Structures and Structural Realism

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Abstract

The 'ontic' form of structural realism (OSR), roughly speaking, aims at a complete elimination of objects of the discourse of scientific theories, leaving us with structures only. As put by the defenders of such a claim, the idea is that all there is are structures and, if the relevant structures are to be set theoretical constructs, as it has also been claimed, then the relations which appear in such structures should be taken to be 'relations without the relata'. As far as we know, there is not a definition of structure in standard mathematics which fits their intuitions, and even category theory seems do not correspond adequately to the OSR claims. Since OSR is also linked to the semantic approach to theories, the structures to be dealt with are (at least in principle) to be taken as set theoretical constructs. But these are 'relational' structures where the involved relations are built from basic objects (in short, the rank of the relation is greater than the rank of the relata), and so no elimination of the relata is possible, although it would be interesting for characterizing OSR. In this paper we present a definition of a relation which does not depend on the particular objects being related in the sense that the 'relation' continues to hold even if the relata are exchanged by other suitable ones. Although there is not a 'complete' elimination of the relata, there is an elimination of 'particular' relata, and so our definition might be viewed as an alternative way of finding adequate mathematical 'set-theoretical' frameworks for describing at least some of the intuitions regarding OSR.

1 Introduction

Philosophers have recently distinguished between two forms of structural realism (SR), namely, the epistemic form and the ontic form (Ladyman 1998, French & Ladyman forthcoming, French 2003, which are indicated for details not recalled here). In short, the epistemic form of SR sustains that the objects of our scientific theories (like electrons etc.) remain epistemologically inaccessible and that all we know are the structural elements (structures) of our theories. But the idea of epistemologically inaccessible objects raises a lot of philosophical problems regarding, for instance, the underdetermination whereby quantum mechanics supports both a metaphysics of individuals and a metaphysics of non-individuals (French op. cit.). To surmount these (and other) concerns, Ladyman (op. cit.) has proposed the *ontic* form of SR, which entails (at least ideally) the complete elimination of objects, leaving us with structures only. As put by French, "the idea is that it is not just that all we *know* are the structures, but that all that there *is* are the structures" (op. cit.).

A natural question arises: what is a structure in the SR 'ontic' sense? In the above mentioned papers, no 'definition' of structure is given, although the question regarding the *understanding* of these structures has being invoked (French & Ladyman op. cit.). As far as we know, no satisfactory discussion on this important topic was provided yet.[1] So, how can we understand what the involved people mean by a 'structure'? Independently of the answer, the above guidelines would entail that the concept of structure should not depend on the particular objects being structured.

Pushing a little bit the proposals of the ontic SR, we find Ladyman suggesting that his view should be developed within the context of the semantic approach to theories, for in this approach a scientific theory is better understood throughout a family of models (which are structures of a kind), and hence this should fit quite nicely the intended proposal of looking at structures only. In addition, he has also suggested the use of 'partial' structures, which generalize the usual 'total' structures, which we recall are also set theoretical constructs.[2]

But, following these guidelines, we should keep aware that both total and partial structures are mathematical constructs built within set theory and, as such, these 'set-theoretical' structures were not born as *structures* from the beginning. Instead, structures are constructed from basic operations on *sets*, according to a well known process provided by axiomatic set theory. In short, a structure results to be a n-tuple whose elements are sets and relations on these sets. This fits the idea that they are to be regarded always "in relational terms" (French & Ladyman op. cit.) (from now on, when mentioning set theoretical structures, we shall keep with first order structures, but of course a more general characterization can be obtained as it is necessary for discussing structures adequate for physics, where the notion of species of structures is relevant). But, given structuralism, these structures and relations, the mentioned authors claim, should not depend of the relata, and hence we arrive at the problem they face: "how can we have structures without objects?" (ibid.). As it is well known, in

[1]It is not clear either if the philosophy of the ontic SR aims at to 'define' structure in some way or intends to take it as a primitive idea. In this second case, relevant axioms should be provided (see below).

[2]We shall not discuss the reasons for such uses here, to which we suggest the reading of their papers. But let us recall that 'standard' structures are *total* in the sense that all relations are *total* in the following sense: given a binary relation R on a set A, then for every ordered pair $\langle x, y\rangle \in A \times A$, either the pair belongs to R or does not. In other words, R can be seen as a pair $R = \langle R_1, R_2\rangle$ where R_1 is the collection of all pairs which belong to R and R_2 is the collection of all pairs which do not. *Partial* relations can be viewed instead as triples $R = \langle R_1, R_2, R_3\rangle$, where R_1 and R_2 are as above and R_3 is the collection of pairs to which we don't know whether they belong to R or not. The use of partial structures in philosophy of science can be seen in da Costa & French 2003.

(extensional) set theoretical terms, no relation (as sets of ordered n-tuples) can be built out of the related elements.

One may guess that instead of set theory one could make use of category theory, which at least in principle should deal with structures in its basic ontology, so apparently being in accordance with the very idea of the ontic SR. The reasons the mentioned philosophers don't use category theory is still not clear to me,[3] but perhaps this is due to the fact that from an intuitive point of view a category is nothing more than an ordered pair (hence a set) whose elements are a collection of objects (the structures) and a collection whose elements are called morphisms (both concepts are of course subjected to adequate postulates). That is, even in category theory we are not completely free from (intuitive) set theory. This of course does not entail that category theory cannot be useful for the philosophical discussions on SR, mainly if we use another kind of approach not so 'linked' to set-theoretical ideas, like the category theory developed by Obiña (1969). But this is something to be investigated further. An interesting topic to be also investigated further in connection to Ladyman's ontic SR would be the development of a 'partial category theory', where the objects would be partial structures, and not 'total' ones, as in standard category theory.

