Paraconsistent Quasi-Set Theory

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Abstract

Paraconsistent logics are logics that can be used to base inconsistent but non-trivial systems. In paraconsistent set theories, we can quantify over sets that in standard set theories (that are based on classical logic), if consistent, would lead to contradictions, such as the Russell set, $R = \{x : x \notin x\}$. Quasi-set theories are mathematical systems built for dealing with collections of indiscernible elements. The basic motivation for the development of quasi-set theories came from quantum physics, where indiscernible entities need to be considered (in most interpretations). Usually, the way of dealing with indiscernible objects within classical logic and mathematics is by restricting them to certain structures, in a way so that the relations and functions of the structure are not sufficient to individuate the objects; in other words, such structures are not rigid. In quantum physics, this idea appears when symmetry conditions are introduced, say by choosing symmetric and anti-symmetric functions (or vectors) in the relevant Hilbert spaces. But in standard mathematics, such as that built in Zermelo-Fraenkel set theory (ZF), any structure can be extended to a rigid structure. That means that, although we can deal with certain objects as they were indiscernible, we realize that from outside of these structures these objects are no more indiscernible, for they can be individualized in the extended rigid structures: ZF is a theory of individuals, distinguishable objects. In quasi-set theory, it seems that there are structures that cannot be extended to rigid ones, so it seems that they provide a natural mathematical framework for expressing quantum facts without certain symmetry suppositions. There may be situations, however, in which we may need to deal with inconsistent bits of information in a quantum context, even if these informations are concerned with ways of speech. Furthermore, some authors think that superpositions may be understood in terms of paraconsistent logics, and even the notion of complementarity was already treated by such a means. This is, apparently, a nice motivation to try to merge these two frameworks. In this work, we develop the technical details, by basing our quasi-set theory in the paraconsistent system $\mathscr{C}_1$. We also elaborate a new hierarchy of paraconsistent calculi, the *paraconsistent calculi with indiscernibility*. For the finalities of this work, some philosophical questions are outlined, but this topic is left to a future work.

Keywords: paraconsistent logic, quasi-set theory, indistinguishable quanta, quantum physics.

1 Introduction: motivation

It is well known that quantum objects are strange and that quantum physics presents lots of puzzles to the philosophical reflection on the nature of these entities. One of the most debated aspects of quantum objects that has been presented to the literature concerns a possible reading of their ontological status; can quantum objects be regarded as *individuals* on a pair with their "classical" counterparts, or would them be considered as *non-individuals*, as it has been suggested by various specialists, among them some of the forerunners of quantum, theory such as Heisenberg, Schrödinger, Born, and Weyl? (the whole history can be found in [9]). It is quite obvious that in order to address these and other related questions, one needs to precise some involved concepts, as for instance; (1) what should we understand by "quantum objects"?;(2) what shold we understand by an "individual"?; (3) are there conditions to individuate quantum objects in whatever situation? Quantum theory, or quantum physics, as we use these words, refer a cluster of theories that cover a certain domain of knowledge. So we can distinguish among the "old" quantum mechanics of Planck, Einstein and Bohr, from the "orthodox" quantum mechanics, or "quantum mechanics" tout court, developed by Heinsenberg, Born, Jordan and Schrödinger (although in distinct fashions), and also from quantum field theory (QFT), started by Dirac and up today forming the core of present day standard model of particle physics.

All these theories refer to quantum objects in one way or another, usually termed "quantum particles", or "elementary particles" for short. Despite the common name, elementary particles are different kinds of entities in all these theories (for a panorama of these differences, see [8, chap.6]). We will not revise all these distinctions here, for they do not import for our discussion. We will assume that there is a discourse about quantum objects in all these theories; the apparently most problematic case is of course concerning QFT, for the most basic entities dealt with by the theory are quantum fields, and "particles" arise from quantum fields as certain field excitations. Anyway, physicists still speak of protons, electrons, photons and so on, for instance as they being accelerated, so it is licit (so we think) to discuss about their ontological status, and even to look whether quantum theory itself (in whatever form we consider) entails that they would obey some "logic" other than classical logic.[1]

In particular, quantum particles are considered as indistinguishable in some situations, in a way that nothing in the theory can distinguish them [19, chap.11

[1]Thus, we are asking for a different type of "quantum logic" than those of standard literature.

and 12] (or, as some say, not even in *mente Dei* they can be distinguished, as said Dalla Chiara and Toraldo di Francia [5]). Indistinguishability, as is well known, is a characteristic trait of quantum physics, without which there would be no quantum physics at all. But there is a puzzle here: quantum physics is described by using standard mathematics, encompassing classical logic, and in such a framework there are no indistinguishable objects. Classical mathematical framework is Leibnizian in the sense that some form of Leibniz famous Principle of the Identity of Indiscernibles is a theorem of classical logic. The standard way to deal with indiscernible objects within such a logical basis is by the introduction of symmetry conditions. Really, indistinguishability, or indiscernibility, usually means invariance by permutations. Thus, being F an n-ary predicate, then $F(x_1, \ldots, x_n)$ means the same as $F(x_{\pi(1)}, \ldots, x_{\pi(n)})$ for any permutation π of $\{1, \ldots, n\}$. But in order to express that, it is necessary to label the particles first; that is, to begin by supposing that they were *named*, hence individualized, by x_1, x_2 etc., and then we postulate that permutations among these particles do not conduce to distinct physical situations. Of course this works quite well both from the physical and from the mathematical point of view, but from a philosophical perspective, it seems that there is something lacking here.

Years ago (in 1963), Heinz Post guessed that quantum indiscernibility should not be "made" this way, starting with individuals (entities in principle always individuated as being *that individual*) and going entities devoid of individuality by weakening the identity conditions; he suggested that the indiscernibility of quantum objects should be taken *right at the start* (for a detailed discussion on all these historical points, see [9]). The direction according to which the indiscernibility is "made from the top down" (taking the "top" as designating standard objects of our surroundings) is encapsulated in the discourse of those above mentioned forerunners, who spoke in *the lost of individuality* (or even in *the lost of identity*) of quantum objects. That means that, at the bottom level, in the quantum realm, physical objects behave differently, obeying other kinds of "statistics", which *make them* non-individuals. But Post is suggesting something different, by positing that we should *begin* with indiscernibility. His ideas had an independent echo in the first mathematical problem posed by Yuri Manin to a new List of Mathematical Problems presented at the American Mathematical Association, as a continuation of the celebrated list of Mathematical Problems advanced by Hilbert in 1900 (see [2]; Manin's problem is at page 36). There, Manin asked for the development of a theory of "sets" (his emphasis) which would enable us to deal with collections of indiscernible objects, which obvioulsy do not obey the standard postulates of set theory; for instance, they do not obey extensionality, so as they, in general, would not have an associated ordinal, and so on. More recently, the same issue was considered by John Stachel in two papers [16] and [17]. There Stachel introduced the idea used above of the top to down look, where the identity is lost (according to the standards), so as the view from the bottom to the top, which we understand as equivalent to Post's and Manin's suggestions, that is, when we begin with quantum entities properly.

