Prefinitely axiomatizable modal and intermediate logics

Marcus Kracht *

II. Math. Institut
Arnimallee 3
1000 Berlin 33

*This work was partly supported by the project NF 102/62 - 356 ('Structural and Semantic Parallels in Natural Language and Programming Language') funded by the Netherlands Organization for the Advancement of Research (N. W. O.).

Abstract. A logic Λ bounds a property P if all proper extensions of Λ have P while Λ itself does not. We construct logics bounding finite axiomatizability and logics bounding finite model property in the lattice of intermediate logics and the lattice of normal extensions of **K4.3**.

Key Words. Modal Logics, Intermediate Logics, Bounded Properties, Finite Axiomatizability, Finite Model Property.

Mathematics Subject Classification. 03 B 45, 03 B 55.

0. Introduction. In [Schumm, 1981] a logic is said to **bound** a property P if all proper extensions have P while the logic itself lacks P. If all extensions of a logic have P then we say that this logic has P **essentially**. For some properties such as being tabular bounding logics have been found; for other their existence has been proved without a particular example being known. We will construct logics bounding various properties in the lattice of normal modal logics and in the lattice of intermediate logics. In [Schumm, 1981] the prime examples of modal logics bounding certain properties are non-normal logics and thus our results will be a definite improvement on this. We concentrate on properties of finite axiomatizability and finite model property (fmp). The first to note that finite axiomatizability is a bounded property was [Rautenberg, 1979]. Shortly after that [Wroński, 1979] constructed a logic bounding finite axiomatisability which is based on a 3-valued matrix. The case of normal modal logics was still open. Our first example is an extension of $\mathsf{K4.3}$, and it bounds fmp as well; we have thus proved that not all extensions of $\mathsf{K4.3}$ have fmp, unlike the case of $\mathsf{S4.3}$. We will also construct an intermediate logic bounding finite axiomatizability thus solving a problem posed in [Rautenberg, 1979] and also logics bounding fmp and other completeness properties. In order to prove these results a number of theorems had to be established concerning the modal theory of infinite frames and eliminability of points in frames. I guess these auxiliary results have made the investigation into these rather obscure logics worthwile. I wish to thank first of all Vladimir Rybakov for his extreme care in checking this manuscript and Dick de Jongh for reading parts of an earlier version. If errors have remained, it is of course my own responsibility. Wolfgang Rautenberg has helped me greatly with his knowledge of the field. Furthermore, I wish to thank Sun Ra, Abdus Salam and the Kageyama School of Go for the inspiration.

1. Notation. In this essay all logics are transitive, that is, extensions of $\mathsf{K4}$. We will assume familiarity with the notions of modal logic and we will keep our notation standard. A **frame** is as usual a pair $f = \langle f, \lhd \rangle$ where $\lhd$ is a binary relation on f. No distinction is made between a frame and its set of worlds. We write an ordinary arrow $p : f \to g$ if p is a p-morphism. If in addition p is injective we denote this by $p : f \rightarrowtail g$ and if p is a surjective p-morphism we denote this by $p : f \twoheadrightarrow g$. A frame g is called an **extract** of f if g is the p-morphic image of a generated subframe of f. We say that f **omits** g if g is not an extract of f. If p embeds f as a subframe in the sense of [Fine, 1985] then we write $f \overset{\subset}{\to} g$. If f is a transitive frame we call t a **weak successor** of s if either $s = t$ or $s \lhd t$. A successor is called **strong** if it is not a weak successor. A frame f is **one-generated** if there is a point $s \in f$ such that every point $t \in f$ is a **weak successor** of s. All logics considered in this essay will be of finite width; to be more precise, they will all be of width 2. Logics of finite width have been defined and closely studied in [Fine, 1974b]. They are known to be complete with respect to Kripke-frames; moreover, the Kripke-frames can be chosen such that the function assigning depth to points can be extended over the whole frame (see [Kracht, 1991]). In general, the depth of a point is therefore an ordinal number, possibly infinite. If $s \in f$ is a point of depth α (we write $dp_f(s) = \alpha$) and t a strong successor then $dp_f(t) < \alpha$. If t is only a weak successor then $dp_f(t) \leq \alpha$. Taking the usual definition of an ordinal number as the set of all smaller ordinals, this allows us to define the depth via $dp_f(s) = \{dp_f(t) : s \lhd t \ntriangleleft s\}$; note that by this definition terminal points have depth 0 but this is rather welcome for our purposes. For a frame f we let $dp(f) = \{dp_f(s) : s \in f\}$, and so a one-point frame is of depth 1.

For axiomatizing logics we use two tools. That of a **splitting** ([Rautenberg, 1980] and [Kracht, 1990]) and that of a FINE-splitting ([Fine, 1985]). If Λ is a logic containing $\mathsf{K4}$ and f a finite, one-generated frame we denote by Λ/f the logic obtained by splitting f from Λ—which is the smallest logic Θ containing Λ such that $f \notin Md(\Theta)$—and by $\Lambda\{f\}$ the smallest logic containing Λ and the subframe logic $\mathsf{K4}_f$. (We are not using the subscript notation of [Fine, 1985] here in order to avoid small print.) Extensions considered here are usually of the kind $\mathsf{K4}M/N$ where M and N are (possibly infinite) sets of finite, one-generated frames. $\mathsf{K4}M/N$ simply denotes the splitting of the subframe logic $\mathsf{K4}_M$ by the frames of N.

For a property P a logic is said to have P **essentially** if all extensions have P. A logic is said to **bound** P or to be **pre-**P if all proper extensions have P but the logic itself is not. For finite model property and finite axiomatizability two important facts can be proved. If Λ_1, Λ_2 are transitive logics which are essentially finitely axiomatizable (have fmp essentially) then $\Lambda_1 \cap \Lambda_2$ is essentially finitely axiomatisable (has fmp essentially). Both are seen using the next lemma. A property P of logics is said to be **intersective** if from the fact that Λ_1 and Λ_2 both have P we can infer that $\Lambda_1 \cap \Lambda_2$ has P as well.

Lemma 1 *Suppose that both Λ_1 and Λ_2 have P essentially and that P is intersective. Then $\Lambda_1 \cap \Lambda_2$ has P essentially. In other words, to have P essentially is intersective as well.*

Proof. Suppose that $\Theta \supseteq \Lambda_1 \cap \Lambda_2$. Then $(\Theta \cup \Lambda_1) \cap (\Theta \cup \Lambda_2) = \Theta \cup (\Lambda_1 \cap \Lambda_2) = \Theta$, by distributivity. By hypothesis, both $\Theta \cup \Lambda_1$ and $\Theta \cup \Lambda_2$ have P and since P is intersective, Θ has P as well. $\square$

By this lemma, to have fmp essentially is intersective. Moreover, to be finitely axiomatisable is intersective if we concentrate on extensions of $\mathsf{K4}$; thus to be essentially finitely axiomatisable is intersective for transitive logics.

Some particular notations for frames will also be useful. A reflexive point is denoted by $\bullet$ and an irreflexive point by x. The box $\square$ stands for either $\bullet$ or x.

2. Strings and Decompositions. In most cases it is not easy to decide whether a particular frame can be mapped p-morphically onto another and to see that a given map is a p-morphism is mostly also not straightforward. The story of this paper had to be rewritten a number of times because a certain p-morphism has been overlooked. In order to have some more rigorous methods for checking, two tools will be introduced here. The first is the decomposition of p-morphisms. Call a p-morphism $\pi : f \twoheadrightarrow g$ **minimal** if it is not an isomorphism and for every factorization $f \twoheadrightarrow h \twoheadrightarrow g$ h is isomorphic either to g or to f. Likewise a p-morphism $\iota : f \rightarrowtail g$ which is not an isomorphism is **minimal** if for every factorization $f \rightarrowtail h \rightarrowtail g$ h is isomorphic to either g or f. Here is a lemma that shows the importance of minimal morphisms in our context. It is an adaptation of a result originally found in [de Jongh and Troelstra, 1966] and rediscovered in [Bellissima, 1988].

Lemma 2 *Suppose that f, g are Grz-frames without ascending chains. Then $\pi : f \twoheadrightarrow g$ is minimal iff there is exactly one nontrivial fibre $\pi^{-1}(x)$ and it contains two points. $\iota : f \rightarrowtail g$ is minimal iff $\sharp(g - \iota(f)) = 1$.*

Proof. In each case the conditions on minimality are sufficient. That they are also necessary will be shown. Let $\pi : f \twoheadrightarrow g$ be minimal. Then take a point s of minimal depth such that there is a $t \neq s$ with $\pi(t) = \pi(s)$. If both are of equal depth then the map ρ identifying just s with t is a p-morphism; for if $\rho(s) \lhd \rho(x)$ then either $\rho(x) \lhd \rho(s)$ in which case $x = s$ and so $s \lhd x$ or $\rho(x) \ntriangleleft \rho(s)$ in which case $\pi^{-1}(x) = \{x'\}$ since $\pi^{-1}(x)$ must contain points of lesser depth than s (and t). But π was a p-morphism and so $t \lhd x'$ as well. Similarly for the remaining cases of $\rho(x) \lhd \rho(y)$. If, however, the depth of t is greater than the depth of s then take an immediate predecessor x of s. By the same methods show that the map ρ identifying x with s is a p-morphism. If π was not minimal, then it factors through ρ.

Now if $\iota : f \rightarrowtail g$ is minimal, let $M = g - \iota(f)$. Since M has no ascending chains there is a maximal point $m \in M$. Now $h = \iota(f) \cup \{m\}$ is a generated subframe of g and ι clearly factors through the embedding $h \rightarrowtail g$. $\square$

It is clear that between such frames p-morphisms are decomposable into the elementary operations of adding a point, conflating two points or dropping a point. (The latter two are not the same.)

