Dunn's Relevant Predication, Real Properties and Identity
Author(s): Philip Kremer
Source: *Erkenntnis (1975-)*, Vol. 47, No. 1 (Jul., 1997), pp. 37-65
Published by: Springer
Stable URL: http://www.jstor.org/stable/20012785
Accessed: 23/05/2009 19:09

PHILIP KREMER

DUNN'S RELEVANT PREDICATION, REAL PROPERTIES AND IDENTITY

ABSTRACT. We critically investigate and refine Dunn's *relevant predication*, his formalisation of the notion of a real property. We argue that Dunn's original dialectical moves presuppose some interpretation of relevant identity, though none is given. We then re-motivate the proposal in a broader context, considering the prospects for a classical formalisation of real properties, particularly of Geach's implicit distinction between real and "Cambridge" properties. After arguing against these prospects, we turn to relevance logic, re-motivating relevant predication with Geach's distinction in mind. Finally we draw out some consequences of Dunn's proposal for the theory of identity in relevance logic.

According to a certain classical ontology, there are two fundamental kinds of being: (i) individuals; and (ii) properties and relations. There is a further intuition that some of an individual's properties have an especially intimate connection to it. The pre-eminent distinction between more and less privileged properties is Aristotle's essential-accidental distinction. But other distinctions have been proposed. Plato's image of the philosopher attempting to carve the world at its joints (*Phaedrus* 265e, *Statesman* 287c) is taken up in contemporary discussions of the scientist trying to discover the scientifically fundamental kinds. St. Thomas's distinction between "real" relations "which exist in the nature of things" and merely "logical" relations, which "are to be found only in the apprehension of the reason comparing one thing to another" (*Summa Theologica*, Q. 28, Art. 1), is echoed in Bealer's 1982 distinction between genuine qualities, which "determine the logical, causal, and phenomenal order of reality", and mere concepts, which "pertain, primarily, not to the world, but instead to thinking taken in the broadest sense". The moderns distinguish primary from secondary qualities. Moore 1919 considers whether all relations are, in the terminology of Bradley 1893, *internal* relations, which "affect" or "modify" or "pass into" the "being of their terms". And Goodman 1955 inspired a distinction between projectible properties and their gruesome relatives.

In a sequence of papers (1987, 1990, 1990a, 1990b, 199+ and 199+a) J.M. Dunn places himself in this tradition – many of the above historical citations come from his papers. Dunn's own contribution is his notion of

Erkenntnis **47:** 37–65, 1997.

relevant predication: a formalisation, in the context of first order relevance logic with identity, of the notion of a "real" property, i.e. a property that has an intimate life with objects.

In the present paper, we critically study Dunn's notion of relevant predication; we broaden the logico-philosophical context in which it was first motivated, by considering alternative strategies for formalising the distinction between real and hokey properties; and we draw some of the consequences relevant predication has for relevance logic. But first we note that relevant predication is a notion well worth pursuing: it formalises a kind of distinction of obvious significance; and it is an important advance in the relevance logic enterprise. Among other things, it can be used to begin a project that relevance logicians have yet satisfactorily to pursue: the project of giving a clear analysis of identity in relevance logic. One of the goals of the present paper is to begin such an analysis.

We begin by considering Dunn's original motivation for relevant predication (§1). This motivation relies on intuitions concerning what follows relevantly from claims of the form $x = y$.[1] We call these intuitions into question, and argue that they must be supplemented by an interpretation of identity in the relevance context.

Our criticism of relevant predication's original motivation leads us to provide independent motivation, in a broader philosophical context. We consider the prospects of using classical logic for articulating a distinction between real and hokey properties (§2, §3). After narrowing our task (§3, §4), we argue that classical logic is not up to the task (§5). So we move to the less traditional setting of relevance logic (§6–§8), re-motivating Dunn's central definitions (§9–§11), and fleshing out the interpretation of relevant identity that makes the most sense of Dunn's definitions (§12). We conclude by considering the consequences of this interpretation of identity (§13). (This interpretation and its consequences are further developed in Kremer 1994.)

Note: the expression 'real property' is not meant to evoke the debate over the ontologically standing of properties among realists, conceptualists, nominalists, et al. Even ontological parsimonious nominalists might agree that some predicates are somehow special or fundamental. And Dunn's project trades talk of real properties in for talk of relevant predication.[2]

1. DUNN'S ORIGINAL MOTIVATION

Dunn 1987 considers the following pair of statements:

(1) Socrates is such that he is wise.

(2) Reagan is such that Socrates is wise.

The classical logician might treat the logical structure exhibited by (2) as a degenerate case of the subject-predicate structure exhibited by (1). But Dunn takes (1) and (2) to be quite different, as is brought out by the "strict analogy" between them and (3) and (4):

(3) If anyone is Socrates then he is wise.

(4) If anyone is Reagan then Socrates is wise.

The arguments for (3) and (4) are taken to be (5) and (6):

(5) Socrates is wise. Therefore, if (x = Socrates) then x is wise.

(6) Socrates is wise. Therefore, if (x = Reagan) then Socrates is wise.