The distinctive fact is that Stachel makes explicit reference to QFT, where there are not “particles” in the standard (“classical”) sense. But for sure we can say that, yet in QFT there are not “classical” particles, but this does not entails that there are not “particles” at all. But let us see Stachel himself, noting that he speaks of indiscernibility even within the context of QFT:

> Looking upward from the perspective of relativistic field theory [QFT], classically there is no particle concept associated with a field. In relativistic quantum field theory, the closest analog to the particle concept is that of 'field quantum', and one is struck by the limited range of applicability of this concept: only certain states of a quantum field diagonalize the occupation number operator for the field; and, even if the system is in such a state, one cannot attribute individuality to units that are truly field quanta. They come in different kinds; but within a kind they manifest no inherent individuality. As noted above, they possess *quiddity* but not *haecceity*. [16, p.210]

What can we learn from these remarks? Firstly, that even in the quantum field approach, that entities which are called ‘particles’, the epiphenomena of fields, present indiscernibility, having not individuality in the standard sense. But if this is so, we have a blatant contradiction with the underlying mathematics which describes them! Quantum fields, from the mathematical perspective, are “classical mathematical objects”, hence they are individuals for they obey the rules of standard theory of identity which, as we have said, is Leibnizian. As we said above, the standard trick is to keep “confined” to a certain mathematical structure, characterized by some chosen relations under which the considered objects are permutational invariant. In some sense, this is equivalent to that schema advanced by Quine in using just few predicates and relations and so defining *identity* (so he thinks) by agreement with respect to all these predicates and relations [14]. Thus, standard frameworks (read: standard set theories like ZF) enable us to deal with indistinguishable objects, but at the price of restricting the considered predicates/relations, that is, at the price of being confined to a certain structure. In other words, all we have is indiscernibility with respect to a structure [12]. Can we deal (mathematically) with truly indiscernible objects, assuming that they exist?

Thus, we think that the restriction to a certain structure, albeit it satisfies the physics, does not resolves the philosophical problem, because it corresponds to a restriction to a certain relational structure, built as above within standard set theories. But in a set theory like ZF (which we can suppose underlies our discussion), *every* structure can be extended to a rigid structure, that is, to a structure that has the identity function as the only automorphism (while Quine's approach seem to permit other as well). In short, in the whole universe of sets, any entity is an individual, and really and trully non-individuals (yet objects of some kind) cannot exist.

But there is a second remark. The talk of quiddities and haecceities concerns Stachel's claim that quantum objects have the first but not the latter, which is in agreement with Dalla Chiara and Toraldo di Francia's concept of *nomological* *objects*, namely, objects given by physical law [5]. The fact that some quantum objects of a same kind may be *absolutely* indistinguishable brings interesting problems regarding logical foundations. Let us give an example. Simon Saunders has sustained that although indiscernible, fermions obey Pauli's principle, being distinguished by an irreflexive relation, say, "having opposite direction of spin" (see for instance [13] for an updated approach). French and Rickles think that such an hypothesis put the data (quantum objects) as having ontological priority over relations, in the sense that "the former can be said to 'bear' the latter" [10]. Thus, they say, this is to be questioned under their preference for the "ontological structural approach", which intends to see the *structures* as having ontological priority over anything. But, in standard extensional set theories (with the foundation axiom), any relation is built out of the relata; thus, in order to have, say, a binary relation involving the objects a and b, we need to begin with a and b, then going to $\{a\}$ and $\{a, b\}$ (which can be done by the pair axiom), then to $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$ (again by the pair axiom) and finally to collections of such pairs. One way to cope with this problem within the scope of set theories would be to look for an intensional set theory suited to cope with quantum physics, still to be developed; another one would be to look for non-well-founded sets, or try to develop a notion of structure using, for instance, Tarski's logic of relations and the corresponding set theory without variables [18]. But our logic of indistinguishability may offer an alternative route, quite closer to the intuitions regarding physics itself, so out of these rather artificial set theoretical constructs. Let us sketch the basic idea.

Take for instance a molecule of water, H_2O. It does not matter what are the particular Hidrogen and Oxigen atoms that enter in the formation of a particular molecule; its physical properties are independent of what particular atoms are enrolled. If we look at them as individuals, then there would be a difference between two molecules of water.[2] This situation is quite different from a rugby team, where the exchange of one player by another may cause an immense difference to the final result. Once we fix the structural aspect of the molecule, which import due to the existence of isomers, the individuality of the components does not matter. Of course the same happens with most basic entities, like electrons and so on. Thus, even eventually being discerned by irreflexive relations, they cannot be regarded as *individuals* in the standard sense, despite they are "objects" of some kind (can be values of variables of some adequate language), and hence they are suitable to be mapped on some metaphysics.

The true talk of non-individuals, here understood as entities that do not obey the standard theory of identity, can be done only outside of classical framework. We guess that this might be a real situation where a quantum logic (in the sense of the logic of quantum objects) might be vindicated. Quasi-set theory is one of the possibilities. In this paper, we develop a paraconsistent version of this theory, hoping that it can be useful for further discussions not only about in-

[2]The talk that the spatio-temporal location distinguishes them —as it was said by Kant himself in a well known passage— but it does not change the quantum mechanical fact that whatever permutation of the two molecules keep the physical result absolutely the same.

discernibility, but also in those situations which involve possible contradictions, if there are some. Of course we should advance some of the applications we envisage of such a theory, but we shall leave this task for a forthcoming paper. In this one, we just sketch the theory. Anyway, something on this respect can be seen at chapter 9 of [9], and in [7].