In this paper, we shall sketch the main ideas of a way of defining structures which, in a sense, do not depend of the objects being related by the relations they involve. The meaning of this 'in a sense' will be made clear below. So, we are not discussing the philosophical foundations of the ontic SR, but just trying to answer French & Ladyman's question: "how can we have structures without objects?", put by them in the following alternative form: "how can we have an effect without a something which is doing the effecting?" (op. cit.). As we shall see, the answer depends on the understanding of the *something* in this phrase. In a future work, we intend to investigate the idea of taking the notion of 'structure' as a primitive concept, subjected to adequate axioms, which perhaps will be closer to French & Ladyman's interests in finding a way of expressing 'pure' structure (without elements) within a mathematical framework. Hence, this paper can be seen as a first attempt to approach the subject by using an alternative mathematical device.

It should be remarked that this paper, although motivated by French and Ladyman's intuitions, does not intend to cover his ideas *in totum*. What we offer here is a way of looking to relational structures where the involved relations do not depend on the particular objects being related, but only of their 'kind' (or sort). As a motivation for what we shall present next, let us pose the basic question from the following point of view: we could ask, for instance, either it is possible to have an effect without something which produces it. Without entering in metaphysical disputes, we can say that once having an effect, we tend to accept that there is something which has produced such an effect; but in regarding quantum entities, which are the case we shall look more closely here, the fact is that a certain 'effect' produced by a certain entity (say, an electron)

[3]Recently, it seems that Elaine Landry has getting some results in this direction.

could be produced by whatever 'similar' one (by 'another' electron). Something 'produced' by a microscopic entity could be produced by whatever microscopic entity of the same species, or sort (electrons, say). So, may be we don't need to have the very object which has produced a certain effect, and all we need is the general structure of 'something' that has produced it (say, by knowing that it was an electron, and *whatever* electron could do the job as well). It is in this sense that our definition given below works. We shall provide the grounds for defining structures in a way so that by adequate exchange of its basic elements, the 'skeleton' of the relevant relations shall be preserved. These relations can then be useful for constructing structures which might be interesting for sustaining at least some aspects of OSR ('ontic' structural realism).

Insisting a little bit on the differences of approaches, we remark that Ladyman's original proposal has motivated the developments given here, but he is looking for a notion of structure *without objects* in the relevant relations, and it seems difficult to say how this can be achieved in set theoretical terms, for a relation is always constructed from the objects it relates (in extensional contexts).[4] Anyway, our definition given below is so that the relations must be based on 'kinds of particulars', and not on specific elements, and in a certain sense it seems to be closer to Eddington's and Cassirer's structuralism, which are based on quantum indistinguishability, although we shall also not pursue these relationships here (but see French 2003; French & Krause forthcoming, Chap. 3.).

We shall proceed as follows: in the next section we recall in brief the objections to the standard (set-theoretical) definition of relation and structure for the ontic form of SR. In recalling them, we can arrive at an interesting motivation which will guide the developments given in the remaining of the paper. After this digression, we shall introduce the basic intuition we shall pursue latter, taken from some simple examples from chemistry. Finally, we introduce the basic ideas of the mathematical stuff we shall be dealing with for the characterization of *relations without the (specific) relata* in the final sections. Some ideas involving 'structural objects' shall be mentioned at the end.

2 The structure of a relation

Since the ontic SR intends to make use of the semantic approach, structures should be set-theoretical constructs, that is, mathematical objects of the form $\mathfrak{A} = \langle A, R \rangle_{i \in I}$, where A is a set obtained by set-theoretical operations on some 'base sets' (in the sense of Bourbaki 1968; see also da Costa forthcoming) and the R_i form a sequence of relations on A.[5] Here, for simplicity, we shall take first order structures of the form $\mathfrak{A} = \langle A, R \rangle$, where A is a non empty set and R is a binary relation on A, although, some other cases may be used in the examples below. As already mentioned above, in doing physics one needs more

[4]A work in this sense is been delineated jointly with N. da Costa and S. French.

[5]In mentioning Bourbaki, we are not claiming that we are accepting here his 'syntactic' approach to mathematics.

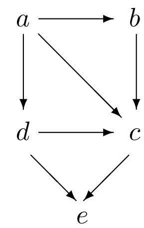


Figure 1: The relation R

than first order structures, but the considerations to be made here can be seen as a first sketch of an idea which can be generalised for higher order structures; this generalisation is left to be investigated further.

In standard mathematics (read: an extensional set theory like ZFC with the axiom of foundation), in order to define a relation R on a set A, we need that the set A had been constructed before. Technically, we may say that the rank of R is greater than the rank of A in the cumulative hierarchy.[6] Although Russell's concept of the *structure* of a relation (Russell 1974, Chap. 6) is already part of history and has been precisely described within the set-theoretical framework (as mentioned below), we may gain in having a look at some of his ideas, for the involved intuitions fit in much what we are trying to say here.[7] Given a certain relation R as a collection of ordered pairs, say $R = \{(a,b),(a,c),(a,d),(b,c),(c,e),(d,e)\}$ defined on $A = \{a,b,c,d,e\}$, we may characterize the 'structure' of R by representing R as a 'map', as showed in the Figure 1 and then 'abstracting' the elements of A, keeping with the schema shown in the Figure 2.