The next tool is that of a linear decomposition of frames. If f and g are frames, let $f \oslash g$ denote the frame obtained by putting f before g. To be precise, $f \oslash g = \langle f + g, \lhd_f \cup \lhd_g \cup f \times g\rangle$ with $f + g$ being the disjoint union. Any sequence $\oslash_{i\in\alpha} f_i$ with $\alpha \in Ord^{op}$, the converse well-orders, is called a **string** and the f_i are the **segments**. Segmentation plays a role in the decomposition of p-morphisms. The operation $\oslash$ produces chains of frames, while the disjoint union $\oplus$ produces what is sometimes called an **anti-chain**.

Lemma 3 *$\pi : f \oslash g \twoheadrightarrow d$ is a p-morphism iff $\pi \restriction f$ and $\pi \restriction g$ are p-morphisms. $\iota : f \oslash g \rightarrowtail d$ is a p-morphism iff $\iota \restriction g$ is an isomorphism and $\iota \restriction f$ a p-morphism.* □

Moreover, if $f \oslash g \rightarrowtail d$ then $d \cong f' \oslash g$ for some f'. For surjections $\twoheadrightarrow$ this need not hold. But for minimal p-morphisms we can get a clear picture of the possibilities. Let $\pi : f \oslash g \twoheadrightarrow d$ a minimal p-morphism such that the nontrivial fibre $\pi^{-1}(x)$ is not contained in either f or g. Then, as $\pi^{-1}(x)$ has two points, s, t say, one of them is in f the other in g. Let then $s \in f, t \in g$. We have $s \lhd t \not\lhd s$. It then follows that g must be one-generated and therefore $g \cong \bullet \oslash g'$. Thus π may only conflate and end point of f with the generator of g. If $\oslash_{i\in\alpha} f_i$ is a **maximal decomposition** if every f_i cannot be decomposed into two segments, then the following holds.

Lemma 4 *Suppose that $f = \oslash_{i\in\alpha} f_i$ is a maximal decomposition and $\pi : f \twoheadrightarrow g$ a minimal p-morphism. Then π is either of type $\pi_i : f_i \twoheadrightarrow f_i'$ or of type $\pi_i : f_i \oslash \bullet \twoheadrightarrow f_i' \oslash \bullet$. In the first case π is said to be **decomposable**. In the second case we call π a **fusion**.* □

Finally a word about subframe axioms. In [Fine, 1985] it is shown that for most frames f the subframe axiom for f reduces to a non-embeddability condition for a set F of frames. In the special case of axioms that we are considering, this set reduces to $\{f\}$. Namely, these are frames of the type $\bullet \oslash (\phi_m \oplus \phi_n)$ where ϕ_n, ϕ_m are linear. Moreover, if $\bullet \oslash g$ is indecomposable non-embeddability of $\bullet \oslash g$ into a string $\bullet \oslash \Lambda$ can be checked segment-by-segment by looking whether g embeds into a segment of Λ; again, our frames have this property.

3. Homogenization and dropping points. We will make heavy use of the *homogenization technique* as developed in [Kracht, 1991]. The ideas, which were extracted from

[Fine, 1974b] and [Fine, 1985], are as follows. Given the sentence letters $\mathbb{P}_n = \{p_i : i \in n\}$ and a $k \in \omega$ let $Fm(k, n)$ denote the set of formulas based on $\mathbb{P}_n$ and of modal degree $\leq k$. $Fm(k, n)$ is a boolean algebra whose atom set we denote by $\mathrm{At}(k, n)$. For the rest of this essay, k and n will remain fixed throughout and $P \in Fm(k, n)$. Suppose now that there is a model $\langle g, \gamma, s\rangle \models P$ with $\mathrm{dom}(\gamma) = \mathbb{P}_n$. Then, as it was noted in [Fine, 1974b] and [Fine, 1985], a much simpler model can be constructed for P. Call $u \in g$ **A-maximal** in $\langle g, \gamma\rangle$ if $\langle g, \gamma, u\rangle \models A$ and for all $x \rhd u$ such that $\langle g, \gamma, x\rangle \models A$ we have $x \lhd u$. Call u **maximal** if it is A-maximal for some atom $A \in \mathrm{At}(k, n)$. This A is called the **atom** of u in $\langle g, \gamma\rangle$. Since we are working with frames without strictly ascending chains of points we know that for every $x \in g$ such that $\langle g, \gamma, x\rangle \models A$ there exists a maximal weak successor x^μ with atom A. There is now an important observation on 'dropping' points from a model. Let g^μ denote the subframe of maximal points in g, let h be a subframe of g such that $g^\mu \overset{\subset}{\to} h \overset{\subset}{\to} g$; then by induction it can be shown that for $P \in Fm(k, n)$

$$\langle g, \gamma, s\rangle \models P \Leftrightarrow \langle h, \gamma, s^\mu\rangle \models P,$$

where $\gamma : \mathbb{P}_n \to 2^h$ is the natural restriction of $\gamma : \mathbb{P}_n \to 2^g$ (see [Kracht, 1991] for a proof). Thus we can drop any set of non-maximal points from a model for P and still we retain a model for P. Finally, if g is one-generated and a frame for $\mathsf{Grz}\{wd(\ell)\}$ then $\sharp g^\mu < \ell \times \sharp\, \mathrm{At}(k, n)$. This is so because if $x^\mu \lhd y^\mu$ then the atom of x^μ must be different from the atom of y^μ. Therefore a strictly ascending chain in g^μ contains at most $\sharp\, \mathrm{At}(k, n)$ points. Moreover, for every A there can be at most ℓ maximal points with atom A. Maximal points can be distributed quite arbitrarily in a frame. However, note that points of depth 0 are always maximal. This is quite worth remembering.

The method of homogenization developed in [Kracht, 1991] is not sophisticated enough to yield the results we need. What is called for in our context is a result which allows to 'move' the subframe of maximal points into a certain position. There is a rather simple theorem telling us when this can be achieved. Let $g^\sigma \overset{\subset}{\to} g$ be a subframe of g. We call g^σ **m-compatible** with g^μ if there exists an isomorphism $\iota : g^\mu \to g^\sigma$ such that for every $x \in g$ there is a $\widetilde{x} \in g$ such that for the sets $x^S := \{y \in g^\sigma : x \lhd y\}$ and $\widetilde{x}^M := \{y \in g^\mu : \widetilde{x} \lhd y\}$ we have $x^S = \iota[\widetilde{x}^M]$. We define $\widetilde{x}$ on g by letting $\widetilde{x} = \iota^{-1}(x)$ if $x \in g^\sigma$ and else choose $\widetilde{x}$ such that $x^S = \iota(\widetilde{x}^M)$. Next we define $x^\sigma := \iota(\widetilde{x}^\mu)$. The next theorem tells us that there is a valuation $\widetilde{\gamma}$ such that g^σ is the subframe of maximal points of $\langle g, \widetilde{\gamma}\rangle$ and that x and x^σ have the same atom in $\langle g, \widetilde{\gamma}\rangle$.

Theorem 5 *Let g^σ be m-compatible with g^μ. Then there exists a valuation $\widetilde{\gamma}$ such that for all $P \in Fm(k,n)$*

(†) $\quad x \in \widetilde{\gamma}(P) \Leftrightarrow \widetilde{x} \in \gamma(P)$

Consequently, g^σ is the subframe of maximal points of $\langle g, \widetilde{\gamma}\rangle$ and x and x^σ have the same atom.

Proof. Define $\widetilde{\gamma}$ by $x \in \widetilde{\gamma}(p) \Leftrightarrow \widetilde{x} \in \gamma(p)$ for $p \in \mathbb{P}_n$. (†) is now proved by induction. It follows that (‡) : $x \in \widetilde{\gamma}(P) \Leftrightarrow x^\sigma \in \widetilde{\gamma}(P)$. For $x \in \widetilde{\gamma}(P) \Leftrightarrow \widetilde{x} \in \gamma(P) \Leftrightarrow \widetilde{x}^\mu \in \gamma(P) \Leftrightarrow \widetilde{x^\sigma} \in \gamma(P)$(since $\widetilde{x^\sigma} = \widetilde{y}^\mu$) $\Leftrightarrow x^\sigma \in \widetilde{\gamma}(P)$. Now for the proof of (†). The only critical step is $P = \Diamond Q$. If $x \in \widetilde{\gamma}(\Diamond Q)$ then $y \in \widetilde{\gamma}(Q)$ for some $y \rhd x$. By IH, $\widetilde{y} \in \gamma(Q)$ and $\widetilde{y}^\mu \in \gamma(Q)$ and so $\widetilde{x} \in \gamma(\Diamond Q)$ since $\widetilde{x} \lhd \widetilde{y}^\mu$. (This is so because $y^\sigma \in x^S$ and thus $\widetilde{y}^\mu = \widetilde{y^\sigma} \in \iota^{-1}[x^S] = \widetilde{x}^M$.)

Conversely, assume $\widetilde{x} \in \gamma(\Diamond Q)$. Then $y \in \gamma(Q)$ for some $y \rhd \widetilde{x}$. We can assume that $y = y^\mu$ and so $y = \widetilde{z}^\mu$ for $z = \iota(y)$. By IH, $z \in \widetilde{\gamma}(Q)$ since $\widetilde{z} = y$. But $\widetilde{x} \lhd \widetilde{z} = \widetilde{z}^\mu$ and so $\widetilde{z}^\mu \in \widetilde{x}^M$ from which $z \in x^S$ and consequently $x \lhd z$. Thus $x \in \gamma(\Diamond Q)$.

