An Argument for Minimal Logic

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Abstract

The problem of negative truth is the problem of how, if everything in the world is positive, we can speak truly about the world using negative propositions. A prominent solution is to explain negation in terms of a primitive notion of metaphysical incompatibility. I argue that if this account is correct, then minimal logic is the correct logic. The negation of a proposition A is characterised as the minimal incompatible of A composed of it and the logical constant $\neg$. A rule based account of the meanings of logical constants that appeals to the notion of incompatibility in the introduction rule for negation ensures the existence and uniqueness of the negation of every proposition. But it endows the negation operator with no more formal properties than those it has in minimal logic.

> There is implanted in the human breast an almost unquenchable desire to find some way of avoiding the admission that negative facts are as ultimate as those that are positive.
>
> Russell

1 Introduction

Negation can be defined in terms of implication and an absurd proposition, often denoted by $\perp$ and called *falsum*: $\neg A$ if and only if A implies $\perp$. Minimal logic results from the positive intuitionist calculus – the calculus formalised by the usual introduction and elimination rules for the constants $\rightarrow$, $\wedge$, $\vee$, $\exists$ and $\forall$ in a system of natural deduction – by adding $\perp$ but not adding any rules for it. As there are no rules for $\perp$, there are no rules for establishing it as true: nothing counts as a proof of $\perp$. However, precisely because there are no rules for $\perp$, minimal logic does not say anything about the nature of the absurd and remains silent on the content of $\perp$.

Contrast with intuitionist logic, where $\perp$ is governed by an elimination rule only, *ex falso quodlibet*:

$$(\perp E)\ \frac{\perp}{C}$$

In intuitionist logic, $\perp$ is the ultimate absurdity: everything follows from it. As some propositions, such as $0 = 1$, are false, and even necessarily so, in intuitionist logic $\perp$ must also be false, and even necessarily so.

While it is a prominent, if not widely adopted, position to reject classical logic and opt for intuitionist logic,[1] minimal logic has not had many champions in the contest for the correct logic.

The received view is that minimal logic has too little to say about the symbol $\perp$, and hence the symbol $\neg$, for them to formalise an absurd proposition and a logical constant worthy of the name 'negation'. In minimal logic, any proposition can play the role of $\perp$, as long as it is considered to be one that cannot be established as true. Many agree with Milne's verdict: 'Minimal logic emerges as the logic of personal prejudice: $\neg A$ is asserted when A entails the unacceptable.' (Milne, 1994, 66) For minimal logic it is all the same whether $\perp$ is 'Beetroot is delicious' or '0 = 1'.

In this paper I take up the challenge and argue for minimal logic as the correct logic. More precisely, I will argue that no more than minimal logic is forthcoming as the correct logic for an explication of negation in terms of a primitive notion of *metaphysical incompatibility*, which is popular with metaphysicians grappling with the problem of *negative truth*: If everything in the world is positive, if there is only what there is, not also what there is not, then how can we speak truly about the world using negative propositions? The incompatibilist account of negation answers that the positive properties of things and what they are incompatible with determine the truths of negative propositions. To give an example, it is true that the apple is not red, because it is green, and being red is incompatible with being green. I will argue that this account of negation gives no grounds for attributing any more *formal* properties to negation than those it enjoys in minimal logic. Whether two propositions are incompatible depends on their *content*. Formal logic has little to say about the content of 'red' and 'green'. Similarly, it has little to say about incompatibility.[2] The appeal to incompatibility is an extra-formal element of the account of negation. It provides principles for excluding frivolous propositions from playing the role of $\perp$. Thus despite only minimal logic being forthcoming as the logic for an explication of negation in terms of metaphysical incompatibility, the resulting concept of negation is a substantial one, as it is supported by a substantial metaphysics.[3]

2 The Problems of Falsity and Negative Truth

If I say truly that the apple is green, then this is because the apple has that colour. But how can I say truly that the apple is not red? There is only the colour the apple has, not also the colour it does not have. If I say truly that there is a hippopotamus in the zoo, then this is because of the presence of a hippo there. But how can I say truly that there is no hippopotamus in this room? There are only the presences of things in the room, not also the absence of a hippo. This,

[1]See the work of Dummett and Prawitz, for instance (Dummett, 1978a), (Dummett, 1978b), (Prawitz, 1972), (Prawitz, 1977), (Prawitz, 2006). Weiss provides an overview (Weiss, 2002).

[2]Aspects of incompatibility can be formalised, such as that if p and q are incompatible, then it is not possible that p and q. This can only be done once negation is in place, and so cannot be the first step for anyone wishing to explain negation in terms of incompatibility.

[3]This article uses material presented in (Kürbis, 2019b, Ch 4) for a rather different purpose.

in a nutshell, is the problem of negative truth. If everything in the world is positive, if there are only presences, not also absences, if there is only what there is and not also what there is not, then how can negative propositions be true of the world?

Put slightly differently, nothing corresponds to negation and that is how it should be. By using negation, we want to say what is not, and if negation did correspond to something, we should be saying what is. But now we have a problem. To speak truly is to say what is: truth corresponds to reality. So how can we speak truly using negation?

The problem of negative truth is rooted in the ancient problem of falsity: how can meaningful speech be false? The problem was raised by Parmenides, who points out that 'you could not know what is not – that cannot be done – nor indicate it.' (Fragment 2.7-8)[4] If speech is a kind of indicating – in speech we indicate what we are talking about – then we cannot speak about what is not, as then there is nothing to be indicated. But that, it seems, is exactly what false statements attempt to do. Parmenides concludes: 'What is there to be said and thought must needs be: for it is there for being, but nothing is not.' (Fragment 6.1-2) We cannot think about what is not, as it is not there to be thought about, and neither can we speak about it. Parmenides warns us that there is only what there is, not also what there is not: 'Never shall this be forcibly maintained, that things that are not are, but you must hold back your thought from this way of enquiry, nor let habit, born of much experience, force you down this way, by making you use an aimless eye or an ear and tongue full of meaningless sound: judge by reason the strife-encompassed refutation spoken by me.' (Fragment 7) 'Knowledge of what is not' is no knowledge at all. 'Indicating what is not' is no indicating at all. False speech would be speech about what is not, and so, if speech is to be meaningful, 'false speech' is no speech at all.

For Plato, the solution to the problem of falsity lies at the heart of semantics, epistemology and metaphysics. The line of thought that all meaningful speech is true speech and that falsity is impossible is elaborated in Socrates' encounter with the sophists Euthydemus and Dionysodorus in Plato's *Euthydemus*. Euthydemus proposes a notion of truth that is not unreasonable: 'The person who speaks what is and things that are speaks the truth.' (284a)[5] Speech needs a subject matter. To speak meaningfully is to speak about something. To speak truly is to speak of what is. There is only what is, not also what is not. False speech purports to be about what is not, but then it lacks a subject matter and is not about anything, and nothing has been said. Euthydemus concludes that 'if Dionysodorus really does speak, he speaks the truth and things that are' (284c). Meaning and truth coincide.[6]

The *Theaetetus* spells out some of the consequences of the problem of falsity for epistemology. If we cannot think falsely, but only truly, we cannot disagree either, as disagreement requires that at least one of us is wrong. The argument of the *Euthydemus* has the consequence that there is no such thing as contradicting

[4]I'm quoting Parmenides after (Kirk et al., 1983).

[5]I'm quoting Plato after (Plato, 1997).

[6]The sophists' *logoi* are what MM McCabe calls *chopped*: sayings correspond one-to-one to what they are of, their *pragma*, and truth and meaning are explicated at once: 'What the *logos* means, it says, and what it says, is so.' (McCabe, 2019) My comment on McCabe's paper (Kürbis, 2019a) contains some more thoughts on the relation between ancient and contemporary questions relating to falsity.

each other. If I attempt to contradict you, then, if you spoke falsely, there is nothing to contradict, as you did not speak meaningfully, and if you speak truly, then either I end up speaking falsely, and hence not meaningfully, or I speak meaningfully, hence truly, and so I must speak about something other than you do. Socrates takes this to be the foundation of Protagoras' doctrine that man is the measure of all things. Its refutation is at the centre of Socrates' and Theaetetus' investigation into what knowledge is (170ff): without the possibility of getting things wrong, there can be neither knowledge nor opinion.

In the *Sophist*, Plato tackles the metaphysics of the problem. The middle section of the dialogue culminates in 'the patricide of father Parmenides'. The Eleatic Stranger is lead to 'insist by brute force both that *that which is not* somehow is, and then again that *that which is* somehow is not' (241d). He concludes that *that which is* has a share *the same* and *the different*. 'It seems that when we say *that which is not*, we don't say something contrary to *that which is*, but only something different from it.' (275b) Things are the same in some respects, but different in others. An appeal to the notion of difference in metaphysics avoids the embarrassment of admitting non-being.

The Eleatic Stranger's investigation also provides another fundamental aspect of the solution to the problem of falsity: Meaning is compositional and there are different functions to different parts of speech. The Eleatic Stranger observes that 'there are two ways to use your voice to indicate something about being. [...] One is called names, and the other is called verbs. [...] A verb is the sort of indication that's applied to an action. [...] And a name is the kind of spoken sign that's applied to things that perform the actions. [...] No speech is formed just from names spoken in a row, and also not from verbs that are spoken without names.' (261e-262a) Names and verbs have different functions and speech is formed by weaving together names and verbs (262c-d). 'Theaetetus' names Theaetetus, 'sits' indicates the activity of sitting, 'flies' indicates the activity of flying, and so all three are meaningful, and so are sentences formed by putting them together in the right way. The Eleatic Stranger concludes that 'if someone says things about you [Theaetetus], but says different things as the same or not beings as being, then it definitely seems that false speech really and truly arises from that kind of putting together verbs and names' (263d). 'Theaetetus flies' is, after all, about something, that is, Theaetetus and flying, but it is false because the activities of Theaetetus are different from flying.[7]

The problem of how false statements can be meaningful is different from the problem of negative truth, but they are related. The link is the principle that '$\neg A$' is true if and only if 'A' is false. A solution to one is also a solution to the other.