2 The postulates of the theory $\mathfrak{Q}_P$

Let us call $\mathcal{L}$ the language of the theory $\mathfrak{Q}_P$, the paraconsistent quasi-set theory. It encompasses a list if individual variables (a denumerable one), standard logic connectives ($\neg$, $\wedge$, $\vee$, $\rightarrow$, while $\leftrightarrow$ is defined as usual), quantifiers ($\forall$ and $\exists$), and auxiliary symbols of pontuation. The specific symbols of $\mathcal{L}$ are: four unary predicates m, M, Z, and C, two binary predicates $\equiv$ and $\in$ and an unary functional symbol qc. The terms of $\mathcal{L}$ are the individual variables and the expressions of the form $qc(x)$, where x is an individual variable. Intuitively speaking, $qc(x)$ stands for "the quasi-cardinal of x". Formulas are defined as usual; informally speaking, $m(x)$ says that x is an m-atom (in the intended interpretation, a quantum object), $M(x)$ means that x is an M-atom (these objects act as the ur-elements in the theory ZFU–Zermelo-Fraenkel with *Urelemente*), which in the intended interpretation stands for a "macroscopic atom", and $Z(x)$ says that x is a *set* (a copy of a set of ZFU). Finally, $x \equiv y$ says that x is indistinguishable (or indiscernible) from y and $x \in y$ says that x is an element of y.

Some definitions are in order.

Definition 2.1 *The following concepts are useful:*

1. $\alpha^\circ := \neg(\alpha \wedge \neg\alpha)$ We say that α is *well-beaved*; otherwise, it is *ill-behaved*.

2. $\neg^*\alpha := \neg\alpha \wedge \alpha^\circ$ This is the *strong negation*. It will have all the properties of standard negation.

3. $x \stackrel{*}{=} y := [Q(x) \wedge Q(y) \wedge \forall z(z \in x \leftrightarrow z \in y)] \vee [(M(x) \wedge M(y) \wedge \forall_Q z(z \in x \leftrightarrow z \in y)]$ This is the *strong equality*, or *identity*. It will have all the properties of classical equality. For simplicity, we shall write $x = y$, and read it as "x is certainly identical to y".

4. $x \neq y := \neg^*(x = y)$ We read "x is certainly distinct from y".

5. $Q(x) := \neg m(x) \wedge \neg M(x)$ (x is a quasi-set, or qset for short).

6. $E(x) := Q(x) \wedge \forall y(y \in x \rightarrow Q(y))$ (x is a qset whose elements are also qsets, or x as no atoms as elements)

7. $x \subseteq y := \forall z(z \in x \rightarrow z \in y)$ (subqset) Remark: since the notion of identity ($\stackrel{*}{=}$ does not hold for m-atoms, in general we don't have effective means to know either a certain m-atom belongs of does not belong to a certain qset. But the definition works in the conditional form.)

8. $D(x) := M(x) \vee Z(x)$ (x is a *Ding*, a " classical object" in the sense of Zermelo's set theory, namely, either a set or a macro-ur-element).

The underlying logic of $\mathfrak{Q}_P$ is da Costa's paraconsistent calculus $\mathscr{C}_1^\star$ (see [4]). Thus, we have the following categories of postulates for $\mathfrak{Q}_P$:

2.1 First group of postulates–the underlying logic

($\rightarrow_1$) $\alpha \rightarrow (\beta \rightarrow \alpha)$

($\rightarrow_2$) $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\alpha \rightarrow \gamma))$

($\rightarrow_3$) $\alpha,\ \alpha \rightarrow \beta / \beta$

($\wedge_1$) $\alpha \wedge \beta \rightarrow \alpha$

($\wedge_2$) $\alpha \wedge \beta \rightarrow \beta$

($\wedge_3$) $\alpha \rightarrow (\beta \rightarrow \alpha \wedge \beta)$

($\vee_1$) $\alpha \rightarrow (\alpha \vee \beta)$

($\vee_2$) $\beta \rightarrow (\alpha \vee \beta)$

($\vee_3$) $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \vee \beta \rightarrow \gamma))$

($\neg_1$) $\beta^o \rightarrow ((\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \neg\beta) \rightarrow \neg\alpha))$

($^\circ{}_1$) $\alpha^o \wedge \beta^o \rightarrow (\alpha \wedge \beta)^o \wedge (\alpha \vee \beta)^o \wedge (\alpha \rightarrow \beta)^o$

($^\circ{}_2$) $(M(x))^\circ$, $(Z(x))^\circ$

($\neg_2$) $\alpha \vee \neg\alpha$

($\neg_3$) $\neg\neg\alpha \rightarrow \alpha$

($\forall_1$) $\frac{\alpha \rightarrow \beta(x)}{\alpha \rightarrow \forall x \beta(x)}$

($\forall_2$) $\forall x \alpha(x) \rightarrow \alpha(y)$

($\exists_1$) $\alpha(x) \rightarrow \exists x \alpha(x)$

($\exists_2$) $\frac{\alpha(x) \rightarrow \beta}{\exists x \alpha(x) \rightarrow \beta}$

($^\circ{}_3$) $\forall x(\alpha(x))^o \rightarrow (\forall x \alpha(x))^o$

($^\circ{}_4$) $\forall x((\alpha(x))^o \rightarrow (\exists x \alpha(x))^o$

where the variables x and y and the formulas α and β satisfy the usual restrictions. As we see, from postulate ($^\circ{}_2$), only the formulas $M(x)$ and $Z(x)$ are well-behaved, but not $m(x)$, $x \equiv y$ and $x \in y$. This will enable us to find interesting applications, as we shall show soon.

2.2 The second group of postulates–indiscernibility

The following postulates govern the relation of indiscernibility; the last one, plus axiom ($\in_3$) below, provide that extensional equality has all the characteristics of first-order identity.

($\equiv_1$) $\forall x(x \equiv x)$

($\equiv_2$) $\forall x \forall y(x \equiv y \to y \equiv x)$

($\equiv_3$) $\forall x \forall y \forall z(x \equiv y \wedge y \equiv z \to x \equiv z)$

($=_4$) $\forall x \forall y(x = y \to (\alpha(x) \to \alpha(y)))$, with the standard restrictions (recall that $x = y$ means $x \overset{*}{=} y$).