According to Russell, the 'structure' of R should not depend on the particular terms forming part of the field of the relation, which should be modified without altering its structure (ibid., p. 63).[8] So, in the Figure 2, the $\bullet$'s should stand for 'places' of elements of a set that has the same 'relation number' than R (in Russell's terminology, the relation number of R is the class of all relations *similar* to R, that is, those that have the same structure than R). Then, according to this idea, we would be able to substitute the bullets by elements of another set B by 'preserving the structure' of R. But simply substitutions do not ensure that the structure is preserved. For instance, take $B = \{a',b',c',d',e'\}$. Here a first restriction arises: B must have the same cardinality of A.[9] Let us suppose this for simplicity. If we simply substitute the

[6]For instance, a binary relation R on a set A is a set of ordered pairs of elements of A; so, to obtain R we must go up in the hierarchy of sets, starting with elements x, y of A, then obtaining the sets $\{x\}$ and $\{x,y\}$ and then the pairs $\{\{x\},\{x,y\}\} = \langle x,y\rangle$.

[7]These concepts were developed in deep in Whitehead and Russell's *Principia Mathematica*, but here we need only the basic intuitive motivations, and this is why we keep with Russell's 1974 book as our main reference on these matters.

[8]The field of a relation is the union of its domain with the domain of its inverse. In the present case, it is the set A itself.

[9]This kind of restriction runs in the directions of the criticisms directed to Russell by

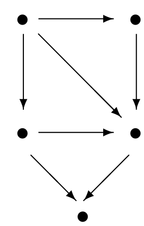


Figure 2: The structure of R

elements of B for the corresponding elements of A, does this entail that a relation $S = \{(a', b'), (a', c'), (a', d'), (b', c'), (c', e'), (d', e')\}$ with 'the same structure' will be found? Not necessarily, of course, for the elements of B may be not related as are the elements of A. The substitution will give the right result if and only if B is already 'structured as A is', but to ensure the success of the substitutions, we need to know that in advance. In the present day language of set theory, we say that $\langle A, R\rangle$ and $\langle B, S\rangle$ are *similar*, or have *the same structure* iff there exists a bijection from $f : A \mapsto B$ such that $(x, y) \in R$ iff $(f(x), f(y)) \in S$. But, let us insist, given A and B, of course there is no reason to guess that such an f exists. Simple substitution of the elements of an arbitrary B for the elements of A does not entail similarity.

We need some care here in speaking about these 'substitutions' for not confusing it with the process of abstraction in mathematics. In mathematics, when a mathematician says "if l and m are lines with slopes k_1 and k_2 such that $k_1.k_2 = -1$, then they are orthogonal", he is not making reference to particular (individual) lines in the plane, but to arbitrary lines provided that the condition of the antecedent of the conditional is fulfilled (hence, l and m act as 'parameters'). If we 'substitute' l and m in this sentence for slopes of particular lines (say, by those of $x - y + 1 = 0$ and $x + y + 1 = 0$), the resulting sentence is of course true. But, it should be remarked, it holds only if l and m are already known to be *lines with slopes* k_1 *and* k_2 *such that* $k_1.k_2 = -1$. This seems to be obvious, but is a distinctive fact which explains what we are proposing. In other words, to preserve the truth of a sentence like that of the slopes, we need to know *previously* the mathematical structure of the involved elements (that is, that they are *lines* with certain well defined slopes and so on) for, if not, the sentence would be false. In Russell's example, we have a set (a collection of objects) which are related in some way by R, and the relation will be 'preserved' not by arbitrary substitutions, but if and only if we substitute the elements of the given set by elements of another set whose elements are already structured *in the same way* as A is, although these elements can be of distinct nature or being themselves distinctly 'structured'.[10] For instance, take $A = \{a, b, c, d\}$ endowed with a reflexive relation whose elements (other than the pairs (x, x) –which are

Newman; see Newman 1928. I thank S. French for mentioning this point.

[10]Here, by 'structured elements' we mean the very structure the objects can have by themselves, and not the structure the set of which they are elements (see the next example).

in the relation– are $(a, b), (a, c), (b, d), (c, d)$. Now take $B = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, with $a \neq b$. Then of course $\langle A, R\rangle$ and $\langle B, \subseteq\rangle$ are similar, but the elements of A and B do not need to have 'the same structure' (the *elements* of A and B themselves don't need to be 'similar'; A may be a set of numbers, and B a set of set of numbers). In short: the simple exchange of elements, without taking them from an similarly structured set, do not necessarily 'preserve the structure', that's what we are insisting to say (the relevance of this 'obvious' fact will be made clear below). But of course if we substitute the elements of A by 'similarly structured' objects, then the structure (on A) will be preserved. For instance, consider substitution of elements of groups by elements of isomorphic groups. But in this case we need to know in advance that the elements to be substituted do have already the desired structure in order the substitution works. It seems that if we can find a kind of relation which enables substitutions of its basic elements so that the structure is preserved, than we could arrive at a relation (and so at structures) which in a sense do not depend on the (particular) elements being related, and give perhaps a different way of looking at the concept of invariance. Let us try to make sense to this idea.