In a classic paper Molnar presents the problem of negative truth as a challenge for a satisfactory theory of truth. Molnar argues that the problem arises because we are prone to accept metaphysical assumptions which 'can be summed up in four theses:

(i) The world is everything that exists.
(ii) Everything that exists is positive.
(iii) Some negative claims about the world are true.
(iv) Every true claim about the world is made true by something that exists.

[7] See Denyer's and McCabe's work for more on the problem of falsity in ancient Greek philosophy. Denyer underlines the banality of Plato's solution. (Denyer, 1991) McCabe argues that there is rather more to the problem and Plato's solution than what is sketched here. See also (McCabe, 2006).

(i)-(iv) jointly imply that all negative truths must have positive truthmakers.' (Molnar, 2000, 84f)[8] This conclusion has an odd, if not paradoxical, ring to it: 'Each of (i)-(iv) is individually plausible, but the quartet may not be co-tenable.' (Molnar, 2000, 72) The problem of negative truth arises because existence seems to be essentially positive, and it is hard to escape the pull of the correspondence theory of truth, at least when talking about worldly items such as apples and hippos. So how can we speak truly using negative propositions, if there is nothing to which they correspond?

The discussion of the *Sophist* already hinted at a solution. The next section presents a particularly attractive way of reconciling Molnar's four metaphysical theses by appealing to a primitive notion of metaphysical incompatibility. Negative truth is explained in terms of what things are and which properties are different from or incompatible with each other. This allows us to steer past the metaphysical Charybdis of which Parmenides has warned us, while not falling prey to the sophists' Scylla.

3 Incompatibility

At some point in his thinking Russell accepted that there are negative facts, as that to which negative true and false statements correspond, in addition to positive facts, which are what true statements correspond to. Russell reports that his view nearly produced a riot at Harvard: 'The class would not hear of there being negative facts at all.' (Russell, 1919b, 42) Many philosophers share Mumford's sentiments towards negative facts: they are 'too mysterious to be taken seriously. [Molnar's principle] (ii) has almost a ring of aprioricity about it. How can these facts exist and be negative? Indeed, how can any existent really be negative?' (Mumford, 2007, 49)[9] One way of avoiding negative facts is to appeal to a primitive notion of metaphysical incompatibility.

Reacting to Russell's Harvard lectures, Demos spelled out an account of negative truths in terms of positive ones and a relation of incompatibility, which he calls 'opposition' or 'contrariety'. Demos proposed that 'a particular and simple negative proposition is of the form "not-p is true," where p is any positive proposition, and "not" means "an opposite or a contrary of". As such, a negative proposition constitutes description of some true positive proposition in terms of the relation of opposition which the latter sustains to some other positive proposition.' (Demos, 1917, 193f). For instance, 'John is not at home' describes a proposition which is a contrary of 'John is at home', such as 'John is at the store': it conveys 'the statement "The true proposition, or the truth as to John's whereabouts, is a contrary of the proposition, "John is at home".' (Demos, 1917, 194) Demos concludes that 'through the definition of negative propositions just offered, the world of positive objects is re-established as the ultimate term of reference in all assertions of a particular nature. Negative propositions refer to positive propositions and positive propositions in their turn assert positive facts.

[8]Molnar's conclusion follows from (ii) and (iv) alone. (iii) ensures that it is not vacuously true. (i) is redundant. I argue elsewhere that maybe (iv) was meant to be 'Every true claim about the world is made true by something in the world'. Then (i) is no longer redundant if it means 'Everything in the world exists'. See (Kürbis, 2018).

[9]Mumford's 'solution is that there are no negative truths.' (Mumford, 2007, 51) I heard of no riots at the Aristotelean Society in the mid 2000s, but I'm not sure a more boisterous audience would have accepted Mumford's presentation quietly.

In both cases there is reference to the latter, but in the first case the reference is indirect, and in the second direct.' (Demos, 1917, 194) As the problem of negative truth arises already for simple or atomic propositions, I will follow Demos and restrict consideration to these cases.

Demos's rejection of the view that there are negative facts is motivated by empiricist epistemology: 'The reason why such a view must not be entertained is the empirical consideration that strictly negative facts are nowhere to be met with in experience, and that any knowledge of a negative nature seems to be derived from perception of a positive kind.' (Demos, 1917, 189) Empiricists before Demos were also concerned with the problem of negative truth. In *Elements of Philosophy*, Hobbes discusses the semantics of negation:

> Positive names are those that are applied because of the similarity, equality or identity of objects thought; negative ones are those that are applied because of distinctness or dissimilarity or difference. [...] But the positive names are prior to the negative ones, because, unless the former existed beforehand, there could be no use of the latter. For when the name 'white' was applied to certain things, and following thereafter the names *black*, *dark coloured*, *diaphanous* etc. to other things, it was not possible to comprise of all things that are dissimilar to *white*, which are infinite in number, in one name, except through the negation of *white*, that is, the name *not white* or something equivalent, in which *white* is repeated (just as it is in *dissimilar to white*). And so by these negative names we call to mind and signify that of which we are not thinking.[10]

Locke concurs and writes that 'negative or privative words cannot be said properly to belong to, or signify no ideas: for then they would be perfectly insignificant sounds; but they relate to positive ideas, and signify their absence.' (Locke, 1979, bk III, ch 1, §4) For Hobbes and Locke, the meanings of negative expressions are derivative of the meanings of positive ones, and so negative truths are secondary to positive truths.

More recently, Price argued that 'the apprehension of incompatibility [is] an ability more primitive than the use of negation' (Price, 1990, 226). He explains negation by proposing that 'it is appropriate to deny a proposition P (or assert $\neg P$) when there is some proposition Q such that one believes that Q and takes P and Q to be incompatible' (Price, 1990, 231). Price accepts that in principle, we never need to have recourse to negative propositions: 'In practice, almost everything we want to say can be expressed in what is overtly a positive form. ("Yes, we are free of bananas, we are totally free of them all", for example.) Of course, this might not be true of every language. But it seems likely to be true of natural languages; which suggests that the point of negation does not lie in extending the expressive power of language.' (Price, 1990, 223) Negation, according to Price, is needed only for pragmatic reasons. But this additional thought is not one to which an incompatibility theorists need to committed. As suggested by Hobbes, there may be no expressions other than ones using negation or something similar that cover the indefinite amount of cases referred to in negative terms.

[10] My translation after (Hobbes, 2000, ch 2, §7).

These epistemological concerns exhibit once more, as did the discussion of Plato, the wide-ranging issues surrounding negation and negative truth. Explaining negation in terms of incompatibility may be advantageous in more than one area of philosophy. In the sections to follow, however, I will focus on the semantics of negation based on the metaphysics of incompatibility.

Some more examples may not come amiss to give further substance to the claim that the things there are and what their properties are incompatible with suffice to explain negative truths.

Sitting in the gymnasium is incompatible with flying. Assuming Theaetetus has followed Theodorus' invitation, his activity of sitting besides Socrates in the gymnasium suffices to guarantee the truth of 'Theaetetus does not fly'.

Concerning the hippopotamus that is not in my room, Cheyne and Pigden argue that the 'great big positive fact (or collection of facts)' *the room as it actually is* 'necessitates or makes true the proposition that there is no hippopotamus in the room.' (Cheyne and Pigden, 2006, 255) Had there been a hippo in the room, that fact would not have existed. Containing intact furniture, books on shelves and an unscathed philosopher is incompatible with a room containing a hippopotamus, and that suffices to guarantee the truth of 'There is no hippopotamus in the room'.

Veber, too, emphasises that large, positive facts are the truthmakers of negative truths. 'If the truth of Q is incompatible with the truth of P then P will entail Not-Q and thus P's truthmaker will function as Not-Q's truthmaker as well. Provided that every negative truth is entailed by some set of positive truths with positive truthmakers, negative truths can be made true by positive facts.' (Veber, 2008, 82) That neither the Great Wall of China nor a golf ball are in my coffee cup can be established by adducing a sufficient amount of positive facts. In the first case, 'that the cup has certain dimensions and that the Wall has certain dimensions are metaphysically incompatible with the Wall being contained in the cup' (Veber, 2008, 83). The dimensions of the cup and the Great Wall of China both constitute positive facts, and they suffice to guarantee the truth of 'The Great Wall of China is not in my coffee cup'. In the second case, 'truths about the distribution of air (or coffee) molecules inside the cup' are incompatible with a golf ball being in it, because golf balls are made of 'rubber or hard plastic' and that 'an air (or coffee) molecule is located in a certain place at a certain time is incompatible with a molecule of rubber or hard plastic being there' (Veber, 2008, 83). These positive facts suffice to guarantee that 'There is no golf ball in my coffee cup' is true.

I admit that I may not have established beyond reasonable doubt that negative truth can be explained in terms of a primitive notion of metaphysical incompatibility. Sceptics are invited to apply *modus tollens*, if their preferred logic is not minimal logic. But the view is well motivated and interesting. The next section takes a closer look at Demos's account of the meaning of negative propositions and a problem that a satisfactory definition of negation in terms of incompatibility must avoid.[11]

[11]For opposition to the present account of negation, see (Armstrong, 2004, 55ff), (Kürbis, 2019b, Ch 4), (Molnar, 2000, Sec II), (Taylor, 1952) and (Taylor, 1953). The last paper is a response to a paper of Ayer's on delineating negative and positive propositions. (Ayer, 1952) For the problem of negative truth to get off the ground no general account of this distinction is needed. One example of a negative truth suffices. Russell argued that Demos's view has no methodological advantage, and in fact some disadvantages, over the view that there are negative facts, and that it is circular, if the

4 Negation as Incompatibility

According to Demos, negative propositions are descriptions of positive ones. This view, however, must be rejected. Negation is a sentential operator, not a description operator. Demos explains the meaning of 'not-p is true' as 'an opposite or contrary of p is true', but he fails to explain the meaning 'not-p'. 'Not-p' cannot mean 'an opposite or contrary of p'. The latter cannot be embedded in contexts where 'not-p' can be embedded, such as 'If q then not-p': 'If q then an opposite or contrary of p' is ill-formed. The phrase 'an opposite or contrary of p' is a description of a proposition, not a proposition. Demos is guilty of confusing two uses of 'is true': It is a predicate in 'an opposite or contrary of p is true' and a sentential operator in 'not-p is true'. If we settle Demos with a use-mention confusion, he can only count as having explained the meaning of propositions of the form 'not-p' when they are *mentioned*, where he should have explained their *use*. It is the use of negation that we are after.

