(\Pi)$  for  $\{\mathfrak{X}(\psi) : \psi \in \Pi\}$ .

Using our Translation Manual, we can now state the conditions under which a three-valued valuation and an  $S5$ -model are translationally equivalent.

DEFINITION 3.3. Let  $v$  be a three-valued valuation on  $\mathcal{P}$  and let  $M$  be an  $S5$ -model. Then  $v$  and  $M$  are 3-equivalent, if for all  $\varphi$  in  $\mathcal{L}_{\mathcal{P}}^{\mathcal{O}}$  it holds that

$$\begin{aligned} v^*(\varphi) &= \mathbf{a} && \text{iff } M \models_{S5} \mathbf{a}(\varphi) \\ v^*(\varphi) &= \mathbf{b} && \text{iff } M \models_{S5} \mathbf{b}(\varphi) \\ v^*(\varphi) &= \mathbf{c} && \text{iff } M \models_{S5} \mathbf{c}(\varphi). \end{aligned}$$

Note that if a three-valued valuation  $v$  and an  $S5$ -model  $M$  are 3-equivalent, it holds for all  $\varphi$  in  $\mathcal{L}_{\mathcal{P}}^{\mathcal{O}}$  and for all subsets  $\mathfrak{X}$  of  $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$  that  $v^*(\varphi) \in \mathfrak{X}$  if and only if  $M \models_{S5} \mathfrak{X}(\varphi)$ .

For each  $S5$ -model there is a 3-equivalent three-valued valuation and for each three-valued valuation there is a 3-equivalent  $S5$ -model. To show this, we need a lemma about the specific type of modal formulas that are generated by our Translation Manual. In fact, all translations of propositional formulas are fully modalized, that is, for all  $\varphi$  in  $\mathcal{L}_{\mathcal{P}}^{\mathcal{O}}$  it holds that  $\mathbf{a}(\varphi)$ ,  $\mathbf{b}(\varphi)$ , and  $\mathbf{c}(\varphi)$  are in the modal sublanguage  $\mathcal{ML}^*$ , where  $\mathcal{ML}^*$  is built from  $\{\Diamond p : p \in \mathcal{P}\} \cup \{\Box p : p \in \mathcal{P}\} \cup \{\perp\}$  using negation ( $\neg$ ), disjunction ( $\vee$ ), and conjunction ( $\wedge$ ). Fully modalized formulas have a special property: they are true *somewhere* in an  $S5$ -model if and only if they are true *everywhere* in that model (a straightforward structural induction proves this):

LEMMA 3.1. Let  $M (= \langle W, V \rangle)$  be an  $S5$ -model. Let  $w \in W$  and  $\varphi \in \mathcal{ML}^*$ . Then

$$M, w \models_{S5} \varphi \quad \text{iff} \quad M \models_{S5} \varphi.$$

LEMMA 3.2. For each  $S5$ -model there is a 3-equivalent three-valued valuation.

PROOF. Let  $M (= \langle W, V \rangle)$  be an  $S5$ -model. We construct a three-valued valuation  $v_M$  by stipulating that for all atomic formulas  $p$  in  $\mathcal{P}$

$$\begin{aligned} v_M(p) &= \mathbf{a} && \text{iff } V(p) = W \\ v_M(p) &= \mathbf{b} && \text{iff } V(p) \neq \emptyset \text{ and } V(p) \neq W \\ v_M(p) &= \mathbf{c} && \text{iff } V(p) = \emptyset. \end{aligned}$$

It is easy to see that  $v_M$  is a three-valued valuation:  $v_M$  assigns to each atomic formula  $p$  exactly one of the truth-values  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ .

We now show by structural induction on  $\varphi$  that  $v_M$  and  $M$  are 3-equivalent.

*Basis.* That the claim holds for atomic formulas follows directly from the definition of  $v_M$ , Definition 2.1, Lemma 3.1, and the semantics of  $S5$ .

