Simulation and Transfer Results in Modal Logic – A Survey

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1 Introduction

Modal logic is by and large the theory of a single normal operator. The great majority of papers that develop the theory of modal logics deal with single operator logics, while papers that are concerned with applications of modal logics tend to use several operators, and occasionally also non-normal operators, dyadic operators, and sorted languages. This defect has been noticed quite early by DANA SCOTT in [38], but his criticism has had little effect on the research in modal logic. Even the rise of temporal logic and dynamic logic has not changed that to a great extent, perhaps for the reason that both were deemed to be too different from 'plain' modal logic to be assimilated with it.

One can only speculate about the reasons for not dealing with several operators. Beyond mere tradition we believe it is due to the great success of modelling intuitionistic logic with modal logic, and the great interest in extensions of **K4** during the 70ies and 80ies. Moreover, to those in the know it must have seemed too ambitious to develop a theory of several operators, for even the lattice of monomodal logics is very complex. Finally, however, the interest in modal logic and its applications was rising sharply in the late 80ies, and with it came the quest for a theory of several operators. Rather than building such a theory from scratch, it seemed worthwile to try to build it upon the already existing theory of monomodal logic. One example is [19]. GOLDBLATT takes the machinery of Stone representation originally used by JÓNSSON and TARSKI in their classic [24] and generalized it to the setting of polymodal, polyadic operators. However, this was from a technical point a straightforward extension of these methods, even though it killed the case of duality theory for modal logics in one blow. What is left with respect to duality theory are only the known unsolved problems in correspondence theory, such as a complete characterization of elementary, modally definable properties. Thus, only the completeness and decidability problems remained as new territory for research in polymodal logic. However, they turned out to be rather involved. Already the simplest case of bimodal logics deserves very careful proofs despite its apparent simplicity. We are alluding here to the case of a bimodal logic which has no axioms that use both of the operators. Such logics were called *stratified* in [16], and *independently axiomatizable* or *fusions* of their monomodal fragments in [28]. For fusions it was shown for many properties $\mathfrak{P}$ that they have $\mathfrak{P}$ iff both of their monomodal fragments have $\mathfrak{P}$. The list of properties includes finite model property, completeness, canonicity and decidability. It does not contain tabularity, however. In fact, it was shown in [22] that even if the fragments of a bimodal logic Λ are tabular, then Λ can have continuously many extensions, and – what is more –, even continuously many maximal ones (i. e. logics of codimension 1 in the lattice of normal extensions of Λ). This result shows that studying a bimodal logic via its monomodal fragments is a rather raw approximation, comparable to studying a monomodal logic via its completion, that is, the smallest complete logic containing the given logic (see [5, 6] and below).

Another way to relate monomodal logics and polymodal logics was found earlier by THOMASON in [42, 44]. He showed that there is a way to code any finite number of modal operators with a single operator such that many negative properties (e. g. undecidability, incompleteness) are being preserved. Using this reduction numerous counterexamples to specific conjectures in monomodal logic have been found by first developing a counterexample with several operators, and then appealing to the properties of THOMASON's simulation. However, THOMASON did not develop the full potential of this simulation. In [27] it is shown that THOMASON's simulation preserves not only negative properties but also positive properties of logics, such as decidability, finite model property, completeness and canonicity. Moreover, the map itself is an isomorphism from the lattice of n-modal logics onto an interval of logics in the lattice of monomodal logics. In a sense, these results justify the exclusive study of monomodal logics *ex post*, because by simulation a statement that is true of all monomodal logics has a great chance of being true of polymodal logics in general.

We have briefly introduced two different methods for comparing classes of modal logics. The first one investigates the *transfer* of properties when we extend the language while the second one defines a *simulation* of the logics of one class by means of another class. As indicated above, the methods are connected by the fact that simulations yield insights into the class of simulated logics only via strong transfer results for the simulation. In this paper we give three examples of transfer problems that have been studied in the past and are of great significance, namely

- the transfer from normal polymodal logics to their fusions,
- the transfer from a normal modal logic to its extension by adding the universal modality,
- the transfer from normal modal logic to its minimal tense extension.

Likewise, we give five examples of simulations of modal logics via modal logics, namely

- simulations of normal polymodal logics by normal monomodal logics,
- simulations of nominals and the difference operator by normal operators,
- simulations of monotonic modal logics by normal bimodal logics,
- simulations of polyadic normal modal logics by polymodal normal logics,
- simulations of intuitionistic modal logics by normal bimodal logics.

Finally, we note that simulations of non–modal systems as modal logics form another powerful tool for investigating modal logics. For instance, showing how hard modal logic generally is let us note as examples the simulation of second order logic [43], word problems for semigroups [40], the Minsky machine [23, 9] and [10], and for complexity results also the tiling problems [41]. However, we shall not deal with those simulations here.

2 Fundamentals of Modal Logic

We assume that the basic notions of modal logics are familiar. Nevertheless, the terminology is explained here briefly. It is more or less identical to [8]. The language of κ-modal logic $\mathcal{L}_\kappa$, κ an cardinal number, consists of denumerably many variables p_i, $i \in \omega$, the booleans $\top$, $\neg$ and $\wedge$, and modal operators $\Box_i$, $i < \kappa$. Other occurring symbols are treated as abbreviations in the standard way. Modal operators are often kept apart by using different symbols rather than subscripts, for example $\blacksquare$, $\boxminus$, $\square$ etc. Elements of $\mathcal{L}_\kappa$ are denoted by lower case Greek letters. Logics are equated with the set of their theorems, and so they are simply subsets of $\mathcal{L}_\kappa$. A (κ-modal) logic is *normal* if it contains the tautologies of the boolean calculus, the so-called *box-distribution axioms* $\Box_i(p \rightarrow q). \rightarrow .\Box_i p \rightarrow \Box_i q$, for every $i < \kappa$, and is closed under substitution, modus ponens and the rule $p/\Box_i p$, for $i < \kappa$. The consequence $\vdash_\Lambda$ associated with a classical logic has as its only rule of inference modus ponens. Thus $\Sigma \vdash_\Lambda \phi$ iff ϕ is derivable from $\Sigma \cup \Lambda$ with the use of modus ponens alone. The smallest κ-modal normal logic is denoted by $\mathbf{K}_\kappa$. Given a normal modal logic Λ and a set X of formulae, then $\Lambda \oplus X$ denotes the smallest normal logic containing Λ and X. A logic Λ is *decidable* if for every formula ϕ it is decidable whether or not $\phi \in \Lambda$. A logic has *interpolation* if for every formula $\phi \rightarrow \psi \in \Lambda$ there exists a formula χ in the variables occurring both in ϕ and ψ such that $\phi \rightarrow \chi \in \Lambda$ and $\chi \rightarrow \psi \in \Lambda$. Λ is *Halldén-complete* if for every pair formulae ϕ, ψ with no variables in common, if $\phi \vee \psi \in \Lambda$ then either $\phi \in \Lambda$ or $\psi \in \Lambda$.

The logics extending the system $\mathbf{K}_\kappa$ form a complete, distributive lattice, denoted by $\mathcal{E}\,\mathbf{K}_\kappa$. An important concept in the study of these lattices is the notion of a splitting. A *splitting* of a lattice $\mathfrak{L}$ is a pair $\langle x, y\rangle$ of elements such that for every element z either $z \leq x$ or $z \geq y$, but not both. In other words, the pair is a splitting iff the lattice is partitioned into the ideal generated by x and the filter generated by y. Given x, y is uniquely determined; likewise, y uniquely determines x. For logics we use the following notation. Given a splitting $\langle \Lambda, \Theta\rangle$ of $\mathcal{E}\mathbf{K}_\kappa$ we say Θ is a *splitting logic* of the lattice $\mathcal{E}\mathbf{K}_\kappa$ and write $\mathbf{K}_\kappa/\Lambda$ for Θ.

A boolean algebra is a quadruple $\mathfrak{A} = \langle A, 1, -, \cap\rangle$ satisfing the standard laws of boolean logic. An *expanded boolean algebra* is a pair $\langle \mathfrak{A}, \langle \blacksquare_i | i < \kappa\rangle\rangle$, where $\blacksquare_i : A \rightarrow A$. The operators are said to be *monadic* in this case. An operator is called a *hemimorphism* if $\blacksquare_i 1 = 1$ and $\blacksquare_i(a \cap b) = \blacksquare_i a. \cap .\blacksquare_i b$. A *$\kappa$-modal algebra* is a pair $\mathfrak{A} = \langle \mathfrak{A}, \langle \blacksquare_i | i < \kappa\rangle\rangle$, where $\blacksquare_i$ are hemimorphisms. (There is an extension of notation and terminology to polyadic operators. Polyadic operators corresponding to normal polyadic logics must be hemimorphisms with respect to all their arguments. To avoid being overly abstract, though, we stick to the unary case.) The theory of an expanded boolean algebra $\mathfrak{A}$ is the set of all formulas ϕ such that $h(\phi) = 1$ for all homomorphisms from the term-algebra over $\mathcal{L}_\kappa$ into $\mathfrak{A}$. The theory of $\mathfrak{A}$ is denoted by $\mathsf{Th}\,\mathfrak{A}$. For a class $\mathfrak{K}$ of algebras, $\mathsf{Th}\,\mathfrak{K} = \bigcap\langle \mathsf{Th}\,\mathfrak{A} | \mathfrak{A} \in \mathfrak{K}\rangle$. A class of algebras is a *variety* iff it is closed under products, subalgebras and homomorphic images. The following is a standard theorem.