Demos also fails to address the problem of falsity for negative propositions. Demos writes: 'In general, in negative assertion I am referring descriptively to that proposition which is true. Thus, when I say that "John is not at home," I have reference to where, as a matter of fact, John is, that is, to the true proposition about John, and my statement is "An opposite of 'John is at home' is true," or, "The true proposition (the truth) is an opposite of 'John is at home'.".' (Demos, 1917, 193) If 'John is not at home' is true, according to Demos it describes the true proposition 'John is at the store', supposing that John is indeed at the store, or some other proposition that truly states his whereabouts. But what does 'not-p' mean when it is false? If 'not-p' is true, we can grant Demos that there is a unique true proposition that is incompatible with p and to which 'not-p' refers. But if 'not-p' is false, there is no true proposition it can refer to. Demos does not have an account of what false negative propositions mean. If the apple is green and I say falsely 'The apple is not green', I cannot have reference to which colour, as a matter of fact, the apple has, that is, to the true proposition about the apple, as the one thing I certainly to not wish to say with 'The apple is not green' is that the apple is green. And because there is, in general, no guarantee that there is only one proposition that is incompatible with p, it is not open to Demos to handle the case where 'not-p' is false by having recourse to his other characterisation of the meaning of 'not-p' as 'An opposite of p is true', as this may refer to more than one proposition. The best Demos can do would be to admit that if the apple is green, 'The apple is not green' may mean 'The apple is red', or 'The apple is blue', or 'The apple is yellow' etc., for any colour incompatible with green. But the negation of 'The apple is green' should be a unique proposition.[12]

Three desiderata for a satisfactory explanation of negation in terms of

aim is to avoid negative notions, as incompatibility is itself negative. (Russell, 1919a, 5f), (Russell, 1919b, 43ff) Ryle also argued for the existence of negative facts (Ryle, 1929). Ironically the original version of Ryle's paper got lost, and it had to be rewritten.

[12]Clearly Demos does not countenance the option that if the apple is green, 'The apple is not green' means the disjunction of the propositions incompatible with it 'The apple is green'. 'The apple is green' is a simple proposition, and anyone relegating negative propositions to the secondary status of having only indirect reference to facts, because negative facts are nowhere to be met with in experience, should treat disjunctive propositions equally: we should expect a true disjunctive proposition to refer indirectly to the fact in virtue of which it is true. Besides, the problem of falsity arises once more for false disjunctions.

incompatibility emerge. (i) The specification of the meaning of a negated proposition must be independent of whether the proposition is true or false. (ii) It must guarantee that ‘not-p’ is a unique proposition, determined by p and what it is incompatible with. (iii) Even though this is often left implicit, it must include or imply that $\neg A$ is incompatible with A.

A definition of negation one finds in the literature on incompatibility, namely that $\neg A$ if and only if for some B, B and B is incompatible with A, does not satisfy the second desideratum. Various propositions incompatible with A can play the role of the definiendum $\neg A$. Being transparent is incompatible with being opaque. So if the glass is transparent, then there is a proposition B such that B and B is incompatible with ‘The glass is opaque’. Conversely, as the glass is either opaque or transparent, if there is a proposition B such that B and B is incompatible with ‘The glass is opaque’, then the glass is transparent. Hence ‘The glass is transparent’ can take the place of ‘The glass is not opaque’ in the definition. Being purple, blue, green, yellow or orange is incompatible with being red. So if the apple is red, then there is a proposition B such that B and B is incompatible with ‘The apple is purple, blue, green, yellow or orange’. Conversely, as the apple is either purple, blue, green, yellow, orange or red, if there is a proposition B such that B and B is incompatible with ‘The apple is purple, blue, green, yellow or orange’, then the apple is red. Hence ‘The apple is red’ can take the place of ‘The apple is not purple, blue, green, yellow or orange’ in the definition. For a satisfactory explanation of negation in terms of incompatibility a few more things need to be said about why the definition of $\neg A$ defines *negation* rather than an operator that picks out some proposition or other amongst the ones incompatible with A.

5 Negation as Minimal Incompatibility

The moral to be drawn from the last section is that there must be more to a specification of the meaning of negation in terms of incompatibility than that $\neg A$ if and only if there is a B such that B and A is incompatible with B, and that $\neg A$ is incompatible with A. The negation of A is not just any incompatible of A, but a distinguished one amongst them.

Useful material for a more precise characterisation of negation has been made explicit by Brandom:

> The notion of incompatibility can be thought of as a sort of vector-product of a *negative* component and a *modal* component. It is *non*-com*possibility*. To use this semantic notion to introduce a negation operator into the object language, we must somehow isolate and express explicitly that negative component. The general semantic model we are working with represents the content expressed by a sentence by the set of sets of sentences incompatible with it. So what we are looking for is a way of computing what is incompatible with negated sentences (and, more generally, with sets of sentences containing them). [...]
>
> Incompatible sentences are Aristotelian *contraries*. A sentence and its negation are *contradictories*. What is the relation between these? Well, the contradictory is *a* contrary: any sentence is incompatible

with its negation. What distinguishes the contradictory of a sentence from all the rest of its contraries? The contradictory is the *minimal* contrary: the one that is entailed by all the rest. Thus every contrary of 'Plane figure *f* is a circle'—for instance '*f* is a triangle,' '*f* is an octagon,' and so on—entails '*f* is not a circle.' *Blue*, *green*, *yellow* all entail *not-red*. [...] *q* is the negation of *p* just in case *q* is the minimal incompatible of *p*: the one entailed by everything else incompatible with it. (Brandom, 2008, 126f)

Peacocke expresses virtually the same thoughts about negation:

> What is primitively obvious to anyone understanding negation is just that $\neg A$ is incompatible with A. [...] It takes further reflection to realize that $\neg A$ is also the weakest condition incompatible with A. [...] It is precisely because $\neg A$ is the weakest condition incompatible with A that ($\neg I$) is a valid rule [...]
>
> $$(\neg I) \qquad \frac{\begin{array}{cc} \overline{A}^{\,i} & \overline{A}^{\,i} \\ \Pi & \Sigma \\ \neg B & B \end{array}}{\neg A}\, i$$
>
> If $\neg A$ were not the weakest condition incompatible with A, then from the fact that from A (and other assumptions) one can derive incompatible propositions, one would not be able to conclude that $\neg A$. It would be a *non sequitur*, since the premises would not have excluded the possibility that something holds which is incompatible with A, but which does not imply this stronger 'negation'. (Peacocke, 1987, 163f)[13]

Peacocke's notion of weakness is equivalent to Brandom's notion of minimality. Peacocke's argument is somewhat compressed, but it can be spelled out. If $\neg A$ was not the weakest incompatible of A, then, if B and $\neg B$ have been derived from A, it may be that, rather than $\neg A$, some other proposition Φ that is incompatible with A should have to be derived and A discharged:

$$\frac{\begin{array}{cc} \overline{A}^{\,i} & \overline{A}^{\,i} \\ \Pi & \Sigma \\ \neg B & B \end{array}}{\Phi}\, i$$

If, however, it can be shown that Φ entails $\neg A$, then $\neg A$ is the weakest proposition incompatible with A, and so $\neg A$ can always be deduced in the circumstance. This is possible by appealing to a more general introduction rule for $\neg$ that Peacocke alludes to in the paragraph following the statement of ($\neg I$): If A entails incompatible propositions B_1 and B_2, then $\neg A$:

$$(\neg I^{Inc}) \qquad \frac{\begin{array}{cc} \overline{A}^{\,i} & \overline{A}^{\,i} \\ \Pi & \Sigma \\ B_1 & B_2 \end{array}}{\neg A}\, i$$

[13] Notation slightly adapted.

Ex hypothesi, Φ is incompatible with A, so by $(\neg I^{Inc})$ it entails $\neg A$:

$$\dfrac{\Phi \qquad \dfrac{}{A}\,{}^{1}}{\neg A}\,{}^{1}$$

This holds for whatever proposition Φ might be, and thus, because $(\neg I^{Inc})$ is valid, $\neg A$ is the weakest proposition incompatible with A: Any proposition incompatible with A entails $\neg A$. Hence if $\neg A$ was not the weakest proposition incompatible with A, $(\neg I^{Inc})$ would be invalid.

Peacocke argues that his account validates classical logic, as 'once a thinker has worked out that $\neg A$ is quite generally the weakest condition incompatible with A, for arbitrary A, he is in a position to infer that double negation elimination is valid. Anything incompatible with $\neg A$, i.e. something which entails $\neg\neg A$, must also entail A too—on pain of $\neg A$ not being the weakest condition incompatible with A.' (Peacocke, 1987, 164) I'll scrutinise Peacocke's argument in due course. In the next section I'll argue that, although Brandom and Peacocke are on the right path to a solution to the problem that ended the previous section, a little more is needed to solve it.

6 Negation as a Logical Constant

According to Brandom and Peacocke, for each proposition, there are propositions incompatible with it and some entailments hold between them. The negation of A is a proposition amongst the propositions incompatible with A such that each of them entails it, and there is exactly one such proposition. The negation of A is the minimal incompatible of A. Brandom and Peacocke need to guarantee that *there is* a minimal incompatible of each proposition and that there is *only one* of them. Call these the existence problem and the uniqueness problem.