*Induction Hypothesis.* Suppose that our theorem holds for all formulas  $\varphi$  with less operators than the formula  $\otimes_n(\varphi_1, \dots, \varphi_n)$ .

*Induction Step.* Consider  $\otimes_n(\varphi_1, \dots, \varphi_n)$ . Suppose  $v_M^*(\otimes_n(\varphi_1, \dots, \varphi_n)) = \mathbf{a}$ . Then  $f_{\otimes_n}(\langle v_M^*(\varphi_1), \dots, v_M^*(\varphi_n) \rangle) = \mathbf{a}$ . Take an arbitrary  $\varphi_i$  and consider  $v_M^*(\varphi_i) = \mathbf{r}_i$ . By the Induction Hypothesis, it must be that  $M \models_{S5} \mathbf{r}_i(\varphi_i)$  for all  $i$  with  $1 \leq i \leq n$ . Hence,  $M \models_{S5} \mathbf{r}_1(\varphi_1) \wedge \dots \wedge \mathbf{r}_n(\varphi_n)$ . It is clear that  $\langle \mathbf{r}_1, \dots, \mathbf{r}_n \rangle \in \mathbf{a}(f_{\otimes_n})$ . Therefore,  $M \models_{S5} \mathbf{a}(\otimes_n(\varphi_1, \dots, \varphi_n))$ .

Suppose  $M \models_{S5} \mathbf{a}(\otimes_n(\varphi_1, \dots, \varphi_n))$ . Then there is an  $n$ -tuple  $\langle \mathbf{r}_1, \dots, \mathbf{r}_n \rangle$  in  $\mathbf{a}(f_{\otimes_n})$  such that  $M \models_{S5} \mathbf{r}_1(\varphi_1) \wedge \dots \wedge \mathbf{r}_n(\varphi_n)$ . Hence,  $M \models_{S5} \mathbf{r}_i(\varphi_i)$  for all  $i$  with  $1 \leq i \leq n$ . By the Induction Hypothesis, it must be that  $v_M^*(\varphi_i) = \mathbf{r}_i$  for all  $i$  with  $1 \leq i \leq n$ . Therefore,  $v_M^*(\otimes_n(\varphi_1, \dots, \varphi_n)) = f_{\otimes_n}(\langle v_M^*(\varphi_1), \dots, v_M^*(\varphi_n) \rangle) = f_{\otimes_n}(\langle \mathbf{r}_1, \dots, \mathbf{r}_n \rangle) = \mathbf{a}$ .

The cases for  $\mathbf{b}$  and  $\mathbf{c}$  are analogous. ■

LEMMA 3.3. *For each three-valued valuation there is a 3-equivalent S5-model.*

PROOF. Let  $v$  be a three-valued valuation. We construct an  $S5$ -model  $M_v$  ( $= \langle W_v, V_v \rangle$ ) by stipulating that (1)  $W_v = \{w, w'\}$  and (2) for all atomic formulas  $p$  in  $\mathcal{P}$

$$\begin{aligned} V_v(p) = W_v & \text{ iff } v(p) = \mathbf{a} \\ V_v(p) = \{w\} & \text{ iff } v(p) = \mathbf{b} \\ V_v(p) = \emptyset & \text{ iff } v(p) = \mathbf{c}. \end{aligned}$$

Obviously,  $M_v$  is an  $S5$ -model. An adaption of the inductive proof of Lemma 3.2 shows that  $v$  and  $M_v$  are 3-equivalent. ■

We now prove our theorem that every truth-functional three-valued propositional logic can be conservatively translated into the modal logic  $S5$ :

THEOREM 3.4. *Let  $\Pi \subseteq \mathcal{L}_{\mathcal{P}}^{\mathcal{O}}$ , let  $\varphi \in \mathcal{L}_{\mathcal{P}}^{\mathcal{O}}$ , and let  $\mathfrak{X}, \mathfrak{Y} \subseteq \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ . Then*

$$\Pi \models_3^{\mathfrak{X}\mathfrak{Y}} \varphi \text{ iff } \mathfrak{X}(\Pi) \models_{S5} \mathfrak{Y}(\varphi).$$