By (‡), x and x^σ have the same atom in $\langle g, \widetilde{\gamma}\rangle$. To see that x^σ is maximal, assume that $x^\sigma \lhd y$ and that both have the same atom in $\langle g, \widetilde{\gamma}\rangle$. Then $x^\sigma \lhd y^\sigma$ and so $\widetilde{x}^\mu \lhd \widetilde{y}^\mu$. Since $\widetilde{x}^\mu$ and $\widetilde{y}^\mu$ have the same atom in $\langle g, \gamma\rangle$, $\widetilde{y}^\mu \lhd \widetilde{x}^\mu$ and so $y^\sigma \lhd x^\sigma$ from which $y \lhd x^\sigma$. $\square$

Theorem 5 has consequences worth reflecting on. First, if we have a model, then this theorem says that we can drop some or all non-maximal points with impunity. However, sometimes dropping points has to be used with care. For if g is a frame for a logic Λ it is not guaranteed that dropping points will yield another frame for Λ. Thus call dropping M from g **safe** if $\mathrm{Th}(g - M) = \mathrm{Th}(g)$; moreover, call dropping M **supersafe** if for every f, h $\mathrm{Th}(f \otimes (g - M) \otimes h) = \mathrm{Th}(f \otimes g \otimes h)$. If $g - M$ is an extract of g, dropping M is safe and if $g - M$ is a p-morphic image of g dropping M is supersafe.

In addition to dropping from a model there is the possibility of dropping from a frame analoguous to [Fine, 1974b]. But the difference is that we can actually give some explicit criteria for when points can be dropped. Let us call a set $N \subset g$ **eliminable** if for every finite subframe $g^\mu \subseteq g$ there is an m-compatible g^σ such that no point of N is a point of g^σ. Then any model for a formula P on g can be made into a model of P on $g - N$. (For by eliminability, for any model for P we can assume that no maximal point is in N since we have finite width and no ascending chains; but N can be dropped from the model.) Hence $\mathrm{Th}(g - N) \subseteq \mathrm{Th}(g)$. However, the following theorems demonstrate that the situation is as good as it can be.

Theorem 6 *Suppose that $N \subset g$ is a set of eliminable points. Then dropping N is safe.*

Proof. We need to show that $\mathrm{Th}(g-N) \supseteq \mathrm{Th}(g)$. Thus let P be consistent with $\mathrm{Th}(g-N)$. Then there is a model $\langle g-N, \gamma, s\rangle \models P$. We will find a δ such that $\langle g, \delta, s\rangle \models P$. To this end let $(-)^{\mu}$ be as usual the function assigning to each $x \in g - N$ a maximal weak successor with the same atom (with respect to γ). Now extend $(-)^{\mu}$ to a function $(-)^{\nu} : g \to g$ by choosing for each $x \in N$ a successor x^{ν} which is also maximal, that is, $x^{\nu\mu} = x^{\nu}$ (by which also $x^{\nu\nu} = x^{\nu}$). (For example, there always is a successor of depth 0 that is maximal.) Now define $x \in \delta(p) \Leftrightarrow x^{\nu} \in \gamma(p)$. We will show by induction on $Q \in Fm(n,k)$ that $x \in \delta(Q) \Leftrightarrow x^{\nu} \in \gamma(Q)$. In particular, it follows that if $x \notin N$, $x \in \gamma(Q) \Leftrightarrow x^{\mu} \in \gamma(Q)$ (by definition of $(-)^{\mu}$) $\Leftrightarrow x^{\nu} \in \gamma(Q)$ (since $x^{\mu} = x^{\nu}$) $\Leftrightarrow x \in \delta(Q)$. After having done the induction we have that $\langle g, \delta, s\rangle \models P$ since $s \in g - N$ and $\langle g - N, \gamma, s\rangle \models P$.

In the induction there is only one critical case, that of $\Diamond$. Let thus $Q = \Diamond R$. If $x \in \delta(\Diamond R)$ then for some successor $y \in \delta(R)$. By IH, $y^{\nu} \in \gamma(R)$ and so $y^{\nu\mu} \in \gamma(R)$. $y^{\nu\mu}$ is a weak successor of y and so $x \lhd y^{\nu\mu}$ from which $x \in \gamma(\Diamond R)$, as $x \lhd y^{\nu}$. Conversely, if $x^{\nu} \in \gamma(\Diamond R)$ then for some successor $y \in \gamma(R)$ from which $y \notin N$ and hence by IH $y^{\mu} \in \gamma(R)$ and so $y^{\nu} \in \gamma(R)$ from which again by IH $y \in \delta(R)$. Now as $x \lhd y$, $x \in \delta(\Diamond R)$. $\Box$

Lemma 7 *Let N be eliminable in g. Then N is eliminable in $f \ogreaterthan g \ogreaterthan h$ for every pair of frames f, h.*

Proof. Suppose $N \subset f \ogreaterthan g \ogreaterthan h$. Let $N_f = N \cap f, N_g = N \cap g, N_h = N \cap h$. By assumption on g, there is a N'_g such that $N'_g \cap M = \emptyset$ and N'_g is m-compatible with N_g in g. We have to show now that in that case N is m-compatible in $f \ogreaterthan g \ogreaterthan h$ with $N' = N_f \cup N'_g \cup N_h$.

To start, we have an isomorphism $\iota_g : N_g \to N'_g$ such that for every $x \in g$ there is a $\widetilde{x}$ so that $x^{N'_g} = \iota_g[\widetilde{x}^{N_g}]$. Now let $\iota : N \to N'$ be defined by $\iota(x) = \iota_g(x)$ if $x \in g$ and $\iota(x) = x$ otherwise. Now define $\widehat{x}$ by $\widehat{x} = x$ if $x \in f \cup h$ and $\widehat{x} = \widetilde{x}$ if $x \in g$. Then ι is first of all an isomorphism as is readily checked; moreover, if $x \in f \ogreaterthan g \ogreaterthan h$ then $\iota[\widehat{x}^{N'}] = x^{N}$. To see this, note three cases. **Case 1:** $x \in h$. Then $\widehat{x} = x$. And so $\iota[\widehat{x}^{N`}] = \iota[x^{N'}] = x^{N'} = x^{N}$. **Case 2:** $x \in g$. Then $\iota[\widehat{x}^{N'}] = \iota[\widetilde{x}^{N'}] = \iota_g[\widetilde{x}^{N'} \cap g] \cup \iota_h[\widetilde{x}^{N'} \cap h] = x^{N} \cap g. \cup .x^{N} \cap h = x^{N}$. **Case 3:** $x \in f$. Then $\iota[x^{N'}] = \iota_f[x^{N'} \cap f] \cup \iota_g[x^{N'} \cap g] \cup \iota_h[x^{N'} \cap h] = x^{N'} \cap f. \cup .\iota_g[x^{N'} \cap g]. \cup .x^{N'} \cap h = x^{N} \cap f. \cup .x^{N} \cap g. \cup x^{N} \cap h = x^{N}$. $\Box$

Corollary 8 *Let N be eliminable in g. Then dropping N is supersafe.* □

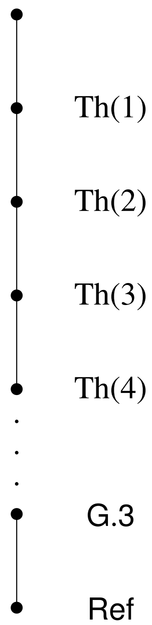
4. A logic bounding finite axiomatizability. If α is a converse well-order, that is $\alpha^{op} \in Ord$, α is a isomorphic to the string $\bigotimes_{i \in \alpha} \mathsf{x}$. (Recall here that x stands by convention for the one-element irreflexive frame.) The logic of all converse well-orders is $\mathsf{G.3}$. Every proper extension of $\mathsf{G.3}$ is finitely axiomatizable and tabular while $\mathsf{G.3}$ has fmp and is finitely axiomatisable. Although this also follows from the subframe theorem of [Fine, 1985] we will give a proof using the dropping technique to make the reader familiar with it. Let α be an infinite converse well-order, that is, $\alpha^{op} \in Ord$. Take any finite $\alpha^{\mu} \subseteq \alpha$. Then α^{μ} is a finite well-order of cardinality k. Take as α^{σ} the points of depth $\leq k$ in α. This subframe is m-compatible with α^{μ} iff $0 \in \alpha^{\mu}$. But if α^{μ} is the subframe of maximal points, $0 \in \alpha^{\mu}$ is guaranteed. By consequence, all points of depth $\geq \omega$ can be dropped. Thus for infinite α, $\mathrm{Th}(\alpha) = \mathrm{Th}(\omega^{op})$. Now we are studying the logic of the frames $\bullet \bigotimes \alpha^{op}$, $\alpha \in Ord$. Let $\mathsf{K4.3}^{\bullet} = \bigcap \langle \mathrm{Th}(\bullet \bigotimes \alpha^{op}) : \alpha \in Ord \rangle$. $\mathsf{K4.3}^{\bullet}$ is a subframe logic; namely, if we add to $\mathsf{K4.3}$ the three following axioms we get $\mathsf{K4.3}^{\bullet}$. (Note that □ matches with either • or x.)

