Structures and Models of Scientific Theories: A Discussion on Quantum Non-Individuality

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Abstract. In this paper we consider the notions of structure and models within the semantic approach to theories. To highlight the role of the mathematics used to build the structures which will be taken as the models of theories, we review the notion of mathematical structure and of the models of scientific theories. Then, we analyse a case-study and argue that if a certain metaphysical view of quantum objects is adopted, namely, that which sees them as non-individuals, then there would be strong reasons to ask for a different mathematical framework for describing the structures that would be the models of the corresponding theory. In departing from the standard frameworks (that is, worked on within standard mathematics), we hope to bring to the scene, within the scope of the semantic approach, the importance of paying attention to some fundamental concepts usually only superficially touched by philosophers of science (if touched).

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1. Introduction: the role of foundational studies

Ever since the end of the nineteenth century, philosophy of science has established itself as a respectful branch of philosophy. The great specialization of scientific disciplines and the overall role of science in contemporary life corroborates the need of philosophical reflection in search of an understanding of many of the intricacies created by scientific practice. Obviously, these reflections involve all aspects of science, such as social and ethical aspects, as

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well as epistemological, ontological, and metaphysical, among possibly others. Related to these issues but with a different set of worries, we may be concerned with the logical foundations of some of these disciplines. Briefly speaking, foundational studies have among their main task the clarification of the ideas and methods involved in the formulation of scientific theories (either in physics, biology, human sciences, etc.). For instance, we may be interested in problems such as what *theories* are, *where* they are formulated (that is, the relevance of a particular mathematics for the 'construction' of a theory), what we mean by ambiguous words such as *model* of a scientific theory, what is the relationship between theories and *reality*, if there is some, etc.

The philosopher seeking foundational studies, if concerned particularly with the *logical* foundations of scientific theories, would address a critical discussion to some of these problems. So, in investigating the way the theories are built, we can, for example, pay attention to their underlying logic and mathematics. There are several different—and not equivalent—logics *and* mathematics; in trying to make more rigorous the exposition of scientific theories, so that the underlying logic and mathematics becomes totally explicit, we need to employ the axiomatic method as a tool for conceptual clarity (but it works also as an heuristic tool in science—[26]). This kind of investigation involves, for example, the analysis of the dichotomy between the informal discourse of the scientists and the corresponding facts described by the theories they consider, and how these notions fit in one among the various possible rigorous reconstructions of the theory.

For example, in quantum theory (say orthodox—non-relativistic quantum mechanics),[1] according to the discourse of certain interpretations, we find philosophers speaking about *particles*, *indistinguishability*, *identity*, and the like, but without attributing precise meaning to these notions. For instance, let us fix for a moment on the notion of indistinguishability, or indiscernibility. It is usually recognized that quantum entities may be indiscernible, and that quantum mechanics cannot provide any distinction among them. A typical example is that of a Bose-Einstein condensate, where we find lots of particles or atoms in the very same quantum state, being indiscernible by all mechanisms provided by quantum theory. Another example is that of the two electrons of a Helium atom, but the situation is a little bit different. In this case, we have two quantum objects (electrons) but they are not in the same state; being fermions, electrons must obey Pauli's Exclusion Principle, and so they present a difference, in this case, the direction of their spin. One of them has spin UP in a certain direction, while the other one has spin DOWN in the same direction. The problem is that nothing can tell us

[1] Really, we should distinguish among several 'quantum mechanics', for there are different interpretations of the formalism, each having its peculiar characteristics, such as matrix mechanics (Heisenberg, Born, Jauch), wave mechanics (de Broglie, Schrödinger), bohmian mechanics (Bohm), and so on. Here, when speaking of 'quantum mechanics' (QM), or 'quantum theory', we are thinking more in the mathematical formalism, which we assume being done by the Hilbert-space approach.

which is which, as this is well known. But quantum mechanics is erected using standard mathematics (and logic), so, the following version of the excluded middle law holds: for *any* objects a and b, $a = b \vee a \neq b$ is a logical truth. Since the two electrons are not the very same entity (which is the meaning of $a = b$), then we need to accept that $a \neq b$. Hence, once we cannot say which is which, it seems that what is lacking is some kind of hidden variables, something that would be added to the formalism of quantum mechanics in order to distinguish them. But this move is not accepted in general, mainly due to the no-go theorems such as Kochen-Specker's [13]. So, how should we to understand the relationship between identity and indistinguishability? Do these concepts coincide (as in standard logic and mathematics), or can they be kept separated? Do identity and indistinguishability always apply to the items treated by the theory or do they have a limited range? Obviously, this is where the philosopher interested in foundations enters the stage, and we shall have more to say on this specific topic later.

In this work, we shall be concerned precisely with this kind of issue. We shall discuss the precise formulation of scientific theories on nowadays most widely accepted view, the semantical approach to scientific theories, and point to some of the consequences of adopting the classical mathematical framework in philosophical discussions. Our paradigmatic case will be the well-known problem of identity and individuality in quantum mechanics. We begin with the notion of structure.

2. Structures, models and theories

In this section we assume first-order ZFC. Our aim is to sketch how we can understand scientific theories according to the so-called *semantic approach* to theories. One of the slogans of the approach is that to present a theory is to present a class of structures, the models of the theory. Here, we start to see a link between many of the notions that should be clarified in foundational studies, according to the previous discussion. We begin with structures, following the general theory exposed in [4]. Notice that the fact that we are working in ZFC has its own consequences, and making the underlying framework explicit is part of the foundational worry with rigor, which will be touched on later.

2.1. Structures

We begin with the notion of structure. The first important definition we shall need is that of types:

Definition 2.1 (Types). *The set $\mathbb{T}$ of types is the least set satisfying the following conditions:*

(a) $i \in \mathbb{T}$ (i is the type of the *individuals*)
(b) if $t_1, \ldots, t_n \in \mathbb{T}$, then $\langle t_1, \ldots, t_n \rangle \in \mathbb{T}$

Thus, i, $\langle i \rangle$, $\langle i, i \rangle$, $\langle \langle i \rangle, i \rangle$, $\langle \langle i \rangle \rangle$ are examples of types. Each type has an order, attributed according to the following definition:

Definition 2.2 (Order of a type). *The order of a type,* $\mathsf{Ord}(t)$*, is defined as follows:*

(a) $\mathsf{Ord}(i) = 0$
(b) $\mathsf{Ord}(\langle t_1, \ldots, t_n \rangle) = \max\{\mathsf{Ord}(t_1), \ldots, \mathsf{Ord}(t_n)\} + 1.$

Thus, $\mathsf{Ord}(\langle i \rangle) = \mathsf{Ord}(\langle i, i \rangle) = 1$, while $\mathsf{Ord}(\langle i, \langle i \rangle \rangle) = 2$. *Relations* will be understood here as both extensional sets (collections of n-tuples) and being of finite rank (that is, having finite weight only). *Unary* relations are sets.