In the empirical sciences, the above schema of simple substitutions by 'preserving the structure' makes sense. Really, we can find examples of situations where 'simple substitutions' can take place without taking the elements from already known similar structures. The only we need to know is that the involved elements must be substituted by *indistinguishable* ones, but not necessarily belonging to a similarly structured set. In these sciences, Russell's claim that "the field [of a 'relation'] can be modified without altering the structure" (Russell op. cit., p. 63) can be realized. Let us consider this case by mentioning some few simple examples, for they will motivate the definition we shall give below.

The structure of R given by the map of Figure 1 above resembles in much structural formulas of chemical compounds. Let us take the ethylic alcohol to exemplify (Figure 3). In this case, instead of the elements of A, we have the symbols H, C and O, which should be seen (as they of course are) as *places* for carbon, hydrogen and oxygen atoms. In other words, what we are expressing with the Figure 3 is not something concerning a set of particular individuals like the set A above; the ethylic molecule is not simple a collection (set) of certain atoms. A chemical substance like C_2H_6O may originate different chemical compounds, which are described by presenting their structural formulas; depending on the arrangement of the atoms (their *structure*, or *form*), the same collection of carbons, hydrogens and oxygens may produce quite different chemical substances, called isomers (and the same happens in several other cases). For instance, C_2H_6O may stand for both $CH_3 - CH_2 - OH$, the ethylic alcohol and $H_3C - O - CH_3$, the methylic ether (Figure 4). So, certain *structures* appear in these situations.

Russell's concept of the 'number of a relation' helps in fixing the idea. But, while (as we have seem) in standard mathematics the simple substitution of the elements of A by other elements does not ensure that the relation is preserved, in chemistry if we substitute the elements by similar ones, we do find the 'same' chemical element, regarded that H atoms are substituted by H atoms

$$
\begin{array}{ccccccccc}
 & & H & & H & & & & \\
 & & | & & | & & & & \\
H & - & C & - & C & - & O & - & H \\
 & & | & & | & & & & \\
 & & H & & H & & & &
\end{array}
$$

Figure 3: Ethylic Alcohol, C_2H_6O

$$
\begin{array}{ccccccccc}
 & & H & & & & H & & \\
 & & | & & & & | & & \\
H & - & C & - & O & - & C & - & H \\
 & & | & & & & | & & \\
 & & H & & & & H & &
\end{array}
$$

Figure 4: Methylic Ether, C_2H_6O

and so on (we are talking in general terms, so we shall not discuss how such a substitution can be performed, but simply assume that it can be done, even in ideal terms. Perhaps more adequate examples can be found by chemists or quantum physicists). This is an important point for the definition we will give latter: we *can* (at least ideally) substitute the H, C and O atoms by 'other' H, C and O atoms respectively by 'preserving the structure'.[11] This fact is nicely exemplified by Roger Penrose in the context of quantum mechanics (which of course would provide most adequate examples of what we are trying to say):

> "according to the modern theory [quantum mechanics], if a particle of a person's body were exchanged with a similar particle in one of the bricks of his house then nothing would happened at all." (Penrose 1989, p. 360)

So, in chemistry, the kind of 'substitutional' property of atoms in a certain structure makes it in certain sense to be *independent* of the particular (individual) involved elements. If we imagine that the structural formula of the ethylic alcohol represents a certain 'relation' among the H, C and O atoms (which is of

[11] Let us recall once more that some care is in need here: in standard mathematics, if $\langle A, R\rangle$ and $\langle B, S\rangle$ are similar, of course we can do the substitution, but given $\langle A, R\rangle$ and a (non structured) set B, there is no way of sustaining that a similar structure $\langle B, S\rangle$ can be found (that is, that a similar relation S on B will be found), as it happens in chemistry. In other words, in substituting the elements of A by the elements of B, we need first to check (so to say, 'by hand') if the elements of B are also in the relation. In chemistry, by the contrary, we know that in advance: substitution of elements by similar (indistinguishable) ones (H by H, C by C etc.) *do preserve the structure.*

course what is happening), then its chemical properties do not depend on the particular atoms being involved. So, there is a 'relation' which can be said to be independent of the individual relata it links (except in what respects their 'nature' –see below). Let us reinforce this idea of the independence of particular elements that enter in a certain effect by giving another example. Take for instance a simple chemical reaction

$$\mathrm{NO} + \mathrm{O}_3 \to \mathrm{NO}_2 + \mathrm{O}_2,$$

where one nitric oxide molecule reacts with one ozone molecule to produce one nitrogen dioxide molecule and one oxygen molecule. We remark that it is not important *what* particular oxygen atom (there are three) was captured by the nitric oxide molecule to form the nitrogen dioxide; the only relevant fact is that the captured element *must be* one oxygen atom. Thus Toraldo di Francia says: "this enable us to put within parentheses the *true* nature of the entities and emphasize the *only* secure property: the number!" (1986, p. 122). Really, every oxygen atom of the ozone molecule plays the same role in the reaction. Only the quantity of them is important, and for sure the same holds in the quantum context in regarding elementary particles. The definitions we shall present below intend to capture such situations in a set-theoretical framework. For understanding them, let us mark the fact that in the exemplified case there is a certain range of possibilities from where the oxygen atom can be taken (namely, the ozone molecule that enter in the reaction). Below we shall speak of the 'surroundings' of a certain collection of elements, which will stand for a zone from which the elements to be exchanged should be 'taken', and the above exemplified kind of exchange shall be captured by a theorem to be presented below (Theorem 3.1). Further examples will also be mentioned below.