The first frame excludes that we have a non-initial reflexive point, while the second excludes proper clusters. Again by [Fine, 1985] $\mathsf{K4.3}^{\bullet}$ has fmp—a fact which the dropping technique can also show nicely. Now it is easy to show that $\mathsf{K4.3}^{\bullet}$ has $2^{\aleph_0}$ extensions and therefore not all extensions can be decidable. Just consider from the powerset of ω into the lattice of normal extensions of $\mathsf{K4.3}^{\bullet}$ denoted by $\mathcal{E}\mathsf{K4.3}^{\bullet}$ the map $\iota : \mathcal{P}(\omega) \to \mathcal{E}\mathbf{K4.3}^{\bullet}$ defined by $\iota : N \mapsto \mathsf{K4.3}^{\bullet}/\{\bullet \bigotimes \alpha^{op} : \alpha \in N\}$. Since for finite α, β $\bullet \bigotimes \beta^{op}$ is not an extract of $\bullet \bigotimes \alpha^{op}$ unless $\alpha = \beta$ the logics $\mathrm{Th}(\bullet \bigotimes \alpha^{op})$ and $\mathrm{Th}(\bullet \bigotimes \beta^{op})$ are incomparable for different numbers; therefore, ι is injective and so $\sharp \mathcal{E}\mathsf{K4.3}^{\bullet} = 2^{\aleph_0}$. (See [Fine, 1974a] for a similar argument.) Now consider the logic $\mathsf{Ref} := \iota(\omega)$. This logic is not finitely axiomatisable; for we have an axiomatization by infinitely many axioms none of which is dispensable. On the other hand, Ref has the same finite models as $\mathsf{G.3}$. Consequently, as $\mathsf{G.3}$ is finitely axiomatisable the two must be different. So Ref lacks fmp. This proves first of all that $\mathsf{K4.3}$ does not possess fmp essentially. But there is more. Using Theorem 6 we can show that for infinite α, $\mathrm{Th}(\bullet \bigotimes \alpha^{op}) = \mathrm{Th}(\bullet \bigotimes \omega^{op})$ by showing that all points of depth $> \omega$ are eliminable except for the reflexive point. $\mathsf{K4.3}^{\bullet} = \bigcap \langle \mathrm{Th}(\bullet \bigotimes \alpha^{op}) : \alpha \in Ord \rangle = \bigcap \langle \mathrm{Th}(\bullet \bigotimes \alpha^{op}) : \alpha \in \omega + 1 \rangle$ and so we have that

Ref = Th($\bullet \ogreaterthan \omega^{op}$) since we have eliminated the finite $\bullet \ogreaterthan \alpha^{op}$. Then no proper extension of Ref can have $\bullet \ogreaterthan \omega^{op}$ among its models, nor any other $\bullet \ogreaterthan \alpha^{op}$. Thus every proper extension includes the logic of the converse ordinals, which is G.3.

Theorem 9 *The logic* Ref = $\mathsf{K4.3}^\bullet/\{\bullet \ogreaterthan \alpha^{op} : \alpha \in \omega\}$ *bounds finite axiomatizability and finite model property. Moreover,* Ref = Th($\bullet \ogreaterthan \omega^{op}$). $\Box$

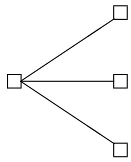
The extension lattice of Ref looks as follows.



This can be interpreted as a splitting result as follows. We observe that $\bullet \ogreaterthan \omega^{op}$ is a Ref-frame and therefore $\Diamond p \wedge \Box(\neg p \vee \Diamond p)$ is consistent with Ref. Hence Ref $\nvdash \Diamond p \to \Diamond(p \wedge \Box\neg p)$, that is, Ref $\nvdash \Box(\Box p \to p) \to \Box p$. But now, since Th($\bullet \ogreaterthan \alpha$) = Th($\bullet \ogreaterthan \beta$) for all infinite converse ordinals, we have that Ref = G.3 $\cap$ Th($\bullet \ogreaterthan \omega^{op}$) = Th($\bullet \ogreaterthan \omega^{op}$). Thus G.3 = Th($\bullet \ogreaterthan \omega^{op}$)/ $\bullet \ogreaterthan \omega^{op}$. Indeed, the algebra of finite and cofinite sets of $\bullet \ogreaterthan \omega^{op}$ is finitely presented by factoring out the equation $a \to \Diamond a = 1$ from the freely one-generated algebra; in symbols, $\mathbf{A}^f(\bullet \ogreaterthan \omega^{op}) \cong \mathcal{F}_{\mathsf{Ref}}(a)/\{a \to \Diamond a\}$. We thus obtain that G.3 = Ref($\Box(\Box p \to p) \to \Box p$) (see [Kracht, 1990]). It is striking that in the presence of this axiom we can forget almost all other axioms; for we have G.3 = $\mathsf{K4.3}^\bullet(\Box(\Box p \to p) \to \Box p)$. So while Th($\bullet \ogreaterthan \omega^{op}$) is obtained by splitting out countably many frames and yet is not finitely axiomatisable, G.3 is obtained by splitting just one more frame and it is finitely axiomatisable. The paradox is quickly resolved if we remind ourselves of the following facts. If N is a finite subset of ω then $\iota(N)$ can be shown to have the finite model property and therefore $\mathsf{A}^f(\bullet \otimes \omega^{op})$ is not finitely presentable and does therefore not induce a splitting. However, as soon as N is cofinite, $\iota(N)$ contains sufficiently many axioms to make $\mathbf{A}^f(\bullet \otimes \omega^{op})$ finitely presentable.

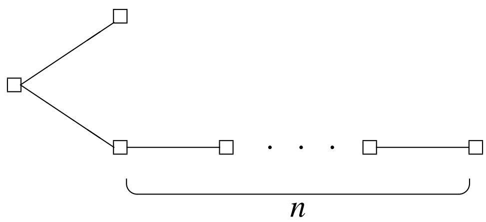
5. The intergalactic research program. The intermediate case is by far more complex. Obviously, one cannot use the example of a linear logic since all extensions of $\mathsf{Grz.3}$ have the finite model property. But we need not go very far beyond that. The logic we are looking for will be of width 2.

$wd(2)$



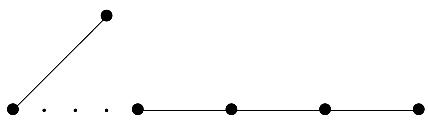
(We are now omitting the arrows; they are assumed to go from left to right.) Frames for $\mathsf{Grz}\{wd(2)\}$ which are one-generated have at most two points of given depth. We call the set $s\ell_f(\alpha) = \{s \in f : dp_f(s) = \alpha\}$ the α**-slice** of f. Following [Kracht, 1991] we say that a logic containing $\mathsf{K4}$ is of **tightness n** if is contains the logic $\mathsf{K4}\{ti(n)\}$ where $ti(n)$ is the following set of frames. (Never mind the confusion between a frame and a set of frames; above $\mathsf{S4}$ this set reduces anyway to a singleton which will then carry the name $ti(n)$as well.)

$ti(n)$



Alternatively, Λ is of tightness n if for every one generated frame f for a point s there does not exist a chain of n points incomparable to a successor of s. For example, $\Lambda \supseteq \mathsf{K4}$ is of tightness 1 iff no point in a one-generated frame is incomparable with any other iff every one-generated frame is linear iff $\Lambda \supseteq \mathsf{K4.3}$. Logics of finite width are complete with respect to frames in which every point has a depth. If f is such a frame and s a point of depth $\alpha = \omega \times k + \beta$ with $\beta < \omega$ then the maximally connected subframe containing

s of points of depth less than $\omega \times (k+1)$ but at least $k \times \omega$ is called the **galaxy** of s. (This is reminiscent of the definition of a galaxy in non-standard analysis.) If $\Gamma, \Delta \subseteq f$ are galaxies of f we write $\Gamma \lhd \Delta$ if for all $g \in \Gamma$ and all $d \in \Delta$ $g \lhd d$; Γ and Δ are called **comparable** if either $\Gamma \lhd \Delta$ or $\Delta \lhd \Gamma$ or $\Gamma = \Delta$. A frame is called a **street** if is a string of galaxies. In a street there is in addition to the notion of a depth also the rather coarse notion of *galactic depth*. A point is said to be of **galactic depth** $k+1$ if it is of depth $k \times \omega + \beta$ for some β. In that case we also say that this point is of **local depth** β; here it pays off to let terminal points have depth 0, since for points of galactic depth 0 local depth and depth are the same. The depth is thus determined by the local depth and the galactic depth. Likewise, a frame is of **galactic depth** k if it is of depth $k \times \omega + \beta$ for some β. It can be shown that logics of finite width and finite tightness are complete with respect to streets. For let $\Lambda \supseteq \mathsf{K4}\{wd(m), ti(n)\}$. Then Λ is complete; thus let f be a one-generated Λ-frame. Let all galaxies of depth $< \beta$ be linearly ordered. Assume that there are two galaxies of depth β, namely Γ and Δ. They must then be incomparable but there is a s such that s precedes both Γ and Δ. Then neither $s \in \Gamma$ nor $s \in \Delta$. In addition, one of Γ, Δ must be an infinite galaxy; if not, s must belong to one of the galaxies. Let Γ be infinite. Then Γ contains a chain of n points none of which is comparable with any point of Δ. Since no member of $ti(n)$ embeds into f, Δ must be empty. So Γ is the only galaxy of depth β. It is perhaps instructive to see an example of a frame with a non-finite and non-initial galaxy in order to understand why the argument is not entirely trivial.



A logic has fmp iff it is complete with respect to frames of galactic depth 1. A logic has **galactic fmp** if it is complete with respect to frames of finite galactic depth.

Theorem 10 *All extensions of* $\mathsf{S4}$ *of finite width and finite tightness have galactic fmp.*

Proof. We prove that dropping a galaxy of non-zero galactic depth is supersafe. It the follows that any street is modally equivalent to its galactically finite substreets. Thus let $(\Sigma \circledcirc)\Gamma \circledcirc \widetilde{\Sigma}$ be a street. (The bracketed segment is optional.) Then $(\Sigma \circledcirc)\Gamma \circledcirc \widetilde{\Sigma} \twoheadrightarrow (\Sigma \circledcirc) \bullet \circledcirc \widetilde{\Sigma}$. But $\mathrm{Th}((\Sigma \circledcirc) \bullet \circledcirc \widetilde{\Sigma}) = \mathrm{Th}((\Sigma \circledcirc)\widetilde{\Sigma})$ by the lemma given below. $\square$

Lemma 11 *Let f be a one-generated S4-frame of finite width and finite tightness. Assume that f has exactly one point w of depth $k \times \omega > 0$. Then $\{w\}$ is eliminable.*

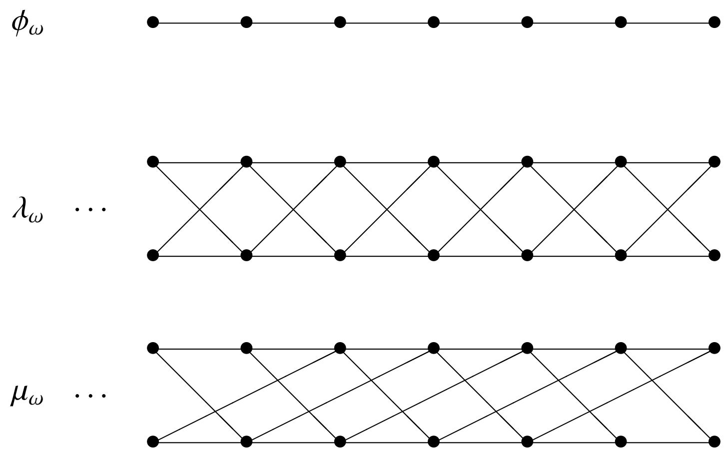
Proof. Suppose that $N \subset f$ is finite. Let $N^+ = \{x \in N : x \lhd w \not\rhd x\}$, $N^- = \{x \in N : w \lhd x \not\lhd w\}$. Then by our assumptions about f, $N = N^+ \cup \{w\} \cup N^-$ and $N^+ = \{x \in N : dp_f(x) > dp_f(w)\}$, $N^- = \{x \in N : dp_f(x) < dp_f(w)\}$.