Proposition 1 *Any normal κ-modal logic is complete with respect to a variety of κ-modal algebras. The correspondence between logics and varieties is a lattice anti-isomorphism with respect to the inclusion (of logics and varieties, respectively).*

A *Kripke-frame* is a pair $\mathfrak{f} = \langle f, \langle \lhd_i | i < \kappa\rangle\rangle$, where $\lhd_i \subseteq f^2$ are the so-called *accessibility relations.* A *valuation* into $\mathfrak{f}$ is a function $\beta : var \to \wp(f)$. A *Kripke-model* is a triple $\langle \mathfrak{f}, \beta, x\rangle$, where $x \in f$ and β is a valuation. By induction on ϕ, $\langle \mathfrak{f}, \beta, x\rangle \models \phi$ is defined. A generalized frame is a pair $\mathfrak{F} = \langle \mathfrak{f}, \mathbb{F}\rangle$ where $\mathbb{F}$ is a system of subsets of f closed under relative complement, intersection and the operations $\blacksquare_i : \wp(f) \to \wp(f)$, defined by

$$\blacksquare_i a := \{y | (\forall z)(y \lhd_i z. \to .z \in a)\}.$$

We call a member of $\mathbb{F}$ an *internal set* of the generalized frame. A generalized model is a triple $\langle \mathfrak{F}, \beta, x\rangle$, where $x \in f$ and $\beta : var \to \mathbb{F}$, a valuation. By assumption on $\mathbb{F}$, the set of all points at which ϕ holds is internal.

A generalized frame $\mathfrak{F} = \langle \mathfrak{f}, \mathbb{F}\rangle$ uniquely determines a modal algebra $\mathfrak{F}_+ = \langle \mathbb{F}, 1, -, \cap, \langle \blacksquare_j | j < \kappa\rangle\rangle$. Moreover, given a modal algebra $\mathfrak{A}$ we can construct a generalized frame $\mathfrak{A}^+$ via Stone-representation, by taking f to be the set of ultrafilters, and $U \lhd_j T$ iff for all $\blacksquare_j a \in U$ we have $a \in T$; and finally, the field of sets is the field of sets of the form $\widehat{a} = \{U | a \in U\}$. It turns out that $(\mathfrak{A}^+)_+ \cong \mathfrak{A}$, while $\mathfrak{F}$ is not necessarily isomorphic to $(\mathfrak{F}_+)^+$; both have the same modal theory, however. Given a logic Λ and a cardinal α, $\mathfrak{Fr}_\Lambda(\alpha)$ denotes the algebra freely generated by α many generators in the variety of all Λ-algebras. Then $\mathfrak{Can}_\Lambda(\alpha) := (\mathfrak{Fr}_\Lambda(\alpha))^+$ denotes the canonical frame over α many sentence letters. We call it the *α-canonical frame* for Λ.

A general frame $\mathfrak{F}$ is *differentiated* if for two points x, y there exists an internal set a such that $x \in a$ but $y \in -a$; $\mathfrak{F}$ is called *tight* if $x \lhd_j y$ iff for all internal sets a we have $y \in a$ if $x \in \blacksquare_i a$; $\mathfrak{F}$ is *compact* if for any ultrafilter U of the boolean algebra of internal sets there is a $x \in \bigcap U$. A frame is *refined* if it is differentiated and tight; and it is *descriptive* if it is refined and compact. Furthermore, a frame is *canonical* if it is isomorphic to an α-canonical frame of some logic Λ. A logic is *g-persistent* if for all generalized frames, if $\langle \mathfrak{f}, \mathbb{F}\rangle$ is a frame for Λ, so is the underlying Kripke-frame $\mathfrak{f}$. A logic is *r-persistent* if for every refined frame $\langle \mathfrak{f}, \mathbb{F}\rangle$ for Λ the underlying Kripke-frame is a frame for Λ as well. Analogously d-persistence is defined with respect to the class of descriptive frames, and canonicity alias c-persistence with respect to the class of canonical frames. Johan van Benthem has proved in [2] that a logic is g-persistent iff it is axiomatizable by constant axioms, and Sambin & Vaccaro [37] that a logic is d-persistent iff it is canonical. A large class of canonical logics is described by the theorem of Sahlqvist. We present it here in the polymodal setting. Call a formula *positive* if it is built from constant formulae, variables and the connectives, $\wedge$, $\vee$, $\Box_j$, $\Diamond_j$. A formula is *strongly positive* if it is composed form variables and constant formulae with the help of $\wedge$ and $\Box_j$ alone.

Theorem 2 (Sahlqvist [36]) *Suppose that $\boxplus(\phi \to \psi)$ is a formula such that (1.) $\boxplus$ is a prefix of modal operators $\Box_j$, (2.) ψ is positive and (3.) ϕ is composed from strongly positive formulae with the help of $\wedge$, $\vee$ and $\Diamond_j$. Then $\mathbf{K}_\kappa \oplus \boxplus(\phi \to \psi)$ is canonical and the class of frames satisfying this axiom is elementary.*

The formulas falling under the conditions of Sahlqvist's Theorem are called *Sahlqvist formulae*. The elementary conditions expressed by them can be characterized. Let the language $\mathcal{L}_\kappa$ consist of variables for worlds, a symbol $\doteq$ for equality, relational symbols $\lhd_j$ for each $j < \kappa$, and quantifiers $\exists$, $\forall$. Define from that the restricted quantifiers

$$\begin{array}{rcl} (\exists y \rhd_j x)\alpha & := & (\exists y)(x \lhd_j y \,.\&.\, \alpha) \\ (\forall y \rhd_j x)\alpha & := & (\forall y)(x \lhd_j y. \to .\alpha), \end{array}$$

as well as the generalized accessibility relations $x \lhd^\sigma y$, where σ is a sequence of elements in κ. Namely, put $x \lhd^{\langle\rangle} y := x \doteq y$, and $x \lhd^{\langle \sigma, j\rangle} y := (\exists z)(x \lhd^\sigma z .\&. z \lhd_j y)$. The language $\mathcal{R}_\kappa$ is the language obtained from $\mathcal{L}_\kappa$ be replacing the unrestricted quantifiers by restricted quantifiers, and admitting atomic subformulae of the form $x \lhd^\sigma y$. Then the following is shown in [26].

Theorem 3 (Kracht [26]) *An elementary condition is definable by means of a Sahlqvist formula iff it is of the form $(\forall x)\alpha(x)$, where $\alpha(x) \in \mathcal{R}_\kappa$ is positive and in every atomic formula $x \lhd^\sigma y$ either x or y is bound by a universal quantifier not in the scope of an existential quantifier.*

We define the *Sahlqvist rank* to be the maximum alternation of quantifiers in which a nonconstant subformula is embedded. Thus, constant formulae and those using only universal quantifier are of rank 0. (A note of caution. The subformulae $x \lhd^\sigma y$ can hide existential quantifiers; this must be taken into account when calculating the rank. Moreover, $(\forall y \rhd_j x)\alpha$ is defined to be positive if α is. As an elementary formula however it is not positive.)

A logic is *complete* if it is the logic of its Kripke-frames. It is *compact* if every consistent set has model based on a Kripke-frame, and *weakly compact* if every consistent set based on a finite set of sentence letters has a model based on a Kripke-frame. A logic has the *finite model property* (fmp) if it is the logic of its finite Kripke-frames. Clearly, g-persistent, r-persistent, and d-persistent logics are all complete, but there are logics which have the finite model property without being d-persistent. Completeness was originally believed to be an abundant property, but it is not. For given Λ, let the *Fine-spectrum* be the set of logics which have the same Kripke-frames as Λ, and let the degree of incompleteness, $\delta(\Lambda)$, be the cardinality of the Fine-spectrum of Λ. Then the following holds.

Theorem 4 (Blok [6]) *For a monomodal logic Λ either $\delta(\Lambda) = 1$ or $\delta(\Lambda) = 2^{\aleph_0}$. The first obtains iff Λ is inconsistent or the (possibly infinite) join of splitting logics of $\boldsymbol{K}_1$.*

Theorem 5 (Blok [6]) *A logic is a splitting of $\boldsymbol{K}_1$ iff it is of the form $\boldsymbol{K}_1/\mathsf{Th}\,\mathfrak{f}$, where $\mathfrak{f}$ is a cycle-free Kripke-frame.*

Let us note that it follows that no consistent tabular logic, no consistent extension of **K4** and no consistent proper extension of $\mathbf{K} \oplus \Diamond\top$ is the (possibly infinite) join of splitting logics. In other words, no standard system with the exception of $\mathbf{K} \oplus \Diamond\top$ is the join of splitting logics. Call Λ *intrinsically complete* if it has degree of incompleteness 1. Define the *co-covering number* $\gamma(\Lambda)$ to be the cardinality of all logics immediately below Λ. Then $\gamma(\Lambda)$ is finite or countable iff $\delta(\Lambda) = 1$, otherwise it is $= 2^{\aleph_0}$. This has also been shown in [6]. For the case of non-intrinsically complete logics this is a by-product of the proof of the first of the theorems. In the case of intrinsically complete logics it follows from the fact that such a logic if consistent is the countable union of splitting logics. The exceptional case is the inconsistent logic. Here, we have the following theorem.

Theorem 6 (Makinson [31]) *$\mathcal{E}\boldsymbol{K}_1$ has exactly two co-atoms, namely the logics of the one point frames.*

3 Transfer Theorems

3.1 From Monomodal to Polymodal

The simplest kind of polymodal logics that one can think of are those in which no axiom uses more than one kind of operator; in other words, the operators are independent of each other. Let us explain this in the case of bimodal logic, with operators $\Box$ and $\blacksquare$. Here, we are working in the language $\mathcal{L}(\Box, \blacksquare)$, which has the sublanguages $\mathcal{L}(\Box)$ and $\mathcal{L}(\blacksquare)$, based each on a single operator. Suppose we have a bimodal logic $\Lambda = \mathbf{K}_2 \oplus (X \cup Y)$, where formulae from X do not contain $\blacksquare$ and formulae from Y do not contain $\Box$. Then we say that Λ is *independently axiomatized.* In this case we can alternatively think of Λ as a kind of join of two monomodal logics, one being $\Lambda_\Box = \mathbf{K}_1 \oplus X = \Lambda \cap \mathcal{L}(\Box)$ the other being $\Lambda_\blacksquare = \mathbf{K}_1 \oplus Y = \Lambda \cap \mathcal{L}(\blacksquare)$. We say that Λ is the *independent join* or *fusion* of $\Lambda_\Box$ and $\Lambda_\blacksquare$, denoted by $\Lambda_\Box \otimes \Lambda_\blacksquare$. Forming fusions of modal logics is the simplest way to construct new logics from old ones. They were studied by FINE & SCHURZ [16] and KRACHT & WOLTER [28]. Fusions have the following properties. The fusion $- \otimes -$ is a map from $(\mathcal{E}\,\mathbf{K}_1)^2$ into $\mathcal{E}\,\mathbf{K}_2$ commuting with (infinite) joins in both arguments. Also, given any bimodal logic Λ we can define $\Lambda_\Box$ and $\Lambda_\blacksquare$ to be intersection of Λ with the languages formed with the operators $\Box$ and $\blacksquare$. These maps commute with (infinite) meets.