So, what we are approaching is a characterization of a concept of structure which mirrors what happens in chemical (and of course in quantum) situations, but we shall try to do it in set-theoretical terms: as we will see, we shall have a certain relation R, given as a 'set' (really, a 'quasi-set') of ordered 'pairs' so that when the elements of these pairs are 'exchanged' by suitable ones (we shall say that they are 'indistinguishable' in a sense to be explicated below) then the relation 'continues to hold', which is of course contrary to the case involving standard relations of usual set theories, due to the axiom of extensionality, as we have seen. But, first, let us turn to the general ideas involving the mathematical framework we shall be working within, namely, quasi-set theory.

3 The mathematical framework: general ideas on quasi-set theory

Intuitively speaking, a quasi-set is a collection of indistinguishable (but not identical) objects.[12] This of course is not a strict 'definition' of a quasi-set, acting more or less as Cantor's 'definition' of the concept of set as "a collection into a

[12]For details, see Krause 1992, 1996, 2003.

whole of distinct elements of our intuition or of our thought" (Cantor 1958, p. 85). Although not precise, this characterization gives an intuitive account of the concept. The quasi-set theory $\mathfrak{Q}$ was developed following Erwing Schrödinger's remark that the concept of identity cannot be applied to elementary particles (Schrödinger 1952, pp. 17-18); this idea is expressed in the theory by assuming that expressions like $x = y$ are generally not well-formed formulas (and likewise for the negation $x \neq y$). But there is a concept of 'indistinguishability' ($\equiv$) that may hold among the entities of the theory. This enable us to consider logico-mathematical systems in which identity and indistinguishability are separated concepts; that is, these concepts do not reduce to one another as in standard (Leibnizian) set theories. So, we could say, paraphrasing Cantor, that a quasi-set is a collection of elements to which we cannot say either they are identical or distinct from one another. An important fact is that this is not an *epistemological* ignorance; it is rather *ontological*, essentially due to the lack of sense in applying the concept of identity to some of the objects of the domain.

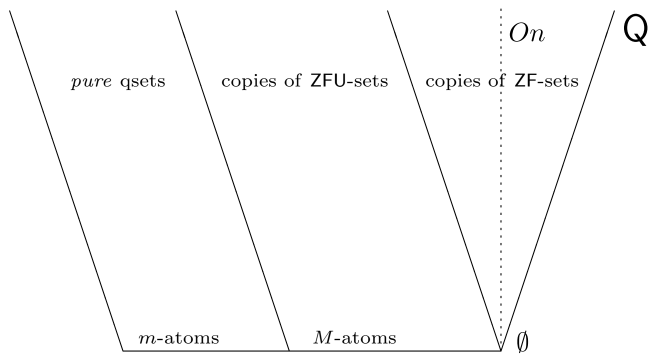
Quasi-set theory $\mathfrak{Q}$ allows two kinds of Urelemente: the m-atoms, whose intended interpretation are the quanta (and perhaps macroscopic things like atoms and molecules), and the M-atoms, which stand for usual objects, to which classical logic is supposed to apply.[13] Quasi-sets are the collections obtained by applying ZFU-like (Zermelo-Fraenkel plus *Urelemente*) axioms to a basic domain composed of m-atoms, M-atoms and aggregates of them.[14] The theory still admits a primitive concept of quasi-cardinal which intuitively stands for the 'quantity' of objects in a collection. The main idea is that the quasi-cardinal of a quasi-set cannot be associated (in the sense of this association being something described in the 'classical' part of $\mathfrak{Q}$ –see Figure 5) to a particular ordinal due to the (absolute) indistinguishability of the m-atoms, and this is the motive for taking this concept as primitive. Notwithstanding this, it is possible to define a translation from the language of ZFU into the language of $\mathfrak{Q}$ in such a way so that there is a 'copy' of ZFU in $\mathfrak{Q}$ (the 'classical' part of $\mathfrak{Q}$). In this copy, all the usual mathematical concepts can be defined, and the 'sets' (really, the '$\mathfrak{Q}$-sets') turn out to be those quasi-sets whose transitive closure (this concept is like the usual one) do not contain m-atoms (see again the Figure 5).[15]

In $\mathfrak{Q}$ there may exist quasi-sets whose elements are m-atoms only, termed 'pure' quasi-sets, whose elements are indistinguishable (in the sense of partaking the primitive indistinguishability relation $\equiv$) and the axioms provide the grounds for saying that nothing in the theory can distinguish the elements of such an x from the others. The axioms of $\equiv$ are those of an equivalence relation. Furthermore, we can define an 'extensional identity' ($=_E$) among all those objects of the domain which are not m-atoms, and so that it will act as the usual

[13]But of course other kinds of logic could be used instead with due adaptations.

[14]Perhaps for some applications it would be interesting to have, say, 'quasi-classes', and so we could use NBG-like axioms.

[15]So, we can make sense to the primitive concept of quasi-cardinal of a quasi-set x (written $qc(x)$) as being a cardinal defined in the 'classical' part of the theory. When the existence of m-atoms is postulated, apparently there is no way of defining a translation from $\mathfrak{Q}$ to ZFC, so the theories are not equiconsistent.



The 'classical' part of $\mathfrak{Q}$.