3 The new hierarchy $\mathscr{C}_n^{\equiv}$, $0 \leq n \leq \omega$ of paraconsistent calculi with indiscernibility

The above postulates enable us to extend the usual procedures used to characterize da Costa's $\mathscr{C}$-systems to introduce a new hierarchy of paraconsistent calculi, the hierarchy $\mathscr{C}_n^{\equiv}$, $1 \leq n \leq \omega$ of *paraconsistent calculi with indiscernibility*, as follows. We shall not develop these systems here, but just to mention the propositional hierarchy. Of course the first calculus $\mathscr{C}_1$ is subsumed in the above axiomatics, but the whole hierarchy is mentioned here just to emphasize this new category of paraconsistent logics. Let us begin with $\mathscr{C}_1^{\equiv}$. Its language is obtained from the language of the predicate calculus $\mathscr{C}_1^{\star}$ by adding a primitive binary symbol $\equiv$ of indistinguishability, subjected to the following postulates:

($\equiv_1$) $\forall x(x \equiv x)$

($\equiv_2$) $\forall x \forall y(x \equiv y \to y \equiv x)$

($\equiv_3$) $\forall x \forall y \forall z(x \equiv y \wedge y \equiv z \to x \equiv z)$

These postulates intuitively say that $\equiv$ has the properties of an equivalence relation. But we do not postulate that substitutivity holds, that is, it is not the case that $\forall x \forall y(x \equiv y \to (\alpha(x) \to \alpha(y)))$ (with the standard restrictions). The motive is that with this property plus ($\equiv_1$), the indistinguishability relation would collapse into identity, and no formal distinction between them both would exist. Anyway, from the intuitive physical point of view, it seems that if x and y stand for indistinguishable micro-objects, their substitution from one another in any context would be accepted. In order to achieve this desired result, we use a roundabout way, namely, the theorem (5.1) presented below. Thus, we reach to the intended result without keeping indistinguishability and identity the same formal concept.

It is of course easy to reach to a new hierarchy of paraconsistent calculi

$$\mathcal{C}_0^{\equiv}, \mathcal{C}_1^{\equiv}, \mathcal{C}_2^{\equiv}, \ldots, \mathcal{C}_n^{\equiv}, \ldots, \mathcal{C}_\omega^{\equiv}. \quad (1)$$

by the same procedures that conduce to other hierarchies of paraconsistent calculi (see [4]). Here, $\mathcal{C}_0^{\equiv}$ might be understood as the classical first order calculus without equality but with a binary primitive predicate symbol $\equiv$ of indistinguishability, subjected to the postulates ($\equiv_1$) to ($\equiv_3$).

4 Axioms for quasi-set

Starting with the system $\mathfrak{Q}$ of quasi-set theory, we can introduce a new hierarchy of paraconsistent quasi-set theories $\mathfrak{Q}_n$, $0 \leq n \leq \omega$. The quasi-set theory $\mathfrak{Q}$ will be called here $\mathfrak{Q}_0$. The language L of $\mathfrak{Q}_1$ is that described in section 2. The syntactic notions of L are obvious adaptations of those of $\mathscr{C}_1^{\equiv}$.

In the theory $\mathfrak{Q}$, we had a postulate saying that no x can be an m and an M atom simultaneously, that is, $\forall x(\neg m(x) \vee \neg M(x))$. Here, due to the ill behaviour of the predicate m, we drop this axiom but introduce another one, specific to the new predicate C (for "crisp", or "sharp"):

(C) $\forall x(m(x) \rightarrow (C(x) \rightarrow M(x)))$

This postulate may sound strange, but it has a strong motivation. A certain m-object may, by some process, described case-by-case by a device described by the predicate C, becomes a M-object, say when a quantum entity becomes 'classical', making a *click* in an experimental device. Thus, C stands for 'crisp", in opposition to 'blurring'. Intuitively speaking, C so to say eliminates the quantum behaviour of x. It expresses a kind of 'collapse' of something related to the quantum entity.[3] The way something becomes crisp of course depends on physics, and it not a matter of logic.

Definition 4.1 (Well-Behaved Membership) $x \in^* y := (x \in y)^\circ$

When $x \in^* y$, we may say that "x certainly belongs to y". Important to realize that $\neg x \in^* y$ is not equivalent to $\neg x \in y$ (that is, to $x \notin y$). Really, we have $\neg x \in^* y \rightarrow \neg x \in y$, but not conversely.

Other postulates are the following ones:

($\in_1$) $\forall x \forall y(x \in y \rightarrow Q(y))$ If something has an element, then it is a qset; in oter words, the atoms have no elements (in terms of the membership relation).

($\in_2$) $\forall_D x \forall_D y(x \equiv y \rightarrow x = y)$ Indistinguishable *Dinge* are extensionally identical. This makes $=$ and $\equiv$ coincide for this kind of entities.

($\in_3$) $\forall x \forall y[(m(x) \wedge x \equiv y \rightarrow m(y)) \wedge (M(x) \wedge x = y \rightarrow M(y)) \wedge (Z(x) \wedge x = y \rightarrow Z(y))]$

[3] In quantum physics, of course that what collapses is the wave function, which describes the state of the system. But we are here working in close analogy to that.

($\in_4$) $\exists y \forall x(\neg^* x \in y)$ This qset will be proved to be a set (in the sense of obeying the predicate Z), and it is unique, as it follows from the weak extensionality (below). Thus, from now own we shall denote it, as usual, by '$\emptyset$'.

($\in_5$) $\forall_Q x(\forall y(y \in x \to D(y)) \leftrightarrow Z(x))$ This postulate grants that something is a set (obeys Z) iff its transitive closure does not contain m-atoms. That is, *sets* in $\mathfrak{Q}$ are those entities obtained in the 'classical' part of the theory (see figure 1).

($\in_6$) $\forall x \forall y \exists_Q z(x \in z \wedge y \in z)$

Postulate ($\in_6$) must be explained, and we shall do it below. The other ones are intuitive. The quasi-set universe can be seen in the following figure:

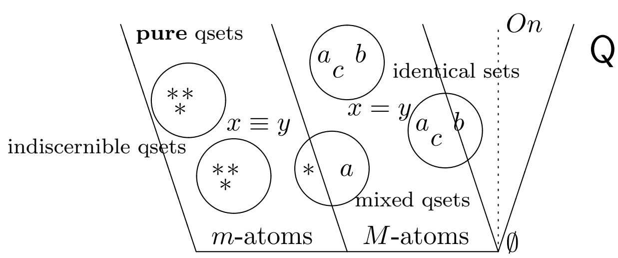

Figure 1: The Quasi-Set Universe: On is the class of ordinals, defined in the 'classical' part of the theory.

($\in_7$) If $\alpha(x)$ is a formula in which x appears free, then

$$\forall_Q z \exists_Q y \forall x(x \in y \leftrightarrow x \in z \wedge \alpha(x)).$$

The qset y of the schema ($\in_7$), the Separation Schema, is denoted by $[x \in z : \alpha(x)]$. When this qset is a set, we write, as usual, $\{x \in z : \alpha(x)\}$.

($\in_8$) $\forall_Q x(E(x) \to \exists_Q y(\forall z(z \in y \leftrightarrow \exists w(z \in w \wedge w \in x)))$. The union of x, writen $\cup x$. Usual notation is used in particular cases.

