Technical Modal Logic

Marcus Kracht *
Fakultät LiLi
Universität Bielefeld
Postfach 10 01 31
D-33501 Bielefeld
marcus.kracht@uni-bielefeld.de

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Abstract

Modal logic is concerned with the analysis of sentential operators in the widest sense. Originally invented to analyse the notion of *necessity* applications have been found in many areas of philosophy, logic, linguistics and computer science. This in turn has led to an increased interest in the technical development of modal logic.

1 Introduction

Modal logic originated in the analysis of necessity. Write "it is necessary that p" in the form "$\Box p$" and "it is possible that p" in the form "$\Diamond p$". While at first the interpretation of "$\Box$" as necessity prevailed, soon it appeared that there are many more ways to read "$\Box$", such as "it will always be the case that", "it ought to be the case that", "it is provable that", "everywhere it is the case that" and so on. If the interpretation of "$\Box$" is thus up for grabs, the logic of this operator is completely undetermined. What emerged from this was the study of modal logic as an abstract discipline where only minimal conditions for the behaviour of "$\Box$" are given at the

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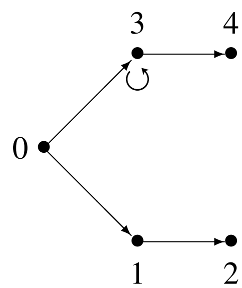
Figure 1: A Kripke Frame and a Valuation



$\beta(p_0) := \{0, 1, 3\}$
$\beta(p_1) := \{1, 2, 4\}$
$\beta(p_2) := \{1, 3, 4\}$
$\beta(p_3) := \varnothing$
$\ldots$

outset. In what follows below I shall outline some important results in the study of modal logic. I shall restrict my exposition to propositional logics, specifically to so-called normal modal logics.

Expositions of modern modal logic can be found in [Chagrov and Zakharyaschev, 1997], [Blackburn *et al.*, 2001], and [Kracht, 1999b]. Furthermore, for a survey on modern developments I refer to [Blackburn *et al.*, 2007].

2 First Steps

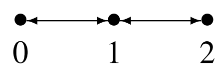
The language $\mathcal{L}_\Box$ of modal logic consists of full propositional logic (formulated here only in $\wedge$ and $\neg$) together with one unary sentential operator, $\Box$. The set of variables is $\{p_i : i \in \mathbb{N}\}$, though I shall use p and q to denote variables if their exact identity is irrelevant. $\Diamond\varphi$ is defined as $\neg\Box\neg\varphi$. A **Kripke-frame** is a pair $\langle W, R\rangle$ where W is a set and R a binary relation on W (see Figure 1, the arrows encode the relation). Let β be a **valuation**, that is, a map from the variables to subsets of W, and let $w \in W$. Then $\langle W, R, \beta, w\rangle \vDash \varphi$ is defined as follows.

$$(1)\qquad \begin{array}{ll} \langle W, R, \beta, w\rangle \vDash p_i & \text{iff } w \in \beta(p_i) \\ \langle W, R, \beta, w\rangle \vDash \neg\varphi & \text{iff } \langle W, R, \beta, w\rangle \nvDash \varphi \\ \langle W, R, \beta, w\rangle \vDash \varphi \wedge \chi & \text{iff } \langle W, R, \beta, w\rangle \vDash \varphi; \chi \\ \langle W, R, \beta, w\rangle \vDash \Box\varphi & \text{iff for all } w'\text{: if } w\,R\,w' \text{ then } \langle W, R, \beta, w'\rangle \vDash \varphi \end{array}$$

In Figure 1 we have, for example

$$(2)\qquad \langle W, R, \beta, 0\rangle \vDash \Diamond\Box p_2$$

Figure 2: A Tolerance Relation



since $0\,R\,3$ and

(3) $\langle W, R, \beta, 3\rangle \vDash \Box p_2$

which in turn is the case since $3\,R\,3, 4$ and $\{3, 4\} \subseteq \beta(p_2)$.