Claim: For every finite set $M \subset f$ of points of depth $< dp_f(w)$ there exists a point of depth $< dp_f(w)$ seeing all points of M.

Assume that f is of tightness ℓ. The proof is by induction on the cardinality of M. The case where $M = \{t\}$ is trivial. Now assume $M = \{t\} \cup M'$. By induction hypothesis, there exists a $s_0 \lhd M'$ with $dp(s_0) < k \times \omega$. Now take any strictly descending chain $s_\ell \lhd s_{\ell-1} \lhd \ldots \lhd s_0$ with $dp(s_{i+1}) = dp(s_i) + 1$. Then s_ℓ is $< k \times \omega$. By tightness, $s_\ell \lhd t$. Thus $s_\ell \lhd M$, as required.

The lemma is now proved by taking w' to be a point of depth $< k \times \omega$ such that $w' \lhd N^-$. Then $N' := N^+ \cup \{w'\} \cup N^-$ is m-compatible with N. □

6. The subatomic research program. The following frames are of particular interest to us.



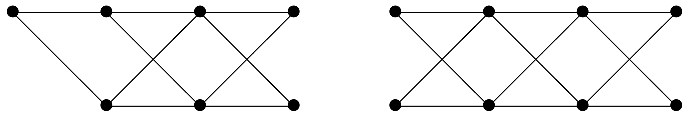
Let us call a generated subframe of ϕ_ω a **photon**, a generated subframe of λ_ω a **lepton** and a generated subframe μ_ω a **meson**. A string is photonic all segments are photons and **leptonic** if all segments are leptonic or photonic and **mesonic** if all segments are either photonic, leptonic or mesonic. Our goal here is to determine the logic of photonic, leptonic and mesonic strings. To do this we will develop a solid arithmetic of p-morphisms for these frames. The photons might not seem worth a discussion, but it is worthwile starting with the simplest case and see what gets lost when we go further down in the lattice of intermediate logics.

Thus let us begin with the photons. They come in a variety ϕ_k where k is the depth of the frame. Note that $\phi_1 = \bullet$ and $\phi_{n+k} = \phi_n \oslash \phi_k$ so that in fact photons decompose completely into strings of $\bullet$. Any photonic string is then a string of $\bullet$, which is the most basic component of 'frame matter'. Minimal p-morphisms are $\phi_k \rightarrowtail \phi_{k+1} \twoheadrightarrow \phi_k$; moreover, $\phi_k \rightarrowtail \phi_\omega \twoheadrightarrow \phi_k$.

Theorem 12 $\mathsf{Pho} = \mathsf{Grz.3} = \mathsf{Grz}\{wd(1)\}$ *is the logic of photonic strings. Pho is pretabular, pre-compact, has fmp essentially and is essentially finitely axiomatisable and essentially decidable.*

We will sketch the proofs of the claims, which are well-known. In this simple case we meet a number of standard arguments. First, if Φ is a photonic string, and $\Phi^\mu \subseteq \Phi$ a finite subset of maximal points, we can supersafely drop non-maximal $\bullet$ (i) if they are behind a $\bullet$ (ii) if they are directly followed by a $\bullet$ (iii) if they are of depth $k \times \omega > 0$. Hence, all points of non-zero depth are droppable from a model. Thus if $\Phi^\mu \neq \emptyset$ everything outside Φ can be supersafely dropped. If $\Phi^\mu = \emptyset$, we can drop everything except one $\bullet$. This shows that for every photonic string $\mathrm{Th}(\Phi)$ has fmp and thus that Pho has fmp essentially. Thus every extension of Pho is a splitting logic. For if $\mathsf{Pho} \subseteq \Lambda$ let N be the set of finite, one-generated photons which are not frames for Λ. Then $\mathsf{Pho}/N \subseteq \Lambda$; but since the logics have the same finite models they are in fact equal. Now, Λ is finitely axiomatizable, if N is finite or if N can be replaced by a finite set. But certainly N is a set of photons; and for $k < \ell$, ϕ_k is an extract of ϕ_ℓ and so $\mathsf{Pho}/\phi_\ell \subset \mathsf{Pho}/\phi_k$. Thus, as the photons are linearly ordered by the order of being an extract of the other, we can replace N in the splitting representation by ϕ_ℓ with ℓ being the least k with $\phi_k \in N$. Hence $\Lambda = \mathsf{Pho}/\phi_\ell$. Decidability follows as well as tabularity.

Now on to the leptons. Leptons of depth k come in two varieties, one-generated and two-generated. Let us write $\lambda_k^\bullet$ for the one-generated lepton of depth k and λ_k° for its two-generated companion. It turns out that λ_ω is best classified as one-generated; logically, this is reasonable since λ_ω and $\bullet \oslash \lambda_\omega$ have the same logical theory. Dropping or adding this point is supersafe.



$\lambda_4^\bullet$ λ_4°

There is a decomposition $\lambda_k^\circ = \bigotimes_{i\in k}\lambda_1^\circ$ and $\lambda_{k+1}^\bullet = \lambda_1^\bullet \otimes \lambda_k^\circ$, thus any leptonic string decomposes into $\lambda_1^\bullet$ and λ_1°. Moreover, $\lambda_1^\bullet = \phi_1 = \bullet$ and $\lambda_1^\circ = \bullet \oplus \bullet$. This leaves, in order to get a full picture of admissible p-morphisms, only two choices. We have *fusion* $\lambda_1^\bullet \otimes \lambda_1^\bullet \twoheadrightarrow \lambda_1^\bullet$ and $\lambda_1^\circ \twoheadrightarrow \lambda_1^\bullet$. On the side of embeddings note $\lambda_1^\bullet \rightarrowtail \lambda_1^\circ$. We conclude this lemma.

Lemma 13 *There are p-morphisms $\lambda_k \twoheadrightarrow \lambda_n^\bullet$ for all $k \geq n$. There are no p-morphisms $\lambda_k \twoheadrightarrow \lambda_n^\circ$, $n < k < \omega + 1$. Every finite leptonic string is an extract of λ_ω.*

Proof. If $k \geq n$ then $k = n + \ell$ for some ℓ. Then $\lambda_k^\bullet = \lambda_\ell^\bullet \otimes \lambda_1^\circ \otimes \lambda_{n-1}^\circ \twoheadrightarrow \lambda_\ell^\bullet \otimes \lambda_1^\bullet \otimes \lambda_{n-1}^\circ \twoheadrightarrow \lambda_1^\bullet \otimes \lambda_{n-1}^\circ = \lambda_n^\bullet$. To see that no p-morphisms $\lambda_k^\circ \twoheadrightarrow \lambda_n^\circ$ exist simply note that every minimal p-morphism produces at least a segment $\lambda_1^\bullet$—of which we can never get rid. The last observation goes as follows. $\lambda_n^\circ \rightarrowtail \lambda_\omega$; if Λ is a leptonic string of depth n, Λ decomposes completely into the leptons $\lambda_1^\bullet, \lambda_1^\circ$. We can now reduce λ_n° to Λ by applying $\lambda_1^\circ \twoheadrightarrow \lambda_1^\bullet$ in each segment where it is necessary. □

Theorem 14 *The logic* $\mathsf{Lep} = \mathsf{Grz}\{wd(2), ti(2)\}$ *is the logic of leptonic strings. Moreover,* $\mathsf{Lep} = \mathrm{Th}(\lambda_\omega)$. *$\mathsf{Lep}$ has fmp essentially, is essentially finitely axiomatisable and essentially decidable.*

Proof. First, Lep is complete with respect to one-generated strings. We have to show that any Lep-string is a leptonic string and vice versa. This is not hard to do. The strategy is now to show that $\mathrm{Th}(\Lambda)$ has fmp for Λ a one-generated leptonic string. If any such logic has fmp then Lep has fmp essentially. Now take a one-generated $\Lambda = \bigotimes_{i\in\alpha}\lambda(i)_1$ with $\lambda(i)_k = \lambda_1^\bullet, \lambda_1^\circ$. Now assume a finite subframe $\Lambda^\mu \subseteq \Lambda$ of maximal points. In any segment that contains two points which are not both maximal we supersafely drop one non-maximal point. This leaves us with finitely many components of type λ_1°. In between these components sit photonic strings which can be supersafely reduced to either $\bullet$ or a photon containing the maximal points. If we cannot drop any more points we end up with a subframe $\Lambda' \supseteq \Lambda^\mu$ of cardinality $\leq 3/2 \times \sharp\Lambda^\mu$. (Check that any non-maximal point must immediately precede two maximal points in order not to be dropped at some stage.)