Definition 2.3 (Order of a relation). *The order of a relation is the order of its type.*

Thus, binary relations of individuals are order–1 relations, and so on. This, together with the previous definition of order of a relation makes clear that the order of a relation should not be confused with the notion of the weight of a relation. As mentioned before, $\mathsf{Ord}(\langle i \rangle) = \mathsf{Ord}(\langle i, i \rangle) = 1$, but the first is the type of an unary relation, while the second is the type of a binary relation. The same goes for n-ary relations, and one should not confuse order with arity.

Now, we shall introduce a function t_D as follows:

Definition 2.4 (Scale based on D). *Let D be a set. We pose:*

(a) $t_D(i) = D$
(b) If $t_1, \ldots, t_n \in \mathbb{T}$, then $t_D(\langle t_1, \ldots, t_n \rangle) = \mathcal{P}(t_D(t_1) \times \ldots \times t_D(t_n))$.
(c) The scale based on D is the union of the range of t_D, and it is denoted by $\varepsilon(D)$.

Thus, t_D is a function that attributes to each type in $\mathbb{T}$ the set of all relations of that type. We say that the elements of $t(a)$, for $a \in \mathbb{T}$, are objects of type a. The set $\varepsilon(D)$ is the set of all relations of all types.

Definition 2.5 (Structure). *A structure $\mathfrak{E}$ based on a set D is an ordered pair*

$$\mathfrak{E} = \langle D, r_\iota \rangle \tag{1}$$

where $D \neq \emptyset$ and r_ι represents a sequence of relations of degree n belonging to $\varepsilon(D)$. These relations are called the primitive elements of the structure.

Thus, the relations in a structure may have as relata not only elements of the domain (these are called *order*-1 *relations*), but also subsets of D and other 'higher' elements. The domain may comprise also several sets, some of then called the *principal basis*, while the others are the *secondary basis* of the structure (this terminology is Bourbaki's, who has an approach closer to ours). The vector-space example below is a typical case, as we shall see in the next subsection.

Definition 2.6 (Order of a structure). *Let $\mathfrak{E} = \langle D, r_\iota \rangle$ be a structure. Its order,* $\mathsf{Ord}(\mathfrak{E})$*, is defined as follows: if there is a greatest order of the relations in r_ι, then the order of the structure is that greater order, and it is ω otherwise.*

If the primitive relations of the structure are relations having individuals of D as relata only, we say that the structure is an *order*–1 structure. These are the structures typical of standard model theory. In general, the structures of scientific theories are not order-1 in this sense; they involve not only relations on the elements of the base sets, but more sophisticated relations (and operations), as we shall see with our example of the non-relativistic quantum mechanics. Even in mathematics we find structures which are not order-1, such as topological spaces, well-orders, ciclic groups and so on. This distinction between orders has important consequences when we consider scientific theories, as we shall see soon.

2.2. Models and theories

Now, how can we understand a scientific theory? According to one possible version of the so-called semantic approach, a theory can be seen as a class of structures, the models of the theory. Obviously, this deserves qualification. First of all, the word 'model' can be taken in so many senses that it is difficult to understand what does it mean to say that a theory is a class of models. Here, for the sake of rigor and simplicity, we shall understand this concept in the sense of some set-theoretical structure (as seen in the last section) satisfying some conditions, the postulates of the theory. This agrees with Suppes' talk of "models in the sense of Tarski" [27, p.20]. Also, taken literally, the purported characterization suffers from an obvious circularity, for it aims at clarifying the concept of theory appealing to the models, but models of what? Well, of the theory itself. This is one of the reasons why we shall not call it a definition, but only a heuristic characterization, which serves to illuminate the main idea (for further discussion, see [24]).

One of the possible ways to throw some light in this problem and which can turn the purported characterization into a workable definition, can be sought in what came to be called in the literature a *set-theoretical predicate*. This is roughly seen as a proposal which marks the beginning of the semantical approach, when in the middle 50s Patrick Suppes initiated a new approach to the axiomatization of scientific theories. According to Suppes, the axiomatization of a scientific theory can be made through a formula of set theory, which specifies what kind of constraints the structures satisfying it must conform to: "to axiomatize a theory is to define a set-theoretical predicate" [27, p.30]. The structures satisfying the predicate, then, are the models of the theory, and so the predicate can be seen as selecting a class of models (but see the discussion in [16] and Suppes reply in [28]).

Suppes approach has obvious advantages over axiomatization in strict sense, that is, axiomatization which proceeds through the usual method of devising a formal apparatus making explicit the vocabulary, syntactic rules of formation and derivation, and so on.[2] Working inside set theory, one has already at hand all the mathematics needed when dealing with empirical theories, and so, it is not necessary to build them all from the bottom, furnishing

[2]We give a detailed description of both methods in [12].

axioms and providing the necessary theorems; besides being counterproductive, this would entail a mathematical prolegomenon which would be big enough to fill an encyclopedia, which would be in obvious contrast with the scientific practice (see also [27]).

In his exposition of the method (see [25, 27]), Suppes did not give a rigorous definition of a set theoretical predicate, but expected that the main idea kept clear from some examples (for a rigorous account of his method, see da Costa and Chuaqui [3]). Besides, working inside ZFC set theory allows him to follow standard mathematical practice,[3] in which one uses every theorem of set theory available without further justification. This kind of procedure has an influence on the sense in which we can say a structure is a model for the postulates: in order to show for instance that the additive group of the integers $\mathcal{Z} = \langle \mathbb{Z}, + \rangle$ is a group, we prove in ZFC that the formulas (of the language of ZFC extended to cope with symbols such as $\mathcal{Z}$, $\mathbb{Z}$, etc.) are true, in the Tarskian sense, in $\mathcal{Z}$, that is, we get a result that can be written as follows:

$$ZFC \vdash (\mathcal{Z} \models A1 \wedge A2 \wedge A3),$$

where $A1, A2, A3$ are the formulas that traduce the group axioms (see the example below). That is, we just derive, with the full resources of set theory, that the objects in the structures, *i.e.*, elements of the domain and the relations composing the structure, have the properties stated in the postulates of the set theoretical predicate.