In case that $\langle W, R, \beta, w\rangle \vDash \varphi$ we say that φ is (**locally**) **true at** w under the valuation β. φ is true **globally** (under β) if it is true everywhere (under β). In the example, $\neg p_3$ is globally true, and so is also $p_0 \vee \Diamond p_1$. φ is **valid** in the frame if it is globally true under all valuations. We write $\langle W, R\rangle \vDash \varphi$. It is easily noted that classical equivalences hold locally, that is, if φ and χ are equivalent in propositional (nonmodal) calculus then

(4) $\langle W, R, \beta, w\rangle \vDash \varphi \quad \Leftrightarrow \quad \langle W, R, \beta, w\rangle \vDash \chi$

But these equivalences hold even in the scope of modal operators. So, $\neg\Box\varphi$ holds at a world iff $\Diamond\neg\varphi$ holds there. (Recall that the latter is short for $\neg\Box\neg\neg\varphi$ and that $\varphi \leftrightarrow \neg\neg\varphi$ is a classical tautology.) Consider the frame $\mathfrak{F} = \langle W, R\rangle$ where $W = \{0, 1, 2\}$ and

(5) $W = \{\langle 0, 0\rangle, \langle 0, 1\rangle, \langle 0, 2\rangle, \langle 1, 0\rangle, \langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle\}\rangle$

(See Figure 2.) The relation is reflexive and symmetric but not transitive. A relation is called a **tolerance** if it is reflexive and symmetric; a tolerance need not be transitive. The relation "is indistinguishable from" is a tolerance. Now let β be a valuation and $\beta(p_0) = \{2\}$. Then we have $\langle \mathfrak{F}, \beta, 0\rangle \vDash \Diamond\Diamond p_0$. For since $\langle \mathfrak{F}, \beta, 2\rangle \vDash p_0$ and $1\,R\,2$ we also have $\langle \mathfrak{F}, \beta, 1\rangle \vDash \Diamond p_0$. (Since p_0 is not true at 1 we have $\langle \mathfrak{F}, \beta, 1\rangle \nvDash \Box p_0$, or $\langle \mathfrak{F}, \beta, 1\rangle \vDash \neg\Box p_0$.) Now, $0\,R\,1$ and so $\langle \mathfrak{F}, \beta, 0\rangle \vDash \Diamond\Diamond p_0$. Yet, $\langle \mathfrak{F}, \beta, 0\rangle \nvDash \Diamond p_0$, and so $\langle \mathfrak{F}, \beta, 0\rangle \vDash \neg\Diamond p_0$, or $\langle \mathfrak{F}, \beta, 0\rangle \vDash \Box\neg p_0$. It follows that

(6) $\mathfrak{F} \nvDash \Diamond\Diamond p_0 \to \Diamond p_0$

Likewise one sees that

(7) $\mathfrak{F} \nvDash \Box p_0 \to \Box\Box p_0$

Consider now an interpretation of "$\Box\varphi$" as "almost certainly φ". On the frame side we think of this as being true in a world w if all worlds indistinguishable from w satisfy φ. The relation of indistinguishability is a tolerance. It is reflexive, and symmetric, but not always transitive. In measurements, you may not be able to distinguish two points that are a tenth of a millimeter apart, but you may be able to tell them apart if their distance is half a millimeter. The properties of the indistinguishability relation reflect the fact that the following are always true: $p \to \Box p$ and $p \to \Box\neg\Box\neg p$ (which is the same as $p \to \Box\Diamond p$). The first now reads "If p then almost certainly p" and the second "If p then almost certainly not almost certainly not p". However, as we have just seen, there is doubt whether "If almost certainly p the almost certainly almost certainly p." There is an intimate relationship between requirements on the relation R and the logical postulates that are valid in the frame. More on that below in Section 5.

Thus, as different interpretations of "$\Box$" yield different requirements on the frame structure, so different formulae are valid under different interpretations. On the one hand we have a class $\mathcal{K}$ of frames that are in some sense "right" for the interpretation, on the other hand we have a set L of formulae that are considered "valid", or **theorems**, under the interpretation. This interplay between semantics and syntax is characteristic of research into modern modal logic. If all goes well, $\mathcal{K}$ and L are a complete match for each other in that L is precisely the set of formulae valid in all frames of $\mathcal{K}$ and $\mathcal{K}$ is the class of frames in which all formulae of L are valid. Unfortunately, contrary to initial hopes in modal logic, this is not always the case. This is the source of many complications (as well as many beautiful results).