Now the theorem is proved if we show that Lep is essentially fnitely axiomatisable, since essential decidability will follow. Since Lep has fmp essentially, every extension of Lep is a splitting of Lep. The question is then whether we can always choose a finite set F such that $\Lambda = \mathsf{Lep}/F$. To this end define a partial order $\preccurlyeq$ on the set $\mathfrak{Lep}$ of one-generated finite leptonic strings by $f \preccurlyeq g$ iff f is an extract of g. The order $\preccurlyeq$ is a well-partial order (wpo) in the sense of [Kruskal, 1960] as we will show below. Recall that a partial order is called a **well partial order** if for all sets N the set $\min M$ of $\preccurlyeq$-minimal elements exists and every set of mutually incomparable elements (**anti-chain**) is finite. If $\sqsubseteq \subseteq \preccurlyeq$ has minima and has no infinite antichains, neither has $\preccurlyeq$. Moreover, if $\preccurlyeq^i$ are wpo's on M^i ($i = 1, 2$) then $\preccurlyeq^1 \cup \preccurlyeq^2$ is a wpo on $M^1 \cup M^2$ and $\preccurlyeq^1 \times \preccurlyeq^2$ a wpo on $M^1 \times M^2$ (see [Kruskal, 1960]).

Proposition 15 *The following are equivalent.*

(i) $\preccurlyeq$ *is a well-partial order.*

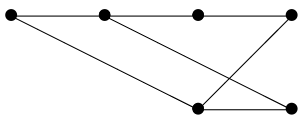
(ii) *Lep is essentially fnitely axiomatisable.*

(iii) *Lep is essentially decidable.*

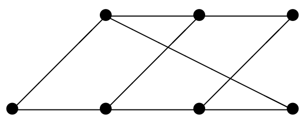
Proof. Clearly, since $\mathsf{Lep}/M = \mathsf{Lep}/\min M$, (i) implies (ii). However, if (i) does not hold there is a set N such that $\min N$ is infinite. Then there is an extension which is not finitely axiomatisable. Thus (i) and (ii) are equivalent. Moreover, in that case for every $M, M' \subseteq \min N$ we have $\mathsf{Lep}/M = \mathsf{Lep}/M' \Leftrightarrow M = M'$ whence Lep has uncountably many extensions; but only countably many of them are decidable. Hence (iii) implies (i). Now if Lep is essentially finitely axiomatisable, it is essentially decidable since it has fmp essentially. This shows (ii) $\Rightarrow$ (iii). $\square$

All that is left to show is that $\preccurlyeq$ is wpo on $\mathfrak{Lep}$. Now let $\Lambda \in \mathfrak{Lep}$. Then $\Lambda = \bigotimes_{i \in n} \lambda^\bullet_{k(i)}$ for some numbers $n, k(i) \in \omega$. If $\widetilde{\Lambda} = \bigotimes_{i \in \widetilde{n}} \lambda^\bullet_{\widetilde{k}(i)}$ is another such frame then $\widetilde{\Lambda} \twoheadrightarrow \Lambda$ if there exists an isotone embedding $\sigma : n \rightarrowtail \widetilde{n}$ with $\sigma(0) = 0$ and $k(i) \leq \widetilde{k}(\sigma(i))$. Thus if we represent members of $\mathfrak{Lep}$ by sequences $\langle k(i) : i \in n \rangle$ and define an order $\sqsubseteq$ according to this definition then $\sqsubseteq$ is almost a wpo according to [Kruskal, 1960]. If we ignore the clause '$\sigma(0) = 0$' then we have exactly the definition of non-branching trees over $\langle \omega, \leq \rangle$, the latter being a wpo, and hence the whole is a wpo by Kruskal's Theorem. The extra clause is a harmless complication which we can in fact ignore (this produces an order which is a direct product of the space of trees-over-$\langle \omega, \leq \rangle$ with $\langle \omega, \leq \rangle$). The uneasy reader may however observe that our order is isomorphic to the order obtained for $\mathsf{S4.3}$-frames ordered also by 'being extract of'. By appealing to the result of [Fine, 1971] that this is a wpo, our case is proved. □

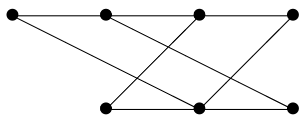
Now we are treating the mesons; their case is much more involved and the decomposition method will do its job rather well here. Again we use the subscript μ_k to denote a meson of depth k and the superscript $\mu^\bullet$ for a one-generated meson and μ° for a two-generated one. But it turns out that this does not determine them completely. Depending on which point generates $\mu^\bullet$ we get a different meson and likewise we have two choices for two-generated mesons. Namely, if μ_k° is two-generated of depth k then the two generating points might be of equal depth or of different depth. This we distinguish by writing $\mu_k^{\circ=}$ in the one case and $\mu_k^{\circ<}$ in the other. Since a one-generated meson $\mu_k^\bullet$ decomposes into $\bullet \oslash \mu_{k-1}^\circ$ this distinction is carried over to the one-generated mesons and we write $\mu_k^{\bullet=}$ for the meson whose generating point has immediate successors of equal depth and $\mu_k^{\bullet<}$ if it has immediate successors of different depth.



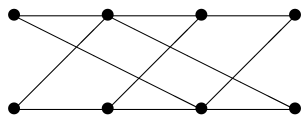
$\mu_4^{\bullet<}$



$\mu_4^{\bullet=}$



$\mu_4^{\circ<}$

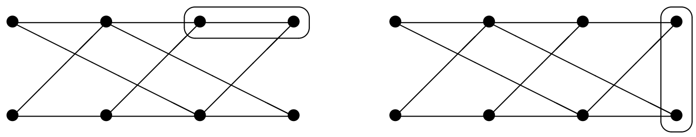


$\mu_4^{\circ=}$

The mesons are indecomposable with the exception of $\mu_k^{\bullet=} = \bullet \oslash \mu_{k-1}^{\circ=}, \mu_k^{\bullet<} = \bullet \oslash \mu_{k-1}^{\circ<}$. They can be generated via minimal embeddings from each other as follows.

$$\begin{array}{llll} \mu_k^{\circ=} \rightarrowtail \mu_{k+1}^{\circ<} & \mu_k^{\circ<} \rightarrowtail \mu_{k+1}^{\bullet<} & \mu_k^{\bullet=} \rightarrowtail \mu_k^{\circ=} & \mu_k^{\bullet<} \rightarrowtail \mu_k^{\circ<} \\ \mu_k^{\circ=} \rightarrowtail \mu_{k+1}^{\bullet=} & \mu_k^{\circ<} \rightarrowtail \mu_k^{\circ=} & & \end{array}$$

No other arrows exist. With respect to minimal p-morphisms we first observe that there exist only two. The best way to see this is to recall that if a minimal p-morphism that identifies two points s, t then s and t share all successors which are not equal to s or t. Then there are two choices. (i) s and t are of equal depth. Then if s or t had a successor, we had decomposability. Thus s and t are of depth 0. (ii) s precedes t. Then s cannot have two immediate successors. Hence s is of depth 1. This gives the following cases.



These p-morphisms produce the following outputs which for beauty's sake are listed in commutative diagrams. By decomposability of the one-generated mesons, we list only the two-generated cases.

$$\begin{tikzcd}
\mu^{\circ<}_{k+2} \arrow[r, tail] \arrow[d, two heads] & \mu^{\circ=}_{k+2} \arrow[r, tail] \arrow[d, two heads] & \mu^{\circ<}_{k+3} \arrow[d, two heads] \\
\mu^{\circ=}_{k} \ogreaterthan \lambda^{\circ}_{1} \arrow[r, tail] & \mu^{\circ<}_{k+1} \ogreaterthan \lambda^{\circ}_{1} \arrow[r, tail] & \mu^{\circ=}_{k+1} \ogreaterthan \lambda^{\circ}_{1}
\end{tikzcd}$$

$$\begin{tikzcd}
\mu^{\circ<}_{k+1} \arrow[r, tail] \arrow[d, two heads] & \mu^{\circ=}_{k+1} \arrow[r, tail] \arrow[d, two heads] & \mu^{\circ<}_{k+2} \arrow[d, two heads] \\
\mu^{\circ<}_{k} \ogreaterthan \lambda^{\bullet}_{1} \arrow[r, tail] & \mu^{\circ=}_{k} \ogreaterthan \lambda^{\bullet}_{1} \arrow[r, tail] & \mu^{\circ<}_{k+1} \ogreaterthan \lambda^{\bullet}_{1}
\end{tikzcd}$$

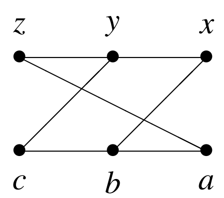
By the above it follows that $\mu^{\circ=}_{k+n+1} \twoheadrightarrow \mu^{\circ<}_{k+n} \twoheadrightarrow \mu^{\circ<}_{k} \ogreaterthan \lambda^{\circ}_{n} \twoheadrightarrow \mu^{\circ<}_{k} \ogreaterthan \phi_{n-1} \twoheadrightarrow \mu^{\circ<}_{k} \ogreaterthan \bullet$. Also, $\mu_{\omega} \twoheadrightarrow \mu_{\omega} \ogreaterthan \lambda^{\circ}_{1} \twoheadrightarrow \mu_{\omega} \ogreaterthan \lambda^{\circ}_{n} \twoheadrightarrow \lambda_{\omega}$. We thus get that any $\mu_{\omega} \ogreaterthan \Lambda$ with Λ a finite mesonic string is a p-morphic image of μ_{ω}. As a result we note that there is no nontrivial p-morphism $\mu \twoheadrightarrow \mu'$ between mesons unless μ' is leptonic because p-morphisms introduce decomposability into a meson and a lepton and the lepton never disappears. Note that there are a few exceptional mesons.

$$\mu_1^{\bullet=} = \mu_1^{\bullet<} = \lambda_1^{\bullet} = \bullet \qquad \mu_2^{\bullet=} = \lambda_1^{\bullet} \circledgt \lambda_1^{\circ}$$
$$\mu_1^{\circ=} = \mu_1^{\circ<} = \lambda_1^{\circ} \qquad \mu_2^{\bullet<} = \lambda_1^{\bullet} \circledgt \lambda_1^{\bullet}$$

A final note. Call a string **one-mesonic** if it is either leptonic or of the type $\mu \circledgt \Lambda$ where Λ is a leptonic string and μ a meson. Our considerations above show that if a string contains n mesons then any extract of that string contains at most n mesons. Hence the class of one-mesonic strings is closed under p-morphic images and generated subframes. Moreover, any finite one-mesonic string is an extract of μ_ω. Now define $p_2^2 = \bullet \circledgt (\phi_2 \oplus \phi_2)$. p_2^2 excludes two parallel two-element chains.