To illustrate these points, let us see some examples of set theoretical predicates in the style of Suppes:

Example 2.1 (Groups). *For a group predicate, we start with a base non-empty set G. We want to axiomatize the structures of the kind: $\mathcal{G} = \langle G, * \rangle$, where $* \in \mathcal{P}(G \times G \times G)$, satisfying ($A1$) associativity, ($A2$) the existence of the identity element, and ($A3$) the existence of inverses. Then, the set-theoretical predicate may be as follows:*

$$\mathbb{G}(x) \leftrightarrow \exists G \exists * (x = \langle G, * \rangle \wedge G \neq \emptyset \wedge * \in \mathcal{P}(G \times G \times G) \wedge (A1) \wedge (A2) \wedge (A3))$$

The structures that satisfy the predicate are the models of $\mathbb{G}$, vis., the groups, for instance, $\mathcal{Z}$ above.

Example 2.2 (Vector spaces). *In the case of vector spaces over a field, we have a base set V (the vectors), and an auxiliary set, K (the domain of the field). The basic operations, despite the usual ones of the field, are the following ones: $+ \in \mathcal{P}(V \times V \times V)$ and $\cdot \in \mathcal{P}(K \times V \times V)$. We shall not write explicitly the axioms for vector spaces, for they are well known. The set theoretical predicate is then:*

$$\mathbb{V}(x) \leftrightarrow \exists V \exists K \exists + \exists \cdot (x = \langle V, K, +, \cdot \rangle \wedge etc.).$$

[3] In his works, Suppes refer to an informal set theory. But, if pressed, he would refer to a standard set theory, such as ZFC.

To present a set-theoretical predicate is equivalent to present the postulates of the theory, as we usually do—really, in practice we of course *do not* write a Suppes predicate, but prefer a shorter notation.

3. A set-theoretical predicate for non-relativistic quantum mechanics

Since this is our case study, we shall take a look also in a set-theoretical predicate for non-relativistic quantum mechanics.

A non-relativistic quantum mechanics ($\mathsf{QM_{NR}}$) can be seen as a structure

$$\mathsf{QM_{NR}} = \langle S, \{H_i\}, \{A_{ij}\}, \{T_{ik}\}\rangle_{i\in I, j\in J, k\in K} \tag{2}$$

where S is a set of physical systems,[4] $\{H_i\}$ is a collection of Hilbert spaces, $\{A_{ij}\}$ is a collection of Hermitian operators on the space H_i and $\{T_{ik}\}$ is a collection of unitary operators on H_i, $\{T_{ik}\} \subset \{A_{ij}\}$, where the following guidelines (usually called 'axioms') are satisfied:

(i) For each physical system $s \in S$, we associate a complex Hilbert space $H_s \in \{H_i\}$. The vectors $|\psi\rangle$ of this space represent the *states* of the physical system. It is called the *state vector* of the system, and stands for all we know about it. The state vectors are normalized, for $k.|\psi\rangle$ (for any complex number k) represents the same state as $|\psi\rangle$.

When we have a system composed by several elements of S, we associate to it the tensor product of the Hilbert spaces of the composing systems (in some order). If the cardinal of the subset of systems is n (call them $s_1, \ldots, s_n$), the Hilbert space is

$$\mathcal{H} = \mathcal{H}_{s_1} \otimes \ldots \otimes \mathcal{H}_{s_n}.$$

A typical vector of this space is written $|\psi_1\rangle \otimes \ldots \otimes |\psi_n\rangle$, or simply $|\psi_1\rangle \ldots |\psi_n\rangle$ for short. When the systems are considered to be indiscernible, we make $\mathcal{H}_i = \mathcal{H}_j$ for any i and j.

(2) Let $|\psi(t)\rangle$ represent the state at time t. Then, for each $|\psi\rangle$ we associate an unitary operator T_s such that for any instant of time t, we have that

$$|\psi(t)\rangle = T_s(t).|\psi(0)\rangle, \tag{3}$$

where $|\psi(0)\rangle$ is the state at time $t = 0$. This represents the *unitary evolution* (in time) of the vector state, and it is called the *Schrödinger equation.*

[4]Newton da Costa has an interesting proposal of seeing quantum objects themselves as structures of the form $s = \langle \mathbb{R} \times \mathbb{R}^3, \mathcal{H}_s, \mathcal{O}, \mathbb{P}\rangle$, where $\mathbb{R} \times \mathbb{R}^3$ is the Galilean space-time, $\mathcal{H}_s$ is a Hilbert space and $\mathcal{O}$ a collection of self-adjoint operators on $\mathcal{H}$, subjected to standard definitions compatible with the axioms presented in this section. By the way, $\mathbb{P}$ is a mapping which plays the role of a probability measure. It is clear that his schema can be incorporated to ours. (da Costa's ideas were presented in a seminar delivered in 22.09.2010)

(3) The eigenvalues of A, that is, those (real) scalars a_i such that $A|\psi_i\rangle = a_i.|\psi_i\rangle$ are the possible results of a measurement of A. It is assumed that the Hermitian operators represent observable physical quantities that can be measured on the system at a certain state. Sometimes we distinguish between the observable (such as mass, energy, momentum, number of particles, etc.) from the corresponding Hermitian operators by writing A for the observable and $\hat{A}$ for the operator. We think that we don't need this distinction here.

(4) It is know that any Hermitian A is diagonalizable, what means that we can find a basis $\{|\alpha_i\rangle\}$ for the considered Hilbert space formed by eigenvectors of A. Thus, for any state $|\psi\rangle$, we can write $|\psi\rangle = \sum_i c_i|\alpha_i\rangle$, where $c_i = \langle\alpha_i|\psi\rangle$ are the Fourier coefficients. Thus, $|c_i|^2 = P_i$ represents the probability that the measurement of A gets the value a_i. This postulate is known as *Born rule*.

(5) If a measurement of A gives the result a_i, the state vector $|\psi\rangle$ becomes $|\alpha_i\rangle$ immediately after the measurement. This is known as the *collapse* of the vector state.

Now, the foundationalist may be interested in investigating what are the implications of using this specific underlying mathematical basis to formulate non-relativistic quantum mechanics. In particular, ontological issues may be in strong conflict with some of the presuppositions of the mentioned mathematical basis. Let us turn to this point now.

3.1. Quantum indistinguishability and non-individuality

In the beginnings of quantum physics, Heisenberg, Born, Schrödinger, Bohr, later Weyl, Hesse, among others, spoke of the *lost of individuality* of quantum entities. They have also referred to these entities as *non-individuals* (for historical references and further information, see [9, chap. 3]). In fact, it seems that there are no differences among atoms of the same species, as well as among other sub-atomic 'particles' of the same kind. In a certain sense, all protons (electrons, neutrons, quarks, ...) are *exactly* alike. So, with no distinguishing feature between two of them, it seems there is no reason to say they can be individuals in any sense. Really, by an individual, informally speaking, we usually mean something that can be identified as such in different circumstances: it has identity. But quantum objects as those mentioned seem to be of a distinct nature; can we identify an electron twice? It is clear that we can't. Thus, quantum objects would be *non-individuals*.