3 Algebraic Semantics of Modal Logics

In this section I look at the interpretation of modal logic. We shall see what it can mean that some proposition is "possible" or "necessary" as opposed to being true. First we fix our language. As it shall turn out, there are various ways to understand these notions. Let $\mathcal{K}$ be an arbitrary class of Kripke-frames (for example, the class of all $\langle W, R\rangle$ such that R is a tolerance relation on W). The set of formulae valid in all members of $\mathcal{K}$ is denoted by $\mathrm{Th}(\mathcal{K})$. It is a normal modal logic, where a **normal modal logic** is a set L of formulae with the following properties.

① L contains all tautologies of the classical propositional calculus;

② L contains the "box distribution" formula $\Box(p \to q) \to (\Box p \to \Box q)$;

③ L is closed under modus ponens: if $\varphi \to \chi \in L$ and $\varphi \in L$ then $\chi \in L$;

④ L is closed under substitution: if $\varphi \in L$ then $s(\varphi) \in L$;

⑤ L is closed under necessitation: if $\varphi \in L$ then also $\Box\varphi \in L$.

As usual, a substitution is the replacement of variables by formulae. We write $\mathbf{K} \oplus \Gamma$ for the least (normal modal) logic containing Γ (the letter 'K' stands for 'Kripke'). $\mathsf{K} \oplus \bot$ is the inconsistent logic: it contains every formula, that is, $\mathsf{K} \oplus \bot = \mathcal{L}_\Box$. The inconsistent logic has no frames. It is easy to check that $\mathsf{K} \oplus p_i$ is also the inconsistent logic. However $\mathsf{K} \oplus \Box\bot$ is *not* inconsistent. The frame $\langle\{0\}, \varnothing\rangle$ is a frame for this logic!

Given a logic L, we associate a consequence relation $\vdash_L$ with L defined as follows. $\Sigma \vdash_L \chi$ iff for some finite subset $\Sigma' \subseteq \Sigma$ we have $\bigwedge \Sigma' \to \chi \in L$. This consequence relation is also called the **local consequence**, to distinguish it from the *global* consequence, of which more below. The definition is a brute reduction to the logic as set; another option is to say that $\Sigma \vdash_L \chi$ if there is a proof of χ from $\Sigma \cup L$ using only modus ponens. This means that there is a sequence $\delta_0, \delta_1, \cdots, \delta_n$ of formulae such that (1) $\chi = \delta_n$, (2) for every $i < n$, either (2a) $\delta_i \in \Sigma \cup L$ or (2b) there are numbers $j, k < i$ such that $\delta_k = \delta_j \to \delta_i$. This means that δ_j and $\delta_j \to \delta_i$ occur before δ_i. The **global** consequence, by contrast, allows as an additional rule also necessitation (even on formulae from Σ). This means that there is a sequence of formulae satisfying (1), and (2′) for every $i < n$: either (2a) or (2b) or (2′c) there is $j < i$ such that $\delta_i = \Box\delta_j$.

Given a logic L and a formula φ, we form the following set

$$C_L(\varphi) := \{\chi : \varphi \leftrightarrow \chi \in L\} \tag{8}$$

This is the equivalence class of φ in L. Algebraically speaking, the relation Θ_L defined by $\varphi\,\Theta_L\,\chi$ iff $\varphi \leftrightarrow \chi \in L$ is a congruence. The congruence classes are the sets $C_L(\varphi)$. Two formulae contained in a set are equivalent in L, and so substitutable in the sense of Leibniz' Criterion: if $\chi \in C_L(\varphi)$, and τ' results from τ by replacing some occurrence of φ by χ then $\tau \leftrightarrow \tau' \in L$. It is therefore algebraically speaking natural to consider each of these classes of expressions as a single object, and this is what the following construction does. Technically, what we do is factor out the congruence. Let $A_L := \{C_L(\varphi) : \varphi \in \mathcal{L}_\Box\}$. The equivalence classes form a boolean algebra with the following operations.

$$\begin{aligned} 1 &:= C_L(\top) \\ -C_L(\varphi) &:= C_L(\varphi) \\ C_L(\varphi) \cap C_L(\chi) &:= C_L(\varphi \wedge \chi) \end{aligned} \tag{9}$$

Furthermore, we can define a unary operator $\blacksquare$ by

(10) $\quad \blacksquare C_L(\varphi) := C_L(\Box\varphi)$

$\blacksquare$ has the following properties: (a) $\blacksquare 1 = 1$, and (b) $\blacksquare(a \cap b) = \blacksquare a \cap \blacksquare b$. The structure $\langle A_L, 1, -, \cap, \blacksquare\rangle$ is called a **boolean algebra with operator** (BAO). For a BAO $\mathfrak{A}$, let β be a map from variables into A. This map can be extended as follows.