Remark on ($\in_6$) As we see, it says that given x and y, there is a qset z which contains both of them, although z may have other elements as well. But, from the separation schema, using the formula $\alpha(w) \leftrightarrow w \equiv x \vee w \equiv y$, we get a subqset of z which we denote $[x, y]_z$, the qset of the indiscernibles of either x or y that belong to z. When $x \equiv y$, this qset reduces to $[x]_z$, called the *weak singleton* of x. Later, with the postulates of quasi-cardinal, we will be able to prove that this qset has a subqset with quasi-cardinal equals to 1 (really, the theory is compatible with the existence of more than one of such sets), which we call a *strong singleton* of x (in z), written $[[x]]_z$ (sometimes the sub-indice will be left implicit). A counter-intuitive fact is that, since the relation $\equiv$ is reflexive and the strong singleton of x (really, *a* strong singleton, for we cannot grant that it is unique) has just one element, we can think that this element *is* x. But this cannot be proven in the theory, for such a proof would demand the identity relation, which cannot be applied to m-atoms. Anyway, we can informally reason *as if* the element of the strong singleton $[[x]]_z$ *is* x, although this must be understood as a way of speech. Perhaps the better way to refer to this situation is (informally) to say that the element of $[[x]]_z$ is *an object of the kind* x.

Remark on the extensional identity Some remarks are in order at this point. We are using just one sort of variables, for we think we can circumvent some of the problems that may appear due to this option. For instance, although we have originally motivated quasi-set theory with the claim that the standard concept of identity would not apply to quantum entities (here represented by the m-atoms), definition 2.1(3) makes things a little bit different from the formal point of view. Really, since we are using a monosorted language, if $m(x)$, then by the definition, we get $\neg(x = y)$ for any y. In particular, $\neg(x = x)$ for any m-object x. The same happens if x is a qset having m-atoms in its transitive closure, that is, being x a qset which is not a set (in the sense of the predicate Z). That is, if $Q(x)$ and $\neg Z(x)$, then $\neg(x = y)$ for any y, and in particular $\neg(x = x)$. Anyway, there are no (as far as we know) formal problems concerning these facts, for we have only "deduced", say, that $\neg(x = x)$ for an m-object x, but we can't go to its identity. Although the third excluded law holds, even being $\neg(x = y) \vee (x = y)$ a theorem of $\mathfrak{Q}_P$, we never get $x = x$ in the case of m-objects. Intuitively, perhaps we can say that, since the concept of identity should make no sense to m-objects, it would be quite natural that they cannot be identical to themselves. But the above theorem would not occur if we have used a many-sorted language.

But let us go back to the postulates of $\mathfrak{Q}_P$.

($\in_9$) $\forall_Q x \exists_Q y \forall z(z \in y \leftrightarrow w \subseteq x)$, the power qset of x, denoted $\mathcal{P}(x)$.

($\in_{10}$) $\forall_Q x(\emptyset \in x \wedge \forall y(y \in x \to y \cup [y]_x \in x))$, the infinity axiom.

($\in_{11}$) $\forall_Q x(E(x) \wedge x \neq \emptyset \to \exists_Q y(y \in x \wedge y \cap x = \emptyset))$, the axiom of foundation, where $x \cap y$ is defined as usual.

Interesting to note that in $\mathfrak{Q}$, so as in $\mathfrak{Q}_P$, we cannot do the simple exercise of obtaining the power qset of a given qset z, say one which has quasi-cardinal 2 and its elements are indistinguishable from either x or y, supposed indiscernible. In other words, we cannot write something like $\mathcal{P}(z) = [\emptyset, [[x]], [[y]], z]$, for to do that we need the identity relation. This is due to the fact that $[[x]] \equiv [[y]]$, as it result from the postulates, and we cannot discern between these two qsets for any tools provided by the theory. Nevertheless, it seems intuitive that such a z should have 4 subqsets. This fact shall be grant by a suitable postulate to be presented later. So, the theory is compatible with the hypothesis that z has 4 subqsets, yet we cannot exemplify this fact by samples, but only from the postulates (which, by the way, is that what imports). The same can be said concerning the axiom of foundation.

Definition 4.2 (Weak ordered pair) *Being z a qset to which both x and y belong, we pose*

$$\langle x, y\rangle_z := [[x]_z, [x, y]_z]_z \tag{2}$$

Then, $\langle x, y\rangle_z$ takes all indiscernible from either x or y that belong to z, and it is called the "weak" ordered pair, for it may have more than two elements. Sometimes the sub-indice z will be left implicit.

Definition 4.3 (Cartesian Product) *Let z and w be two qsets. We define the cartesian product $z \times w$ as follows:*

$$z \times w := [\langle x, y\rangle_{z\cup w} : x \in z \wedge y \in w] \tag{3}$$

Functions and relations cannot also be defined as usual, for when there are m-atoms involved, a mapping may not distinguish between arguments and values. Thus we provide a wider definition for both concepts, which reduce to the standard ones when restricted to classical entities. Thus,

Definition 4.4 (Quasi-relation) *A qset R is a binary quasi-relation between to qsets z and w if its elements are weak ordered pairs of the form $\langle x, y\rangle_{z\cup w}$, with $x \in z$ and $y \in w$.*

More general quasi-relations (n-ary) can be defined easily. Thus, a quasi-function (mapping) between z and w is a binary quasi-relation f between them such that if $\langle x, y\rangle \in f$ and $\langle x', y'\rangle \in f$ and if $x \equiv x'$, then $y \equiv y'$. In other words, a quasi-function maps indistinguishable elements indistinguishable elements. It is easy to define the corresponding concepts of injective, surjective, and bijective quasi-functions.

Here we can see a distinctive characteristics of m-objects. Suppose we have a set with 5 "classical" elements, ordered as $a_1, \ldots, a_5$. Of course a permutation between two of them, say the permutation π_{13} which exchange a_1 and a_3 leads to a different arrangement. But this would not happen if they were indistinguishable m-atoms, for no permutation would distinguish them. Taking an analogy, a queue with John, Mary, and Tom in this order (say, to buy tickets to the best places in a theatre) is different from another with Mary, Tom, and John.

Concerning quantum objects, physicists have used certain "queues" of Calcium ions to transmit information, and of course the information is independent of the position of the particular Calcium ions (which is different of some information—say a social fact involving personalities—transmitted among people, women, say, which can reach to the end of the row differently depending on the order of the involved people—*pace*, girls).[4] Quantum objects, as far as we know, are exactly the same in these aspects too; their permutations do not change the relevant physical facts, say the relevant probabilities (or information).

The last part of the theory we need to explain before going to its paraconsistent aspects deals with the concept of quasi-cardinal.