The main motivation behind the claim that these objects are not individuals is of course a possible way to interpret quantum objects in a metaphysical framework accommodating this kind of objects. 'Quantum objects' may be particles (as in orthodox QM), fields in relativistic quantum mechanics, or any other among many possibilities (field excitations, and so on). In any case, the strange relation of these items to identity (or lack of identity) seems to grant that they can no longer be treated as individuals any more. Think of a

BEC for instance (a Bose-Einstein Condensate). As the temperature becomes 'critic' (*circa* few pico Kelvins), the wavelengths become longer, so that the 'particles' cannot be treated as individuals any more (if they were): they became a *soup* of matter waves. But even so, no physicist will say that this soup, the 'big atom' is composed by just one entity, but by lots of entities in the same quantum state, behaving in *unisono* (unison).

We can associate an interpretation to this phenomena. How can we speak of *individuals* composing a BEC? Since they are in the same quantum state, can the quanta in a BEC be seen as distinct *solo numero*? Otherwise there will be 'properties' they have not included in the formalism of quantum mechanics (hidden variables). Recall that permutations of objects of the same kind lead to the same physical significative values (the same expectation value). So, if properties of quantum entities are understood properly in terms of these values, it seems that they share all their properties. It has been much debated whether this implies that quantum objects violate Leibniz' Principle of the Identity of Indiscernibles, according to which there are no two entities differing *solo numero*. If we accept this idea (as many philosophers seem to hold), then we can either regard quantum objects as non-individuals, or to ground their individuality in some mysterious *substratum*, something not many are willing to do. Paul Teller has interesting arguments to avoid the introduction of substratum, haecceities, thisness and so on in quantum mechanics [22]. In agreement with that, it seems quite 'natural' to pursue (formally) a *metaphysics of non-individuals*, grounded on a possible interpretation of quantum 'objects'. Informally speaking, non-individuals are objects to which the standard notion of identity does not apply. Why? If the standard theory of identity holds for some objects, they can always (in principle) be discerned from any other object.[5] Then they are individuals. Classical theory of identity says that indiscernible things are the very same thing. There are no indistinguishable but not identical objects in the classical realm. Remembering the foundationalist interest in the consequences of the underlying logic to some fundamental questions about the theory, this is one that has far reaching consequences, for it deals with the very nature of the items dealt with by the theory (that is, its accompanying ontology). *Indiscernibility* may be assumed to be a fundamental concept[6] (in QM applications, we shall avoid discussing interpretations such as Bohm-Hiley's –but see French & Krause [9]). Heinz Post, in 1963 proposed that the indiscernibility (non-individuality) of quantum objects should be considered *right at the start* (as a primitive notion, see

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[5] We said *in principle*. Standard mathematics presents us situations where objects (real numbers, say) are distinct, but this distinction cannot be expressed in the languages employed. But, as we shall emphasize also later, 'classical objects' are either *identical* —the same object— or *different*. But in this mentioned case where they are different but their difference is not expressible, there are relations/properties that we prove to exist but which cannot be expressed in the corresponding language, say a well order on $\mathbb{R}$. Of course this cannot be assumed in the quantum case except if we agree in introducing hidden variables.

[6] This can be seen from Gibbs paradox — see [23, Chap.4, p.95], where the author concludes that "Indistinguishability is an experimental property of nature".

[18]). Usually, the formalism of orthodox QM uses symmetrization postulates: symmetric and anti-symmetric vectors/functions express indiscernibility. For two systems labeled 1 and 2 entangled in two possible states a and b, the join system is described by the wave function (in the standard formalism)

$$|\psi_{12}\rangle = \frac{1}{\sqrt{2}}(|\psi_1^a\rangle|\psi_2^b\rangle \pm |\psi_2^a\rangle|\psi_1^b\rangle)$$

Note that we need to label the objects; our languages are *objectual* [29, pp.220ff]. So, in order to grant that the order of the taken objects does not matter, we use symmetric functions (or vectors), with the add of the Indistinguishability Postulate below. This is of course a mathematical trick, for what imports for physics is that the expectation value of the measure of any observable $\hat{O}$ for the system in the state $|\psi\rangle$ does not change after a permutation of the particles. Being P a permutation operator, we express this by means of the Indistinguishability Postulate [19, 20]:

$$\langle\psi_{12}|\hat{O}|\psi_{12}\rangle = \langle P\psi_{21}|\hat{O}|P\psi_{21}\rangle$$

But of course that from the foundational point of view we should try to find a formalism (a logic) for QM without appealing to these artificial labeling of quanta. Redhead and Teller suggest to scape of the *Hilbert tensor product vector space formalism*, which uses labels, by shifting to the Fock space formalism (ibid., [21]). But, as we have seen, this move is done still within standard mathematics, where all objects are, in a sense, individuals. In our opinion, in order to rightly sustain a metaphysics of non-individuals, a different mathematical and logic framework should be used. Once we have got such a framework, we should try to ground a semantics for such a suitable formalism, such as the above, motivated by the metaphysics of non-individuals. Below we shall suggest how this can be done, a procedure we may regard to be consonant with what we call the von Weizsäcker–da Costa's *Principle of Semantic Consistency*, as formulated by von Weizsäcker: "the rules by which we describe and guide our measurement, defining the semantics of the formalism of a theory, must be in accordance with the laws of the theory." (cf. [11, p.156]; [2]).

So, if certain objects are to be considered as 'absolutely indiscernible', a semantics for such a logic will demand a mathematical theory compatible with the hypothesis of indiscernibility, and we saw that ZFC is not such a theory. Hence, there is a sense in speaking a *logic of quantum physics* — really, the logic of quantum objects— in a distinct way from the usual approaches of 'quantum logic' (the study of the orthomodular lattice of the closed sub-spaces of a Hilbert space). One such proposal was made with quasi-set theory [17], but we shall not discuss this issue here. Next, we outline the mathematical framework.

4. Quasi-set theory $\mathfrak{Q}$

We will review now the basic notions of quasi-set theory $\mathfrak{Q}$. More detailed developments can be found in [9, chap. 7] and [10].

4.1. The language of the formal theory

The underlying logic of $\mathfrak{Q}$ is classical first order logic without identity. The postulates we assume are, first, a complete set of postulates for the classical first order calculus, but we do not assume the classical semantics for this calculus, that is, it should not be a formal Tarskian semantics built in ZFC, for this would bring us back classical concepts such as identity into $\mathfrak{Q}$ (for a discussion on these topics see [1]). We take the following symbols as primitive:

(i) propositional connectives,
(ii) quantifiers
(iii) individual variables (a denumerable set)
(iv) two binary predicates $\equiv$ and $\in$,
(v) three unary predicates m, M and Z, and
(vi) an unary functional symbol qc.