PROOF. Suppose that  $\mathfrak{X}(\Pi) \not\models_{S5} \mathfrak{Y}(\varphi)$ . Then there is an  $S5$ -model  $M$  ( $= \langle W, V \rangle$ ) and a  $w$  in  $W$  such that  $M, w \models_{S5} \mathfrak{X}(\psi)$  for all  $\psi$  in  $\Pi$  and  $M, w \not\models_{S5} \mathfrak{Y}(\varphi)$ . By Lemma 3.1 it follows that  $M \models_{S5} \mathfrak{X}(\psi)$  for all  $\psi$  in  $\Pi$  and  $M \not\models_{S5} \mathfrak{Y}(\varphi)$ . By Lemma 3.2 there is a three-valued valuation  $v_M$  such that  $v_M^*(\psi) \in \mathfrak{X}$  for all  $\psi$  in  $\Pi$  and  $v_M^*(\varphi) \notin \mathfrak{Y}$ . Therefore,  $\Pi \not\models_3^{\mathfrak{X}\mathfrak{Y}} \varphi$ .

Suppose that  $\Pi \not\models_3^{\mathfrak{X}\mathfrak{Y}} \varphi$ . Then there is a three-valued valuation  $v$  such that  $v^*(\psi) \in \mathfrak{X}$  for all  $\psi$  in  $\Pi$  and  $v^*(\varphi) \notin \mathfrak{Y}$ . By Lemma 3.3 there is

an  $S5$ -model  $M_v (= \langle W_v, V_v \rangle)$  such that  $M_v \models_{S5} \mathfrak{X}(\psi)$  for all  $\psi$  in  $\Pi$  and  $M_v \not\models_{S5} \mathfrak{Y}(\varphi)$ . By Lemma 3.1 there is a  $w$  in  $W_v$  such that  $M_v, w \models_{S5} \mathfrak{X}(\psi)$  for all  $\psi$  in  $\Pi$  and  $M_v, w \not\models_{S5} \mathfrak{Y}(\varphi)$ . Therefore,  $\mathfrak{X}(\Pi) \not\models_{S5} \mathfrak{Y}(\varphi)$ . ■

In general, the length of the translations produced by our Translation Manual is *exponential*, because a single step in the production of, say, the  $\mathfrak{a}$ -translation of a complex formula built from an  $n$ -ary operator  $\otimes_n$  might comprise  $3^n$  clauses: if  $f_{\otimes_n}(\langle \mathfrak{x}_1, \dots, \mathfrak{x}_n \rangle) = \mathfrak{a}$  for all  $\langle \mathfrak{x}_1, \dots, \mathfrak{x}_n \rangle \in \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}^n$ , then

$$\mathfrak{a}(\otimes_n(\varphi_1, \dots, \varphi_n)) = \bigvee_{\langle \mathfrak{x}_1, \dots, \mathfrak{x}_n \rangle \in \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}^n} (\mathfrak{x}_1(\varphi_1) \wedge \dots \wedge \mathfrak{x}_n(\varphi_n)).$$

This fact raises the question whether there are three-valued logics for which there is a shorter translation into  $S5$ . The answer is affirmative: there is an elegant *linear* conservative translation of the three-valued logics  $LP$  and  $K_3$  into the modal logic  $S5$ .

#### 4. A Linear Translation of $LP$ and $K_3$ into $S5$

Translations of the Logic of Paradox ( $LP$ ) and Strong Three-Valued Logic ( $K_3$ ) into modal logic have been offered in the literature. Batens [1, p. 284] gives essentially the same translation of  $LP$  into  $S5$  as we do, but fails to include a proof. Busch’s [3, p. 72] translation of  $K_3$  into modal logic only applies to the fragment of the language of propositional logic that is built from (negations of) atomic formulas using disjunction and conjunction, and introduces unnecessary  $\Box$ s in each step where a disjunction or a conjunction is treated. We present a *single* linear translation that conservatively translates both  $LP$  and  $K_3$  into  $S5$ .