5 Postulates for quasi-cardinals

In standard approaches to set theory, as it is well known, a cardinal is a particular ordinal. Hence, we must have the ordinal concept defined first in order to go to cardinals. This is not absolutely necessary, but it is the standard way of using these concepts. In our case, the idea is that a pure qset of indiscernible objects may have a cardinal (its *quasi-cardinal*), but not an associated ordinal. Hence, we have chosen to take the concept of quasi-cardinal as primitive, subjected to adequate postulates that grant the operational character of the concept. In the first versions of $\mathfrak{Q}$, we have said that every qset has a cardinal. But Domenech and Holik [6], and independently Arenhart [1], have argued that when we consider relativistic quantum physics, sometime we can't associate a cardinal to any collection. They are right, so we have changed axiom 5 below, just by enabling that *some* qsets may have not an associated cardinal. The postulates are as follows, where we use α, β, etc, for naming representing:

(qc_1) $\forall_Q x(\exists_Z y(y = qc(x)) \rightarrow \exists! y(Cd(y) \wedge y = qc(x) \wedge (Z(x) \rightarrow y = card(x)))$
If the qset x has a quasi-cardinal, then its (unique) quasi-cardinal is a cardinal (defined in the 'classical' part of the theory) and coincides with the cardinal of x stricto sensu if x is a set.

(qc_2) $\forall_Q x(x \neq \emptyset \rightarrow qc(x) \neq 0)$. Every non-empty qset has a non-null quasi-cardinal.

(qc_3) $\forall_Q x(\exists_Z \alpha(\alpha = qc(x)) \rightarrow \forall \beta(\beta \leq \alpha \rightarrow \exists_Q z(z \subseteq x \wedge qc(z) = \beta)))$ If x has quasi-cardinal α, then for any cardinal $\beta \leq \alpha$, there is a sub-qset of x with that quasi-cardinal.

In the remaining axioms, for simplicity, we shall write $\forall_{Qqc} x$ (or $\exists_{Qqc} x$) to mean that the qset x has a quasi-cardinal.

(qc_4) $\forall_{Qqc} x \forall_{Qqc} y(y \subseteq x \rightarrow qc(y) \leq qc(x))$

(qc_5) $\forall_{Qqc} x \forall_{Qqc} y(Fin(x) \wedge x \subset y \rightarrow qc(x) < qc(y))$

It can be proven that if both x and y have a quasi-cardinal, then $x \cup y$ has a quasi-cardinal. Then,

(qc_5) $\forall_{Qqc} x \forall_{Qqc} y(\forall w(w \notin x \vee w \notin y) \rightarrow qc(x \cup y) = qc(x) + qc(y))$

[4] Concerning the experiments with Calcium ions, see the *Scientific American*, March 2006.

In the next axiom, $2^{qc(x)}$ denotes (intuitively) the quantity of subquasi-sets of x. Then,

(qc_6) $\forall_{Qqc} x(qc(\mathcal{P}(x)) = 2^{qc(x)})$

This last axiom is precisely that one which enables us to think that, given a qset with cardinality, say, equal to 4, then it has 16 subqsets. Some of them, as we shall say with the last axiom below, cannot be discerned from one each other, but they *do not count as one!*. This is important for expressing an ontology of quantum theory; suppose an Helium atom in its fundamental state. There are two electrons being considered, so we can represented them by a qset of indiscernible m-atoms with quasi-cardinal 2. The subqsets containing just one electron cannot be discerned from one another, but they of course count as two. Of course the electrons have not all their quantum numbers in common, due to Pauli's Principle, but this does not matter: the important thing is that we cannot (even in principle!) to say which is which (see the next section), and in $\mathfrak{Q}$, the same happens with their unitary qsets.

The last postulate of $\mathfrak{Q}_P$ is the Weak Extensionality Axiom,[5] which intuitively says that qsets with "the same quantity" (expressed in terms of the quasi-cardinals) of elements of "the same kind" (related by $\equiv$) are indistinguishable (are themselves in the relation $\equiv$). In the statement of the postulate below, $Qsim(z, t)$ means that the elements of z and t are indiscernible and that they have the same quasi-cardinal; $x/\equiv$ stands for the quotient qset of x by the relation $\equiv$. In symbols,

$(\equiv_5)$ $\forall_Q x \forall_Q y((\forall z(z \in x/\equiv \rightarrow \exists t(t \in y/\equiv \wedge \wedge QSim(z, t)))) \wedge \forall t(t \in y/\equiv \rightarrow \exists z(z \in x/\equiv \wedge \wedge QSim(t, z))) \rightarrow x \equiv y)$

The following theorem express the invariance by permutations in $\mathfrak{Q}_P$, and with this result we finish our revision:

Theorem 5.1 *Let x be a finite qset such that $\neg^*(x = [z])$ and let z be an m-atom such that $z \in x$. If $w \equiv z$ and $w \notin x$, then there exists $[[w]]$ such that*

$$(x - [[z]]) \cup [[w]] \equiv x$$

Proof: <u>Case 1:</u> $t \in [[z]]$ does not belong to x. In this case, $x - [[z]] = x$ and so we may admit the existence of $[[w]]$ such that its unique element s does belong to x (for instance, s may be z itself); then $(x - [[z]]) \cup [[w]] = x$. <u>Case 2:</u> $t \in [[z]]$ does belong to x. Then $qc(x - [[z]]) = qc(x) - 1$ (by a result not proven here).[6] We then take $[[w]]$ such that its element is w itself, and so it follows that $(x - [[z]]) \cap [[w]] = \emptyset$. Hence, by (qc_6), $qc((x - [[z]]) \cup [[w]]) = qc(x)$. This intuitively says that both $(x - [[z]]) \cup [[w]]$ and x have the same quantity of indistinguishable elements. So, by applying the week extensionality axiom, we obtain the result. ∎

[5]We shall not list there the Replacement Axioms and the qset version of the Axiom of Choice—which intuitively says that from a qset with qsets as elements and 2x2 disjoints, there is a qset having just one (expressed by the quasi-cardinal) element of each one of the elements of the given qset.

[6]But see [9, p.293].