Theorem 7 (Thomason [45]) *The logic $\Lambda \otimes \Theta$ is a conservative extension of Λ iff either Λ is inconsistent or else Θ is consistent.*

THOMASON's proof is based on the fact that the class of atomless boolean algebras is $\aleph_0$-categorical. In order to be able to explain the method of transfer we deliver here a proof based on MAKINSON's theorem. If Θ is consistent then it is contained in the logic of the one point reflexive frame $\boxed{\circ}$, which is $\mathbf{K}_1 \oplus p \leftrightarrow \Box p$, or in the logic of the one point irreflexive frame $\boxed{\bullet}$, which is $\mathbf{K}_1 \oplus \Box\bot$. It is easy to see that $\Lambda \subseteq (\Lambda \otimes \Theta)_\Box$, so it is actually enough if we show that if Θ is one of these logics, then equality holds. Suppose $\Theta = \mathbf{K}_1 \oplus p \leftrightarrow \Box p$. Let $\phi \notin \Lambda$. Then there is a model $\langle \mathfrak{F}, \beta, x\rangle \models \neg\phi$ based on a generalized frame $\langle f, \lhd, \mathbb{F}\rangle$ for Λ. Put $\mathfrak{F}^\circ = \langle f, \lhd, \blacktriangleleft, \mathbb{F}\rangle$, with $\blacktriangleleft = \{\langle x, x\rangle | x \in f\}$. Then this is straightforwardly checked to be a generalized frame for $\Lambda \otimes \Theta$, and we have $\langle \mathfrak{F}^\circ, \beta, x\rangle \models \neg\phi$. In the other case we argue with $\mathfrak{F}^\bullet = \langle f, \lhd, \blacktriangleleft, \mathbb{F}\rangle$ instead, where $\blacktriangleleft = \emptyset$.

A generalization of this construction is the key to the transfer results for fusions. For suppose that we want to build a model for ϕ, $\phi \notin \Lambda$. Then since we only know how to build a model in the monomodal fragments, we build the model in stages, alternating between the operators $\Box$ and $\blacksquare$. First we build a model for the formula viewed as a formula of the language $\mathcal{L}(\Box)$, with each subformula of the form $\blacksquare\psi$ replaced by a new variable $q_{\blacksquare\psi}$. These variables are like promises. Whenever $q_{\blacksquare\psi}$ is true at a point in a model, we are promising to build a model for $\blacksquare\psi$, and if that variable is not true, we are promising to build a model for $\neg\blacksquare\psi$. Thus, at each node of the model already built we then still have to fulfill these subformulas $\blacksquare\psi$. Now we build a model for them in conjunction, that is, at each node we look which variables for $\blacksquare\psi$ are true and which ones are false, and build a model accordingly. However, again we will not do this in one step. Rather, this time we treat the $\blacksquare\psi$ as formulas of the language $\mathcal{L}(\blacksquare)$, with subformulas of the form $\Box\chi$ replaced by new variables $q_{\Box\chi}$. We continue in this fashion, until we have consumed the formulas and no complex formula remains to be fulfilled. This is the naive picture, building the model like a tree. Unfortunately, this strategy can only work if great care is taken. First of all, it does not work with generalized frames, and so almost all the transfer results are conditional on the completeness of the logics. It is possible to refine this technique in such a way that we need only completeness with respect to atomic frames (see [27]; a general frame is *atomic* if the singleton sets are internal). But this is still not a fully general result.

In order to formulate the general transfer theorem for fusions we extend the notion of fusions. For $\alpha \leq \omega$ consider a sequence $\langle \Lambda_i | i < \alpha \rangle$ of normal polymodal logics formulated in languages $\mathcal{L}_i$ such that the modal operators of $\mathcal{L}_i$, $i < \alpha$, are mutually disjoint. Then the *fusion* $\bigotimes \langle \Lambda_i | i < \alpha \rangle$ is the smallest normal polymodal logic in the common language $\bigcup \{\mathcal{L}_i | i < \alpha\}$ containing all Λ_i, $i < \alpha$. It can be proved that the $\mathcal{L}_j$-fragment, $j < \alpha$, of this logic is Λ_j again if all Λ_i, $i < \alpha$, are consistent; otherwise the fusion is also inconsistent.

Theorem 8 (Fine & Schurz [16], Kracht & Wolter [28], Wolter [47]) *Suppose $\langle \Lambda_i | i < \alpha \rangle$ is a sequence of consistent normal polymodal logics and let $\boldsymbol{P}$ be one of the following properties.*

- *g-, r-, d-, c-persistence*
- *being Sahlqvist of rank k*
- *completeness*
- *compactness*
- *finite model property*
- *completeness and decidability*
- *completeness and the interpolation property*
- *completeness and Halldén-completeness*

Then $\bigotimes \langle \Lambda_i | i < \alpha \rangle$ has $\boldsymbol{P}$ iff all Λ_i, $i < \alpha$, have $\boldsymbol{P}$.

(For the finite model property it is required that there is a number n such that each logic has a model of size at most n. This is satisfied e. g. if all of them are monomodal.) We have formulated a version of the theorem which is a bit more general than the one proved in [16] or [28]. The proof of this general version is basically the same – with the exception that the original proof uses Makinson's Theorem, which does not hold for polymodal logics. A way to manage this difficulty can be found in [47]. Transfer of decidability and interpolation for incomplete logics remains an open problem. Although the reduction of decidability and interpolation of the fusion to its fragments can be formulated in a purely syntactical way, the legitimacy of this reduction relies (so far) on the completeness proof for the fusion. It is still open whether this restriction can be dispensed with. Given that decidability transfers in case of completeness the question arises whether the complexity of the decision procedure transfers as well. The answer is negative as is shown in Spaan [41], who gives a complete description of the increase of complexity under fusions.

Given these results, the following seems a worthwile strategy for the analysis of a polymodal logic Λ. First, study the monomodal fragments, and then think of Λ as being obtained from the fusion of these logics via some interaction postulates. This may be practically a good strategy, but can be shown to lead to no significant reduction (at least in principle). Let Λ for simplicity be a bimodal logic. Define the *independent kernel* of Λ to be the fusion $\Lambda_\Box \otimes \Lambda_\blacksquare$. This is the largest independently axiomatizable logic contained in Λ. Call the *independency spectrum* of Λ the set of all bimodal logics with the the same independent kernel as Λ. Let us note two results on independency spectra.

- There exist monomodal tabular logics Λ and Θ of codimension 2 and 3, respectively, in $\mathcal{E}\,\mathbf{K}_1$ such that the independency spectrum of $\Lambda \otimes \Theta$ has cardinality $2^{\aleph_0}$. (See [22].)
- There is a tabular monomodal logic Λ of codimension 2 such that the lattice of extensions of $\mathbf{T} = \mathbf{K} \oplus \Box p \to p$ can be embedded into the lattice of extensions of $\mathbf{S5} \otimes \Lambda$ in such a way that the range of the embedding is a subset of the independency spectrum of $\mathbf{S5} \otimes \Lambda$. (See [50].)

It seems that the first result can be generalized. The specific conjecture is that if both of the logics have codimension at least 2 then the independency spectrum has cardinality $2^{\aleph_0}$.

3.2 Adding the Universal Modality

In [21] the *universal modality* (written here as $\boxtimes$) is introduced. By itself, it is just a standard **S5**-operator. However, this operator, if added to a modal logic Λ, yields a new logic $\Lambda^\boxtimes$ in the language of Λ expanded by the (new) symbol $\boxtimes$. Axioms are those of Λ, **S5** for $\boxtimes$ and for each operator $\Box$ and axiom $\boxtimes p. \to .\Box p$, which induces that the underlying relation for $\boxtimes$ contains all other relations. Therefore, it is an equivalence relation, in which each block is a set of components connected with respect to the old relations. In generated subframes, this reduces to saying that this relation is just the total relation on the frame, every point being accessible to every point. Whence the name for that operator. The special interest in this modality derives from the fact that it is tightly related to a special deducibility relation in modal logic, the *global consequence relation.* Recall that the standard consequence for a logic Θ, $\vdash_\Theta$ or $\vdash$ (with Θ dropped if understood in the context), has only one rule of inference, namely modus ponens; we call it the *local consequence relation.* The global relation for Θ, $\Vdash_\Theta$ or simply $\Vdash$, has in addition to modus ponens also the rule $p \Vdash \Box p$, for all operators $\Box$. Concepts such as decidability, fmp and completeness split into a *local* variant – which is the standard one – and a *global* variant. For example, a logic is *globally decidable* if the problem '$\phi \Vdash \psi$' is decidable, Likewise for the other properties. (Since both consequence relations define the same theorems for a logic, the problems '$\Vdash \phi$' and '$\vdash \phi$' are identical.) The following is proved in [21].

Theorem 9 (Goranko & Passy [21]) *Let $\boldsymbol{P}$ be one of the following properties: decidability, canonicity, finite model property, Kripke-completeness. Λ has*

P *globally iff* $\Lambda^{\boxtimes}$ *has* ***P*** *locally.*

Given this equivalence, the properties of a logic with an added universal modality can be reduced to global properties of the logic itself. Hence it is equivalent to say that a property is *preserved under addition of the universal modality* and to say that *a logic has the property globally if it has that property locally.* Notice in passing that for $\Lambda^{\boxtimes}$ it is equivalent to say that it has a property locally and that it has that property globally, by the fact that '$\phi \Vdash \psi$' is equivalent with '$\boxtimes\phi \vdash \psi$'.

It is clear that the global property implies the corresponding local property, but what about the converse? [41] and [49] have proved that there are logics which have fmp locally but not globally. [21] prove that if Λ admits filtration, then so does $\Lambda^{\boxtimes}$, thus covering a number of significant logics.