(11)
$$\begin{aligned} \overline{\beta}(p_i) &:= \beta(p_i) \\ \overline{\beta}(\neg\varphi) &:= -\overline{\beta}(\varphi) \\ \overline{\beta}(\varphi \wedge \chi) &:= \overline{\beta}(\varphi) \cap \overline{\beta}(\chi) \\ \overline{\beta}(\Box\varphi) &:= \blacksquare\overline{\beta}(\varphi) \end{aligned}$$

A **filter** is a subset F such that (i) if $a \in F$ and $a \leq b$ then $b \in F$, (ii) if $a, b \in F$ then $a \cap b \in F$. An **ultrafilter** U is a maximal filter. We write $\langle \mathfrak{A}, \beta, U\rangle \vDash \varphi$ if $\overline{\beta}(\varphi) \in U$. And we write $\mathfrak{A} \vDash \varphi$ if for all valuations β and all ultrafilters U we have $\langle \mathfrak{A}, \beta, U\rangle \vDash \varphi$. (Equivalently, for all β we have $\overline{\beta}(\varphi) = 1$.)

Let $\mathfrak{A} = \langle A, 1, -, \cap, \blacksquare\rangle$ be a BAO. Then let $U(\mathfrak{A})$ be the set of ultrafilters. For $U, V \in U(\mathfrak{A})$ put $U\,R\,V$ if for all $\blacksquare a \in U$ we have $a \in V$. The pair $\langle W, R\rangle$ is a Kripke-frame. Also, for $a \in A$ put $\widehat{a} := \{U \in U(\mathfrak{A}) : a \in U\}$. The structure

(12) $\quad \langle W, R, \{\widehat{a} : a \in A\}\rangle$

is a so-called **generalized Kripke frame**. This is a Kripke-frame $\langle W, R\rangle$ together with a subset $S \subseteq \wp(W)$ closed under intersection and complement and the operation

(13) $\quad \blacksquare A := \{x : \text{for all } y\text{: if } x\,R\,y \text{ then } y \in A\}$

Valuations may only be into S. The clauses for $\vDash$ are as above with S added. It is immediate that the set of worlds at which a given formula holds is a member of S. A Kripke frame can be seen as a general frame with $S = \wp(W)$. Clearly, we have a BAO $\langle S, W, -, \cap, \blacksquare\rangle$ (notice that the set W plays the role of the unit element). Thus, we have a correspondence between BAOs and generalized Kripke frames ([Jónsson and Tarski, 1951], [Goldblatt, 1976a; Goldblatt, 1976b]).

Let's return to the algebra $\mathfrak{A}_L$. There is a valuation κ defined by $\kappa(p_i) := C_L(p_i)$. With this valuation we get $\overline{\kappa}(\varphi) = C_L(\varphi)$. Also, for every valuation β, there is a substitution s such that $\overline{\beta}(\varphi) = \overline{\kappa}(s(\varphi))$. It follows that $\mathfrak{A}_L \vDash \varphi$ iff $\overline{\kappa}(s(\varphi)) = 1$ for all substitutions s iff $\varphi \in L$. As a result we get the following.

Theorem 1 *For every modal logic L there exists a BAO $\mathfrak{A}$ such that $\mathfrak{A} \vDash \varphi$ if and only if $\varphi \in L$.*

This can be improved somewhat. For a logic L, let $\mathrm{Alg}(L)$ be the class of BAOs $\mathfrak{A}$ such that $\mathfrak{A} \vDash \varphi$ for all $\varphi \in L$. It turns out that $\mathrm{Alg}(L)$ is a **variety**, that is, a class of algebras closed under subalgebras, homomorphic images and products ("HSP Theorem", see [Burris and Sankappanavar, 1981]).