Take, once more, the pair of propositions '*a* is opaque' and '*a* is transparent'. *a* is transparent if and only if *a* is not opaque, so '*a* is transparent' stands in the same inferential relations to other propositions as '*a* is not opaque', and thus the two propositions have the same inferential role. In case the discharged assumption A in Peacocke's $(\neg I)$ is '*a* is opaque', '*a* is transparent' can play the role of the conclusion $\neg A$. Nonetheless, one would probably want to say that '*a* is not opaque' and '*a* is transparent' do not mean the same. There seem to be examples of sentences not containing negation that stand in exactly the same inferential relations to other sentences as the negation of some sentence. Such sentences are as much minimal incompatibles as negations. Nonetheless we may not want to call them negations, as they do not contain the negation operator, just as we may not want to call 'The glass is opaque' and 'The glass is transparent' negations of each other. Similarly for '*a* is not red' and '*a* is purple, blue, green, yellow or orange'. Uniqueness of minimal incompatibles cannot be guaranteed as nothing in this notion rules out different propositions from standing in the relation of minimal incompatibility to a given proposition. Hence, in the general fashion in which it is stated in the first paragraph of this section, the uniqueness problem has no solution.[14]

[14] Wright also raises the issue that a proposition may have more than one minimal incompatible. (Wright, 1993, 124)

The previous paragraph may contain a problem for a crude inferentialism: there are propositions that differ in meaning, but that have the same inferential role. A more sophisticated form of inferentialism, however, can observe that 'a is not opaque' is composed from 'a is opaque' by the addition of the operator 'not', while 'a is transparent' is not composed in this way. That constitutes a salient difference between the two propositions. Similarly for 'a is not red' and 'a is purple, blue, green, yellow or orange'. The differences in the expressions that occur in these propositions account for the differences in their meanings.[15]

The negation of A is not just any minimal incompatible of A, but a minimal incompatible of A that has a certain compositional structure, that is, it is a minimal incompatible of A composed of A and another expression. That expression is the word 'not', the meaning of which we are trying to explicate. If we can take our explication to be one that may, if not actually, then at least in principle, serve to introduce the expression 'not' into a language, we have solved the existence problem. Furthermore, 'not' is a very special word: it is a logical constant. The negation of A is not merely a distinguished minimal incompatible of A, but, crucially, it is one that contains an expression that figures in logical laws. This observation gives a handle on the uniqueness problem: if we can show that any two negations of A are logically equivalent, we can speak of the unique negation of A.[16]

Brandom is explicit that at this stage of his dialectics, we are not entitled to assume that there already is a negation operator in the language, and he proposes a way of introducing one:

> It might happen that in some standard interpretation of the vocabulary to which p belongs, there already is such a q [which is the negation of p]. But we cannot count on every sentence already having such a negation in every interpretation. So we need to introduce new sentences, of the form Np, on the basis of this relation [of primitive incompatibility]. [...] What is incompatible with the negation of p is what is incompatible with *every* set of sentences incompatible with p—that is, the incompatibility set of Np is just the intersection of the incompatibility sets of *everything* incompatible with p. (Brandom, 2008, 127)

Brandom's proposal of how to introduce a negation operator into a language, however, does not work. If one of the alleged relata of a relation does not exist, the relation relates nothing, and it does not help to introduce expressions supposedly standing for the relatum. Consider the following paraphrase of some of what Brandom says here and in the passage quoted earlier: 'q is the youngest brother of p just in case q is the minimal younger brother of p: the one younger than every brother of p. We cannot count on every person already having

[15] Brandom does, I think, neglect the importance of compositionality to the issue at hand. The statement 'q is the negation of p just in case q is the minimal incompatible of p: the one entailed by everything else incompatible with it' is strictly speaking false, as the examples show.

[16] The existence of a minimal incompatible for each proposition is a strong requirement. Why should it be that for any A, there is a proposition entailed by *all* propositions incompatible with it? There are examples of propositions that are incompatible with a given proposition but that stand in no salient inferential relations to each other. Take 'a is a red square'. 'a is green' and 'a is round' are both incompatible with it. Neither entails the other. Why should there be a further proposition entailed by both? Of course once there is negation, there is one: 'a is not a red square' is entailed by both, 'a is green' and 'a is round'.

such a brother in every interpretation. So we need to introduce new names, of the form "the youngest brother of p", on the basis of this relation.' We could introduce such names, I suppose, but that wouldn't guarantee that everyone has a youngest brother. Brandom's proposal of how to introduce a negation operator does not work, because nothing guarantees that the intersection of the incompatibility sets of everything incompatible with p is not empty.

The existence problem may be less of a problem for Peacocke, if his interests lie less in defining the meaning of negation, than in characterising the understanding of a thinker who already grasps it. But Peacocke aims to justify rules of inference for negation. He characterises a speaker's or thinker's understanding of a logical constant as consisting at least partially in finding certain inferences primitively compelling. (Peacocke, 1987, 165) Thus it is plausible that he is after more than merely a blunt stipulation that the negation of A is its minimal incompatible composed of A and a negation operator that is unique up to interderivability.

Wittgenstein writes: 'The introduction of a new expedient in the symbolism of logic must always be an event full of consequences. No new symbol may be introduced in logic in brackets or in the margin – with, so to speak, an entirely innocent face.' (Wittgenstein, 1922, 5.452) We cannot simply introduce a symbol into a language and claim it is negation. The symbol must be introduced in a way that ensures it serves the demands of logic. The introduction of a logical constant into a language is a momentous affair.

The problems encountered can be addressed by appealing to a more general theory of logical constants, a theory of the kind of expressions they are and how their meanings are determined. The quotation from Peacocke hints at one such theory, stemming from Gentzen and developed by Dummett and Prawitz: The logical constants are expressions governed by rules of inference in a system of natural deduction such that their introduction rules, if they are of a certain form, determine their meanings, and their elimination rules are determined relative to them by a principle of harmony. This theory is also congenial to Brandom's outlook. But before discussing it, it is instructive to see why another important theory of the logical constants is not available to an account that aims to explicate negation in terms of incompatibility, namely to introduce logical constants, and determine their meanings, via truth tables.[17]

If we had classical truth tables at our disposal, it would be possible to establish that there is exactly one negation of A, as there is one and only one truth function that satisfies the truth table of negation:

[17]I set aside theories that specify the meanings of logical constants in terms of possible worlds or similar entities. One such account relevant to the present concerns is Berto's, who analyses negations as modal operators, where an accessibility relation on worlds is explained intuitively in terms of compatibility. (Berto (2015)) What counts as a world can be specified in a variety of ways: possible worlds, stages in constructions, incomplete situations or information states. Thus a pluralism of negations emerges. Berto ultimately aims to base his account of negation on a primitive notion of incompatibility, and so he argues that appeal to this notion validates various negations. I am interested in a robust metaphysical notion of incompatibility, which restricts the admissible interpretations of 'world' in Berto's account, and in an account favoured by metaphysicians worried about negative notions. In as much as formulas in Berto's models are evaluated in terms of two independent notions, one positive, one negative, these metaphysicians may not be put at ease by Berto's account, and the considerations in the paragraph to follow apply. Besides, as negation is supposed to be grounded in a primitive notion of incompatibility, and the worlds are ordered by a primitive notion of compatibility, one may be forgiven to wonder what the relation between those two primitive notions might be, and it is difficult to escape the conclusion that it is one of *negation*.

A	$\neg A$
T	**F**
F	**T**

The truth table of negation defines its meaning in terms of the two notions of truth and falsity. As '$\neg A$' is true if and only if 'A' is false, anyone worried about negative truth is not going to take the notion of falsity lightly, and so on the present account, falsity needs to be explained in terms of truth and incompatibility. Assuming the metalanguage to be an extension of the object language by quotation marks that turn sentences of the metalanguage into names of sentences of the object language and a truth predicate applying to such names, falsity can be explained in the following way: 'A' is false if and only if there is a proposition B, such that 'B' is true and A is incompatible with B. But from what has been said so far, it is not possible to reconstruct the truth table for negation. The first row is fine: If 'A' is true, then, as A and $\neg A$ are incompatible, there is a proposition B, i.e. A, such that 'B' is true and $\neg A$ is incompatible with B, and so, by the definition of falsity, '$\neg A$' is false. The problem is the second row: If 'A' is false, then, by the definition of falsity, there is a proposition B, such that 'B' is true and A is incompatible with B, but what has been said so far does not necessitate that $\neg A$ must be such a proposition B, i.e. that '$\neg A$' is true. This is once more the uniqueness problem: what has been said so far does not exclude the option that 'A' is false not because '$\neg A$' is true, but because there is some proposition other than $\neg A$ that is true and incompatible with A.

The previous paragraph may suffice to cast doubt on Peacocke's argument that his account of negation validates double negation elimination. If '$\neg\neg A$' is true, then, using the equivalence that 'C' is false if and only if '$\neg C$' is true, '$\neg A$' is false, and so, by the definition of falsity, there is a proposition B, such that 'B' is true and $\neg A$ is incompatible with B. But nothing in what has been said so far guarantees that A must be such a B, i.e. that 'A' is true. We will, however, only be in a position to judge this question adequately after the rule based account of the meaning of negation, the approach suggested by Peacocke himself, has been spelled out and assessed.