Figure 5: The Quasi-Set Universe

identity of ZFC for those entities to which it applies. Within the theory, the idea that there is more than one entity in x is expressed by an axiom which states that the quasi-cardinal of the power quasi-set of x (the concept of subquasi-set is like that of standard set theory)[16] has quasi-cardinal $2^{qc(x)}$, where $qc(x)$ is the *quasi-cardinal* of x (which is a cardinal obtained in the 'copy' of ZFU just mentioned). Now, one can ask: what does this supposition mean?

Consider the three protons and the four neutrons in the nucleus of a ^{7}Li atom (alternatively we could take the three oxygen atoms of the ozone molecule of our previous example). As far as quantum mechanics goes, nothing distinguishes among these *three* protons. If we regard these protons as (intuitively) forming a quasi-set, its quasi-cardinal should to be 3, and there is no apparent contradiction in saying that there are also 3 subquasi-sets with 2 elements each. In the same vein, it is reasonable to say that there are three 'singletons' and so on, although we can't distinguish among them either (we will say that they, so as their collections with the same quasi-cardinal, are indistinguishable). So, it is reasonable to postulate that the quasi-cardinal of the power quasi-set of x is $2^{qc(x)}$. Whether we can distinguish among these subquasi-sets or not is a matter which does not concern logic (and of course even physically this cannot be done).

In other words, we may reason as if there are three entities in our quasi-set x (in our example above), but x must be regarded as a collection for which it is not possible to discern its elements as individuals. Although we can suppose their existence, the theory does not enable us to *write down* the corresponding singletons, as we do in ZFC when we write $\{a\}$, $\{b\}$ and $\{c\}$, given a, b and c. In $\mathfrak{Q}$, there are no *names* for the m-atoms. The grounds for such kind of

[16]This is what makes a basic difference with fuzzy sets. In fuzzy set theory, as it is well-known, the counter-domains of the characteristic functions are not $\{0, 1\}$, but $[0, 1]$.

reasoning has been delineated by Dalla Chiara and Toraldo di Francia as partly theoretical and partly experimental. Speaking of electrons instead protons, they noted that in the case of the helium atom we can say that there are two electrons because, *theoretically*, the appropriate wave function depends on six coordinates and thus "... we can therefore say that the wave function has the same degrees of freedom as a system of two classical particles" (op. cit., p. 268).[17] Dalla Chiara and Toraldo di Francia have also noted that, "[e]xperimentally, we can ionize the atom (by bombardment or other means) and extract two separate electrons ..." (ibid.).

Of course, the electrons can be counted as two only at the moment of measurement; as soon as they interact with other electrons (of the measurement apparatus, for example) they enter into entangled states once more. It is on this basis that one can say that there are two electrons in the helium atom or six in the 2p level of the sodium atom or (by similar considerations) three protons in the nucleus of a ^{7}Li atom (and it may be contended that the 'theoretical' ground for reasoning in this way also depends on these experimental considerations, together with the legacy of classical metaphysics). On this basis we formulate the axiom of 'weak extensionality' of $\mathfrak{Q}$, which says that those quasi-sets that have the same quantity of elements of the same sort (in the sense that they belong to the same equivalence class of indistinguishable objects) are also indistinguishable.

This axiom has interesting consequences. There is no space here for the details, but let us mention just one of them which is related to the above discussion on the non observability of permutations in quantum physics, which is one of the most basic facts regarding indistinguishable quanta (recall Penrose's quotation given above). In standard set theories (ZFC), if $w \in x$ and $z \notin x$, then of course $(x - \{w\}) \cup \{z\} = x$ iff $z = w$. That is, we can 'exchange' (without modifying the original arrangement) two elements iff they are *the same* elements, by force of the axiom of extensionality. But in $\mathfrak{Q}$ we can prove the theorem below, where $\overline{z}$ (and similarly $\overline{w}$) stands for a quasi-set with quasi-cardinal 1 whose only element is indistinguishable from z (respectively, from w –the reader shouldn't think that this element *is identical to either* z or w, for the relation of equality doesn't apply here;[18] the set theoretical operations can be understood according to their usual definitions):

Theorem 3.1 [Unobservability of Permutations] *Let x be a finite quasi-set such that x does not contains all indistinguishable from z, where z is an m-atom such*

[17] This might be associated to the legacy of Schrödinger, who says that this kind of formulation "gets off on the wrong foot" by initially assigning particle labels and then permuting them before extracting combinations of appropriate symmetry (Schrödinger 1998).

[18] The only we can say is that the element that belongs to $\overline{z}$ is indistinguishable from z. It would be a fallacy to think that since we can talk of *the* only element of $\overline{z}$ which is indistinguishable from z, then we are individuating it. The important fact is that in $\mathfrak{Q}$ we can express set-theoretically the invariance of permutations; that what import is that we could think of *another* indistinguishable from z forming another of such 'singletons', and no appreciable differences between them would be found. So, no identification is possible, and we definitively do not know either we are working with a certain m-atom or with *another* indistinguishable one. As in physics, this does not matter.

that $z \in x$. If $w \equiv z$ and $w \notin x$, then there exists $\overline{w}$ such that

$$(x - \overline{z}) \cup \overline{w} \equiv x$$

The proof makes use of several axioms of quasi-set theory, and shall be not repeated here (but see Krause 2003). Supposing that x has n elements, then if we 'exchange' their elements z by correspondent indistinguishable elements w (set theoretically, this means to perform the operation $(x - \overline{z}) \cup \overline{w}$), then the resulting quasi-set remains *indistinguishable* from the original one in the sense of the weak extensionality axiom. In a certain sense, it is not important whether we are dealing with x or with $(x - \overline{z}) \cup \overline{w}$. This of course gives a 'set-theoretical' sense to Penrose's claim mentioned above. So, within $\mathfrak{Q}$ we can express that 'permutations are not observable'. Perhaps another physical situation may reinforce the claim: suppose that an electron a is absorbed by an atom and becomes entangled with the electrons in the outer shell of this atom (this outer shell is our quasi-set x).[19] Then, some time latter, an electron b is emitted from the atom. Could we say that this 'new' atom is *the same* (in an extensional sense) than the previous one we had before having absorbed the electron? Of course not, for there is no sense in saying that that a is identical to b or that it is not. The most we can say is that 'both atoms' are indistinguishable, which is expressed by Theorem 3.1, an play the same role in whatever application or theoretical consideration.