Theorem 2 *For every normal modal logic L, $\mathrm{Alg}(L)$ is a variety of BAOs. For every variety $\mathcal{V}$ of BAOs, $\mathrm{Th}(\mathcal{V})$ is a normal modal logic. These operators are inverses of each other.*

It so turns out that there is a tight correspondence between BAOs and generalized frames. However, when we turn to Kripke frames, matters are different. Let $\mathrm{Frm}(L)$ denote the class of Kripke frames for L. Conversely, let $\mathrm{Th}(\mathcal{K})$ be the set of formulae valid in all frames of $\mathcal{K}$. These maps are not inverses of each other; there are logics L and L' such that $\mathrm{Frm}(L) = \mathrm{Frm}(L')$. This phenomenon, incompleteness, has meant that the Kripke-frames had to be generalised.

4 The Lattice of Modal Logics

As already said in the introduction there are different interpretations of "$\Box$", and they may give rise to different logics. Thus, even though we assume classical logic as our basic logic we still have a multitude of logics. This multitude has the structure of a lattice. We write $\mathrm{NExt}\,\mathsf{K}$ for the set of all normal modal logics. $\mathrm{NExt}\,\mathsf{K}$ is partially ordered by set inclusion. If L_1 and L_2 are normal modal logics, so is the intersection $L_1 \cap L_2$. Also, the infinite intersection $\bigcap_{i \in I} L_i$ of a family of normal modal logics is again such. Given L_1 and L_2 we write $L_1 \sqcup L_2$ for the least (normal modal) logic containing both L_1 and L_2. (In general this is *not* the set union.) Similarly $\bigsqcup_{i \in I} L_i$ is the least logic containing all L_i. What we have, in mathematical terms, is a lattice $\langle \mathrm{NExt}\,\mathsf{K}, \cap, \sqcup \rangle$. (Since the infinitary operations are also defined, this lattice is called **complete**.) There is a bottom element, K, and a top element, the inconsistent logic, $\mathsf{K} \oplus \bot$.

The lattice $\mathrm{NExt}\,\mathsf{K}$ is distributive. Moreover, it satisfies an infinite law of distribution

(14) $$\left(\bigsqcup_{i \in I} L_i\right) \cap M = \bigsqcup_{i \in I} (L_i \cap M)$$

There is no analogous law relating $\bigcap$ and $\sqcup$, and this means that the lattice lacks a certain duality. For example, there are two logics T_1 and T_2 which are lower neighbours of the inconsistent logics (in other words they are maximally consistent) and every consistent logic is contained in either T_1 or T_2 ([Makinson, 1971]). On the other hand, NExt K has no atoms, that is elements $x > 0$ such that for no y, $0 < y < x$.

An important lattice theoretic concept is that of a *splitting*. A **splitting** is a pair $\langle L, L' \rangle$ so that for every logic M either $M \subseteq L$ or $M \supseteq L'$ but not both. If L is the logic of a single frame $\mathfrak{W}$, L' is then the least logic not having $\mathfrak{W}$ as its frames. An example is $\langle L, \mathsf{K} \oplus \Diamond\top \rangle$ where L is the logic of the frame $\langle \{0\}, \varnothing \rangle$ (which we have identified above as $\mathsf{K} \oplus \Box\bot$). L' has the following property: if $L' \subseteq \mathsf{K} \oplus \Gamma$ then there exists a single $\gamma \in \Gamma$ such that $L' \subseteq \mathsf{K} \oplus \gamma$. It can be shown abstractly that if L is an atom in the lattice NExt K then there is a logic L' such that $\langle L, L' \rangle$ is a splitting. Unfortunately, NExt K possesses no atoms.

Call a logic L **complete** if $L = \mathrm{Th}(\mathrm{Frm}(L))$, that is, whenever $\varphi \notin L$ there is a Kripke-frame $\langle W, R \rangle$ for L such that $\langle W, R \rangle \nvDash \varphi$. [Blok, 1980] investigated the question how widespread completeness is. Say that L' is a **completion** of L if $L' = \mathrm{Th}(\mathrm{Frm}\, L)$, equivalently, if L' is complete and has the same Kripke-frames as L. The **degree of incompleteness** of L is the cardinality of the set $\{L' :$ L and L' have the same completion$\}$. If every logic were complete then the degree of incompleteness would be 1 for all logics. [Blok, 1980] shows that the degree of incompleteness is either 1 or $2^{\aleph_0}$. Moreover, it is 1 only for quite uninteresting logics: the iterated splittings of K, which Blok also characterised (there are only countably many of them). Thus, incompleteness is in some sense the norm rather than the exception.