Terms and formulas are defined as usual. Notice once again that identity is not part of the primitive vocabulary, and that the only terms in the language are variables and items of the form $qc(t)$, where t denotes a term. The intuitive meaning of the primitive symbols is given as follows:

(i) $x \equiv y$ (x is indiscernible from y)
(ii) $m(x)$ (x is a 'micro-object', or an m-atom)
(iii) $M(x)$ (x is a 'macro-object' or an M-atom)
(iv) $Z(x)$ (x is a 'set' – a copy of a ZFU set)
(v) $qc(x)$ (the quasi-cardinal of x)

Now, we introduce some important definitions, with the intuitive interpretation attributed to them.

Definition 4.1.

(i) $Q(x) := \neg(m(x) \vee M(x))$ (x is a qset)
(ii) $P(x) := Q(x) \wedge \forall y(y \in x \rightarrow m(y)) \wedge \forall y \forall z(y \in x \wedge z \in x \rightarrow y \equiv z)$
(x is a pure qset, having only indiscernible m-atoms as elements.)
(iii) $D(x) := M(x) \vee Z(x)$
(x is a *Ding*, a "classical object" in the sense of Zermelo's set theory, namely, either a set or a 'macro *Urelemente*'.)
(iv) $E(x) := Q(x) \wedge \forall y(y \in x \rightarrow Q(y))$
(x is a qset whose elements are qsets.)
(v) $x =_E y := (Q(x) \wedge Q(y) \wedge \forall z(z \in x \leftrightarrow z \in y)) \vee (M(x) \wedge M(y) \wedge \forall_Q z(x \in z \leftrightarrow y \in z))$ (Extensional identity)— we shall write simply $x = y$ instead of $x =_E y$ from now on.
(vi) $x \subseteq y := \forall z(z \in x \rightarrow z \in y)$ (subqset)

That is, $\mathfrak{Q}$ is a theory containing two kinds of ur-elements, the m-atoms and the M-atoms, and also collections of atoms and other collections, the qsets. Some qsets are specially important: when their transitive closure does not contain m-atoms, they contain only what we call 'classical objects' of the theory (objects satisfying D); items fulfilling this condition satisfy the predicate Z and they coincide with the sets in ZFU, with which classical mathematics can be built inside $\mathfrak{Q}$.

The main idea motivating the development of the theory is that some items are non-individuals (roughly speaking, entities for which the standard notion of identity does not apply), and does not obey the notion encapsulated in the definition of extensional identity. As one can see, this concept is not defined for m-atoms, the items which intuitively represent quantum indistinguishable objects. So, on one side, these things 'do not have identity', that is, it does not make sense to say they are identical or different and, on the other side, the indistinguishability relation holds for every item of the theory, so m-atoms may be indistinguishable without being identical. Important to notice that in saying that some entities are non-individuals, we are not supposing that we cannot speak of them; really, we *can* speak of them. For instance, a qset of indiscernible m-atoms may have a q-cardinal greater than one, say 5, and so we can think of five entities in some situation, although they cannot be discerned in any way. Below we shall see that m-atoms a and b may have distinct properties, that is, it may be the case that $a \not\equiv b$.

4.2. The postulates of $\mathfrak{Q}$

Besides postulates for classical first-order logic without identity (which we shall not list here), we introduce the specific postulates for $\mathfrak{Q}$.

($\equiv_1$) $\forall x(x \equiv x)$
($\equiv_2$) $\forall x \forall y(x \equiv y \rightarrow y \equiv x)$
($\equiv_3$) $\forall x \forall y \forall z(x \equiv y \land y \equiv z \rightarrow x \equiv z)$
($=_4$) $\forall x \forall y(x = y \rightarrow (\alpha(x) \rightarrow \alpha(y)))$, with the usual restrictions.

These postulates ensure us that indistinguishability is an equivalence relation. Now, this relation is not necessarily compatible with the other primitive predicate or relations; that this in fact occurs for m-atoms helps us keeping identity and indistinguishability separated. In fact, if x and y are indistinguishable m-atoms, then being z a qset, we have that $x \in z$ does not entail that $y \in z$, and conversely.

Other postulates are:

($\in_1$) $\forall x \forall y(x \in y \rightarrow Q(y))$
If something has an element, then it is a qset; in other words, the atoms have no elements (in terms of the membership relation).

($\in_2$) $\forall_D x \forall_D y(x \equiv y \rightarrow x = y)$
Indistinguishable *Dinge* are extensionally identical. This makes $=$ and $\equiv$ coincide for this kind of entities.

($\in_3$) $\forall x \forall y[(m(x) \land x \equiv y \rightarrow m(y)) \land (M(x) \land x = y \rightarrow M(y)) \land (Z(x) \land x = y \rightarrow Z(y))]$

($\in_4$) $\exists x \forall y (\neg x \in y)$
This qset can be proved to be a set (in the sense of obeying the predicate Z), and it is unique, as it follows from the axiom of weak extensionality we shall see below. Thus, from now own we shall denote it, as usual, by $\emptyset$.

($\in_5$) $\forall_Q x (\forall y (y \in x \rightarrow D(y)) \leftrightarrow Z(x))$
This postulate grants that something is a set (obeys Z) iff its transitive closure does not contain m-atoms. That is, *sets* in $\mathfrak{Q}$ are those entities obtained in the 'classical' part of the theory.

($\in_6$) $\forall x \forall y \exists_Q z (x \in z \wedge y \in z)$ (pair axiom)

($\in_7$) If $\alpha(x)$ is a formula in which x appears free, then

$$\forall_Q z \exists_Q y \forall x (x \in y \leftrightarrow x \in z \wedge \alpha(x)).$$

This is the Separation Schema. We represent the qset y as follows:

$$[x \in z : \alpha(x)].$$

When this qset is a set, we write, as usual, $\{x \in z : \alpha(x)\}$.

($\in_8$) $\forall_Q x (E(x) \rightarrow \exists_Q y (\forall z (z \in y \leftrightarrow \exists w (z \in w \wedge w \in x))))$.
The union of x, written $\bigcup x$. Usual notation is used in particular cases.

4.3. Some basic concepts

From ($\in_6$): $\forall x \forall y \exists_Q z (x \in z \wedge y \in z)$, using $\alpha(w) \leftrightarrow w \equiv x \vee w \equiv y$, we get a subqset of z which we denote

$$[x, y]_z$$

which is the qset of the indiscernibles of either x or y that belong to z. When $x \equiv y$, this qset reduces to

$$[x]_z$$

called the qset of the indiscernibles from x that belong to z. The qset $[x, y]_z$ does not have necessarily only *two* elements (that is, we may have $qc([x, y]_z) > 2$, for there may be more than just one indistinguishable from x or y in z. Given the qset z and one of its elements, x, the collections $[x]$ and $[x]_z$ stand for *all* indiscernible from x and the qset of the indiscernible from x that belong to z respectively. (Usually, $[x]$ is too big to be a qset.)