6 Paraconsistency

There are certain aspects of the behaviour of quantum objects that defy our standards, which usually are grounded on classical logic, classical mathematics, and classical physics. In a certain sense, all these marvelous fields were built taking into account the way we understand (or understood) our environment (at our scale). To simplify, we can say that our immediate world is a world of individuals, of well defined and distinct objects, each one having its identity, having properties and being in relations. But quantum objects are different.[7] For instance, sometimes we can distinguish between two quantum objects, say two fermions, for they obey an irreflexive and symmetric relation, as the two electrons of an Helium atom in the fundamental state, which have all the same quantum numbers, except the spin: one of them is UP, while the another one is DOWN. The problem is that (as far as we believe in quantum mechanics) we *never* can say which is which. If they were classical objects, we would be able, at least in principle, to identify them, and even name them Peter and Paul, as we do with two identical twins. Furthermore, the physical situation changes when two macroscopic objects are permuted, even if they are 'indiscernible': one thing to say that Peter is in the kitchen and Paul is in the garden and another situation is the opposite one. But, concerning fermions, since the irreflexive relation "... has spin in the opposite direction to ..." is also symmetric, it is indifferent which has spin UP and which has spin DOWN. Technically, the join system is described by an anti-symmetric function ψ_{12} which changes sign when the objects are permuted (getting ψ_{21}), but their square, which gives the relevant probability, lead to the same value ($|\psi_{12}|^2 = |\psi_{21}|^2$).

In short, we can say that quantum objects have, for instance, the following characteristics, some of them partaken with standard objects:

1. They can be collected into amounts. We can say that we can form collections with quantum objects having certain cardinalities (which we can suppose are finite cardinals). But contrariwise to standard sets (sets of standard set theories have most of the properties we ascribe to collections of macro-objects), two collections of such entities may be indiscernible from one another, as two molecules of a composite, or then like to protons, which are composed by quarks.

2. Yet sometimes quantum objects can be distinguished by a quantum number, as the two electrons just mentioned, they do not have *individuality*, in the sense that, say, any permutation of them does not entail to any physical difference, contrariwise to the case of the two identical twins Peter and Paul of the preceding example. By the way, we should not confuse distinguishability, which is something an object may have with other objects, and individuality, which seems to be something of its own, as we have learnt ever since the Scholastics (see [9]).

[7] We shall just emphasize some of these differences, for most of them are discussed in the literature.

3. As our theorem 5.1 has expressed, if we have a certain collection of quantum objects and by some device we exchange one of the collection with *another* one from the outside which is indistinguishable from *that* one (in the sense of being an entity of the same kind, say two protons, or electrons, or even atoms and certain molecules), nothing detectable is achieved. Some examples can be given with ionization processes and with some chemical reactions. In ionization, for instance, a certain neutral atom can lose one electron, turning to a negative ion, and then this ion can absorb an electron, in order to turn again a neutral atom. As far as we know, there are no differences between the two permuted electrons, although we cannot say neither that they are different nor that they are identical (yet we can express that in the language), nor between the first and the second neutral atoms. Of course these collections should be not modeled by sets of standard set theories.

We could continue to list the main characteristics of quantum objects in distinction to 'classical' ones, but we think that perhaps these few ones suffice.

In the theory presented above, there are no distinctions between m-atoms and M-atoms, except that no object can be both at the same time.[8] Furthermore, if m-objects are intended to represent quantum objects in some sense, it would be interesting to distinguish at least between two basic categories, which by resemblance to the physical case we can call *fermions* and *bosons*. Here we shall introduce a definition inspired in Muller & Saunders [13], which helps us to distinguish fermions as those m-objects that obey an irreflexive but symmetric relation (these authors call these objects *weakly discernible*). Thus,

Definition 6.1 (Fermions) *Let x be an m-atom. We say that x is a fermion iff there exists an m-atom y and an irreflexive and symmetric relation R such that $R(x, y)$ holds.*

Of course the name *fermions* does not intend to provide a definition of the physical objects in quantum theory. It is here just a name used in resemblance with the physical situation.

It results from the symmetry of the relation that y is also a fermion. In set theoretical terms, that is, within the framework of standard (extensional) set theory, if we say that there is an irreflexive and symmetric relation on a set A and nothing else, the only think we can deduce is that A has at least two elements. Thus, the existence of such a relation is a way to say that there are more than one element in A, although they are not being individualized in the structure $\langle A, R\rangle$, being R as above. The problem with regard quantum physics is that, as we have said, it is not possible, even *in mente Dei*, to distinguish between the two fermions of the He atom, but the god of standard mathematics is more complacent, and in a set theory like ZF we can always extend the

[8] In our opinion, we need to add to the axioms of quasi-set theory some kind of mereology linking the atoms, as we did in [11]. But there are some problems to be overcome, as we have mentioned in that paper.

considered structure $\langle A, R\rangle$ to a rigid one where the individuality of the elements is enlighten. This apparently poses a strong evidence for the fact that if we want to work with *legitimate* (whatever sense we give to this word) quantum objects, we must leave the classical standards. If not, the only thing we can achieve is indiscernibility relative to certain properties/relations we consider in our relevant structures. In other words, if we are not satisfied with the rather artificial mathematical trick of simply ignoring that there are rigid structures playing the same role than our chosen structures and wish to work with quantum objects as they are to be sometimes, namely, absolutely indistinguishable and, in the case of fermions, weakly discernible only, we see no way other than to consider something like quasi-set theory.

We emphasize that this conclusion is not a simple desire to go against classical logic, which we consider fundamental and always 'true' in its particular domain of application, but as a fact. An analogy may help here. It is well known that in informal set theory, with the full Principle of Abstraction, we can easily derive the existence of the so-called *Russell set*, namely, $\mathcal{R} = \{x : x \notin x\}$, which conduces to a contradiction. In the axiomatic versions, like ZF, this 'set' cannot even more be obtained, due to the restrictions imposed to abstraction. But we may wish to study mathematical objects like $\mathcal{R}$, and of course this cannot be done *within* ZF (supposed consistent). But we can do that for instance in some paraconsistent set theories [4]. The same seems to happen here. If we wish to study *absolute* indiscernible objects, or weakly discernible objects without admitting that can be shown to be individuals by some additional (hidden) relation or property, classical logic and set theory again seem to need re-analyses.