While the lattice of all logics is quite intricate, parts of it can be rather well behaved. Write NExt L for the sublattice of all logics containing L. The logic Alt_1 is characterised by the axiom $\Diamond p \to \Box p$. The lattice NExt Alt_1 contains only complete logics; it is countable, and all logics have the finite model property and are decidable. Another much studied lattice is NExt $\mathsf{K4}$, where $\mathsf{K4} = \mathsf{K} \oplus \Diamond\Diamond p \to \Diamond p$. The interest in $\mathsf{K4}$ is due in part because of the close connection with intuitionistic logic.

Recall the so-called Gödel translation of intuitionistic formulae.

$$(15) \qquad \begin{aligned} p^\tau &:= \Box p \\ (\neg\varphi)^\tau &:= \Box(\neg\varphi^\tau) \\ (\varphi \wedge \chi)^\tau &:= \varphi^\tau \wedge \chi^\tau \end{aligned}$$

This translation sends an intuitionistic formula to a modal formula. The image of an intermediate logic is not a modal logic (some formulae are missing). (Recall that an intermediate logic is a logic containing Int and being contained in the classical calculus.) So we take as the image of a *logic* under this translation the least modal logic containing all translations of the formulae. The image of Int is the logic Grz, known as Grzegorczyk's logic.

$$\mathsf{Grz} := \mathsf{K4} \oplus \{p \to \Diamond p, \Box(\Box(p \to \Box p) \to p) \to p\} \tag{16}$$

Grz is the logic of all Kripke-frames where the converse of R is the reflexive closure of a well order. If I and I' are distinct intuitionistic logics then $I^\tau \neq (I')^\tau$. Moreover, every logic containing Grz is the image of some intermediate logic. This means that the map is an isomorphism of the lattice of intermediate logics onto the lattice $\mathrm{NExt}\,\mathsf{Grz}$ of extensions of Grz ([Blok, 1976]).

5 Expressivity

An important question is how much we can express about our frames using modal formulae. As is easily seen, modal logic is a fragment of monadic second order logic. Namely, a modal formula φ can be translated into a second order formula as follows. Assume for each variable p_i a distinct monadic predicate letter P_i. Read "$P_i(x)$" as "p_i is true at x". Now put

$$\begin{aligned} p_i^\sigma(y) &:= P_i(y) \\ (\neg\varphi)^\sigma(y) &:= \neg(\varphi^\sigma)(y) \\ (\varphi \wedge \chi)^\sigma(y) &:= \varphi^\sigma(y) \wedge \chi^\sigma(y) \\ (\Box\varphi)^\sigma(y) &:= (\forall z)(y\,R\,z \to \varphi^\sigma(z)) \end{aligned} \tag{17}$$

Given a formula φ we can now express the fact that there is a valuation that makes φ true at some point x by the following second order formula:

$$(\exists P_1)(\exists P_2)\cdots(\exists P_n)\varphi^\sigma(x) \tag{18}$$

Hence, the condition that a modal formula is valid in a frame is also equivalent to a (monadic) second order formula. Sometimes it happens that this formula is equivalent to a first order formula. For example, $\Diamond\Diamond p \to \Diamond p$ is valid in a frame iff R is transitive. For if R is not transitive, say $x\,R\,y\,z$ but not $x\,R\,z$ we put $\beta(p) := \{z\}$ and $\Diamond\Diamond p$ and $\neg\Diamond p$ are true at x. However, if R is transitive and x a point at which $\Diamond\Diamond p$ is true, there is a y with $x\,R\,y$ at which $\Diamond p$ is true, and a point z with $y\,R\,z$ at which p is true. Now $x\,R\,z$ and so $\Diamond p$ is true at x. Similarly, $\Diamond p \to \Box p$ is valid in $\langle W, R\rangle$ iff R is a partial function on W. [Sahlqvist, 1975] describes a large class of modal formulae that axiomatise logics with elementary frame conditions. These logics also have another property, called *canonicity*. Let $\mathfrak{A}_L$ be the BAO constructed above for L, and let $\langle W_L, R_L, S_L\rangle$ be the generalised Kripke frame for that algebra. L is called **canonical** if also $\langle W_L, R_L\rangle$ is a frame for L. Evidently, in that case L is complete. This result has been treated extensively, among other in [van Benthem, 1983], [Sambin and Vaccaro, 1989], [Kracht, 1993], [Goranko and Vakarelov, 2006] (the latter managed to improved the result even). [Fine, 1975] observed that if a logic is complete for an elementary class of Kripke-frames then it is canonical. He asked about the converse. [Goldblatt *et al.*, 2004] gave a negative answer: there is a canonical logic not characterised by an elementary class of Kripke-frames. Given that we can express certain elementary formulae, which ones are they? A characterisation is given in [Kracht, 1999b] for the Sahlqvist class. There is however another way to look at the matter. In order to express a complex elementary condition it may be necessary to add further operators, or alternatively, to model the elementary relations indirectly. In that case it may be possible to encode more first order conditions. Indeed, as [Kracht, 1999a] notes it is possible to go as high as Π^1_0 (this includes first order set theory with true second order comprehension, for example).