Dummett's and Prawitz's theory of the logical constants stipulates that the introduction and elimination rules governing a logical constant **c** define its meaning if they are *in harmony*: the grounds for asserting a formula with **c** as main operator balance the consequences of asserting it. Intuitively, we cannot take more out of A**c**B by applying one of the elimination rules of **c** than we put into it by applying one of its introduction rules. As Prawitz puts it, 'an elimination rule is, in a sense, the inverse of the corresponding introduction rule: by an application of an elimination rule one essentially only restores what had been already established if the major premise of the application had been inferred by an introduction rule.' (Prawitz, 1965, 33) According to Dummett's proof-theoretic justification of deduction, introduction rules count as self-justifying if they satisfy the following criteria: in their schematic representation, only the constant for which it is an introduction rule occurs, and it occurs only once and as the main operator of the conclusion. Thus the premises and discharged hypotheses of an introduction rule are less complex than the conclusion, and the constant itself does not occur in its premises and discharged hypotheses. The elimination rules for a constant are determined relative to its introduction rules by the principle of harmony.[18]

We establish that the rules for a logical constant **c** are in harmony by showing that steps that derive the major premise $A\mathbf{c}B$ of an application of **c** elimination by an application of **c** introduction can be removed from deductions. As an example, consider $\to$, which is governed by conditional proof and *modus ponens*:

$$(\to I) \qquad \dfrac{\begin{array}{c}\overline{A}^{\,i}\\ \Pi \\ B\end{array}}{A \to B}\,i \qquad\qquad (\to E) \qquad \dfrac{A \to B \qquad A}{B}$$

An application of $(\to I)$ that concludes with the major premise of $(\to E)$ can be removed from a deduction by replacing the steps on the left by those on the right:

$$\dfrac{\dfrac{\begin{array}{c}\overline{A}^{\,i}\\ \Pi \\ B\end{array}}{A \to B}\,i \qquad \begin{array}{c}\Sigma\\ A\end{array}}{B} \qquad \rightsquigarrow \qquad \begin{array}{c}\Sigma\\ A\\ \Pi\\ B\end{array}$$

The rules for $\to$ are in harmony. The other rules for the connectives of the positive intuitionist calculus are also in harmony.[19]

To formalise classical logic, it is sufficient to add one of the following four rules to intuitionist logic:[20]

$$\dfrac{\begin{array}{c}\overline{\neg A}^{\,i}\\ \Pi \\ \bot\end{array}}{A}\,i \qquad\qquad \dfrac{\neg\neg A}{A} \qquad\qquad \dfrac{\begin{array}{c}\overline{A}^{\,i}\\ \Pi \\ C\end{array} \qquad \begin{array}{c}\overline{\neg A}^{\,i}\\ \Xi \\ C\end{array}}{C}\,i \qquad\qquad \overline{A \vee \neg A}$$

The two rules on the left make *ex falso quodlibet* redundant, so it suffices to add them to minimal logic.

None of these four rules satisfy the criteria imposed on introduction rules: they contain constants other than negation or negation occurs twice or negation occurs in a discharged hypothesis. But as negation does not occur in the

[18]Strictly speaking, the converse of harmony should also hold, which is Dummett's notion of stability, but we need not go into the details here. The restrictions on the form of introduction rules are connected to Dummett's notion of molecularity in the theory of meaning. In a full development of proof-theoretic semantics, it may be possible to relax them, but not in a way that would interfere with anything that is to follow in this paper. For details, see (Dummett, 1993a, 223ff, 256ff) and (Dummett, 1993b, 44). For a brief overview of the rule based account of the meanings of the logical constants, see (Kürbis, 2015a). For doubt whether Dummett and Prawitz provide a satisfactory account of the meaning of negation, see (Kürbis, 2015b).

[19]For details, see (Prawitz, 1965, 35ff, 49ff). What about $\bot$? It is sometimes said that *ex falso quodlibet* is harmonious with the empty introduction rule. This is debatable. Besides, my aim here is to investigate the prospects of a definition of negation in terms of a primitive notion of metaphysical incompatibility found in the literature on negative truth, and to this questions concerning $\bot$ are tangential. There may be a *different*, i.e. intuitionist, approach to negation that appeals to $\bot$, but that is not my issue.

[20]Gentzen initially opts for the fourth (Gentzen, 1934, 190), later for the second (Gentzen, 1936, 515), Prawitz for the first (Prawitz, 1965, 20), and Tennant for the third (Tennant, 1978). There are axioms and rules not involving $\neg$ or $\bot$ that have the same effect, such as $\vdash A \vee (A \to B)$, $A \to (B \vee C) \vdash (A \to B) \vee C$, or from $\Gamma, A \to B \vdash A$, infer $\Gamma \vdash A$.

conclusion, these are not introduction rules anyway. Treating them as elimination rules, however, reveals that they are not in harmony with the introduction rules for negation, be they derived or primitive. One example suffices: in classical deductions, there may be formulas of the form $\neg\neg A$ that have been derived by an introduction rule and that are premises of double negation elimination, but that cannot be removed from the deduction. In general, whatever the rules for classical negation, they will upset the harmony of the system as a whole: classical logic permits the deduction of more consequences from $\neg\neg A$ than are justified by the grounds for asserting it. The rules for classical negation are not in harmony.[21]

This, too, may be seen to casts doubt on Peacocke's argument that his account of the meaning of negation validates double negation elimination. However, in the four rules for classical negation no appeal is made to the notion of incompatibility. Hence Peacocke does, potentially, command the resources for an argument in favour of classical logic. This question will be assessed in Section 9.

A more serious issue at this point is that Peacocke's ($\neg I$) cannot define the meaning of negation, as negation occurs in one of its premises. A rule that does satisfy the restrictions on introduction rules, however, is the rule ($\neg I^{Inc}$) that was appealed to in Section 5 in the reconstruction of Peacocke's argument that $\neg A$ is minimally incompatible with A. This is a first step to solving the existence problem for Brandom and Peacocke: negation is the logical constant governed by the self-justifying introduction rule that licenses the derivation of $\neg A$ if metaphysically incompatible propositions have been deduced from A.[22] This is more or less what Tennant proposes, who also adopts a largely Dummettian framework. Tennant provides a concrete example of a theory that aims to explicate the meaning of negation in terms of rules of inference governing it and a primitive notion of metaphysical incompatibility. The next section is dedicated to an assessment of how well it fares. I will argue that, whereas Brandom and Peacocke need a solution to the existence problem, Tennant's account suffers from the complementary short-coming and needs a solution to the uniqueness problem. I will argue that both accounts can be fruitfully amalgamated and the result provides a satisfactory account of the meaning of negation in terms of a primitive notion of metaphysical incompatibility.

7 How to Eliminate Negation?

Tennant proposes 'a rule based account of negation.' (Tennant, 1999, 199) Its meaning is specified relative to 'metaphysico-semantical fact[s] of absurdity' (Tennant, 1999, 202), or incompatibility, such as '*a* is red and *a* is green', '*a* is here and *a* is over there simultaneously', '*e* is both earlier and later than *f*', 'You and I are the same person'. Tennant generalises the notion of incompatibility to allow not only two, but any finite number of propositions to stand in that

[21] For further details, see (Dummett, 1993a, 296ff) and (Prawitz, 1965, 34f).

[22] Going back to the example of opaqueness and transparency: '*a* is not opaque' contains the logical constant 'not', an expression that may be introduced into the language and its meaning determined in a specific way, namely by rules of inference alone, while '*a* is transparent' does not. This not an objection to inferentialism: the concept of transparency is not introduced by rules of inference alone, but in relation to transparent objects.

relation. Having arrived at incompatible propositions $p_1 \ldots p_n$ in a deduction, we write $\bot$ to mark the event:

$$(\bot^T) \qquad \frac{p_1 \qquad \ldots \qquad p_n}{\bot}$$

The incompatibility of $p_1 \ldots p_n$ 'arises by virtue of what the sentences mean and *various ways that we understand the world simply cannot be*' (Tennant, 1999, 217). Speakers of a language grasp primitively that certain atomic propositions are incompatible with each other.

According to Tennant, $\bot$ is not an absurd proposition, but a 'structural punctuation marker' (Tennant, 1999, 199). As such it cannot be a subformula, and so negation cannot be defined as $A \to \bot$. Tennant suggests that $\bot$ marks in natural deduction what the empty succedent is used for in a sequent calculus. Instead of $\bot$, we could write nothing at all and leave an empty space. (Tennant, 1999, 205f) $\bot$ is not governed by rules (Tennant, 1999, 216), in particular not by rules that impart a specific meaning on it, such as *ex falso quodlibet*. Tennant's preferred method is to eliminate $\bot$ from natural deduction by formulating rules for how to operate with empty spaces, but he allows its use as a matter of convenience.[23]

Tennant appeals to $\bot$ in the rules for $\neg$:

$$(\neg I^T) \qquad \frac{\begin{array}{c}\overline{A}^{\,i} \\ \Xi \\ \bot\end{array}}{\neg A}\,i \qquad\qquad (\neg E^T) \qquad \frac{\neg A \qquad A}{\bot}$$

$(\neg I^T)$ licenses the deduction of $\neg A$ if A, possibly together with other assumptions, entails an absurdity. $(\neg E^T)$ is determined from $(\neg I^T)$ by the principle of harmony. Thus A and $\neg A$ are incompatible.

If $p_1 \ldots p_n$ are incompatible, $\neg(p_1 \wedge \ldots \wedge p_n)$ is a theorem of Tennant's system. This is not true for all interpretations of $p_1 \ldots p_n$. Tennant's logic is part of an interpreted language, not a purely formal calculus. He accepts that his specification of the meaning of negation is dependent on the presence of primitively incompatible propositions in the language. According to Tennant it is essential to a language that it contains such propositions, as the contrast between the truth conditions of propositions that cannot be true together is essential to the learnability of language. The provability of $\neg(p_1 \wedge \ldots \wedge p_n)$ encapsulates a deep feature of the language. Given the roots of incompatibility in metaphysics, it is not surprising that Tennant is explicit that his logic is part of an interpreted language. Incompatibility is not a formal notion, but connected to the specific contents of atomic propositions. Metaphysics enters the foundations of logic.[24]

A logic for an interpreted language may contain rules for the derivation of atomic formulas from other atomic formulas. Dummett calls these *boundary*

[23]Notice, however, that the explanation of the validity of a sequent $\Gamma \Rightarrow \Sigma$, and the ensuing use of empty succedents, cannot be appealed to in the present context: if all of Γ are true, then not all of Σ are not true, so that if Σ is empty, one of Γ is not true. The question we are trying to answer is what it means that something is not true: The problem of negative truth is also the question of how something can fail to be true.

[24]Tennant's preferred logic is an intuitionist relevant logic. It is in some respects weaker than minimal logic, in others it is stronger. Its consequence relation is not transitive. As it is rather idiosyncratic, I set it aside.

rules. (Dummett, 1993a, 254ff) For instance, some free logics contain a one-place predicate E, interpreted as *exists*, governed by the boundary rule that Et_i may be inferred from the atomic formula $Rt_1 \dots t_n$, for $1 \leq i \leq n$. A language containing words for colours and bodies may have the boundary rule that 'a is extended' follows from 'a is red'. Arithmetic contains the boundary rule that $a = b$ follows from 'The successor of a = the successor of b'. Geometry contains boundary rules for the inferential relations between atomic propositions about points and lines.