Let us go back to the two electrons example to see some possible paraconsistent analogies. We can say that they are distinguishable by the irreflexive relation: "X has opposite direction of spin to Y". This relation distinguishes the two fermions, so, $\neg(x \equiv y)$ in $\mathfrak{Q}_P$. But $\mathfrak{Q}_P$ has all theorems of the paraconsistent calculus $\mathscr{C}_1$, in particular, $\beta^\circ, \alpha \to \beta \vdash \neg\beta \to \neg\alpha$ [4, Th.2.1.8]. Furthermore, axiom ($\in_2$) states that $x = y \to x \equiv y$, so, since $x \equiv y$ is not well-behaved,

$$\mathfrak{Q}_P \nvdash \neg(x \equiv y) \to \neg(x = y). \tag{4}$$

This result may be interpreted as follows. The fact that two fermions are weakly discernible does not entail that they are distinct individuals. So, although counting as more than one, they may continue to be indiscernible in a sense, the sense according to which we don't have any criterion to say which is which. Really, since $x \equiv y$ is ill-behaved, and since $\neg$ is the paraconsistent negation, the formula $(x \equiv y) \wedge \neg(x \equiv y)$, being a theorem of $\mathfrak{Q}_P$, does not trivialize the system, and seems to be in complete agreement with some claims posed by quantum physics, namely, that the two electrons of an He atom in its fundamental state *are* discernible for there exists such an irreflexive and symmetric relation, but continue to be indiscernible for they cannot be identified as individuals (in the standard sense). Furthermore, since in $\mathfrak{Q}_P$ the notion of identity lacks sense for m-atoms, we cannot extend the relational structure $\langle A, R\rangle$ (let us take a minimal one, composed by a qset A with the two electrons and the irreflexive and symmetric relation R) to a rigid one encompassing the identity predicates *"to be identical to ..."*. In this sense, the theory $\mathfrak{Q}_P$ seems to serve to express some traits in quantum physics.

A second way to relate quasi-sets and paraconsistency may be the following one. Recall that the formula $m(x)$ is ill-behaved, thus, we can have $m(x) \wedge \neg m(x)$ without trivialization (where $\neg$ is the paraconsistent negation). How can a certain object x be 'at the same time' an m-object and a 'non'-m-object? Of course this does not make sense for standard objects of our surroundings, but some experiments made with certain physical substances *near* to macroscopic scale, such as fullerene C_{60} and C_{70}, fluorinated fullerene molecules $C_{60}F_{48}$, and *arobenzenes*, show that these "macro molecules" present typical quantum behaviour, namely, interference in two-slit type experiments [**?**]. It should be not so strange that the near future shows the same with still most massive and large substances. Thus, the fact that our theory enables both $m(x)$ and $\neg m(x)$ (being $\neg$ the paraconsistent negation), it is not so extraordinary even from the point of view of empirical sciences.

The last example makes reference to the axiom (C). In says that in the presence of some device expressed by the predicate C (which of course may be the conjunction of several other predicates, depending on the particular model being considered), an m-object may turn to me an M-object. The action of C impedes that an m-object may have a blur behavious. This is again not so strange if we give it an adequate interpretation, say by considering collections of m-objects and the phenomenon of decoherence (but this is far from our intentions here).

When a micro-object becomes a macro-object? A quick answer is: in the collapse, in our formalism, when it satisfies the predicate C. The wave function describing a quantum system, say a system of indiscernible ("identical" in the physicists' jargon) quantum particles is a linear combination of a basis of eingenvectors of some Hermitian operator on the relevant Hilbert space. When a measurement is made, the wave function *immediately* collapses to one of the eingenvectors of the basis. Experimentally, we observe a "click" in the apparatus. This is a distinctive signal, a macroscopic signal that indicates that the quantum system does not exist any more and that we have now something "classical".

Of course these examples are put here as pure expeculations, but we hope some other interested people can help us in making the connections between quantum physics and paraconsistency.

References

[1] Arenhart, J. R. B. [2012], 'Finite Cardinals in Quasi-set Theory'. Forthcoming in *Studia Logica*.

[2] Browder, F. (ed.) [1976]: *Mathematical Problems Arising from Hilbert Problems*, (Proceedings of Symposia in Pure Mathematics, Vol. XXVIII), Providence: American Mathematical Society.

[3] da Costa, N. and de Ronde, C. [2012], 'On the Physical Representation of Quantum Superpositions', In *Anais do VII Simpósio Internacional Principia*, Florianópolis, NEL/UFSC, Col. Rumos da Epistemologia, Vol. 11, 118-131.

[4] da Costa, N. C. A., Krause, D. and Bueno, O. [2007]: 'Paraconsistent logics and paraconsistency', in D. Jacquette (ed.), *Philosophy of Logic*, Elsevier. Handbook of the Philosophy of Science, v. 5, pp. 655-781.

[5] Dalla Chiara, M. L. and Toraldo di Francia, G. [1993]: 'Individuals, kinds and names in physics', in Corsi, G. *et al.* (eds.), *Bridging the gap: philosophy, mathematics, physics*, Kluwer Ac. Pu., pp. 261-283.

[6] Domenech, G. and Holik, F. [2007]: 'A discussion on particle number and quantum indistinguishability', *Foundations of Physics* **37** (6), 855-878.

[7] Domenech, G., Holik, F. and Krause, D. [2008]: 'Q-spaces and the foundations of quantum mechanics', *Foundations of Physics*

[8] Falkenburg, B. [2007]: *Particle Metaphysics: A Critical Account of Subatomic Reality*, Berlin: Springer.

[9] French, S. and Krause, D. [2006]: *Identity in Physics: A Historical, Philosophical, and Formal Analysis*, Oxford: Oxford Un. Press.

[10] French, S. and Rickles [2003]: 'Understanding permutational symmetry', http://arxiv.org/abs/quant-ph/0301020v1

[11] Krause, D. [2011]: 'A calculus of (non-)individuals: ideas for a quantum mereology', forthcoming in Dutra, L. H. de A. e Mortari, C. A. (orgs) [2011]: *Anais do VII Simpósio Internacional Principia*, Florianópois, NEL/UFSC.

[12] Krause, D. and Coelho, A. M. N. [2005]: 'Identity, indiscernibility, and philosophical claims', *Axiomathes* **15**, pp.191-210.

[13] Muller, F. A. and Saunders, S. [2008]: 'Discerning fermions', *British Journal for the Philosophy of Science* **59**, pp.499-548.

[14] Quine, W. V. [1986]: *Philosophy of Logic*, Massachusetts and London, 2a. ed: Harvard Un. Press.

[15] Saunders, S. [2006]: 'Are quantum particles objects?', *Analysis* **66**, pp.52-63.

[16] Stachel, J. [2005]: 'Structural realism and contextual individuality', in Y. Ben-Menahem (ed.), *Hilary Putnam*, Cambridge: Cambridge Un. Press.

[17] Stachel, J. [2006]: 'Structure, individuality, and quantumgravity', in Rickles, D. and French, S. (eds.), *The Structural Foundations of Quantum Gravity*, Oxford, Oxford Un. Press, 2006, pp. 53-82.

[18] Tarski, A. and Givant, S. [1987]: *A Formalization of Set Theory without Variables*, (Colloquium Publications), American Mathematical Society.

[19] van Fraassen, B. [1991]: *Quantum Mechanics: An Empiricist View*, Oxford: Clarendon Press.