Later, with the postulates of quasi-cardinal, we will be able to prove $[x]_z$ has a subqset whose quasi-cardinal equals to 1:

$$[[x]]_z$$

We call it the *strong singleton* of x (really, *a* strong singleton of x, for we cannot grant that it is unique). It has just one element, and we can think of this element *as if* it were x, but in fact, it follows from the definition that all we can know about it is that $[[x]]_z$ contains *one object of the 'species' x*. That is, $qc([[x]]_z) = 1$, there is one item indistinguishable from x in this qset.

4.4. Other postulates and definitions

($\in_9$) $\forall_Q x \exists_Q y \forall z (z \in y \leftrightarrow w \subseteq x)$,
The power qset of x, denoted $\mathcal{P}(x)$.

($\in_{10}$) $\forall_Q x (\emptyset \in x \wedge \forall y (y \in x \to y \cup [y]_x \in x))$,
The infinity axiom.

($\in_{11}$) $\forall_Q x (E(x) \wedge x \neq \emptyset \to \exists_Q y (y \in x \wedge y \cap x = \emptyset))$,
The axiom of foundation, where $x \cap y$ is defined as usual.

Definition 4.2 (Weak ordered pair).

$$\langle x, y \rangle_z := [[x]_z, [x, y]_z]_z \quad (4)$$

Then, $\langle x, y \rangle_z$ takes all indiscernible from either x or y that belong to z, and it is called the "weak" ordered pair, for it may have more than two elements. Sometimes the sub-indice z will be left implicit.

Definition 4.3 (Cartesian Product). *Let z and w be two qsets. We define the cartesian product $z \times w$ as follows:*

$$z \times w := [\langle x, y \rangle_{z \cup w} : x \in z \wedge y \in w] \quad (5)$$

Functions and relations cannot also be defined as usual, for when there are m-atoms involved, a mapping may not distinguish between arguments and values. Thus we provide a wider definition for both concepts, which reduce to the standard ones when restricted to classical entities. Thus,

Definition 4.4 (Quasi-relation). *A qset R is a binary quasi-relation between to qsets z and w if its elements are weak ordered pairs of the form $\langle x, y \rangle_{z \cup w}$, with $x \in z$ and $y \in w$.*

Definition 4.5 (Quasi-function). *f is a quasi-function among q-sets A and B if and only if f is quasi-relation between A and B such that for every $u \in A$ there is a $v \in B$ such that if $\langle u, v \rangle \in f$ and $\langle w, z \rangle \in f$ and $u \equiv w$ then $v \equiv z$.*

In words, a quasi-function maps indistinguishable elements to indistinguishable elements. An interesting question concerns the more specific kinds of functions, that is, injections, surjections and bijections. One can, with some restrictions, define the corresponding concepts, but we shall not do that here (see [9, chap. 7]).

4.5. Postulates for quasi-cardinals

One must notice that in $\mathfrak{Q}$ the standard notion of identity is not defined for some entities. Now, the identity concept is essential to define many of the usual set theoretic concepts of standard mathematics, such as well order, the ordinal attributed to a well ordered set, and the cardinal of a collection. Since identity is to be senseless for some items in $\mathfrak{Q}$, how can we employ these notions? One alternative would be to look for different formulations employing methods that do not rely on identity. Another possibility would be to introduce these concepts as primitive and give adequate postulates for them. Concerning the notion of cardinal, there are interesting issues we should acknowledge. First of all, in $\mathfrak{Q}$, there are no well-orders on quasi-sets of m-atoms. Really, a well-order would imply, for example, that there is a least element relative to this well order, a notion which could only be formulated if identity was defined for m-atoms, for this element would be different from any *other* element in the quasi-set. Second, the usual claim that aggregates of quantum entities can have a cardinal but not an ordinal demands a distinction between the notions of ordinal and of cardinal of a quasi-set; this distinction is made in $\mathfrak{Q}$ by the introduction of cardinals as a primitive notion, called quasi-cardinals.[7]

Let us see the postulates for quasi-cardinals; for details and motivations, see [9, Chap.7], [10]. Here α, β, ... stand for cardinals (defined as usual in the classical part of the theory, that is, in the theory $\mathfrak{Q}$ when we rule out the m-atoms):

(qc_1) $\forall_Q x(\exists_Z y(y = qc(x)) \rightarrow \exists! y(Cd(y) \wedge y = qc(x) \wedge (Z(x) \rightarrow y = card(x)))$

In words, if the qset x has a quasi-cardinal, then its (unique) quasi-cardinal is a cardinal (defined in the 'classical' part of the theory) and coincides with the cardinal of x stricto sensu if x is a set.

(qc_2) $\forall_Q x(\exists y(y = qc(x) \rightarrow x \neq \emptyset \rightarrow qc(x) \neq 0))$.

Every non-empty qset that has a quasi-cardinal has a non-null quasi-cardinal.

(qc_3) $\forall_Q x(\exists_Z \alpha(\alpha = qc(x)) \rightarrow \forall \beta(\beta \leq \alpha \rightarrow \exists_Q z(z \subseteq x \wedge qc(z) = \beta)))$

If x has quasi-cardinal α, then for any cardinal $\beta \leq \alpha$, there is a subqset of x with that quasi-cardinal.

In the remaining axioms, for simplicity, we shall write $\forall_{Q_{qc}} x$ (or $\exists_{Q_{qc}} x$) for quantifications over qsets x having a quasi-cardinal.