Theorem 16 *The logic* $\mathsf{Mes} = \mathsf{Grz}\{wd(2), ti(3), p_2^2\}$ *is the logic of mesonic strings.* Mes *has fmp essentially.*

Proof. It suffices to study the one-generated strings. Since the subframes $wd(2), ti(3), p_2^2$ are of the form $\bullet \circledgt g$ for an indecomposable g, we can check by segmentwise inspection whether Mes is the logic of mesonic strings. Now take a frame $\bullet \circledgt \mu$ such that μ is indecomposable. If μ is a meson (lepton, photon) then it is a Mes-frame; thus the converse needs to be established. Thus assume that μ is not a lepton; then it has at least three points, and so there is some slice $\{x, a\}$ of local depth $n \in \omega$ (see picture below). We now investigate the points behind this slice. Suppose we have a point y immediately preceding x. If a has no predecessors (in μ) then neither has y, by non-embeddability of $ti(3)$. Thus if we have not exhausted the points behind x or a, there must be at least a predecessor of a. Now since p_2^2 is not embeddable, either $y \lhd a$ or $b \lhd x$. By symmetry, we may only deal with one case, say $b \lhd x$. If there is still another point, y has a predecessor. Otherwise let there be only a predecessor $c \lhd b$. Then $c \ntriangleleft y$ implies embeddability of $ti(3)$ and thus $c \lhd y$, which was excluded. So, indeed there is an immediate predecessor $z \lhd y$. Then we must have $z \lhd a$ by $ti(3)$ but we cannot have $z \lhd b$; for otherwise μ was decomposable, for any $c \lhd b \ntriangleleft c$ must also satisfy $c \lhd y$ as we have seen.



Now, n was completely arbitrary. If we start with $n = 0$ we see inductively that μ is in fact a meson $\mu_k^{\circ=}, \mu_k^{\circ<}$ for some k.

Now let M be a mesonic string and let $M^\mu \subseteq M$ be a finite subset. We know by previous proofs that leptonic and photonic segments can be made rare (at most $\sharp M^\mu$ such segments) by supersafe dropping. In addition, mesons without maximal points can be reduced to $\bullet$ and almost always be dropped, which leaves us with finitely many mesons. Thus the only problem we have is that there might be a galactic meson μ_ω. But here comes a surprise.

Lemma 17 *In $\mu_\omega \otimes \mu_\omega$ the first galaxy is eliminable.*

Proof. Assume $g^\mu \subset g$. Let g_0^μ be the part of g^μ containing all points of infinite depth in $\mu_\omega \otimes \mu_\omega$ and let g_1^μ contain all the points of finite depth. g_1^μ is finite and all points are of depth, say, $< n$. Then we can shift g_0^μ into the finite part of $\mu_\omega \otimes \mu_\omega$ by mapping each point of depth $\omega + k$ into a point of depth $n + k$. It is not hard to see that this map satisfies the conditions of Theorem 5. □

By this lemma, for any meson μ, $\mathrm{Th}(\mu \otimes \mu_\omega) = \mathrm{Th}(\mu_\omega)$ and for every lepton $\mathrm{Th}(\lambda \otimes \mu_\omega) = \mathrm{Th}(\mu_\omega)$. Thus if M contains μ_ω, we may forget all points seeing μ_ω. Consequently, $\mathrm{Th}(M) = \mathrm{Th}(\mu_\omega \otimes M')$ for some finite mesonic string M'. And so $\mathrm{Th}(M)$ has fmp for every M. □

Theorem 18 $\mathsf{Mes}^1 = \mathsf{Mes}/\{\bullet \otimes \bullet \otimes \mu_2^{\circ<}, \bullet \otimes \bullet \otimes \mu_2^{\circ<} \otimes \bullet\}$ *is the logic of one-mesonic strings. Moreover,* $\mathsf{Mes}^1 = \mathrm{Th}(\mu_\omega)$.

Proof. If M is not one-mesonic, let $M = \bullet \otimes (M^1 \otimes)\mu(\otimes M^2) \otimes \mu'(\otimes M^3)$ be such that μ' is an indecomposable meson. Then $M \twoheadrightarrow \bullet \otimes \mu^\circ(\otimes M^2) \otimes \mu'^\circ(\otimes M^3) \twoheadrightarrow \bullet \otimes \bullet \otimes \mu^\circ(\otimes \bullet) \twoheadrightarrow \bullet \otimes \bullet \otimes \mu_1^{\circ<}(\otimes \bullet)$. This was excluded. But if M is indeed a one-mesonic string then it omits the depicted frames since they are not one-mesonic. The last claim follows from the fact that a finite one-mesonic string $\mu \otimes \Lambda$ is an extract of μ_ω, which itself is one-mesonic. □

7. An intermediate logic bounding finite axiomatizability. Consider the set $\mathfrak{Mes}^1$ of one-generated, finite strings ordered by $f \preccurlyeq g \Leftrightarrow f$ is an extract of g. Call $\mu \otimes \Lambda$ **thick** if μ is not a lepton and $\Lambda = \lambda_n^\circ$ for some $n \in \omega$. $\mathfrak{Th}$ is the set of thick frames.

Lemma 19 $\preccurlyeq$ *is a well partial order on* $\mathfrak{Mes}^1 - \mathfrak{Th}$.

Proof. $\preccurlyeq$ is a well-partial order on the leptonic strings of $\mathfrak{Mes}^1$; thus it suffices to look ot the non-mesonic ones. Take any two $\mu \otimes \Lambda, \mu' \otimes \Lambda'$. Then $\mu \otimes \Lambda$ is an extract of $\mu' \otimes \Lambda$ if only $\Lambda' \twoheadrightarrow \Lambda$ and μ is an extract of μ'. The one-generated mesons are linearly ordered by inclusion. Moreover, $\sqsubseteq$ defined $f \sqsubseteq g$ iff $g \twoheadrightarrow f$ is a wpo on the finite leptons which are not of the form λ_n°. Now the product of two wpo's is again a wpo; hence $\preccurlyeq$ is a wpo. □

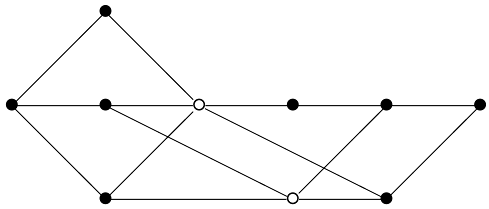
Lemma 20 $\preccurlyeq$ *is not a wpo on* $\mathfrak{Th}$. *In particular,* $\{\mu_3^{\bullet<} \otimes \lambda_n^\circ : n \in \omega\}$ *is an infinite antichain.* □

Theorem 21 $\mathsf{Mes}^1(3) = \mathsf{Mes}^1/\{\mu_4^{\bullet=}, \mu_4^{\bullet<}\}/\{\mu_3^{\bullet<} \otimes \lambda_n^\circ : n \in \omega\}$ *bounds finite axiomatizability. Moreover,* $\mathsf{Mes}^1(3) = \mathrm{Th}(\mu_3^{\bullet<} \otimes \lambda_\omega)$.

Proof. We have seen that the set of splitting frames is an infinite antichain and hence $\mathsf{Mes}^1(3)$ is not fnitely axiomatisable. Yet for any proper extension $\mathsf{Mes}^1(3) \subsetneq \Lambda$ we must have $M \notin \mathrm{Fr}(\Lambda)$ for some finite one-mesonic string M; moreover, M is not thick. But then since $M = (\mu_3^{\bullet<} \otimes)\Lambda$ for some leptonic string Λ, M is an extract of almost all thick frames. Hence $\mathsf{Mes}^1(3)/M$ is finitely axiomatisable. Any extension of Λ is characterized by non-thick frames and by Lemma 19 fnitely axiomatisable over Λ. □

The logic $\mathsf{Mes}^1(3)$ nevertheless has fmp and from that it follows that it is $\bigsqcap$-reducible in the lattice of normal modal logics. This refutes a plausible conjecture that logics bounding certain properties invariably are $\bigsqcap$-irreducible.

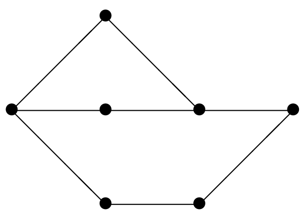
8. Logics bounding fmp and other types of completeness. The logic Ref was not only pre-finitely axiomatisable but also pre-fmp whereas our example of a pre-fmp logic still has fmp essentially. If we want to find a logic bounding fmp we have to descend further in the lattice of intermediate logics. It turns out that logics of frames $\beta \otimes \mu_\omega$ fail to have fmp if β is not mesonic. There is an easy way to show this using an idea that goes back to [Fine, 1972]. Consider the following frame.