Formally (5) and (6) are of the form (7) and (8):

(7) $Fc \vdash (x = c \rightarrow Fx)$.

(8) $p \vdash (x = c \rightarrow p)$.

(7) is an instance of the indiscernibility of identicals, plausibly a relevant principle. Meanwhile, (8) is "nervously close to the dread relevance destroyer" (Dunn 1987, p. 451)

(9) $p \vdash (q \rightarrow p)$.

Dunn argues that we cannot count as a relevant theorem the indiscernibility of identity in its unrestricted form:

(10) $(Ac \rightarrow (x = c \rightarrow Ax))$.

For example, if A is the formula p then (10) would yield (8). We must also rule out indiscernibility for some formulas that *do* contain free occurrences of x: for example, if A is the formula $(p \,\&\, (p \vee Fx))$, then again (10) would yield (8), since A is relevantly equivalent to p.

Which instances of indiscernibility should we accept as theorems or as non-logical truths? Given (7), it seems all right to postulate indiscernibility in some cases – in particular, when Ax is a relevant property of c. In fact, it seems that the *relevant* properties are just those properties expressed by formulas Ax for which the indiscernibility of identity holds. Furthermore, it seems that Fx is true of c in a particularly intimate way just in case the conclusion of (7) is true. These are the main lines of Dunn's original motivation for the following two object language definitions:

(11) 'c relevantly has the property of being (an x) such that A' $=_{\text{df}}$ $\forall x(x = c \to A)$

(12) 'A is of a kind to determine relevant properties (with respect to x)' $=_{\text{df}}$ $\forall x(Ax \to \forall y(y = x \to Ay))$, where y is not free in A.

Note that formulas satisfying (12) are the formulas for which indiscernibility holds. We will render (11) as 'Ax is a relevant property of c', and (12) as 'Ax is a relevant property tout court'.[3]

The central move in the preceding dialectic is the rejection of $(p \to (x = c \to p))$, based on its similarity to $(p \to (q \to p))$. The appropriateness of this rejection depends on the intended interpretation of identity, though Dunn's motivation does not explicitly rely on any particular interpretation. Rather, it relies on our typographical intuitions.

Typographical intuitions are useful and underlie much of the motivation for relevance logic.[4] But these intuitions are defeasible. Consider the atomic propositional constant, $\boldsymbol{t}$, which is often added to relevance logics. $\boldsymbol{t}$ is standardly interpreted as the conjunction of all theorems. Despite the typographical similarity of $(p \to (\boldsymbol{t} \to p))$ to $(p \to (q \to p))$, the former is accepted as a theorem of the relevance logic **R**: we rely in the end on the *interpretation* of the new logical vocabulary.

Like '$\boldsymbol{t}$', '=' is a piece of logical vocabulary, open to interpretation. One interpretation is motivated by the following meta-linguistic principle, supported by a weighty tradition:

(13) '$a = b$' is true iff 'a' and 'b' are intersubstitutible, *salve veritate*, in all contexts.[5]

The interpretation underlying (13) can be articulated in a way similar to the interpretation of $\boldsymbol{t}$: where a and b are any two terms, $a = b$ is interpreted as the infinite conjunction of all formulas of the form $(A[a/x] \leftrightarrow A[b/x])$, where the formula Ax provides a context for x. Suppose we sharpen this interpretation with the assumption that the formula p provides a vacuous context. Then the theoremhood of $(x = y \to (p \to p))$ begins to look plausible: the formula $x = y$ is relevant to $(p \to p)$, since $x = y$ is a conjunction of formulas, among which is $(p \leftrightarrow p)$. This leads, in the relevance logic **R**, to the theoremhood of $(p \to (x = y \to p))$.

Unfortunately, on this interpretation of identity, $\forall x(Ax \to \forall y(y = x \to Ay))$ is a theorem for every formula A. This wreaks havoc with Dunn's central definitions since *every* formula would be of a kind to determine relevant properties.

The moral to be drawn is that Dunn's notion of relevant predication implicitly relies on some weaker interpretation of identity. One might try to buttress Dunn's notion with an interpretation of identity that rules out vacuous, or otherwise ill-behaved, contexts: perhaps '$a = b$' is true iff 'a' and 'b' are intersubstitutible, *salve veritate*, in all *relevant* contexts. But the notion of a relevant *context* is almost indistinguishable from the notion of relevant predication. This seems too tight a circle to draw.

In the end, we will urge such an interpretation of identity. But we will loosen the tight circle with frankly metaphysical considerations concerning both identity and relevant implication. Further, we will argue that our interpretation of identity, the interpretation that makes the best sense of relevant predication, has non-trivial consequences. Before that, however, we provide independent motivation for Dunn's definitions. This motivation begins with a consideration of classical logic strategies for explicating real properties.