(qc_4) $\forall_{Q_{qc}} x \forall_{Q_{qc}} y(y \subseteq x \rightarrow qc(y) \leq qc(x))$

(qc_5) $\forall_{Q_{qc}} x \forall_{Q_{qc}} y(Fin(x) \wedge x \subset y \rightarrow qc(x) < qc(y))$

It can be proven that if both x and y have a quasi-cardinal, then $x \cup y$ has a quasi-cardinal. Then,

(qc_6) $\forall_{Q_{qc}} x \forall_{Q_{qc}} y(\forall w(w \notin x \vee w \notin y) \rightarrow qc(x \cup y) = qc(x) + qc(y))$

In the next axiom, $2^{qc(x)}$ denotes (intuitively) the quantity of subquasi-sets of x. Then,

(qc_7) $\forall_{Q_{qc}} x(qc(\mathcal{P}(x)) = 2^{qc(x)})$

This last axiom enables us to think of subqsets of a given qset in the usual sense; for instance, if $qc(x) = 3$, the axiom says that there exists $2^3 = 8$ subqsets, and axiom (qc_3) enables us to think that there are subqsets with 1, 2 and 3 elements. Furthermore, as we have seen above, in $\mathfrak{Q}$ we can prove that given any object a (either an m-atom, M-atom or quasi-set) we may obtain the 'singleton' of a, termed $[[a]]$ whose qcardinal is 1 (this is the *strong*

[7] As shown by Domenech and Holik, we can define quasi-cardinals for finite qsets in $\mathfrak{Q}$, without resulting that the qset will have an associated ordinal in the usual sense; see [6].

singleton of a). Important to say that there is no sense of saying, within $\mathfrak{Q}$, that a is the only element of $[[a]]$, for in order to prove that we need identity. Anyway, $\mathfrak{Q}$ is consistent with this idea. So, we can reason within $\mathfrak{Q}$ that we may have *a certain m-atom*, without keeping it specified in some form, except that it has some characteristics, and not others. That m-atoms may have different properties can be seen from the fact of $\mathfrak{Q}$ that $\mathfrak{Q}$ *doesn't prove* the Substitutivity of Indiscernibles, that is,

$$\mathfrak{Q} \nvdash a \equiv b \to \forall_Q z(a \in z \leftrightarrow b \in z).$$

To prove this result, suffice to take $[[a]]$. Since $qc(a) = 1$, a and b cannot belong both to this qset, except if $a = b$, which cannot be assumed in the case of m-atoms. So, in an extensional context (and $\mathfrak{Q}$ is an extensional theory, although this should be qualified), we can read $a \in z$ as a having a certain 'property' (whose 'extension' would be z). So, even indistinguishable m-atoms may have distinct properties.

4.6. The Weak Extensionality Axiom

Our next goal is to present the weak extensionality axiom, which generalizes the usual extensionality axiom. Intuitively, it grants us that two q-sets with the same quantity of the same kinds of elements are indistinguishable. For that, we need two extra definitions, the notion of similarity between q-sets, denoted by Sim, and the notion of Q-similarity, denoted Qsim. Intuitively speaking, similar q-sets have elements of the same kind, and q-similar q-sets have elements of the same kind, and in the same quantity:

Definition 4.6.

(i) $\mathrm{Sim}(x, y) := \forall z \forall w(z \in x \wedge w \in y \to z \equiv y)$;
(ii) $\mathrm{Qsim}(x, y) := \mathrm{Sim}(x, y) \wedge qc(x) = qc(y)$.

The weak extensionality axiom reads as follows:

($\equiv_{12}$) $\forall_Q x \forall_Q y((\forall z(z \in x/{\equiv} \to \exists t(t \in y/{\equiv} \wedge \mathrm{Qsim}(z, t)))) \wedge \forall t(t \in y/{\equiv} \to \exists z(z \in x/{\equiv} \wedge \wedge \mathrm{Qsim}(t, z))) \to x \equiv y)$

Intuitively speaking, qsets that have 'the same quantity' (given by their q-cardinals) of elements of the same kind are indiscernible.

The following theorem express the invariance by permutations in $\mathfrak{Q}$, and with this result we finish our revision:

Theorem 4.1 (Invariance by Permutations). *Let x be a finite qset such that $\neg(x = [z]_t)$ for some t and let z be an m-atom such that $z \in x$. If $w \in t$, $w \equiv z$ and $w \notin x$, then there exists $[[w]]_t$ such that*

$$(x - [[z]]_t) \cup [[w]]_t \equiv x$$

Proof: See [9]. ∎

In words, two indiscernible elements z and w, with $z \in x$ and $w \notin x$, expressed by their strong-singletons $[[z]]_t$ and $[[w]]_t$, are 'permuted' and the resulting qset x remains indiscernible from the original one. The hypothesis that $\neg(x = [z]_t)$ grants that there are indiscernible from z in t which do not belong to x. This theorem has a 'physical' interpretation: the qset x must be a neutral atom which is to be ionized by realizing an electron in order to become a negative ion. Thus the m-object z would represent an electron in the outer shell, while w is 'another' electron not in the atom (these words are to be understood metaphorically). Thus, the electron z is realized and, in another experiment, an electron is captured again so that the atom becomes neutral again. The question is: is this last neutral atom *the same* (identical) to the first one? Of course, this would be so if and only if the captured electron is, *ceteris paribus*, exactly the same as the realized one. But, there is any sense in saying that the realized electron is *identical* with the captured one? Quasi-set theory escapes from this dilemma by assuming that the basic notion is that of indiscernibility; the electrons are indiscernible, so as the neutral atoms. And this is enough for physics. Philosophically, we advance a thesis: the notion of identity is a useful notion. It simplifies in much our discourse and mathematics, but it is fragile; as Hume showed, the identity of objects is something attributed by habit. Indistinguishability suffices. But we shall not develop this thesis here.

5. The semantic approach, quasi-sets, and non-individuals

How could the theory $\mathfrak{Q}$ helps the philosopher in the investigation of foundational aspects of quantum mechanics? In order to discuss this point, yet here only superficially, let us consider once more the schema given by the structure (2), presented at page 7. Using $\mathfrak{Q}$, the set S can be (perhaps more appropriately for the quantum mechanical case) represented by a quasi-set, maybe composed by non-individuals only, representing elementary particles, either indiscernible or not.[8] This qset has a cardinal, in general (in the physical applications) finite (so, suppose $qc(S) = n$). Then, even (eventually) without being able to discern the elements of S, we may associate to each of them a Hilbert space in the class $\{H_i\}$,[9] and the rest of the schema follows in an obvious way. In this sense, we are considering structures such as (2) not in ZFC, but in $\mathfrak{Q}$.

What is the importance of this move from ZFC to $\mathfrak{Q}$? We may justify this as follows. The structure (2), built in ZFC, can be extended to a rigid structure, that is, to a structure whose only automorphism is the identity function. This is a theorem of ZFC: *any structure can be extended to a rigid structure* [4]. A rigid structure is a structure whose only automorphism is the identity function. Since the notion of indiscernibility is given relative to a structure by means of invariance by one of its automorphisms,[10] this implies that we may have objects which, *within* the structure, are indiscernible in the sense that there exists a non-trivial (i.e., distinct from the identity function)

[8] We remark that the construction of structures and set theoretical predicates carries through to $\mathfrak{Q}$.

[9] In the case of indiscernible m-atoms, we take the same Hilbert space for all cases.