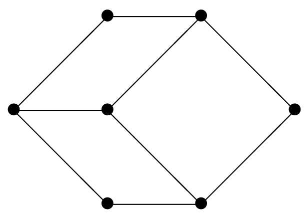


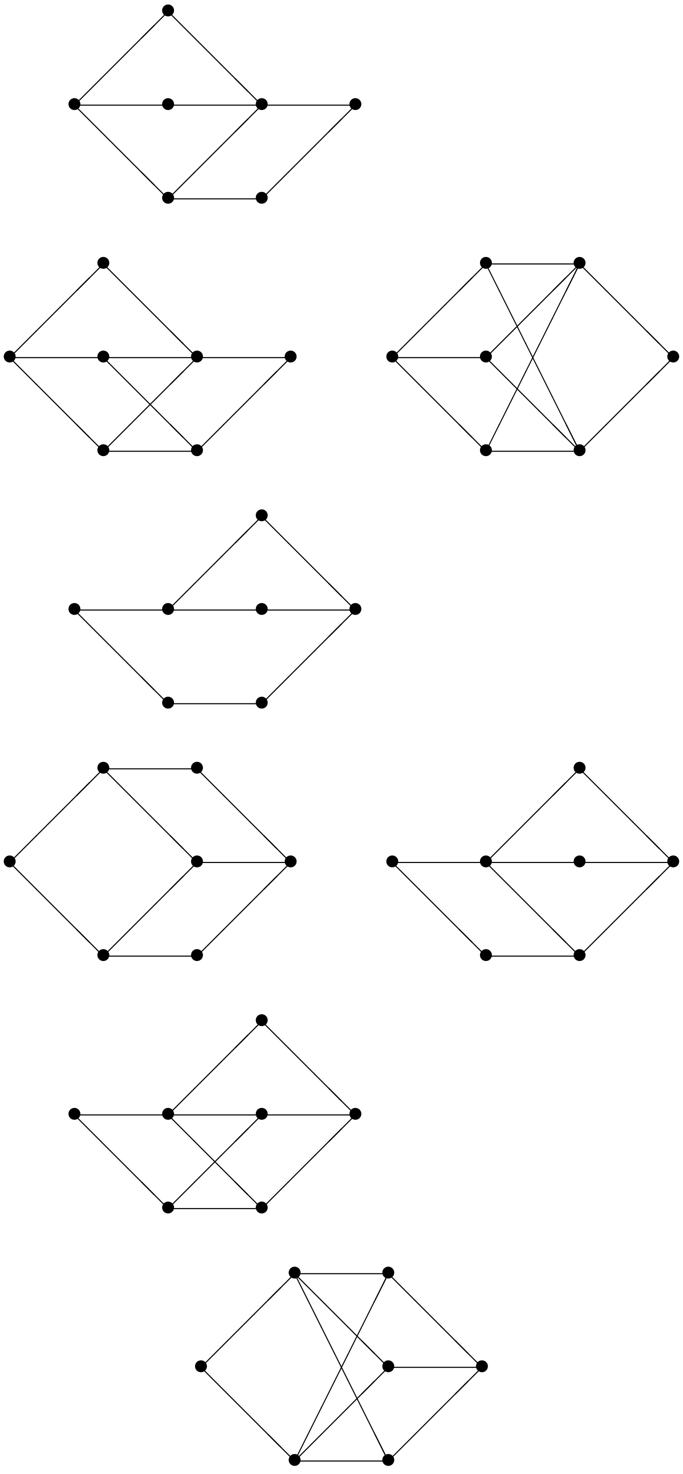
Let A be an axiom saying that whenever the subframe of blobs is embeddable, so is the frame with the circled points added. If our logic contains such an axiom and moreover if the frame of the blobs can be embedded, the logic fails to have fmp because the construction ensures that it is continuously reproduced and we end up with a frame at least containing $wd(2) \ogreaterthan \mu_\omega$. Abstractly, this situation is characterized as a map $id_\rho \ogreaterthan \iota : \rho \ogreaterthan \sigma \stackrel{\subset}{\to} \rho \ogreaterthan \tau$ where $\iota : \sigma \stackrel{\subset}{\to} \tau$ is an embedding. There is a requirement that the points of τ that are new (i. e. not in $\iota[\sigma]$) should be definable in terms of what old points they can see. If that is so then such a map corresponds to an axiom saying that any embedding $\rho \ogreaterthan \sigma \stackrel{\subset}{\to} g$ factors through $id_\rho \ogreaterthan \iota$. Such a map as well the frame it generates are called a **monkey ladder**. (Observe that failure of fmp for the logic Ref can also be attested with a monkey ladder.) The frames $\beta \ogreaterthan \mu_\omega \ogreaterthan \Lambda$ all satisfy a monkey ladder axiom analoguous to the one depicted above. Hence in order to find a frame whose logic bounds fmp we just have to find frames that are minimal with respect to allowing such a monkey ladder. It turns out that $\Lambda = \bullet$ and $\beta = wd(2), ti(3), p_2^2$ all are possible choices. $\beta = wd(2)$ is best suited for our purposes.

Theorem 22 $\mathrm{Th}(wd(2) \ogreaterthan \mu_\omega \ogreaterthan \bullet)$ *bounds fmp.*

Proof. Define $\mathsf{Mon}(1,0) = \mathsf{Grz}\{wd(3), \bullet \ogreaterthan wd(2), ti(3), p_2^2\}/\{\bullet \ogreaterthan \bullet \ogreaterthan \mu_2^{\circ<} \ogreaterthan \bullet, wd(1)\}/M$ where M is the set of the following ten frames. (Not all of them are necessary in this context, but we will need the set as it is later. Observe that the frames of M collect all convergent frames with a 2-slice following or being followed by a 3-slice.)







Let then f be a $\mathsf{Mon}(1,0)$-frame; it can be assumed to be a one-generated street. Consider the case where $wd(2) \ogreaterthan \mu_2^{\circ<}$ is embeddable. Then the embdding is first of all such that $wd(2)$ is initial in the frame by exclusion of $\bullet \otimes wd(2)$; moreover, the frames of M forbid that this antichain of three points is immediately followed by two points. Thus f is decomposable into $wd(2) \otimes g \otimes \bullet$ where g is one-generated and of width 2. It follows that g is one-mesonic by splitting of $\bullet \otimes \bullet \otimes \mu_2^{\circ<} \otimes \bullet$. Moreover, g can, by the same splitting frame, not be finite since it is not a leptonic string. Thus by familiar arguments $\mathrm{Th}(g \otimes \bullet) = \mathrm{Th}(\mu_\omega \otimes \bullet)$ and that had to be proved. Now consider the case when $wd(2) \otimes \mu_2^{\circ<}$ is not embeddable. Then either f is of width 2 in which case it is one-mesonic and so an extract of $wd(2) \otimes \mu_\omega \otimes \bullet$ by which $\mathrm{Th}(f)$ has fmp; or it is not of width 2. In that case we cannot embed $ti(2)$ and so f is completely decomposable and $f = wd(2) \otimes \Lambda$ where Λ is a leptonic string. Finally, the frames of M have excluded that Λ is two-generated. Thus f is again an extract of $wd(2) \otimes \mu_\omega \otimes \bullet$ and $\mathrm{Th}(f)$ has fmp. All this together yields the proof. □

Now that we have shown that there is a logic bounding fmp there still remains the question of how big models must be; up to now, models of galactic depth 2 were sufficient. Now call a logic (k, ℓ)-**complete** if is complete with respect to models of depth $\leq k \times \omega + \ell$. Then we know that all logics of finite width and finite tightness are $(\omega, 0)$-complete so one does not need to go higher. But the next theorem shows that one cannot do better. Proofs from now on are only sketched since they use similar arguments to the ones we have used quite often now.

Theorem 23 $\mathsf{Mon}(\omega, 0) = \mathsf{Grz}\{wd(3), ti(3), p_2^2\}/\{wd(1)\}/M$ *is complete with respect to extracts of iterated monkey ladders* $\otimes_{i \in n} \xi . \otimes . \bullet$, $\xi = wd(2) \otimes \mu_\omega$. *Moreover,* $\mathsf{Mon}(\omega, 0)$ *bounds* $(\omega, 0)$*-completeness.*

Proof. By the splitting axioms of M, if a $\mathsf{Mon}(\omega, 0)$-frame contains an anti-chains with three points then it must be a segment separated by a buffer segment of type $\bullet$ from the other segments. Prove that finite segments are leptonic strings and that only the galactic meson μ_ω is allowed as a segment. This shows the completeness part. Consider now the formula saying that there exist a point seeing n different monkey ladders; for this formula a model must have at least galactic depth n. On the other hand, any proper extension must contain an axiom that forbids than there can be more than a given number n of monkey ladders. But any such axiom forces that any model can be reduced to a model of galactic depth $\leq n + 1$. □

We can fine-tune this method. First observe the following.

Lemma 24 *The logics* $\mathsf{Mon}(0, \ell) = \mathsf{Grz.3}\{\phi_{\ell+1}\}$ *bound* $(0, \ell)$*-completeness.* $\square$

Lemma 25 *The logics* $\mathsf{Mon}(k, \ell + 1) = \mathsf{Mon}(\omega, 0)\{\phi_{\ell+1} . \circledcirc . \circledcirc_{i \in k} \xi\}$ *bound* (k, ℓ)*-completeness for* $\ell > 0$*. The logics* $\mathsf{Mon}(k, 0) = \mathsf{Mon}(k, 1) = \mathsf{Mon}(\omega, 0)\{\circledcirc_{i \in k+1} \xi\}$ *bound* $(k, 0)$*- as well as* $(k, 1)$*-completeness.*

Proof. Consider formulas stating that ℓ steps ahead from here we can still see k different monkey ladders. Such formulas can only be realized on a model with depth at least $k \times \omega + \ell$. For the lemma it is enough to show that such a formula is satisfiable on a frame f iff f is modally equivalent to the frame $\phi_\ell . \circledcirc . \circledcirc_{i \in k} \xi$; and if this formula is not satisfiable on f then f is modally equivalent to a frame of lesser depth. $\square$

Theorem 26 *The logics* $\mathsf{Mon}(k, \ell)$ *bound* (k, ℓ)*-completeness.* $\square$

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