Theorem 10 (Kracht & Wolter [29]) *The following properties of logics are undecidable for modal logics*

- *local decidability*
- *global decidability, given local decidability*
- *local fmp*
- *global fmp, given local fmp*

Some of the results concerning local properties have been shown elsewhere, but the proof method here is uniform and rather straightforward. It is interesting in the present context for several reasons. The first is that it reflects the problems of transferring properties of monomodal logics to logics which extend the fusion by just a margin, namely in this case the axiom(s) $\boxtimes p \to \Box_j p$. So, we have particular cases in which specific properties of logics get lost when we add a single axiom to the independent join of logics. Second, the proof is actually obtained using a detour. The easiest examples by which this theorem can be proved are word problems in semi–groups. It is known that one cannot decide whether a finite presentation of a semi–group using two generators and finitely many relations presents the one element semi–group. A presentation can be written as an axiomatic description of the semi–group viewed now as a Kripke frame with two accessibility relations. Thus, each presentation gives rise to a logic, containing the fusion of $\mathbf{K}.\mathbf{Alt}_1.\mathbf{D} = \mathbf{K} \oplus \Diamond p \to \Box p \oplus \Diamond\top$ with itself. The decision problem of the semigroup is directly translatable into a decision problem of the logic. Fine-tuning this method, all the results above can be established for bimodal logics. Using the results on Thomason simulation in § 4.1 we can show that the same undecidability results hold even for monomodal logics. Moreover, [27] shows that given local fmp, global completeness is undecidable. The proof is based on logics with five operators, but again the simulation theorems establish an analogous undecidability result for any number of operators. Thus also for monomodal logics.

3.3 From Modal to Tense Logic

For a normal monomodal logic Λ describing the class of frames $\mathsf{Gfr}(\Lambda)$ it is natural to form the *minimal tense extension* $\Lambda^+.t$ of Λ, which is defined to be the bimodal theory of the class of frames $\langle g, \lhd, \rhd, \mathbb{G}\rangle$, with $\rhd = \lhd^{\smile}$ (i. e. $\rhd$ is the relational converse of $\lhd$), such that $\langle g, \lhd, \mathbb{G}\rangle \in \mathsf{Gfr}(\Lambda)$. The syntactical definition of $\Lambda^+.t$ is quite simple. If we denote the two modal operators of $\Lambda^+.t$ by $\boxminus$ and $\boxminus$ then $\Lambda^+.t$ is the smallest normal bimodal logic containing Λ formulated in $\boxminus$ and both $p \to \boxminus\Diamond p$ and $p \to \boxminus\Diamond p$. Tense logics are the bimodal logics containing the two axioms above. If compared with lattices of monomodal logics lattices of tense logics quite often behave differently. For instance, it is shown in [25] that both the lattice of extensions of $\mathbf{K}.t$ and the lattice of extensions of $\mathbf{K4}.t$ have only the trivial splitting. In contrast to fusions and adding the universal modality $\Lambda^+.t$ is not always a conservative extension of Λ. In [48] it is shown that there exist $2^{\aleph_0}$ logics whose minimal tense extensions coincide with $\mathbf{K4}^+.t$. Now define the *tense indeterminacy spectrum* of a logic Λ to be the set $\{\Theta | \Theta^+.t = \Lambda^+.t\}$ and call the cardinality of this set the *degree of tense indeterminacy of* Λ. Applying the technique of [48] to the frames defined in [6] yields the following classification.

Theorem 11 *If $\Lambda \neq \boldsymbol{K}$ is consistent and is not a join of splitting logics, then the degree of tense indeterminacy of Λ is $2^{\aleph_0}$. Otherwise it is 1.*

Note that for complete logics the minimal tense extension is readily seen to be a conservative extension. So the phenomenon that minimal tense extensions are not always conservative extensions is closely related to the phenomenon of incompleteness in monomodal logic. In fact, the theorem above states that the degree of incompleteness of a logic Λ coincides with the degree of tense indeterminacy of Λ.

Let us look again at the transfer of properties from Λ to $\Lambda^+.t$. Here we restrict our attention to logics above $\mathbf{K4}$. Again the hope for a general result is destroyed by an example of a logic above $\mathbf{K4}$ with fmp such that the minimal tense extension is not complete with respect to Kripke semantics (see [53]). Nevertheless, such an example is as complicated as the construction of incomplete logics above $\mathbf{K4}$ as is shown by the following transfer results.

Theorem 12 (Wolter [51]) *Let Λ be a logic above $\boldsymbol{K4}$.*

- *If Λ has finite depth, then $\Lambda^+.t$ has the fmp.*
- *If Λ has finite width (in the sense of [15]), then $\Lambda^+.t$ is complete.*
- *If Λ is a cofinal subframe logic (in the sense of [56]), then $\Lambda^+.t$ is complete.*

For the class of cofinal subframe logics there is a remarkable connection between first order definability and completeness.

Theorem 13 (Wolter [52]) *For a cofinal subframe logic Λ the following are equivalent.*

- *The Λ-frames are first order definable.*
- *Λ is d-persistent.*
- *$\Lambda^{+}.t$ has the fmp.*
- *Λ is compact.*

It is an open problem whether decidability transfers from Λ to $\Lambda^{+}.t$, in general. We note, however, the following general positive result, which covers all natural logics containing **K4**.

Theorem 14 (Wolter [51]) *$\Lambda^{+}.t$ is decidable, for all finitely axiomatizable cofinal subframe logics Λ.*

4 Simulation

4.1 From Polymodal to Monomodal Logic

For simplicity, we will show how to simulate two operators, $\blacksquare$ and $\square$, by a single operator, $\boxminus$. The idea goes back to Thomason [42]. Let a bimodal Kripke-frame $\mathfrak{f} = \langle f, \lhd, \blacktriangleleft \rangle$ be given. Then define $f^{sim} = \{\infty\} \cup f^{\circ} \cup f^{\bullet}$, where f° and $f^{\bullet}$ are disjoint sets, each of cardinality equal to the cardinality of f. Any point $x \in f$ is associated with two twins, $x^{\circ} \in f^{\circ}$ and $x^{\bullet} \in f^{\bullet}$. We have a relation corresponding to $\boxminus$, denoted by $<$. It is defined as follows. (i.) We have $x^{\circ} < \infty$ for all $x \in f$, but $x^{\bullet} \not< \infty$. (ii.) We have $x^{\circ} < x^{\bullet} < x^{\circ}$ for all $x \in f$, but for distinct x, y we have that both $x^{\circ} \not< y^{\bullet}$ and $y^{\bullet} \not< x^{\circ}$. (iii.) We have $x^{\circ} < y^{\circ}$ exactly when $x \lhd y$ and $x^{\bullet} < y^{\bullet}$ exactly when $x \blacktriangleleft y$. This defines the frame $\mathfrak{f}^{sim} = \langle f^{sim}, < \rangle$. If a set $S \subseteq f$ is given, we let $S^{\circ} = \{x^{\circ} | x \in S\}$, and likewise $S^{\bullet} = \{x^{\bullet} | x \in f\}$. Notice that no matter what $\mathfrak{f}$ looks like, both f° and $f^{\bullet}$ can be defined by constant formulae, namely f° by $white = \diamondminus \boxminus \bot$ and $f^{\bullet}$ by $black = \neg \diamondminus \boxminus \bot \wedge \neg \boxminus \bot$. Also, $\{\infty\}$ is defined by $\boxminus \bot$. Now given a (generalized) frame $\mathfrak{F} = \langle f, \lhd, \blacktriangleleft, \mathbb{F} \rangle$ with $\mathbb{F}$ closed under the usual operations, $\mathfrak{F}^{sim} = \langle f^{sim}, <, \mathbb{F}^{sim} \rangle$ is a (generalized) frame, where $\mathbb{F}^{sim}$ consists of sets of the form $\Omega \cup S^{\circ} \cup T^{\bullet}$, for $\Omega \subseteq \{\infty\}$ and $S, T \subseteq f$. When given a bimodal logic Λ, we let Λ^{sim} be the logic of the simulations of Λ-frames. Thus, $\Lambda^{sim} = \mathsf{Th}(\mathsf{Gfr}\,\Lambda)^{sim}$. Alternatively, suppose $\Lambda = (\mathbf{K} \otimes \mathbf{K}) \oplus X$, which denotes the smallest normal logic containing X, X a set of formulae. Then

$\Lambda^{sim} = \mathbf{K} \oplus X^{sim}$ with $\phi^{sim} = \mathit{white} \rightarrow \phi^s$. ϕ^s is defined inductively as follows.

$$\begin{array}{lcl} p^s & = & p \wedge \mathit{white} \\ (\neg\phi)^s & = & \mathit{white} \wedge \neg(\phi^s) \\ (\phi \wedge \psi)^s & = & \phi^s \wedge \psi^s \\ (\Diamond\phi)^s & = & \mathit{white} \wedge \diamondsuit\phi^s \\ (\blacklozenge\phi)^s & = & \mathit{white} \wedge \diamondsuit(\mathit{black} \wedge \diamondsuit(\mathit{black} \wedge \diamondsuit(\mathit{white} \wedge \phi^s))) \end{array}$$

As it turns out, the logic of all simulation frames $\mathfrak{F}^{sim}$ can be finitely axiomatized; let us call it **Sim**. If we denote by $\mathbf{2}^{\varnothing}$ the logic of the irreflexive singleton, we have the following theorem.

Theorem 15 (Kracht [27]) *The map $\Lambda \mapsto \Lambda^{sim}$ is an isomorphism from the lattice $\mathcal{E}(\boldsymbol{K} \otimes \boldsymbol{K})$ onto the interval $[\boldsymbol{Sim}, \boldsymbol{2}^{\varnothing}]$ reflecting and preserving*

- *g-, r-, d- and c-persistence*
- *being Sahlqvist of rank k*
- *elementarity*
- *(local/global) completeness*
- *compactness, weak compactness*
- *(local/global) finite model property*
- *(local/global) decidability*
- *interpolation*

The proof is similar to the one in [29] and is given in full generality in [27]. The improvement consists in the fact that the simulation of a variety of bimodal algebras (defined analogously) is again a variety, and that unsimulating a variety of monomodal algebras we get a variety of bimodal algebras. The simulation map is then easily shown to be bijective on the varieties, and therefore an order isomorphism of the lattices, whence also an isomorphism of the lattices. Using the method of unsimulating in the quoted manuscript we find that decidability is preserved both ways, completeness and persistence properties likewise. Notice that we have to add $\mathbf{2}^{\varnothing}$ as a top element and not the inconsistent logic. This follows if we define Λ^{sim} syntactically, since $\bot^{sim} = \mathit{white} \rightarrow \bot$, which is consistent and has the one point irreflexive frame as a model. Based on simulations of the models we get the same effect if we accept as a frame also the frame on the empty set, for then $\emptyset^{sim} = \{\infty\}$.