$LP$  and  $K_3$  evaluate formulas and arguments from a propositional language  $\mathcal{L}_{\mathcal{P}}^{\mathcal{O}}$ , where  $\mathcal{P} = \{p, p', \dots\}$  and  $\mathcal{O} = \{\neg_1, \vee_2, \wedge_2\}$ .  $LP$  adds a third truth-value ‘both’ to the classical pair ‘false’ and ‘true’. An  $LP$ -valuation is a function  $v_{LP}$  from the set  $\mathcal{P}$  of atomic formulas to the set  $\{\{0\}, \{1\}, \{0, 1\}\}$  of truth-values ‘false’, ‘true’, and ‘both’.  $K_3$  adds a third truth-value ‘none’ to the pair ‘false’ and ‘true’. A  $K_3$ -valuation is a function  $v_{K_3}$  from  $\mathcal{P}$  to the set  $\{\emptyset, \{0\}, \{1\}\}$  of truth-values ‘none’, ‘false’, and ‘true’. An  $LP$ -valuation  $v_{LP}$  (a  $K_3$ -valuation  $v_{K_3}$ ) is extended to a valuation  $v_{LP}^*$  (a valuation  $v_{K_3}^*$ ) on  $\mathcal{L}_{\mathcal{P}}^{\mathcal{O}}$  as follows (where  $X$  stands for both  $LP$  and  $K_3$ ):

$$\begin{aligned} 1 \in v_X^*(p) & \text{ iff } 1 \in v_X(p) \\ 0 \in v_X^*(p) & \text{ iff } 0 \in v_X(p) \end{aligned}$$

$$\begin{array}{ll}
1 \in v_X^*(\neg\varphi) & \text{iff } 0 \in v_X^*(\varphi) \\
0 \in v_X^*(\neg\varphi) & \text{iff } 1 \in v_X^*(\varphi) \\
1 \in v_X^*(\varphi \vee \psi) & \text{iff } 1 \in v_X^*(\varphi) \text{ or } 1 \in v_X^*(\psi) \\
0 \in v_X^*(\varphi \vee \psi) & \text{iff } 0 \in v_X^*(\varphi) \text{ and } 0 \in v_X^*(\psi) \\
1 \in v_X^*(\varphi \wedge \psi) & \text{iff } 1 \in v_X^*(\varphi) \text{ and } 1 \in v_X^*(\psi) \\
0 \in v_X^*(\varphi \wedge \psi) & \text{iff } 0 \in v_X^*(\varphi) \text{ or } 0 \in v_X^*(\psi).
\end{array}$$

An argument from a set  $\Pi$  of premises to a conclusion  $\varphi$  is  $X$ -valid (notation:  $\Pi \models_X \varphi$ ), if and only if for each  $X$ -valuation  $v_X$  it holds that if  $1 \in v_X^*(\psi)$  for all  $\psi$  in  $\Pi$ , then  $1 \in v_X^*(\varphi)$ . Note that in  $LP$  the set of designated truth-values for both the premises and the conclusion is  $\{\{1\}, \{0, 1\}\}$ , and that in  $K_3$  the set of designated truth-values for both the premises and the conclusion is  $\{\{1\}\}$ .