6 Consequence

Up to now we have focussed on logics as sets of formulae. However, often we need to generalise this and study what is known as *consequence relations*. It so turns out that the same consequence relation can be related to several sets of theorems. A **consequence relation** is a relation $\succ$ between sets of formulae and a single formula subject to the following requirements. [1]

- If $\gamma \in \Gamma$ then $\Gamma \succ \gamma$.
- If $\Gamma \succ \varphi$ and $\Gamma \subseteq \Gamma'$ then $\Gamma' \succ \gamma$.
- If $\Gamma \succ \delta$ for every $\delta \in \Delta$ and $\Delta \succ \gamma$ then also $\Gamma \succ \gamma$.
- If $\Gamma \succ \gamma$ and s is substitution then $s(\Gamma) \succ s(\gamma)$.

A **tautology** of $\succ$ is a formula γ such that $\varnothing \succ \gamma$. $\mathrm{Taut}(\succ)$ denotes the set of tautologies of $\succ$. A **modal consequence relation** is a consequence relation $\succ$

[1] This would more exactly be called a structural, normal consequence relation.

such that Taut($\succ$) is a normal modal logic, and such that $\{p, p \to q\} \succ q$. Above we have seen the relation $\vdash_L$. This is the least modal consequence relation whose set of tautologies is L. There are more. For example, a consequence relation is called **global** if $\{p\} \succ \Box p$. The least global consequence with tautologies L is denoted by $\Vdash_L$. It is distinct from $\vdash_L$ just in case $p \to \Box p$ is not in L. A **rule** is a pair $\rho = \langle \Gamma, \gamma \rangle$, where Γ is a set of formulae and γ a single rule. ρ is a **derived rule** of $\succ$ if $\rho \in \succ$. It is **admissible** for L if for every substitution s if $s(\Gamma) \subseteq L$ then $s(\gamma) \in L$. The rule of denecessitation, $\langle \{\Box p\}, p \rangle$, is admissible for K. For assume that $\Box\varphi$ is a theorem. Then it holds in all frames. Let $\langle W, R \rangle$ be a frame; we need to show that $\langle W, R \rangle \vDash \varphi$. Add to W a new world w_0 and expand R to R_0 such that $w_0 \, R_0 \, w$ for all $w \in R$. Then $\langle W_0, R_0 \rangle \vDash \Box\varphi$ by assumption. It follows that $\langle W, R \rangle \vDash \varphi$ (for a syntactic proof see [Williamson, 1993]). The rule is however not derived in either $\vdash_{\mathsf{K}}$ or $\Vdash_{\mathsf{K}}$. The necessitation rule is admissible in any modal consequence relation (by definition of a logic) but it is derived only if that relation is global. Given a logic L with infinitely many constant formulae there are $2^{\aleph_0}$ many consequence relations whose set of tautologies is L.

[Rybakov, 1997] discusses the problem of characterising the admissible rules for a logic. In the above terms, the objective is to characterise in terms of explicit rules the maximal consequence relation $\succ$ such that $\tau(\succ) = L$. This question is quite different from the axiomatization of L. For example, $\mathsf{S4}$, the logic of transitive and reflexive frames, is clearly finitely axiomatisable and decidable. Yet, the admissible rules are not axiomatisable by finitely many rules, though it is decidable whether a given rule is admissible.