This theorem has numerous consequences. Let us mention a few. Recall from § 3.2 the logics based on word problems for semi–groups. These logics are axiomatizable by single letter axioms of Sahlqvist rank 0. The same holds of their simulations. Hence, we have plenty examples of undecidable, elementary logics which determine a class of Kripke frames of very low complexity, almost universal. We moreover know that the decidability and the finite model property of such logics is undecidable, even if we restrict ourselves to single–letter axioms. Using a result of [22] that the inconsistent bimodal logic has $2^{\aleph_0}$ co-covers, we have derived once again the fact that the theory of the irreflexive singleton has $2^{\aleph_0}$ many co-covers. What is interesting is of course not the result per se but the new way in which it can be obtained. Moreover, with a little bit of sophistication it can be shown that it is undecidable for a (finitely axiomatized) bimodal logic whether or not it is inconsistent. Thus it is undecidable whether or not a (finitely axiomatized) monomodal logic is the logic of $\mathbf{2}^{\varnothing}$. By Makinson's theorem, consistency *is* decidable, so this result is the best possible. In this way the simulation theorems allow in a natural way to see why the lattice of monomodal logics is so complex – because it has in it also the lattices $\mathcal{E}\,\mathbf{K}_n$ for any n.

4.2 Nominals and the Difference Operator

The variables in modal logics are variables over sets of worlds and not worlds. In many applications it is desirable to have in addition to the standard variables also variables which range exclusively over worlds. Such an approach has been offered by Solomon Passy and Tinko Tinchev [34] and Patrick Blackburn [3]. They introduce special variables, called *nominals*, whose interpretation must be singleton sets, that is, sets of the form $\{x\}$, x a world in the frame. Nominals are denoted by i, j etc. They add expressive power to modal logic. For example, the property that the relation $\lhd$ is irreflexive can now be defined, using the formula $i \rightarrow \Box\neg i$. For if the value of i can only be a singleton then this formula is refutable iff the relation is not irreflexive. [12] uses a different tool to define the difference. Namely, he introduces the *difference operator*. It is written here $[\neq]$. The accessibility relation underlying $[\neq]$ shall always be the inequality between worlds. So, $[\neq]\phi$ is true at w iff at all worlds $v \neq w$ ϕ is true at v. Since this relation is evidently not closed under p-morphisms, the theory of the difference operator is not a normal modal logic. Both nominals and the difference are rather nonstandard devices which work fine on Kripke-structure but present special problems for generalized frames. We will not pursue this theme, however. Notice that with the difference operator the universal modality becomes definable. Namely, we have $\boxtimes\phi \equiv \phi \wedge [\neq]\phi$. On the other hand, given the universal modality and nominals, we can also define the difference operator by taking $[\neq]\phi \equiv i \wedge \boxtimes(\neg i \rightarrow \phi)$, for i not occurring in ϕ. Finally, if a formula ϕ contains a nominal i, then we can replace i by a standard variable p not already occurring in ϕ, using the following equivalence

$$\phi(i) \equiv (p \wedge [\neq]\neg p) \vee \langle\neq\rangle(p \wedge [\neq]\neg p) \rightarrow \phi(p).$$

Theorem 16 (Goranko & Gargov [20]) *The language of the difference operator and the language of the universal operator together with nominals are intertranslatable (with respect to Kripke–frames).*

The effects of both the nominals and the difference operator can be achieved using normal operators as follows. We add two operators, $\Box^{+}$ and $\Box^{-}$, which are tense duals. The logic of these operators is $\mathbf{WOrd} := \mathbf{K4.3}^{+}.t.\mathbf{G.3}^{-}$. That is to say, the relation on which both are based is linear and transitive in both directions and is conversely well-founded. In other words, it is a well-order. So, suppose a Kripke-frame satisfies the logic **WOrd**. Then it is well-ordered by the relation for $\Box^{+}$. The difference operator is then definable by $\langle\neq\rangle\phi := \Diamond^{+}\phi \vee \Diamond^{-}\phi$. The language of **WOrd** is expressively stronger than either the language of the difference operator or that of the universal modality and nominals, so no exact correspondence should be expected. Also, **WOrd** is not canonical.

Nevertheless, even though all of the three logics above are in some way not canonical, it is possible to establish some Sahlqvist type results. The interesting situation is when we have more than one relation, in fact a whole collection of them and the difference operator is added merely for increasing the definability strength of the other operators, such as defining their intransitivity or the like. Then the relation underlying the difference is used for an extraneous purpose. Suppose therefore we let the relations underlying $\langle\neq\rangle$ or $\Diamond^{+}$ be just auxiliary relations, that is, we only need to achieve that a property holds for the relations other than $\neq$ or $<$. Then it is possible to use part of the definability hierarchy. Call the operators $\langle\neq\rangle$ and $\Diamond^{+}$, $\Diamond^{-}$ as well the relations on which they are based *auxiliary* while the other operators and relations are called *main operators (relations).* One can show that every first-order condition $(\forall\vec{x})\phi(\vec{x})$, where ϕ is built using restricted quantifiers over main relations and basic formulae $u = v$, $u \neq v$, $u \lhd^{\sigma} v$, where σ uses main relations only, is Sahlqvist on condition that u is quantified universally by a quantifier not in the scope of an existential quantifier. Notice that there is a certain asymmetry in that negations $u \not\lhd^{\sigma} v$ are not generally admitted. Only inequations may be used. One way to show this is using the method of [26]. There internal descriptions are built using an asymmetric sequent like calculus, pairing elementary and modal formulae. Another tool that is needed is that of substitutions or *decisive sets*. We will not go into the details. Suffice the following indication. To show that a point is different from another we only need to make a formula p true at one point and false at another. To show that they are equal, we must be sure that p is only true at a single world, and that it is true at both worlds under consideration. Since ordinary variables are variables over sets, the latter cannot be achieved. If they are nominals, however, then it can be achieved by definition of nominals. This is the deeper reason for the fact that the addition of nominals (and the universal modality) allows to define properties which are not necessarily positive, but contain inequations, subject to the condition on variables in atomic subformulas that the Sahlqvist theorem makes. Similar considerations can be made for the difference operator and the well-ordering operators.

4.3 From Monotonic Modal Logics to Normal Logics

A monotonic (mono-)modal logic is a set of modal formulas containing all classical tautologies and which is closed under substitutions, modus ponens and $p \to q/ \boxplus p \to \boxplus q$. These logics are known to be complete with respect to general neighbourhood-frames (see e.g. [11]). As compared with normal modal logics monotonic logics have been investigated only fragmentarily. Nevertheles, there are well known applications which require the weakness of monotonic systems, e.g. in deontic logic. Recall that a *neighbourhood frame* for a monotonic logic is a pair $\langle g, N\rangle$ where g is a set and N is a function assigning a set $N(x) \subseteq \wp(g)$ of so-called *neighbourhoods* to every point $x \in g$. Valuations are as in Kripke-frames. $\boxplus\phi$ is accepted at $w \in g$ given a valuation if there is a neighbourhood $S \in N(x)$ such that ϕ is accepted throughout S. A generalized neighbourhood frame is a triple $\langle g, N, \mathbb{G}\rangle$, where $\langle g, N\rangle$ is a neighbourhood frame and $\mathbb{G} \subseteq \wp(g)$ a system of subsets of g closed under intersection, complement and $\boxplus$. We will use two modal operators to simulate $\boxplus$. The bimodal frame simulating $\langle g, N\rangle$ will contain points representing both the points of g as well as the neighbourhoods. Thus, N is turned into a relation between points representing members of g and points representing neighbourhoods. One operator will represent this relation. To recover the membership relation between points and neighbourhoods we need the second operator. Define a translation t from the monomodal language with $\boxplus$ into the bimodal language with $\Box$ and $\blacksquare$ as follows:

$$\begin{array}{rcl} p^t & = & p \\ (\phi \wedge \psi)^t & = & \phi^t \wedge \psi^t \\ (\neg\phi)^t & = & \neg\phi^t \\ (\boxplus\phi)^t & = & \Diamond\blacksquare\phi^t \end{array}$$

Denote by $\mathbf{M}$ the smallest monotonic logic and denote by $\mathbf{M} + \Gamma$ the smallest monotonic logic containing ($\mathbf{M}$ and) Γ.

Theorem 17 (Kracht & Wolter [29]) *For all formulas ϕ, $\phi \in \boldsymbol{M} + \Gamma$ iff $\phi^t \in (\boldsymbol{K} \otimes \boldsymbol{K}) \oplus \Gamma^t$.*

The proof from left to right is easy. We give the basic idea of the proof of the other direction as given in [29], where general neighbourhood frames are simulated as general bimodal frames. Define for a neighbourhood frame $\langle g, N\rangle$ the bimodal frame $\langle g, N\rangle^{ms} = \langle h, \lhd, \blacktriangleleft\rangle$ by

$$h = g \cup \mathcal{C} \text{ where } \mathcal{C} = \{C \subseteq g | (\exists x \in g)(C \in N(x))\},$$

$$x \lhd y \text{ iff } x \in N(y),$$

$$y \blacktriangleleft x \text{ iff } x \in y.$$

This simulation of frames corresponds in the obvious way to the translation of formulas. Let us note that there are some technical difficulties with the new points in $\mathcal{C}$ and with simulations of general neighbourhood frames. The interested reader is referred to the proof in [29]. For a monotonic logic $\Lambda = \mathbf{M} + \Gamma$ denote by Λ^{ms} the logic $\mathbf{K}_2 \oplus \Gamma^t$. The following is shown in [29].