The status of $\bot$ in Tennant's account deserves closer scrutiny. It is due to how $\bot$ figures in $(\neg I^T)$ that $\neg A$ is a proposition that can be derived if A entails incompatible propositions.[25] It is due to how $\bot$ figures in $(\neg E^T)$ that A and $\neg A$ count as incompatible propositions. For $(\neg I^T)$ to be applicable, Tennant must specify some way in which to arrive at $\bot$ in a deduction. One such way is given by $(\neg E^T)$. If we could assume the negations of propositions of the language to be given, this rule would suffice. Tennant, however, aims to specify the meaning of negation on the basis of $\bot$. Thus there must be other ways of arriving at $\bot$ in a deduction. Those ways are given by $(\bot^T)$.

If it only had $(\neg I^T)$ and $(\neg E^T)$, Tennant's account would be circular: $\bot$ could only be understood in terms of $\neg$, $(\neg E^T)$ being the only way that specifies how to arrive at $\bot$ in a deduction, and $\neg$ could only be understood in terms of $\bot$, its introduction rule being $(\neg I^T)$. $(\bot^T)$ breaks the circle: it gives an independent way of arriving at $\bot$ in a deduction. But it is precisely this fact that shows that it is not true that there are no rules for $\bot$.[26]

For all intents and purposes, $(\bot^T)$ is an introduction rule for $\bot$. $\bot$ follows from any collection of incompatible atomic propositions of the language. $\bot$ is effectively the disjunction of their conjunctions. Let $p^1_1 \dots p^1_{n_1} \dots p^m_1 \dots p^m_{n_m}$ be all the collections of primitively incompatible atomic propositions. Then the elimination rule for $\bot$ harmonious with $(\bot^T)$ is:

$$(\bot E^T) \qquad \dfrac{\bot \qquad \begin{array}{c}\underbrace{\overline{p^1_1}^{\,i} \quad \overline{p^1_{n_1}}^{\,i}} \\ \Pi_1 \\ C\end{array} \qquad \dots \qquad \begin{array}{c}\underbrace{\overline{p^m_1}^{\,i} \quad \overline{p^m_{n_m}}^{\,i}} \\ \Pi_m \\ C\end{array}}{C}\, i$$

This rule requires rather demanding conditions for its application. In order to deduce something from $\bot$, it requires that a proposition be derived from all the collections of incompatible atomic propositions of a language. C would have to be derived from propositions about shape, from propositions about colour, from propositions about time, from propositions about space, etc., for all the domains of discourse in which there are primitively incompatible atomic propositions. Unless C is a very complex disjunction, it is hard to see how such a feat should be performed. But it would be odd to claim that only very complex disjunctions follow from $\bot$. One would expect some atomic propositions to follow from $\bot$, for instance those that can be the premises in $(\bot^T)$.

The meaning of $\bot$ as characterised by $(\bot^T)$ and $(\bot E^T)$ varies with the incompatible atomic propositions present in a language. So, then, does the

[25] In Tennant's intuitionist relevant logic, it also figures in the elimination rules for $\wedge$, $\vee$ and $\supset$, each of which specifies a way in which to proceed when $\bot$ has been reached.

[26] Contrast Tennant's system with minimal logic: in minimal logic, there really are no rules for $\bot$.

meaning of negation defined in terms of it. Tennant can claim that the meaning of negation is fixed for any given language for which the incompatible atomic propositions are fixed. But he cannot claim that the meaning of negation is the same across languages, unless those languages contain the same incompatible atomic propositions. Notice also that in as much as languages can be extended, there will be new ways of introducing $\perp$ and $(\perp E^T)$ receives an open ended character: as new incompatible atomic propositions are added to a language, new side deductions of C are required for its application.

This is unsatisfactory. Negation, being a logical constant, has a meaning that is invariant and independent of what expressions a language contains. From the logical perspective, no unique meaning is attached to the symbol $\neg$ in Tennant's account. Tennant needs a solution to the uniqueness problem.

Despite the rules for $\perp$, it is formally inert in the sense that in a purely formal calculus, rather than one for an interpreted language, the presence of $\perp$ does not allow the deduction of any new theorems. The appeal to $\perp$ can be eschewed from Tennant's system. Instead of using $\perp$ in the negation rules, we can literally write nothing at all, not even an empty space, and, instead of recording that we've reached incompatible propositions in a separate step, proceed directly to discharging an assumption and deriving its negation. Doing so brings out more starkly the resources that Tennant would require for his account of negation to be adequate.

We can adjust $(\neg I^{Inc})$ by generalising it to allow any number of incompatible propositions $p_1 \dots p_n$ to occur as premises:

$$(\neg I^{IncG}) \qquad \dfrac{\begin{matrix} \overline{A}^{\,i} \\ \Pi_1 \\ p_1 \end{matrix} \qquad \dots \qquad \begin{matrix} \overline{A}^{\,i} \\ \Pi_n \\ p_n \end{matrix}}{\neg A}\, i$$

It has a harmonious elimination rule in the style of $(\perp E^T)$, where $p_1^1 \dots p_{n_1}^1 \dots p_1^m \dots p_{n_m}^m$ are all the collections of primitively incompatible propositions of the language:

$$\dfrac{\neg A \qquad A \qquad \begin{matrix} \underbrace{\overline{p_1^1}^{\,i} \quad \overline{p_{n_1}^1}^{\,i}} \\ \Pi_1 \\ C \end{matrix} \qquad \dots \qquad \begin{matrix} \underbrace{\overline{p_1^m}^{\,i} \quad \overline{p_{n_m}^m}^{\,i}} \\ \Pi_m \\ C \end{matrix}}{C}\, i$$

This rule, like $(\perp E^T)$, requires conditions rather too demanding for its application.

Another popular elimination rule for negation is *ex contradictione quodlibet*:

$$\dfrac{\neg A \qquad A}{r}$$

This rule, however, even restricted to atomic conclusions, is too general. It allows us to derive *too much* from $\neg A$ given what is put into it by an application of $(\neg I^{IncG})$, namely *any* atomic proposition, whereas $(\neg I^{IncG})$ demands only a restricted number of specific incompatible ones. Steps introducing and eliminating $\neg A$ cannot be removed from a deduction:

$$\dfrac{\dfrac{\begin{matrix}\overline{A}^{\,i} \\ \Pi_1 \\ p_1\end{matrix} \qquad \cdots \qquad \begin{matrix}\overline{A}^{\,i} \\ \Pi_n \\ p_n\end{matrix}}{\neg A}\,i \qquad \begin{matrix}\Sigma \\ A\end{matrix}}{r}$$

There is no guarantee that we can replace r for one of the p_i that are the premises of ($\neg I^{IncG}$). The deduction Π_i of which p_i is the conclusion may contain applications of boundary rules, and there may be no boundary rules that allow the derivation of r from any of the propositions occurring in Π_i.

The disharmony could be remedied by adding a special boundary rule *ex adversis propositionibus quodlibet* that allows the derivation of arbitrary atomic propositions from incompatible ones:

$$\dfrac{p_1 \qquad \cdots \qquad p_n}{r}$$

But this rule is *ad hoc*. Boundary rules connect atomic propositions within a region of discourse, not arbitrary ones. Atomic propositions about shape or number do not entail atomic propositions about colour.[27]

The consequences of an assertion of $\neg A$ stipulated by *ex contradictione quodlibet* do not match the grounds of its assertion as specified by ($\neg I^{IncG}$). Harmony requires a weaker negation elimination rule. We should only be able to derive from $\neg A$ the incompatible propositions required to derive it. Such a rule is difficult to characterise formally, but the required resources may be found in a theory of meaning.

Dummett argues that a theory of meaning specifies, for each proposition, a set of canonical grounds for its assertion. The canonical grounds for an assertion of $A \land B$, for instance, are the canonical grounds for an assertion of A and the canonical grounds for an assertion of B combined. The canonical grounds for an assertion of $A \to B$ is an argument for B on the assumption A. The canonical grounds for an assertion of $A \lor B$ are the canonical grounds for an assertion of A or for an assertion of B. There are also grounds for asserting a proposition A that are not canonical, which we may call the indirect grounds for an assertion of A. The indirect grounds of A derive from its canonical grounds. It is another aspect of harmony that any assertion of A based on indirect grounds can be turned into an assertion of A based on its canonical grounds. Following Brandom, the meaning of a proposition is tied to propositions it is incompatible with. Then, just as every proposition of a language is associated with canonical grounds for its assertion, maybe we can make good the claim that every proposition is also associated with what we can call its canonical incompatibles for applications of ($\neg I^{IncG}$): collections of propositions such that an argument for them on the assumption that A constitutes the canonical grounds for an assertion of $\neg A$. Then the elimination rule harmonious with ($\neg I^{IncG}$) is that from A and $\neg A$ we can derive A's canonical incompatibilities for ($\neg I^{IncG}$). Then a proposition derived by ($\neg I^{IncG}$) and eliminated by that rule can be removed from a deduction. Transform the deduction concluding with the premises of ($\neg I^{IncG}$) into one for canonical incompatibilities for ($\neg I^{IncG}$) of A (which *ex hypothesi* exists). Amongst them are the canonical incompatibles derived by the application of the elimination for $\neg$.

[27] It is for this reason that I do not take into account Brandom's notion of incompatibility entailment: it is a trivial consequence of its definition that incompatible propositions entail everything.

Take the deduction with it as conclusion and append to its open assumption A (if there is one) the argument for A that leads to the minor premise of the elimination rule of $\neg$. The result is a deduction of a canonical incompatible for $(\neg I^{IncG})$ of A that does not take the detour through $\neg A$ from the same (or fewer) assumptions as the original deduction, which is what we wanted.