The theory has other applications, for instance in deriving quantum statistics without the needs of postulating certain symmetry conditions, but these developments shall be not mentioned here (see Krause et al. 1999; Krause 2003).

4 Relations without the relata

Keeping with quasi-set theory in mind, we may turn to a characterization of certain relational structures in which the involved relations do not depend on the (particular) elements being related. In this section we shall be working within the theory $\mathfrak{Q}$, and we will emphasize the case involving m-atoms only. A quasi-relation on a quasi-set A is a quasi-set R whose elements are ordered 'pairs' that belong to A. These 'pairs' should also be understood in a right way. Since the identity relation cannot be used here, an ordered 'pair' $\langle z, w, \rangle$ is something like the collection (quasi-set) of the indistinguishable from z (denoted $[z]$) and the collection of the indistinguishable from either z or w (denoted $[z, w]$) that belong to A; in symbols, $\langle z, w, \rangle =_{\mathsf{df}} [[z], [z, w]]$, which resembles Wiener-Kuratowski's definition. So, each 'pair' may contains more than two elements (the word 'pair' here looks like 'pair of kinds'). So, a (binary) quasi-relation R on A is a quasi-set which obeys the following predicate $\Re$:

$$\Re(R) =_{\mathsf{df}} \forall z(z \in R \rightarrow \exists u \exists v(u \in A \wedge v \in A \wedge z =_E \langle u, v \rangle)).$$

[19] We remark that this 'nomination' of the electrons with a's and b's are only a way of speech. We still remark that, as in standard set theory, $(x - \overline{z}) \cup \overline{w} \equiv (x \cup \overline{w}) - \overline{z}$.

Then the question we would like to discuss may be summed up as follows (formulated for more general relations):

Question: given a certain n-ary q-relation R on a pure qset A, if $R(x_1, \ldots, x_n)$ holds, does $R(x'_1, \ldots, x'_n)$ also hold if $x_i \equiv x'_i$? In other words, are the relations 'preserved' when the relata are exchanged by indistinguishable ones?

The first and direct answer to the above question is that it depends on the relation. If R is membership, then the intended result fails, for if $x \in y$ and $x \equiv x'$, $y \equiv y'$, nothing in the axioms of the theory entails that $x' \in y'$ (this is one of the basic results that make the primitive relation of indistinguishability distinct from identity). Membership is the only primitive relation of $\mathfrak{Q}$ which does not enable substitutivity by indistinguishable (Krause 2003). So, let us take R to be whatever relation distinct from membership; furthermore, we shall work with binary relations only for simplicity and we will be paying attention to relations on quasi-sets whose elements are m-atoms only. So, the question, put in a simple form, is: if R is a binary relation (distinct from membership) and if $R(x,y) \wedge x' \equiv x \wedge y' \equiv y$, does this entail that $R(x', y')$ holds as well? The most interesting case is of course when both x and y are m-atoms (if there are no m-atoms involved, then $\equiv$ becomes (extensional) identity and the answer is a straightforward yes).

Let us suppose for simplicity that R is defined on a finite pure qset, which suffice for our purposes. But $R(x,y)$ means $\langle x, y \rangle \in R$, that is, $[[x],[x,y]] \in R$. Recall that $[x]$ is the qset of all indistinguishable from x (which may have more than one element) and that $[x,y]$ is the qset of the indistinguishable of either x or y and that x and y are not playing the role of names for objects of the domain;[20] instead, they act as generalized names, meaning something like 'some' indistinguishable from x or y respectively. So, a binary relation in the theory $\mathfrak{Q}$ is not a well 'defined' (by its extension) collection of ordered pairs of the elements of some set. If $R(x,y)$ holds, we are not necessarily saying that *that* specific x and *that* specific y are in the relation, but that *some* indistinguishable from x is in the relation with *some* indistinguishable from y. But the problem now is to explain the sense of being R defined on a certain A, for if x' and y' are indistinguishable respectively from x and y, how can we ensure that, being $R(x,y)$ true, the same happens with $R(x', y')$? (for x' and y' may be not members of A). So, the apparent answer to our Question would be *no*.

But there is a sense in preserving this result (that is, in answering it with an 'yes') if we consider what we shall term the *surroundings* of the qset A.[21] The surroundings of A is defined relatively to a qset D which contains A; the definition is $Sur_D(A) =_{\mathsf{df}} [y \in D : y \equiv x \wedge x \in A]$. In words, it is the qset of the elements of D which are indistinguishable from the elements of A. Intuitively,

[20] To distinguish quasi-sets from usual sets, in writing quasi-sets we use '[' and ']' instead of '{' and '}'; so, our 'classifier' is written [:].