Theorem 18 (Kracht & Wolter [29]) *The map $\Lambda \mapsto \Lambda^{ms}$ reflects completeness w.r.t. neighbourhood semantics, the finite model property and decidability.*

This result can be used in order to derive completeness results (for neighbourhood semantics) for most of the monotonic standard systems by establishing the completeness of the bimodal simulation via Sahlqvist's Theorem, e.g. for $\mathbf{M} + \Box p \to \Box\Box p$ and $\mathbf{M} + \Box p \to p$.

In contrast to the simulation of the previous section we encounter a difficulty in axiomatizing the logic of the frames $\langle g, N\rangle^{ms}$. The problem lies in the fact that we see no way to secure by way of an axiomatization the extensionality of neighbourhoods. For, since neighbourhoods are sets of worlds, two neighbourhoods are equal if they contain the same worlds. In the simulating frame, elementhood is an accessibility relation, and to our knowledge there is no formula with which to define extensionality for this relation. Furthermore, the simulating logics are rather awkward, and we know of no property which is preserved under simulation. This may also be due to the fact that the simulated logic may admit unintended structures, unlike in the case of polymodal logics. Notice finally that the translation contains an unhealthy combination of diamonds and boxes which is known to be a limit for Sahlqvist's Theorem.

4.4 From Polyadic Operators to Monadic Operators

In the extensive study [19], Robert Goldblatt has shown that any algebra gives rise to an algebra of complexes alias subsets and that each n-ary function gives rise to an n-ary function of complexes. These new functions behave like modal operators, more exactly like diamonds, since they distribute over unions of sets in any of their arguments. These are called *polyadic operators* to emphasize that they may take more than one argument. Goldblatt has extended standard duality theory and correspondence theory to arbitrary polymodal, polyadic languages.

For our simulation of polyadic normal modal logics as polymodal logics we assume that the polyadic logic is formulated in the language $\mathcal{L}(\diamond)$ with one operator $\diamond$, corresponding to the diamond operator. For simplicity we have chosen $\diamond$ to be a dyadic operator, written in infix notation. The smallest dyadic logic in $\mathcal{L}(\diamond)$ is denoted by $\mathbf{Dy}$. All results established here for extensions of $\mathbf{Dy}$ hold for polyadic logics with several operators of arbitrary adicity. A frame for $\mathcal{L}(\diamond)$ is a pair $\langle g, S\rangle$ where S is a *ternary* relation on g. Valuations are defined as usual. The acceptance clauses for boolean connectives are also standard. A world w accepts $\phi \diamond \psi$ under a valuation if there are worlds x, y with $S(w, x, y)$ such that x accepts ϕ and y accepts ψ under that valuation. We propose the following simulation. Take the original set of worlds and add worlds for all *pairs* of worlds in g. S can then be analysed as a binary relation, between worlds standing for g-worlds and worlds for pairs of g-worlds. To make this work, we add the two projections as relations, thus ending up with a total of three operators.

To implement this idea, we define translations from generalized frames with a ternary relation to generalized frames with three binary relations and vica versa. For $\mathcal{G} = \langle g, S, A\rangle$ put $\sigma\mathcal{G} = \langle \sigma g, \lhd, \lhd_1, \lhd_2, \sigma A\rangle$, where

$$\begin{array}{rcl} \sigma g & = & g \cup (g \times g) \\ \lhd_1 & = & \{\langle\langle x, y\rangle, x\rangle \mid x, y \in g\} \\ \lhd_2 & = & \{\langle\langle x, y\rangle, y\rangle \mid x, y \in g\} \\ \lhd & = & \{\langle x, \langle y, z\rangle\rangle \mid x, y, z \in g, S(x, y, z)\} \end{array}$$

and σA is the closure of A under the boolean operations, $\Box$, $\Box_1$ and $\Box_2$. It is not difficult to show that

$$\{g \cap b | b \in \sigma A\} = A.$$

Conversely we shall first find some axioms valid in the generalized frames $\sigma\mathcal{G}$. Notice that the relations $\lhd$, $\lhd_1$ and $\lhd_2$ allow no iteration beyond the first successor, and that $\lhd_1$ and $\lhd_2$ are partial functions. Thus we have the postulates

$$\mathbf{st} = \{\Box_1\Box_1\bot,\ \Box_2\Box_2\bot,\ \Box\Box\bot\} \text{ and } \mathbf{par} = \{\Diamond_1 p \to \Box_1 p,\ \Diamond_2 p \to \Box_2 p\}.$$

The points which denote the singleton worlds are definable by $\neg\Diamond_1\top$, or alternatively by $\neg\Diamond_2\top$. Since one projection is defined iff the other is as well, we must have the axiom

$$\mathbf{eq} = \Diamond_1\top \leftrightarrow \Diamond_2\top.$$

We shall also need the following postulates.

$$\mathbf{qe} = \{\Diamond\top \to \neg\Diamond_1\top,\ \Box\Diamond_1\top\}.$$

The first one says that no world in g has a projection (thus only pairs of worlds do) while the second one says that all $\lhd$-successors of worlds in g have projections. Denote by **Bin** the normal logic axiomatized by the collection of all those axioms. Certainly **Bin** is d-persistent, by Sahlqvist's Theorem. However, one easily constructs frames for **Bin** which are not of the form $\sigma\mathcal{G}$. We shall come back to this point later. Now consider a frame $\mathcal{H} = \langle h, \lhd, \lhd_1, \lhd_2, B\rangle$ validating **Bin**. Put $\rho\mathcal{H} = \langle \rho h, S, \rho B\rangle$, where

$$\rho h = \Box_1\emptyset,$$

$$S(x, y, z) \Leftrightarrow (\exists z' \in h)(x \lhd z' \wedge z' \lhd_1 y \wedge z' \lhd_2 z),$$

$$\rho B = \{b \cap \rho h | b \in B\}.$$

Using the axioms above it is a bit tedious but straightforward to show that $\rho\mathcal{H}$ is a frame by proving that

$$a \diamond b = \Diamond(\Diamond_1 a \cap \Diamond_2 b) \in \rho B,$$

for all $a, b \in \rho B$. Using the equation $A = \{g \cap b | b \in \sigma A\}$ it is clear that $\rho\sigma\mathcal{G} = \mathcal{G}$, for all generalized frames $\mathcal{G}$ for **Dy**. The translation t of formulas in $\mathcal{L}(\diamond)$ is defined by putting

$$\begin{aligned} p^t &= p \\ (\phi \wedge \psi)^t &= \phi^t \wedge \psi^t \\ (\neg\phi)^t &= \neg\phi^t \\ (\phi \diamond \psi)^t &= \Diamond(\Diamond_1\phi^t \wedge \Diamond_2\psi^t) \end{aligned}$$

The following Lemma follows by induction.

Lemma 19 *For all $\phi \in \mathcal{L}(\diamond)$, generalized ternary frames $\mathcal{G}$ and generalized **Bin**-frames $\mathcal{H}$,*

$$\mathcal{G} \models \phi \Leftrightarrow \sigma\mathcal{G} \models \Box_1\bot \to \phi^t,$$

$$\rho\mathcal{H} \models \phi \Leftrightarrow \mathcal{H} \models \Box_1\bot \to \phi^t.$$

With the help of this lemma we immediately obtain

Theorem 20 *For all formulas ϕ,*

$$\phi \in \mathbf{Dy} \oplus \Gamma \textit{ iff } \Box_1\bot \to \phi^t \in \mathbf{Bin} \oplus \{\Box_1\bot \to \phi^t | \phi \in \Gamma\}$$

Define for $\Lambda = \mathbf{Dy} + \Gamma$ the logic

$$\Lambda^\sigma = \mathbf{Bin} \oplus \{\Box_1\bot \to \phi^t | \phi \in \Gamma\}$$

Obviously the map $\Lambda \mapsto \Lambda^\sigma$ reflects decidability, completeness w.r.t. Kripke semantics and the finite model property. Some more work has do be done in order to prove that it reflects d-persistency.

Theorem 21 *If Λ^σ is d-persistent then so is Λ. If the Λ^σ-frames are first order definable then so are the Λ-frames.*

Proof. Suppose that Λ^σ is d-persistent and that $\mathcal{G} = \langle g, S, A\rangle$ is a descriptive Λ-frame. Consider the Λ^σ-frame $\sigma\mathcal{G}$. Unfortunately, $\sigma\mathcal{G}$ is not descriptive, in general. However, it is readily checked that the underlying Kripke-frame of $\rho((\sigma\mathcal{G})_+)^+$ is isomorphic to $\langle g, S\rangle$. Hence, by Lemma 19 above and since Λ^σ is d-persistent it follows that $\langle g, S\rangle$ is a frame validating Λ. The second statement is clear. $\Box$

With this result at hand we can translate Sahlqvist's Theorem to polyadic logics. Note however, that the syntactic characterization in Sahlqvist's Theorem cannot be carried over blindly. A counterexample due to Maarten de Rijke can be found in [46]. Define the dual operator $p \,\Box\, q = \neg(\neg p \diamond \neg q)$ whose translation is equivalent to $\Box(\Box_1 p \vee \Box_2 q)$. Since in this translation we find a disjunction in the scope a box, only those formulae not containing a dyadic box in the antecedent translate into Sahlqvist formulae. Notice that there is a better behaved dyadic box (with a different interpretation), namely

$$p \circ q = p \,\square\, q \wedge \neg(\neg p \diamond q) \wedge \neg(p \diamond \neg q)$$

whose translation is equivalent to $\Box(\Box_1 p \wedge \Box_2 q)$. Thus we get the following theorem.