This sounds like a tough call. It gets even tougher if we take into account that presumably propositions may be associated with various canonical incompatibilities for $(\neg I^{IncG})$. It should make no difference whether you derive '*a* is red' and '*a* is green' or '*a* is blue' and '*a* is yellow' from A to have canonical grounds for asserting $\neg A$. An argument for 'Fred is green' and 'Fred is red' and an argument for 'Fred is round' and 'Fred is square' from some assumption A (say about the colour and shape of Fred) provide equally good grounds for the conclusion that $\neg A$. But there is little reason to assume that each can be transformed into the other, as we may assume that the boundary rules employed in one argument relate to colours, those employed in the other to shapes. However, for there to be a unique negation, i.e. for it to be possible to show that all negations of A derived by $(\neg I^{IncG})$ from various incompatibilities are interderivable, it would be required that all the collections of canonical incompatibilities for $(\neg I^{IncG})$ of a proposition A should be equivalent in the sense that a deduction of one such set from A can be transformed into a deduction of any other such set from A. This is such a strong demand that we are justified in looking elsewhere for a suitable elimination rule for negation.

8 A Solution to the Uniqueness and the Existence Problems

A rule based account of the logical constants, such as Tennant's, solves the existence problem. $(\neg I^{IncG})$ counts as a self-justifying introduction rule: it introduces the logical constant negation on the basis of the independently given notion of incompatibility between atomic propositions and specifies under which conditions the negation of a proposition may be deduced in a way that does not rely on a prior notion of negation. Tennant, however, needs a suitable elimination rule for negation that ensures its uniqueness.

To formulate an elimination rule that is harmonious with $(\neg I^{IncG})$, I propose to go back to Brandom and Peacocke. Tennant does not make use of their observation that the negation of A is a minimal incompatible of A. Appeal to this notion solves the issue, as I will argue now.

The introduction rules for a logical constant specify the conditions under which a proposition with that constant as main operator may be deduced. For it to count as an introduction rule that specifies the meaning of the constant, the constant must not occur in the premises or discharged hypotheses of the rule.

The elimination rules for a logical constant specify the conditions under which a proposition can be derived from a proposition with that constant as main operator. Although the logical constant must not occur in the premises and discharged hypotheses of an introduction rule, if that rule is to count as specifying the meaning of the constant, there is no complementary injunction on elimination rules, that is, no injunction that the constant must not occur in the conclusion of the elimination rule. The only condition imposed on the elimination rules is that they are harmonious with the introduction rules. That the introduction and elimination rules for a logical constant are in harmony is established by showing that a formula with the constant as main operator that is concluded by one of its introduction rules and is the major premise of one of its elimination rules can be removed from deductions.

The introduction rule for negation ($\neg I^{IncG}$) specifies that the canonical grounds for asserting $\neg A$ are that A entails incompatible propositions. What is a corresponding elimination rule? We need to specify what follows from $\neg A$ in a way that is harmonious with the introduction rule. $\neg A$ is the minimal incompatible of A. Thus A and $\neg A$ can play the role of the premises of ($\neg I^{IncG}$), so that if a proposition B entails them both, we can derive $\neg B$. This, then, specifies what follows from $\neg A$. In fact, it justifies Peacocke's ($\neg I$) as the elimination rule for negation. So let's rename it ($\neg E$).

To show that ($\neg E$) is harmonious with ($\neg I^{IncG}$), it suffices to observe that an application of ($\neg I^{IncG}$) followed by ($\neg E$) can be removed by transforming the deduction on the left into the deduction on the right:

$$\dfrac{\dfrac{\begin{matrix}\underbrace{\overline{A}^{\,1}\ \ \overline{B}^{\,2}} \\ \Pi_1 \\ p_1\end{matrix} \quad \cdots \quad \begin{matrix}\underbrace{\overline{A}^{\,1}\ \ \overline{B}^{\,2}} \\ \Pi_n \\ p_n\end{matrix}}{\neg A}\,{\scriptstyle 1} \qquad \begin{matrix}\overline{B}^{\,2} \\ \Sigma \\ A\end{matrix}}{\neg B}\,{\scriptstyle 2} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\underbrace{\begin{matrix}\overline{B}^{\,2} \\ \Sigma \\ A\end{matrix}\ \ \overline{B}^{\,2}} \\ \Pi_1 \\ p_1\end{matrix} \quad \cdots \quad \begin{matrix}\underbrace{\begin{matrix}\overline{B}^{\,2} \\ \Sigma \\ A\end{matrix}\ \ \overline{B}^{\,2}} \\ \Pi_n \\ p_n\end{matrix}}{\neg B}\,{\scriptstyle 2}$$

Any worries that ($\neg E$) takes on some of the characteristics of an introduction rule can be assuaged by observing that two consecutive applications of ($\neg E$) can be removed from deductions:

$$\dfrac{\dfrac{\begin{matrix}\underbrace{\overline{A}^{\,1}\ \ \overline{B}^{\,2}} \\ \Pi_1 \\ \neg C\end{matrix} \qquad \begin{matrix}\underbrace{\overline{A}^{\,1}\ \ \overline{B}^{\,2}} \\ \Pi_2 \\ C\end{matrix}}{\neg A}\,{\scriptstyle 1} \qquad \begin{matrix}\overline{B}^{\,2} \\ \Sigma \\ A\end{matrix}}{\neg B}\,{\scriptstyle 2} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\underbrace{\begin{matrix}\overline{B}^{\,2} \\ \Sigma \\ A\end{matrix}\ \ \overline{B}^{\,2}} \\ \Pi_1 \\ \neg C\end{matrix} \qquad \begin{matrix}\underbrace{\begin{matrix}\overline{B}^{\,2} \\ \Sigma \\ A\end{matrix}\ \ \overline{B}^{\,2}} \\ \Pi_2 \\ C\end{matrix}}{\neg B}\,{\scriptstyle 2}$$

Furthermore, any two negation operators governed by ($\neg I^{IncG}$) and ($\neg E$)are interderivable, and so the negation defined by these rules is unique.[28]

We have solved Brandom's, Peacocke's and Tennant's problems by combining their views. However, adding ($\neg I^{IncG}$) and ($\neg E$) to the positive intuitionist calculus only yields minimal logic.

Peacocke claims that his account validates double negation elimination. We must now scrutinise Peacocke's argument and see if we can get anything stronger out of incompatibility than we have been able to retrieve so far. If Peacocke is right, *ex contradictione quodlibet* is easily derived, as, using vacuous discharge, by ($\neg E$) contradictions entail every negation, so by double negation elimination, they entail everything.

[28]($\neg I^{IncG}$) and ($\neg E$) are not stable, but this does not matter: the meaning of negation is not specified purely by rules of inference governing it, but by rules that appeal to the metaphysical notion of incompatibility.

9 Peacocke's Argument for Classical Logic

Peacocke's argument that double negation elimination is validated by his account of negation is, in Wright's words, terse. Wright reconstructs the argument, where (IC) is the principle that the negation of a proposition is a weakest proposition incompatible with it:

> Write $\mathbf{W}(A, B)$ for: B is a weakest statement incompatible with A.
>
> Plainly, if $\mathbf{W}(A, B)$, then any statement which entails that A is not true, and thereby via (IC) entails $\neg A$, entails B. So in particular, $\neg A$ itself entails B. Now suppose the relation $\mathbf{W}(A, B)$ is symmetric. Then whenever $\mathbf{W}(A, B)$ holds, so does $\mathbf{W}(B, A)$, and hence, by the reflection just mooted, $\neg B$ entails A. Since (IC) ensures that $\mathbf{W}(A, \neg A)$, it follows—taking $\neg A$ for "B"—that $\neg\neg A$ entails A. (Wright, 1993, 128)

We can make the argument even less terse and more perspicuous by formalising it. I restrict consideration, as does Peacocke, to the binary case. I write $A \dagger B$ for 'A is incompatible with B'. $\dagger$ has the following rules:

$$(\neg I^{\dagger})\ \frac{B \dagger C \qquad \begin{matrix}\overline{A}^{\,i}\\ \Pi\\ B\end{matrix} \qquad \begin{matrix}\overline{A}^{\,i}\\ \Sigma\\ C\end{matrix}}{\neg A}\,i \qquad\qquad (\neg E^{\dagger})\ \frac{}{A \dagger \neg A} \qquad\qquad (S^{\dagger})\ \frac{A \dagger B}{B \dagger A}$$

$(\neg I^{\dagger})$ is $(\neg I^{Inc})$ with the statement of the incompatibility of the propositions derived from A added as a premise. Wright's principle (IC) is easily derived:

$$\frac{A \dagger C \qquad \overline{A}^{\,1} \qquad C}{\neg A}\,1$$

The incompatibility of A and $\neg A$ is encapsulated by $(E^{\dagger})$. $(\neg E)$, the rule Peacocke calls $(\neg I)$, is derivable from $(\neg I^{\dagger})$ and $(\neg E^{\dagger})$. The symmetry of incompatibility, or the more general claim that it is irrelevant in which order incompatible propositions are listed, is left implicit in $(\neg I^{Inc})$ and $(\neg I^{IncG})$, as the premises can be arranged in any order. The binary case is captured by $(S^{\dagger})$. $(\neg I^{\dagger})$ is effectively an elimination rule for $\dagger$. There is no introduction rule for $\dagger$, and for the fundamental case there cannot be one: whether two propositions are primitively incompatible is immediate from their content and not derived from other propositions.