7 Polymodal Logic

For many applications of modal logic we wish to have not just one but several operators. Examples are constituted by tense logic, which uses two operators (sharing the same accessibility relation), mixtures of tense and modality (as in Montague Semantics), and Propositional Dynamic Logic ([Segerberg, 1977] and [Fischer and Ladner, 1979]). Again, if the interpretation of the operators is arbitrary we are led to the study of the lattice $\mathrm{NExt}\,\mathsf{K}_\alpha$ of normal modal logics with α many operators. It turns out that if α is finite there is an complete reduction of polymodal logic to monomodal logic. I shall sketch how this works for $\alpha = 2$.

I will not rehearse the definitions of logics and frames. The operators of the bimodal logic are now called $\Box_1$ and $\Box_2$ while the operator of the monomodal logics carries no index. Given two monomodal logics L_1 and L_2, the logic $L_1 \otimes L_2$

is formed by interpreting the operator of L_1 by $\Box_1$ and the operator of L_2 by $\Box_2$. This is the independent fusion of [Kracht and Wolter, 1991]. Unless L_1 or L_2 are inconsistent, this operation is conservative. This is to say that if we look at the theorems of $L_1 \otimes L_2$ containing no occurrences of $\Box_2$ we recover L_1 (under the interpretation of $\Box$ as $\Box_1$) and similarly for L_2. The operation also has a number of other properties. If L_1 and L_2 are complete, so is $L_1 \otimes L_2$; if on the other hand $L_1 \otimes L_2$ is complete, so are both L_1 and L_2. The same holds for a number of other properties (decidability, elementarity, having the finite model property and more). Recall that if L is a monomodal logic the tense extension is defined by $L^t := (L \otimes \mathsf{K}) \oplus \{p \to \Box_1 \Diamond_2 p, p \to \Box_2 \Diamond_1 p\}$. The lattice $\mathrm{NExt}\,\mathsf{K}_2$ contains the lattice of tense logics, though not as a sublattice or interval. Moreover, the map is not conservative ([Wolter, 1997]).

It is also possible to go the other way: to a bimodal frame $\langle W, R_1, R_2 \rangle$ we can associate a monomodal frame $\langle W^\mu, R^\mu \rangle$ in the following way. W^μ is defined as $\{\star\} \cup W \times \{1, 2\}$ and $x\, R^\mu\, y$ if and only if (i) $x = \langle u, 1 \rangle$, $y = \langle u, 2 \rangle$, (ii) $x = \langle u, 2 \rangle$, $y = \langle u, 1 \rangle$, (iii) $x = \langle u, 1 \rangle$, $y = \langle v, 1 \rangle$ and $u\, R_1\, v$, (iv) $x = \langle u, 2 \rangle$, $y = \langle v, 2 \rangle$ and $u\, R_2\, v$, (v) $x = \langle u, 1 \rangle$ and $y = \star$. This is called the **simulation frame**. It is possible to express the modal operators $\Box_1$ and $\Box_2$ in the new language. The set $W \times \{1\}$ is definable by the formula $\underline{w} := \Diamond \Box \top$, the set $W \times \{2\}$ by $\underline{b} := \Diamond \top \wedge \Box \Diamond \top$. We can thus recover R_1 as follows. $x\, R_1\, y$ iff x and y satisfy $\underline{w}$ and $x\, R\, y$. Similarly, $x\, R_2\, y$ iff x and y satisfy $\underline{b}$ and $x\, R\, y$. This can be lifted to a construction on general frames. Finally, given a bimodal logic L, L^s is defined by the logic of the simulation frames of frames for L. It turns out that the map $L \mapsto L^s$ is a lattice isomorphism from $\mathrm{NExt}\,\mathsf{K}_2$ onto an interval in the lattice $\mathrm{NExt}\,\mathsf{K}$. Again, many properties are preserved back and forth under this isomorphism ([Kracht and Wolter, 1999]). I shall briefly return to the question of expressivity. As we have seen, we need not model binary relations by the one relation R, we can also define them by formula χ involving R. This allows a single binary relation to encode any finite number of binary relations. Set theory is based on a single binary relation as well, $\in$. By introducing enough additional relations we can encode set theory by means of binary relations based on modal operators.

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