Theorem 22 *Suppose $\phi \to \psi \in \mathcal{L}(\diamond, \square, \circ)$ and (a.) ψ is composed from variables and constants with the help of $\wedge$, $\vee$, $\diamond$, $\square$ and $\circ$ and (b.) ϕ is composed form variables and constants in such a way that no positive occurrence of a variable is*

1. *in a subformula of the form $\chi_1 \,\square\, \chi_2$ and*
2. *in a subformula of the form $\chi_1 \vee \chi_2$ or $\chi_1 \diamond \chi_2$ in the scope of $\circ$.*

Then **Dy** $\oplus\ \phi \to \psi$ *is d-persistent and $\phi \to \psi$ is effectively equivalent to a first-order formula.*

This allows to deduce some useful facts about modalities for the categorial analysis of language. If concatenation $\bullet$ is viewed as a ternary relation on strings, we get a dyadic modal operator $\diamond$ (see [35]). Any algebraic equation involving $\bullet$ translates straightforwardly into a modal axiom for $\diamond$, by replacing $=$ by $\leftrightarrow$, $\bullet$ by $\diamond$ and variables for elements by variables for propositions. For example, associativity of this operator is captured by

$$p \diamond (q \diamond r). \leftrightarrow .(p \diamond q) \diamond r$$

(See [32].) Likewise for any other algebraic signature. It is seen immediately that these formulae are Sahlqvist in the general sense, since they involve only diamond-like modalities. It follows that they are canonical and determine first-order properties on frames. This has been claimed in [32] but without any proof. Notice also that undecidability results of rather strong form can easily be obtained with dyadic operators. For example, if this operator is associative and there are 'enough' algebras then [30] have shown the logic to be undecidable. (See Corollary 0.12 in the quoted paper.)

As mentioned above, frames for the logic **Bin** are not necessarily of the form $\sigma\mathcal{G}$. The reason is that there is no axiom expressing existence and uniqueness of the pair $\langle x, y\rangle$ given x and y. However, using the results of §4.2 we can axiomatize these frames using alternatively nominals with the universal modality, the difference operator or a well-ordering. Namely, the existence and uniqueness of the pairing function are expressed by the first-order sentences

$$(\forall x_1, x_2)(\exists y)(y \lhd_1 x_1 \ \&\ y \lhd_2 y_2)$$

$$(\forall x_1, x_2)(\forall y, y')(y, y' \lhd_1 x_1 \ \&\ y, y' \lhd_2 x_2. \to .y = y').$$

Both are expressible e. g. if nominals are added.

4.5 From Intuitionistic Modal Logic to Bimodal Logic

GÖDEL's translation of intuitionistic formulas into modal formulae gives a well-known simulation of intermediate logics as extensions of **S4**. This simulation can be extended to a simulation of intuitionistic modal logics by normal bimodal logics. Denote by $\mathbf{IntK}_\Box$ the smallest logic in the propositional language $\mathcal{L}_\Box$ with primitive symbols $\wedge, \neg, \vee, \rightarrow, \Box$, which contains all intuitionistic tautologies and $\Box(p \rightarrow q) \rightarrow \Box p \rightarrow \Box q$ and is closed under modus ponens, substitutions and $p/\Box p$. This logic has been introduced by BOSIĆ & DOŠEN in [7]. We call an extension of $\mathbf{IntK}_\Box$ which is closed under those rules a $\mathrm{IM}_\Box$-logic. Extensions of $\mathbf{IntK}_\Box$ are investigated in [33], [13] and [54]. In $\mathcal{L}_\Box$ the operator $\Box$ is the only primitive modal operator. $\Diamond$ may be defined as $\neg\Box\neg$, but note that $\Diamond p \leftrightarrow \neg\Box\neg p$ does not hold in intuitionistic logic under the standard interpretation of $\Box$ and $\Diamond$ as $\forall$ and $\exists$, respectively.

Another type of modal intuitionistic logics with two primitive modal operators $\Box$ and $\Diamond$ and weaker connecting axioms was introduced by G. FISCHER SERVI in [17] and [18]. Denote by $\mathcal{L}_{\Box\Diamond}$ the language $\mathcal{L}_\Box$ extended by $\Diamond$. A $\mathrm{IM}_{\Box\Diamond}$-logic is a subset Λ of $\mathcal{L}_{\Box\Diamond}$ which contains $\mathbf{IntK}_\Box$ and

$$\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q \quad \text{and} \quad \neg\Diamond(p \wedge \neg p),$$

and the connecting axioms

$$\Diamond(p \rightarrow q) \rightarrow (\Box p \rightarrow \Diamond q) \quad \text{and} \quad (\Diamond p \rightarrow \Box q) \rightarrow \Box(p \rightarrow q)$$

and which is closed under the rules for $\mathrm{IM}_\Box$-logics and $p \rightarrow q/\Diamond p \rightarrow \Diamond q$. See also [1] and [14] for a motivation as well as results on $\mathrm{IM}_{\Box\Diamond}$-logics.

As concerns simulations by normal bimodal logics we start with $\mathrm{IM}_\Box$-logics. For a set Γ of formulas in $\mathcal{L}_\Box$-let $\mathbf{IntK}_\Box + \Gamma$ denote the smallest $\mathrm{IM}_\Box$-logic containing Γ. SHEHTMAN in [39] extends the GÖDEL translation to a translation t from $\mathcal{L}_\Box$ into the bimodal language with $\Box_I$ and $\Box_M$ as follows

$$\begin{array}{rcl} p^t & = & \Box_I p \\ (\phi \circ \psi)^t & = & \Box_I(\phi^t \circ \psi^t) \\ (\neg\phi)^t & = & \Box_I \neg\phi^t \\ (\Box\phi)^t & = & \Box_I \Box_M \phi^t, \end{array}$$

for $\circ \in \{\wedge, \vee, \rightarrow\}$, and shows

Theorem 23 (Shehtman [39]) *For all $\phi \in \mathcal{L}_\Box$,*

$$\phi \in \boldsymbol{IntK}_\Box + \Gamma \Leftrightarrow \phi^t \in (\boldsymbol{S4} \otimes \boldsymbol{K}) \oplus \Gamma^t \Leftrightarrow \phi^t \in (\boldsymbol{Grz} \otimes \boldsymbol{K}) \oplus \Gamma^t \oplus \mathit{Mix}$$

where $\mathit{Mix} := \Box_I \Box_M \Box_I p \leftrightarrow \Box_M p$.

Using this result and the results on fusions the following can be shown.

Theorem 24 (Wolter & Zakharyaschev [54]) *If an intermediate logic* $\boldsymbol{Int}+\Gamma$ *has one of the properties*

- *the finite model property;*
- *decidability and Kripke completeness;*
- *Kripke completeness,*

then the $IM_\Box$*-logics* $\boldsymbol{IntK}_\Box + \Gamma$, $\boldsymbol{IntK}_\Box + \Gamma + \Box p \to p$ *and* $\boldsymbol{IntK}_\Box + \Gamma + \Diamond\top$ *also have the same property.*

In [4] it is shown that the lattice of intermediate logics is isomorphic to the lattice $\mathcal{E}\mathbf{Grz}$ of extensions of GRZEGORCZYK's logics. This isomorphism is known as the BLOK-ESAKIA isomorphism. [55] extend this isomorphism to an isomorphism between the lattice of $\mathrm{IM}_\Box$-logics onto the lattice of extensions of $\mathbf{Grz} \otimes \mathbf{K} \oplus \mathit{Mix}$. Namely, for $\Lambda = \mathbf{IntK}_\Box + \Gamma$ put $\Lambda^{is} = (\mathbf{Grz} \otimes \mathbf{K}) \oplus \mathit{Mix} \oplus \Gamma^t$.

Theorem 25 (Wolter & Zakharyaschev [55]) *The map* $\Lambda \mapsto \Lambda^{is}$ *is an isomorphism from the lattice of* $IM_\Box$*-logics onto the lattice of normal extensions of* $(\boldsymbol{Grz} \otimes \boldsymbol{K}) \oplus \mathit{Mix}$ *preserving the fmp and reflecting decidability and fmp.*

Given this result several transfer problems arise, e.g. let $\Lambda = \mathbf{Int} + \Gamma$ be an intermediate logic with fmp.

1. Does $\mathbf{IntK}_\Box + \Gamma + \Box p \to \Box\Box p$ have the fmp?
2. Does $\mathbf{IntK}_\Box + \Gamma + \Box p \to \Box\Box p + \Box p \to p$ have the fmp?

A partial answer is given in [54], where it is shown that 1. and 2. hold if no formula in Γ contains disjunction or negation.

Now we come to the simulation of $\mathrm{IM}_{\Box\Diamond}$-logics, as described in [17] and [18]. Denote by $\mathbf{IntK}_{\Box\Diamond}$ the smallest $\mathrm{IM}_{\Box\Diamond}$-logic. Extend the translation t defined above to $\mathcal{L}_{\Box\Diamond}$ by putting $(\Diamond\phi)^t = \Diamond_M \phi^t$. Now consider the normal bimodal logic

$$\mathbf{FS} = (\mathbf{S4} \otimes \mathbf{K}) \oplus \Diamond_I \Box_M p \to \Box_M \Diamond_I p \oplus \Diamond_M \Diamond_I p \to \Diamond_I \Diamond_M p.$$

We call a $\mathrm{IM}_{\Box\Diamond}$-logic $\Lambda = \mathbf{IntK}_{\Box\Diamond} + \Gamma$ simulated$_{\mathbf{FS}}$ if, for all $\phi \in \mathcal{L}_{\Box\Diamond}$,

$$\phi^t \in \mathbf{FS} \oplus \Gamma^t \quad \Leftrightarrow \quad \phi \in \Lambda.$$

Contrary to the simulations described so far it is not known whether all $\mathrm{IM}_{\Box\Diamond}$-logics are simulated$_{\mathbf{FS}}$. So, the technical use of this simulation is limited so far. However, many natural $\mathrm{IM}_{\Box\Diamond}$-logics are known to be simulated$_{\mathbf{FS}}$, consult [18], [1], and [14], and the interpretation of the modal connectives under this simulation is quite natural.

5 Conclusion

We would like to close with some remarks on the overall philosophy behind these transfer results. First of all, the reduction of theories to others is quite a standard technique, also referred to as *interpretation* of theories. The general emphasis here is not in effecting such an interpretation from one logic to another but in maximizing two things: (i) the algebraic properties of the map this interpretation induces from the extension lattice of the interpreted logic into the lattice of extensions of the interpreting logic, and (ii) the properties preserved and reflected by this map. Only with such results in hand the method will give significant insights. For example, it is S. Thomason who has discovered that there is a simulation of polymodal logics by monomodal logics and that this simulation reflects completeness of various kinds and decidability, and this was enough to prove his point. However, for more sophisticated counterexamples in modal logic, more was needed, namely also the fact that in addition it preserved these properties as well. This is far less easy to see, but in effect it led to a great simplification in the study of modal logic. If it is *negative* examples one wants to produce, one can now start with any number of operators and produce such an example. The undecidability of properties of logics, questions about the cardinality of certain intervals etc. can in many cases decided by making a detour into the land of polymodal logics. We also see that it is next to hopeless to expect any significant, global result on decidability and completeness for logics extending **K** analogous to the situation above **K4**. A posteriori we learn that transitivity is a very strong restriction, and why progress has been relatively easy when compared with the study of *all* extensions.