## 5. Linear $X$ -translations

Our translation transforms propositional formulas from  $\mathcal{L}_P^O$  into modal formulas from  $\mathcal{ML}$ . Each propositional formula  $\varphi$  has both an  $LP$ -translation  $\top_{LP}(\varphi)$  and a  $K_3$ -translation  $\top_{K_3}(\varphi)$ . Both  $X$  and  $Y$  may stand for  $LP$  and  $K_3$ , but if  $X$  is the one logic, then  $Y$  is the other. In the course of finding a propositional formula's  $X$ -translation, we may have to use  $Y$ -translations of its subformulas.<sup>3</sup> Our translation is given by the following rules:

$$\begin{array}{ll}
\top_{LP}(p) & = \Diamond p \\
\top_{K_3}(p) & = \Box p \\
\top_X(\neg\varphi) & = \neg\top_Y(\varphi) \\
\top_X(\varphi \wedge \psi) & = \top_X(\varphi) \wedge \top_X(\psi) \\
\top_X(\varphi \vee \psi) & = \top_X(\varphi) \vee \top_X(\psi).
\end{array}$$

If  $\Pi$  is a set of propositional formulas,  $\top_X(\Pi)$  abbreviates  $\{\top_X(\psi) : \psi \in \Pi\}$ .

Let us give two examples:  $\top_{LP}(p \wedge \neg p) = \Diamond p \wedge \neg\Box p$  and  $\top_{K_3}(p \wedge \neg p) = \Box p \wedge \neg\Diamond p$ . An  $X$ -translation  $\top_X(\varphi)$  hence leaves a propositional formula  $\varphi$ 's negations, disjunctions, and conjunctions untouched, and only affects its atomic formulas: if an atomic formula  $p$  in  $\varphi$  is under the scope of an even number of negations, then its translation is  $\top_X(p)$ . If  $p$  in  $\varphi$  is under the scope of an odd number of negations, then its translation is  $\top_Y(p)$ . Hence,

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<sup>3</sup>Our translation procedure using two interrelated types of translations of propositional formulas is similar to van Benthem's [16, p. 234] method for translating 'data formulas' into modal formulas.



the length of an  $X$ -translation  $\top_X(\varphi)$  equals the length of  $\varphi$  plus the number of atomic formula occurrences in  $\varphi$ . Therefore,  $X$ -translations are *linear* translations.

Using our  $X$ -translations, we can now state the conditions under which an  $X$ -valuation and an  $S5$ -model are translationally equivalent (again,  $X$  and  $Y$  may stand for both  $LP$  and  $K_3$ , but if  $X$  is the one logic, then  $Y$  is the other):

DEFINITION 5.1. Let  $v_X$  be an  $X$ -valuation on  $\mathcal{P}$  and let  $M$  be an  $S5$ -model. Then  $v_X$  and  $M$  are  $\top_X$ -equivalent, if for all  $\varphi$  in  $\mathcal{L}_{\mathcal{P}}^{\mathcal{Q}}$  it holds that

$$\begin{aligned} 1 \in v_X^*(\varphi) & \text{ iff } M \models_{S5} \top_X(\varphi) \\ 0 \in v_X^*(\varphi) & \text{ iff } M \not\models_{S5} \top_Y(\varphi). \end{aligned}$$

To show that the three-valued logics  $LP$  and  $K_3$  can be embedded in the modal logic  $S5$ , it suffices to show that for every  $S5$ -model there is a  $\top_X$ -equivalent  $X$ -valuation and that for every  $X$ -valuation there is a  $\top_X$ -equivalent  $S5$ -model. The embedding then follows easily.

LEMMA 5.1. For each  $S5$ -model there is a  $\top_X$ -equivalent  $X$ -valuation.

PROOF. Let  $M (= \langle W, V \rangle)$  be an  $S5$ -model. We construct a three-valued  $X$ -valuation  $v_{X_M}$  by stipulating that for all atomic formulas  $p$  in  $\mathcal{P}$

$$\begin{aligned} v_{X_M}(p) &= \{1\} & \text{ iff } V(p) &= W \\ v_{X_M}(p) &= \begin{cases} \{0, 1\} & \text{ if } X = LP \\ \emptyset & \text{ if } X = K_3 \end{cases} & \text{ iff } V(p) \neq \emptyset \text{ and } V(p) \neq W \\ v_{X_M}(p) &= \{0\} & \text{ iff } V(p) &= \emptyset. \end{aligned}$$

It is easy to see that  $v_{X_M}$  is an  $X$ -valuation:  $v_{X_M}$  assigns to each atomic formula  $p$  exactly one of  $X$ 's three truth-values. By simultaneous structural induction and Lemma 3.1 it is easy to show that  $v_{X_M}$  and  $M$  are  $\top_X$ -equivalent. ■

LEMMA 5.2. For each  $X$ -valuation there is a  $\top_X$ -equivalent  $S5$ -model.