[21] In Krause 2003, we have termed it the 'cloud' of A. But the term 'surroundings' seems to be more adequate in expressing from where the elements to be exchanged by the elements of A are to be taken. In physics, this stand for the experimental apparatus of another chemical compound which may provide the relevant elements for the exchanges.

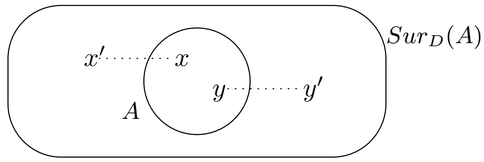


Figure 6: A and its surroundings.

$Sur_D(A)$ acts as the surroundings from where A can 'exchange' elements (see Figure 6). Let us suppose that $\widehat{R}$ is the extension of R to $Sur_D(A)$, that is $\widehat{R}$ is the qset of all 'pairs' $\langle x, y\rangle$ with x and y in $Sur_D(A)$ such that $\langle x, y\rangle \in R$ when $x, y \in A$. Then we can prove in $\mathfrak{Q}$ the following result (the terminology is as above):

Theorem 4.1 *If $A \subseteq D$, $x, y \in A$ and $R(x, y)$, where R is a quasi-relation on A, then there exist $x', y' \in D$ such that $x' \equiv x$ and $y' \equiv y$ so that $\widehat{R}(x', y')$.*

The proof is straightforward: if $\neg\widehat{R}(x', y')$, since $\forall x(x \equiv x)$ is an axiom of $\mathfrak{Q}$, then we would have that $R(x, y)$ but $\neg\widehat{R}(x, y)$, which is impossible by the definition of $\widehat{R}$.

Intuitively, the theorem says that if $R(x, y)$ holds for $x, y \in A$, then if x' and y' are indistinguishable from x and y respectively and belong to a quasi-set D which includes A, then $\widehat{R}$ holds for these elements (that is, $\widehat{R}(x', y')$ holds). We remark that there would be no mathematical sense in saying (in the general case) that $R(x', y')$ holds, for x' and y' may do not belong to A, and R is a quasi-relation defined on A. The extension $\widehat{R}$ of R plays the role of R for the elements of the surroundings of A and coincides with R within A. So, in saying that $\widehat{R}(x', y')$ holds, we are in a certain sense granting that the relation R is maintained (though $\widehat{R}$) when the elements it relates are exchanged by suitable (indistinguishable) ones (say, taken from its neighborhood, like the measurement apparatus), and hence it does not depend on the particular relata it relates (as individuals). It seems to me that this is precisely what the chemical situations involving ionization and others among the above exemplified cases are suggesting us.

5 Structures and Structural Objects

The consideration of quasi-sets and their surroundings in the above sense unable us to consider relational structures (partial relations and structures could be also considered in a suitable extension of these ideas) in the usual sense (that is, as collections of quasi-sets and relations among them which can be 'extended' to their surroundings as above). So, we may arrive at a definition of structures which do not depend on the particular relata they link (as individuals). In other words, in this approach we can have the effects, described by the relevant relations, without a something (as an individual) which is doing the effect, to use French & Ladyman's mentioned phrase. It seems that it is a case like this one that chemistry postulate: the effect of, say, the methylic ether molecule can be seen as independent of whether the molecules are composed by this or by that particular C, H and O atoms. Another example mentioned above also helps here: in the chemical reaction $NO_2 + O_3 \rightarrow NO_3 + O_2$, we have the 'effect' of the creation of an NO_2 molecule but in this process it is not relevant the very individualistic nature of the particular oxygen atom (taken from the O_3 molecule) that enters in the reaction. Of course there is *something* doing the effect, but only its nature (as an oxygen atom) as an entity of a certain kind (or sort) is relevant.

The (perhaps) important remark to be made here is that our above definition enables us to consider these situations from a (quasi)set-theoretical point of view, making sense to the intuitive idea that a body is something like a structured collection of something, although it cannot be regarded as a standard set, as a collection of well defined and distinct objects. The idea that we may have something being preserved in a thing (its structure) without any commitment to the importance of the *material* of which it is composed, is implicit in Schrödinger's philosophy. According to him, among the basic features of an object, there is its *form* or *shape* (Schrödinger 1952; Bitbol 1996, pp. 160ff). As he says,

> "... in palpable bodies, composed of many atoms, individuality arises out of the structure of their composition, out or shape or form, or organization, as we might call it in other cases. The identity of the *material*, if there is any, plays a subordinate role. You may see this particularly well in cases when you speak of 'sameness' though the material has definitively changed. A man returns after twenty years of absence to the cottage where he spent his childhood. ... The shape and the organization of the whole place have remained the same, in spite of the entire 'change of material' in many of the items ... including, by the way, our traveller's own body itself!" (op. cit., p. 20)

By 'shape' here he means something given by sets of invariant properties, like group transformations, but the invariants give us only kinds of entities (electrons, for instance), and although in the everyday world we of course individuate the objects by space-time trajectories, or substance or other means, such individuation must be abandoned in the quantum realm, yet permanence of structure (form) of the composed objects still remains. So, Schrödinger displays elements of a structuralist tendency, although has accepted the non-individuality of the basic compound elements (quantum particles) and has incorporated it within a structuralist metaphysics. Perhaps quasi-set theory can capture at least part of this approach in a mathematical alternative way. Anyway, to pursue the relationships between structural realism and Schrödinger's ideas is also a topic for a future work.

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