Furthermore, there is a certain trade-off between the ease with which a translation is defined and the use it will have in discovering new facts. It is straightforward to see how monomodal logic can be embedded into bimodal logic, or tense logic but it is rather hard to gain any new insights from such an embedding. In fact, one would expect the minimal tense extension to have similar properties than the original logic; but this turns out to be false, as was described in § 3.3. So, the embedding is obvious, but the properties transferred under that embedding are quite hard to discover. In contrast to that, take the simulation of bimodal logic in monomodal logic. It is not obvious that it can be done at all, and to understand the method requires some sophistication. Yet, this is balanced by the ease with which it allows to transfer properties back and forth. In a similar light one may also see correspondence theory. It is obvious that modal logic can be seen as second-order predicate logic, but very little is gained, even though recent trends seem to suggest the contrary. Intuitively, we would simply expect (monadic) second-order logic to be harder if the translation is so easy – and indeed it is. On the other hand, correspondence with first-order properties is difficult to establish (and sometimes false) but the gain from this is considerable and has led to a rich theory.

References

[1] Giambatista Amati and Fiora Pirri. A uniform tableau method for intuitionistic modal logics I. *Studia Logica*, 53:29 – 60, 1994.

[2] Johan van Benthem. *Modal and Classical Logic*. Bibliopolis, 1983.

[3] Patrick Blackburn. Nominal tense logic. *Notre Dame Journal of Formal Logic*, 34:56 – 83, 1993.

[4] Wim Blok. *Varieties of interior algebras*. PhD thesis, University of Amsterdam, 1976.

[5] Wim Blok. The lattice of modal logics: An algebraic investigation. *Journal of Symbolic Logic*, 45:221–236, 1980.

[6] Wim Blok. On the degree of incompleteness in modal logics and the covering relation in the lattice of modal logics. Technical Report 78–07, Department of Mathematics, University of Amsterdam, 1980.

[7] M. Bosić and Kosta Došen. Models for normal intuitionistic logics. *Studia Logica*, 43:217 – 245, 1984.

[8] Robert A. Bull and Krister Segerberg. Basic modal logic. In Dov M. Gabbay and Franz Guenthner, editors, *Handbook of Philosophical Logic*, volume 2, pages 1–88. Reidel, 1984.

[9] Alexander Chagrov and Michael Zakharyaschev. The undecidability of the disjunction property and other related problems. *Journal of Symbolic Logic*, 58:967–1002, 1993.

[10] Lilia A. Chagrova. Undecidable problems in correspondency theory. *Journal of Symbolic Logic*, 56:1261–1272, 1991.

[11] Brian Chellas. *Modal logic, an introduction*. Cambridge University Press, 1980.

[12] Maarten de Rijke. *Extending Modal Logic*. PhD thesis, Universiteit van Amsterdam, 1993.

[13] Kosta Došen. Models for stronger intuitionistic modal logics. *Studia Logica*, 44:39 – 70, 1985.

[14] W. B. Ewald. Intuitionistic tense and modal logic. *Journal of Symbolic Logic*, 51:166 – 179, 1986.

[15] Kit Fine. Logics containing K4, Part I. *Journal of Symbolic Logic*, 39:229–237, 1974.

[16] Kit Fine and Gerhard Schurz. Transfer theorems for stratified modal logics. In *Proceedings of the Arthur Prior Memorial Conference*, Christchurch, New Zealand, 1991.

[17] Gisele Fischer Servi. Semantics of a class of intuitionistic modal calculi. In L. Dalla Chiara, editor, *Italian Studies in the Philosophy of Science*, pages 59 – 72. Reidel, 1980.

[18] Gisele Fischer Servi. Axiomatizations of some intuitionistic modal logics. *Rend. Sem. Mat. Univers. Polit.*, 42:179 – 194, 1984.

[19] Robert I. Goldblatt. Varieties of complex algebras. *Annals of Pure and Applied Logic*, 38:173–241, 1989.

[20] Valentin Goranko and George Gargov. Modal logic with names. *Journal of Philosophical Logic*, 22:607–636, 1993.

[21] Valentin Goranko and Solomon Passy. Using the universal modality: Gains and Questions. *Journal of Logic and Computation*, 2:5–30, 1992.

[22] Carsten Grefe. Modale Logiken funktionaler Frames. Master's thesis, Department of Mathematics, FU Berlin, 1994.

[23] Stephen Isard. A finitely axiomatizable undecidable extension of K. *Theoria*, 43:195–202, 1977.

[24] Bjarni Jónsson and Alfred Tarski. Boolean algebras with operators. *American Journal of Mathematics*, 73:891–939, 1951.

[25] Marcus Kracht. Even more about the lattice of tense logics. *Archive of Mathematical Logic*, 31:243–357, 1992.

[26] Marcus Kracht. How completeness and correspondence theory got married. In Maarten de Rijke, editor, *Diamonds and Defaults*, Synthese, pages 175–214. Kluwer, 1993.

[27] Marcus Kracht. *Tools and Techniques in Modal Logic.* Number 142 in Studies in Logic. Elsevier, Amsterdam, 1999.

[28] Marcus Kracht and Frank Wolter. Properties of independently axiomatizable bimodal logics. *Journal of Symbolic Logic*, 56:1469–1485, 1991.

[29] Marcus Kracht and Frank Wolter. Normal monomodal logics can simulate all others. *Journal of Symbolic Logic*, 64:99–138, 1999.

[30] Á. Kurucz, I. Németi, I. Sain, and A. Simon. Undecidable Varieties of Semilattice-ordered Semigroups, of Boolean Algebras with Operators, and Logics extending Lambek Calculus. *Bulletin of the IGPL*, 1(1):91 – 98, 1993.

[31] David C. Makinson. Some embedding theorems for modal logic. *Notre Dame Journal of Formal Logic*, 12:252–254, 1971.

[32] Michael Moortgat. Multimodal linguistics inference. *Journal of Logic, Language and Information*, 1995. to appear.

[33] Hiroakira Ono. On some intuitionistic modal logics. *Publication of the Research Institute for Mathematical Sciences*, 13:687 – 722, 1977.

[34] Solomon Passy and Tinko Tinchev. An essay in combinatory dynamic logic. *Information and Computation*, 93:263–332, 1991.

[35] Dirk Roorda. Dyadic Modalities and the Lambek Calculus. In Maarten de Rijke, editor, *Diamonds and Defaults*, pages 215 – 253. Kluwer, Dordrecht, 1993.

[36] Hendrik Sahlqvist. First and second order semantics for modal logic. In Stig Kanger, editor, *Proceedings of the Third Scandinavian Logic Symposium*, pages 15–31. North-Holland, 1975.

[37] Giovanni Sambin and Virginia Vaccaro. Topology and duality in modal logic. *Annals of Pure and Applied Logic*, 37:249–296, 1988.

[38] Dana Scott. Advice on Modal Logic. In Lambert, editor, *Philosophical Problems in Logic. Some Recent Developments*, pages 143–174. Reidel, Dordrecht, 1970.

[39] Valentin Shehtman. Kripke type semantics for propositional modal logics with intuitionistic base. In V. A. Smirnov, editor, *Modal and Tense Logics*, pages 108–112. Institute of Philosophy, USSR Academy of Sciences, 1979.

[40] Valentin Shehtman. Undecidable propositional calculi (russian). In *Problemy kibernetiki. Neklassicheskije logiki i ikh primenenije*. Akademia Nauk SSSR, 1982.

[41] Edith Spaan. *Complexity of Modal Logics*. PhD thesis, Department of Mathematics and Computer Science, University of Amsterdam, 1993.

[42] S. K. Thomason. Reduction of tense logic to modal logic I. *Journal of Symbolic Logic*, 39:549–551, 1974.

[43] S. K. Thomason. Reduction of second-order logic to modal logic. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 21:107–114, 1975.

[44] S. K. Thomason. Reduction of tense logic to modal logic II. *Theoria*, 41:154–169, 1975.

[45] S. K. Thomason. Independent propositional modal logics. *Studia Logica*, 39:143–144, 1980.

[46] Yde Venema. *Many-Dimensional Modal Logics*. PhD thesis, Universiteit van Amsterdam, 1991.

[47] Frank Wolter. Eigenschaften unabhängig axiomatisierbarer polymodaler Logiken. Master's thesis, II. Mathematisches Institut, FU Berlin, 1991.

[48] Frank Wolter. *Lattices of Modal Logics*. PhD thesis, FU Berlin, 1993.

[49] Frank Wolter. Solution to a problem of Goranko and Passy. *Journal of Logic and Computation*, 4:21–22, 1994.

[50] Frank Wolter. What is the Upper Part of the Lattice of Bimodal Logics? *Studia Logica*, 53:235–242, 1994.

[51] Frank Wolter. Completeness and decidability of tense logics closely related to logics above K4. to appear in the Journal of Symbolic Logic, 1995.

[52] Frank Wolter. The finite model property in tense logic. *Journal of Symbolic Logic*, 60:757 – 774, 1995.

[53] Frank Wolter. A Counterexample in Tense Logic. *Notre Dame Journal of Formal Logic*, 37:167–173, 1996.

[54] Frank Wolter and Michael Zakharyaschev. Intuitionistic Modal Logics as Fragments of Classical Bimodal Logics. forthcoming, 1994.

[55] Frank Wolter and Michael Zakharyaschev. On the relation between intuitionistic and classical modal logics. forthcoming, 1994.

[56] Michael Zakharyaschev. Canonical formulas for K4, Part I: Basic results. *Journal of Symbolic Logic*, 57:1377–1402, 1992.