Next we formulate principles for Wright's $\mathbf{W}$. If $\mathbf{W}(A, B)$, then A and B are incompatible, and if $\mathbf{W}(A, B)$ and A is incompatible with C, then C entails B:

$$(\mathbf{W}E^{1})\ \frac{\mathbf{W}(A, B)}{A \dagger B} \qquad\qquad (\mathbf{W}E^{2})\ \frac{\mathbf{W}(A, B) \qquad A \dagger C \qquad C}{B}$$

B is a weakest incompatible of A if it is incompatible with A and entailed by any proposition incompatible with A:

$$(\mathbf{W}I)\ \frac{A \dagger B \qquad \begin{matrix}\underbrace{\overline{A \dagger C}^{\,i} \quad \overline{C}^{\,i}}\\ \Pi\\ B\end{matrix}}{\mathbf{W}(A, B)}\,i$$

where C is parametric, i.e. it does not occur in B nor in any assumption it depends on other than $A \dagger C$ and C. Using propositional quantification, $\mathbf{W}(A, B)$ is equivalent to $A \dagger B \wedge \forall q[(A \dagger q) \rightarrow (q \rightarrow B)]$.[29]

We derive $\mathbf{W}(A, \neg A)$, i.e. $\neg A$ is a minimal incompatible of A:

$$\dfrac{\dfrac{}{A \dagger \neg A} \qquad \dfrac{\dfrac{}{A \dagger C}\,{\scriptstyle 2} \quad \dfrac{}{A}\,{\scriptstyle 1} \quad \dfrac{}{C}\,{\scriptstyle 2}}{\neg A}\,{\scriptstyle 1}}{\mathbf{W}(A, \neg A)}\,{\scriptstyle 2}$$

So far, although $\dagger$ is symmetric, $\mathbf{W}$ is not. Interpret $A \dagger B$ as $\neg(A \wedge B)$ and $\mathbf{W}(A, B)$ as $\neg(A \wedge B) \wedge \forall q[\neg(A \wedge q) \rightarrow (q \rightarrow B)]$ in intuitionist logic with propositional quantification. Then all the rules of inference for $\dagger$ and $\mathbf{W}$ are valid, but, as $\forall q[\neg(A \wedge q) \rightarrow (q \rightarrow B)]$ is not equivalent to $\forall q[\neg(B \wedge q) \rightarrow (q \rightarrow A)]$, $\mathbf{W}(A, B)$ and $\mathbf{W}(B, A)$ do not entail each other.

The problem with Peacocke's argument for double negation elimination, as Wright points out, lies with the assumption that $\mathbf{W}$ is symmetric. If $\mathbf{W}$ is assumed to be symmetric, we can derive double negation elimination:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \dagger \neg A} \qquad \dfrac{\dfrac{}{A \dagger C}\,{\scriptstyle 2} \quad \dfrac{}{A}\,{\scriptstyle 1} \quad \dfrac{}{C}\,{\scriptstyle 2}}{\neg A}\,{\scriptstyle 1}}{\mathbf{W}(A, \neg A)}\,{\scriptstyle 2}}{\mathbf{W}(\neg A, A)} \qquad \dfrac{}{\neg A \dagger \neg\neg A} \qquad \neg\neg A}{A}$$

That incompatibility is symmetric can be accepted as uncontroversial. It is not the notion of incompatibility that is at issue. Neither is it at issue what 'minimal' means. What is at issue is whether combination of the two notions that we are interested in is symmetric. Peacocke has given no reason that it is.

Peacocke does not appeal explicitly to the symmetry of minimal incompatibility. He appeals to the claim that A is a minimal incompatible of $\neg A$. On this assumption, the rule more immediately justified than the symmetry of $\mathbf{W}$ is a rule complementary to $(\neg I^{\dagger})$:

$$(C^{\dagger}) \qquad \dfrac{B \dagger C \qquad \begin{matrix} \dfrac{}{\neg A}\,{\scriptstyle i} \\ \Pi \\ B \end{matrix} \qquad \begin{matrix} \dfrac{}{\neg A}\,{\scriptstyle i} \\ \Sigma \\ C \end{matrix}}{A}\,{\scriptstyle i}$$

Together with the rules for $\mathbf{W}$ and $(S^{\dagger})$, it entails the symmetry of $\mathbf{W}$:

$$\dfrac{\dfrac{\dfrac{\mathbf{W}(A, B)}{A \dagger B}}{B \dagger A} \qquad \dfrac{\dfrac{}{B \dagger C}\,{\scriptstyle 2} \qquad \dfrac{\mathbf{W}(A, B) \quad \dfrac{}{A \dagger \neg A} \quad \dfrac{}{\neg A}\,{\scriptstyle 1}}{B} \qquad \dfrac{}{C}\,{\scriptstyle 2}}{A}\,{\scriptstyle 1}}{\mathbf{W}(B, A)}\,{\scriptstyle 2}$$

[29] By an application of ($\mathbf{W}I$) with vacuous discharge, if B is true and incompatible with A, then it is a weakest incompatible of A. We might try to formulate further restrictions on the application of the rule to avoid this consequence. But with the resources at hand it is not evident what this could be, if we do not want to tamper with the consequence relation, as Tennant would advocate. As it stands, ($\mathbf{W}I$) is another reminder that requiring $\neg A$ to be the minimal incompatible of A (rather than the minimal incompatible composed of A and the unique negation) may be asking too much.

We should treat ($\neg I^{\dagger}$) and ($C^{\dagger}$) the same. Just as ($\neg I^{\dagger}$) is instrumental in establishing that $\neg A$ is a minimal incompatible of A, ($C^{\dagger}$) is instrumental in establishing Peacocke's assumption that A is a minimal incompatible of $\neg A$. The symmetry of **W** is a consequence of ($C^{\dagger}$), not the other way round. Eschewing use of † and moving to the general case, the rule is

$$\dfrac{\begin{matrix}\overline{\neg A}^{\,i} \\ \Pi_1 \\ p_1\end{matrix} \qquad \cdots \qquad \begin{matrix}\overline{\neg A}^{\,i} \\ \Pi_n \\ p_n\end{matrix}}{A}\,i$$

where $p_1 \ldots p_n$ are incompatible, or, reintroducing $\bot$:

$$\dfrac{\begin{matrix}\overline{\neg A}^{\,i} \\ \Pi \\ \bot\end{matrix}}{A}\,i$$

These two rules can play no role in an explanation of negation in terms of a primitive notion of incompatibility: they appeal to negation in the discharged premises and create a circular dependence of meaning. Thus the same is the case for ($C^{\dagger}$). We might as well take negation as primitive.

I conclude that Peacocke's argument for classical logic is fallacious, and hence we are left with minimal logic as the logic of minimal incompatibles.[30]

10 Minimal Logic as the Logic of Minimal Incompatibilty

The current investigation has lead to the conclusion that the explication of the meaning of negation in terms of a primitive notion of metaphysical incompatibility is best carried out in within a rule-based framework for the logical constants. The harmonious pair of introduction and elimination rules ($\neg I^{IncG}$) and ($\neg E$) define the meaning of negation in terms of an independently given notion of metaphysical incompatibility. ($\neg I^{IncG}$) is a self-justifying introduction rule for negation. Appealing to the incompatibility of A and $\neg A$, the elimination rule harmonious with ($\neg I^{IncG}$) is ($\neg E$). The two rules guarantee the existence and uniqueness of the negation of every proposition. The negation of A is the minimal incompatible of A that is composed of A and the operator defined by these two rules.

From the purely formal perspective, ($\neg I^{IncG}$) and ($\neg E$) do not endow negation with any properties that go beyond those of the operator $\neg$ in minimal logic. But as the formal rules are backed up by a substantial metaphysics, we can answer the critics of minimal logic.

It is true that formally, any propositions can play the role of $p_1 \ldots p_n$ of ($\neg I^{IncG}$). This is the reason why not many philosophers seem to think that minimal logic is the correct logic. Its negation is rather too weak. Of course

[30]Notice, incidentally, that on the present account, $\neg\neg A$ means that A's implying incompatibilities implies incompatibilities. It is questionable that that should imply A, especially when considering that the incompatibilities that A implies might be from an area of discourse different from the incompatibilities implied by A's implying incompatibilities.

mere unacceptability according to idiosyncratic standards is too weak a notion to serve in the definition of a respectable negation. The present account, however, turns to metaphysics to provide an extra-formal criterion for excluding merely idiosyncratically unacceptable propositions from playing the role of the premises $p_1 \ldots p_n$ of $(\neg I^{IncG})$. They must be metaphysically incompatible. Their conjunction is false as a matter of metaphysical necessity and not merely idiosyncratically unacceptable. Metaphysics provides an account of what the former have that the latter lack. The appeal to metaphysics ensures that what is formally merely minimal logic has a proper negation after all, as it provides informal criteria that restrict the range of propositions that can play the role of the premises $p_1 \ldots p_n$ of $(\neg I^{IncG})$. Metaphysics lets us draw a principled distinction between the kind of propositions required in a satisfactory definition of the meaning of negation and mere idiosyncratic unacceptability. The content of negation is not given by rules alone, but also by metaphysics.

Metaphysics also provides independent reasons for the view that there are primitive incompatibilities. The explanation of negation in terms of incompatibility is well motivated as a solution to the problem of negative truth. It avoids an ontology of negative facts, absences, or otherwise shadowy entities, and respects the intuition that everything that exists is positive, while allowing that there are negative truths.

The argument that establishes $(\neg I^{IncG})$ and $(\neg E)$ as a pair of harmonious rules for negation, a pair that provides a satisfactory definition of the meaning of negation in terms of a primitive notion of metaphysical incompatibility, ensuring the existence and uniqueness of the negation of every proposition, does not establish any more than those two rules, and they only give negation the formal force it has in minimal logic. In particular, no argument for *ex contradictione quodlibet* is forthcoming, as that principle would allow the derivation of *more* propositions from $\neg A$ than are justified by the canonical grounds for asserting $\neg A$ as specified by $(\neg I^{IncG})$. No argument for double negation elimination is forthcoming either.

We are left with the unexpected conclusion that minimal logic provides the logic for an account that explicates negation in terms of a metaphysically primitive notion of incompatibility.[31]

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[31] Ancestors of this paper or of the ideas therein have profited from many discussions with Dorothy Edgington, Keith Hossack, Guy Longworth and Mark Textor, and with audiences at the II Physis-Filosofía y Análisis Workshop in Granada and in Madrid. I am very much indebted to MM McCabe for discussions of the problem of falsity in ancient Greek philosophy. The referees for *dialectica* provided immensely helpful, constructive criticism that was instrumental in giving the article its final shape.

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