PROOF. Let  $v_X$  be an  $X$ -valuation. We construct an  $S5$ -model  $M_{v_X} (= \langle W_{v_X}, V_{v_X} \rangle)$  by stipulating that (1)  $W_{v_X} = \{w, w'\}$  and (2) for all atomic formulas  $p$  in  $\mathcal{P}$

$$\begin{aligned} V_{v_X}(p) &= W_{v_X} & \text{ iff } v_X(p) &= \{1\} \\ V_{v_X}(p) &= \{w\} & \text{ iff } v_X(p) \neq \{0\} \text{ and } v_X(p) \neq \{1\} \\ V_{v_X}(p) &= \emptyset & \text{ iff } v_X(p) &= \{0\}. \end{aligned}$$

Obviously,  $M_{v_X}$  is an  $S5$ -model. An adaption of the inductive proof of Lemma 5.1 shows that  $v_X$  and  $M_{v_X}$  are  $\top_X$ -equivalent. ■

We now prove our theorem that our  $X$ -translations conservatively translate  $LP$  and  $K_3$  into the modal logic  $S5$ :

**THEOREM 5.3.** *Let  $\Pi \subseteq \mathcal{L}_P^\mathcal{O}$ , let  $\varphi \in \mathcal{L}_P^\mathcal{O}$ , and let  $X$  be  $LP$  or  $K_3$ . Then*

$$\Pi \models_X \varphi \quad \text{iff} \quad \top_X(\Pi) \models_{S5} \top_X(\varphi).$$

**PROOF.** Suppose  $\top_X(\Pi) \not\models_{S5} \top_X(\varphi)$ . Then there is an  $S5$ -model  $M$  ( $= \langle W, V \rangle$ ) and a  $w$  in  $W$  such that  $M, w \models_{S5} \top_X(\psi)$  for all  $\psi$  in  $\Pi$  and  $M, w \not\models_{S5} \top_X(\varphi)$ . By Lemma 3.1, it must be that  $M \models_{S5} \top_X(\psi)$  for all  $\psi$  in  $\Pi$  and  $M \not\models_{S5} \top_X(\varphi)$ . By Lemma 5.1, there is an  $X$ -valuation  $v_{X_M}$  such that  $1 \in v_{X_M}^*(\psi)$  for all  $\psi$  in  $\Pi$  and  $1 \notin v_{X_M}^*(\varphi)$ . Therefore  $\Pi \not\models_X \varphi$ .

Suppose that  $\Pi \not\models_X \varphi$ . Then there is an  $X$ -valuation  $v_X$  such that  $1 \in v_X^*(\psi)$  for all  $\psi$  in  $\Pi$  and  $1 \notin v_X^*(\varphi)$ . By Lemma 5.2, there is an  $S5$ -model  $M_{v_X}$  ( $= \langle W_{v_X}, V_{v_X} \rangle$ ) such that  $M_{v_X} \models_{S5} \top_X(\psi)$  for all  $\psi$  in  $\Pi$  and  $M_{v_X} \not\models_{S5} \top_X(\varphi)$ . By Lemma 3.1, it must be that there is a  $w$  in  $W_{v_X}$  such that  $M_{v_X}, w \models_{S5} \top_X(\psi)$  for all  $\psi$  in  $\Pi$  and  $M_{v_X}, w \not\models_{S5} \top_X(\varphi)$ . Therefore  $\top_X(\Pi) \not\models_{S5} \top_X(\varphi)$ . ■