[10] The automorphisms of a structure form the Galois group of the structure —see [4].

automorphism of the structure which leads one object in another object. But the above theorem says that, in the extended structure, they are no more indiscernible: the only object indiscernible from a certain object is the object itself. Since we can *always* extend a structure to a rigid one, indiscernible elements (from the point of view of a certain structure) will be always individualized outside the structure, say in the extended one. That is, yet masked for *within* a certain structure we are *not seeing* their individual characteristics for they are veiled by the existence of non trivial automorphisms, they are individuals, objects that obey the classical theory of identity. This is essentially what happens when we represent indiscernible quantum objects by symmetric or anti-symmetric vectors. Two (or more) quantum objects whose join state is described by either a symmetric or by an anti-symmetric wave function are indiscernible, but they remain indiscernible only in the context of a certain structure, but by force of the underlying mathematics (say, ZFC), there will exist a difference among them, yet sometimes we cannot point this difference, for due to the above instance of the excluded middle law, either they are the very same object ($a = b$) or they are distinct ($a \neq b$). But, if they are not the same entity, as we tend to agree concerning indistinguishable quantum objects forming a collection with cardinality greater than 1, the mathematical representation of quantum objects within standard mathematics (built in ZFC) seems to suggest the existence of hidden variables of some kind. In other words, within ZFC, although indiscernible from the point of view of quantum mechanics, quantum objects *are* individuals (again: they obey the classical theory of identity). But, if we aim at to deal with an alternative metaphysical package, namely, by considering them as non-individuals, we have two options: either to confine the discourse to within a certain non-rigid structure, and this can be done within ZFC proper, and which is a quite artificial move, or we may employ something such as the theory $\mathfrak{Q}$, where the indiscernibility is treated in a quite more natural way.

Thus, it seems that when we are concerned with individuality and identity problems, $\mathfrak{Q}$ looks better for the development of a formulation of quantum mechanics in which indistinguishable entities appear as such *right from the start*, as demanded by Post, in the sense we have seen already. The idea that these items are non-individuals is encapsulated in $\mathfrak{Q}$ with its m-atoms and collections thereof, which are now available for mathematical work in developing a different version of quantum theory. Important to notice that this does not imply that we cannot think of them as 'entities' of some sort. They *are* objects, but objects of a different kind than those postulated by classical logic, actually non-individual objects. Really, a collection of m-atoms has a (quasi)cardinal (say, 5), even if we cannot distinguish them, thus, we may reason *as if* five entities were being treated; in this sense, we are in agreement with the standard informal view of electrons, protons, or even atoms, when they cannot be distinguished from one another, although *being not the very same entity.*

In Domenech et al. (see [7] and [8]), the non-classical part of $\mathfrak{Q}$ was used to define a Hilbert space (called $\mathfrak{Q}$-space) whose vectors refer only to occupation numbers, while permutation operators act as the identity operator on them, reflecting in the formalism the unobservability of permutations. By maintaining both quantum indistinguishability and antisymmetry *without* resort to a symmetrization postulate, spin values in a two-value fermionic system were derived, obtaining identical results as those obtained in the standard Fock space formalism. The main difference to the standard approach is that, although some philosophers think that the (standard) Fock space formalism is free from individual labels (see [19, 20], [21]), it is not. In fact, dealt with within standard mathematics, the represented quanta are individuals, as we have seen, and then Fock spaces just act as another 'structure' where the individuality of the represented objects is blurred in some way, masked by some kind of veil. The only way to deal with truly indiscernible entities is by employing a different mathematical framework. Thus, the use of theories such as $\mathfrak{Q}$ (is there any other?), we can trace the first steps to found a quantum mechanics involving indiscernibility as a metaphysical hypothesis. This way, we shall be working in agreement with the semantic approach, but now in the framework of $\mathfrak{Q}$, and we can look for a *quasi-set theoretical predicate* for quantum theory. Such a predicate would include, among its models, the *intended model*, built by considering m-atoms as representing quantum objects. In this sense, the theory may deal with indistinguishable but non identical items without the need of mathematical tricks, such as the introduction of symmetrization conditions.

An alternative approach could also be considered. Still working in $\mathfrak{Q}$, we can retain classical mathematics involved with the set theoretical predicate presented at section 3, but now we recognize that among the models of the predicate, some may have as its domain a qset with m-atoms, that is, the set of physical systems is now a pure qset. This is a different approach than the just mentioned one, in which one bases the mathematical formalism itself on collections of m-atoms. Here, we restrict ourselves to the classical part of $\mathfrak{Q}$ to do the mathematics necessary for QM, but we employ the resources of non-classical qsets to bring the entities this formalism deals with. In this case, of course, those unwilling to keep with a classical mathematics that uses symmetrization conditions to represent the non-individual entities may stick to the $\mathfrak{Q}$-spaces. But we shall leave these details and pay attention to some kind of difficulties that arise in considering the present discussion.

The first point to be noticed is that the Hilbert space formalism, commonly used in QM, is compatible with at least two ways when it comes to metaphysical issues regarding the entities it deals with: they can be seen either as individuals or as non-individuals. Both readings will rely on how we interpret the symmetrization postulate (see 3.1): one can understand it as implying that only symmetric and anti-symmetric states exists or, on an alternative reading, we can say that every observable must be compatible with permutations, that is, the Hermitian operators representing observable quantities must commute with permutation operators, representing permutations on the labels of particles. The first reading is chosen by the non-individuals package, the second one allows one to interpret the items dealt with by the formalism as individuals, since in this case the asymmetric states that *could* distinguish the particles are simply inaccessible to the particles, but they *are there* (even though they have no physical significance). This is in conformity with the two metaphysical packages dealt with by French and Krause in [9]; the first consider non-individual quanta; the second one consider them as individuals, on a pair with their classical twins (described by classical physics), but at the expenses of introducing restrictions on the available states/observables, as implied by the restrictions imposed by the symmetrization postulate.

What is at issue here is, to our view, is a simple manifestation of the so-called thesis of the *underdetermination* of the metaphysics by the physics (see [9, chap. 4]). In fact, the formalism of quantum mechanics is compatible with (at least) two kinds of metaphysics, the individuals package and the non-individuals package. This compatibility notwithstanding, one must also advance a second step on the discussion of which underlying metaphysics will be adopted, a step we think has not been completely recognized: one should only effectively choose one of the metaphysical packages if the mathematical framework is chosen accordingly. In fact, each metaphysical package has its own accompanying logico-mathematical framework, which is most adequate for its purposes ([9, p.244]). So, it seems that if we want to 'prove' that quantum mechanical entities are individuals or, otherwise, that they are not individuals, there will be not a neutral base to do that, and we need to choose from some of the available mathematical basis one to work within; but then, once we have already made a choice of our preferred metaphysics, we need to choose the corresponding mathematical package to deal with. The move is purposeless, that is, *metaphysics comes first*, for it is already manifest in the very choice of our mathematical framework.

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