Which technical results and philosophical insights can be gained from these linear conservative translations of  $LP$  and  $K_3$  into  $S5$ ? As for the technical results, our theorem carries over some properties of modal logic to  $LP$  and  $K_3$ . Let us give one example from complexity theory. From the fact that the complexity of the satisfiability problem for  $S5$  is NP-complete [12] and the fact that our  $X$ -translations of  $LP$  and  $K_3$  are linear, it follows immediately that the complexity of  $LP$ -validity and of  $K_3$ -validity is in co-NP.<sup>4</sup>

As for the philosophical insights, our theorem contributes to the debate on the ‘proper’ interpretation of  $LP$ . It shows that if we want a formal semantics for  $LP$ , we do not need to endorse any form of dialetheism whatever (dialetheism is the claim that there are true contradictions):  $LP$  has a classical, two-valued semantics and therefore need not be interpreted under the assumption of dialetheism.<sup>5</sup> Although negation, disjunction, and conjunction in  $LP$  can all be rendered classically, atomic formulas in  $LP$  may have different meanings, depending on whether they are under the scope of an even or an odd number of negations. It is this holistic interpretation of atomic formulas in  $LP$  that is the fundamental difference between classical

<sup>4</sup>Cadoli and Schaerf [4] show this for  $LP$ .

<sup>5</sup>Brown [2], Lewis [13], and Cobreros et al. [6] also argue that  $LP$  can be brought within the range of classical logic. Their approach differs from ours in that, in order to make sense of  $LP$ , they tentatively accept a weak form of  $LP$ ’s central tenet, that some formulas can be both true and false.

logic and  $LP$ . Priest's [15] arguments to the contrary notwithstanding, we showed that negation in  $LP$  can be understood as a negation that "satisfies all the proof-theoretic rules of classical negation" [15, p. 203], *provided* we determine the meaning of  $LP$ 's atomic formulas by way of our  $X$ -translation of  $LP$  into  $S5$ . As a consequence, Priest tells only half the story, as far as  $LP$ 's "theoretical account of negation" is concerned, when he says: "Dialetheic logic, unlike modal logic, does [...] provide a genuine rival theory to that provided by classical logic" [15, p. 210]. He might just as well have said that dialetheic logic provides a 'rival theory' of the interpretation of atomic formulas.

## 6. Conclusion

Our Translation Manual can easily be adapted to handle  $n$ -valued logics as long as we have a modal logic in which there are  $n$  mutually exclusive and jointly exhaustive possibilities for each atomic formula  $p$ . For example, every truth-functional four-valued propositional logic can be conservatively translated into a modal logic interpreted on minimal models  $M = \langle W, N, V \rangle$  where  $N$  is a universal neighbourhood such that for each atomic formula  $p$  it holds that (1) either  $\llbracket p \rrbracket \in N$  or  $\llbracket p \rrbracket \notin N$ , and (2) either  $\neg\llbracket p \rrbracket \in N$  or  $\neg\llbracket p \rrbracket \notin N$ .<sup>6</sup> Our adapted Translation Manual then maps the four possible truth-values  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$ , and  $\mathbf{d}$  of any atomic formula  $p$  to these four possibilities:  $\mathbf{a}(p) = \Box p \wedge \Diamond p$ ,  $\mathbf{b}(p) = \Box p \wedge \neg\Diamond p$ ,  $\mathbf{c}(p) = \neg\Box p \wedge \Diamond p$ , and  $\mathbf{d}(p) = \neg\Box p \wedge \neg\Diamond p$ . The adapted Translation Manual's handling of complex formulas and the subsequent proof that the adapted Translation Manual conservatively translates any truth-functional four-valued propositional logic into minimal modal logic closely follow the lines of our treatment of three-valued logics.

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<sup>6</sup>See [5, Chapter 7] for an account of minimal models for modal logics.

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