2. CLASSICAL LOGIC STRATEGIES FOR ACCOUNTS OF REAL PROPERTIES

We begin with a broadly characterised task: to provide some formal account of some notion of real properties. We will narrow our task as we proceed. In the formal tradition, "metaphysical" talk of *properties* often gives way to putatively clearer syntactic talk of predicates and semantic talk of predicates' semantic correlates, *sets*. The framework of classical first order logic has given the concepts of *predicate* and *set* a precise and rich explication. So we begin with the prospects for using classical logic to formalise the kind of distinction, between real and hokey properties, that we have in mind.

How might we proceed? On the semantic side, we want a way to pick out the real sets of objects. One strategy is to develop an explication that does not depend on syntactic concepts. Such a strategy might be motivated by the thought that we are trying to characterise *natural* kinds, with the understanding that which kinds nature privileges should not depend on the language we speak. So, on such a strategy, we want to characterise the real sets of objects in a way that does not depend on which predicates are available.

Depending on the universe of discourse, various privileged classes of sets come to mind: the finite sets (in any universe of discourse); the recursive sets (of natural numbers); and the well-founded sets (of sets). The intuitions privileging these classes are powerful in some contexts, but do not accord with pre-theoretic intuitions concerning real properties. The intuitions favouring finite sets stem from doubts concerning our ability to reason with infinite totalities, but the claim that the real properties are the finitely instantiated properties is implausible. The intuitions favouring recursive sets stem from considerations concerning our computational abilities, but which sets nature privileges should not depend on these abilities. Finally, the intuitions favouring well-founded sets arise out of the iterative conception of sets provided by Zermelo–Fraenkel set theory. But these intuitions are inappropriate in the present context, since we are considering sets in so far as they play a role in the semantics for classical first order logic, and since the operative conception of sets here is not the iterative conception, but the naive conception, according to which a set is the kind of thing that can be the extension of a predicate. (See Boolos 1971.)

Maybe we should approach real properties syntactically. We could stipulate that the real predicates are the open atomic formulas of our fundamental physical or mathematical theory, for example 'x is an electron', 'x and y are simultaneous events', or '$x > y$'. We might include predicates such as $(Fx \;\&\; Gx)$ where Fx and Gx are real; for example, we might count 'x has mass m and charge c' as a real predicate, with respect to x. So we might specify which complex formulas are to count as real predicates, with recursive rules like 'if Ax and Bx are real predicates, then so is $(Ax \;\&\; Bx)$'.

Such considerations would generate an account according to which a predicate (i.e. an open formula), Ax, is real (with respect to variable x) just in case x occurs in Ax in the right sort of way, where the right sort of way is spelled out grammatically. Given such a grammatical explication of real predication, it is a straightforward matter to pick out the real sets: the extensions of the real predicates.

One objection to this grammatical approach is that it is too grammatical: whether x is *really* F would depend on facts of grammar and not on facts about the structure of the world. But, the objection continues, we are after properties "which exist in the nature of things", and not which "are to be found only in the apprehension of the reason ... ". (St. Thomas) In reply it could be argued that the pertinent grammar is the grammar of a privileged language, whose structure is isomorphic to that of the world. So, the reply continues, whether x is really F does ultimately depend upon facts about the structure of the world.

Let us provisionally grant the possibility of such a privileged language. The grammatical approach has a different kind of deficiency, to which we now turn.

3. METALANGUAGE AND OBJECT LANGUAGE APPROACHES

In §2 we considered approaches to the problem of real properties that might be called metalanguage approaches: they pick out the privileged object language formulas – those expressing real properties – in a *metalanguage*. A *grammatical* metalanguage approach would pick out formulas with the right grammatical structure, and a *semantic* metalanguage approach would pick out formulas with the right semantic features. In either case, on a metalanguage approach, the claim 'Ax expresses a real property' is a metalinguistic claim, mentioning but not using the object language formula Ax.

Dunn's approach is, by contrast, an *object language* approach: on an object language approach the claim 'Ax expresses a real property' is formalised as an object language claim, using but not mentioning the formula Ax.[6] A successful object language approach would have a distinct advantage over metalanguage approaches: we could obtain a better understanding of the place of *real property* claims (r.p. claims) in the inferential networks of object language theories. By embedding an r.p. claim within conditionals, for example, we could see what other claims, in the language of some theory, imply it or are implied by it. If our object language theories are provisional rather than final, then moves such as finding out what follows from the claim that F is a real property of c are part of the task of theory revision and improvement. Thus, an object language approach allows us an account of r.p. claims in the absence of any elusive final theory or privileged language.

So we narrow our task to one of providing an *object language* account of real properties. We have not yet dismissed the prospects of providing such an account within the confines of classical logic. Before we consider these prospects (§5), we narrow our task further by considering a particular version of the difference between real and hokey properties, suggested by Geach 1969.

4. GEACH AND CAMBRIDGE PROPERTIES

Geach's worry is not about real and hokey properties, but real and "Cambridge" change.[7] He is unsatisfied with the "Cambridge" criterion for change: the thing, c, has changed if, for some formula Ax, Ac is true at time t, and Ac is false at time $t' > t$. According to this criterion, things undergo all sorts of Cambridge changes that are not intuitively real. For example, suppose that (i) Fx stands for 'x is tall'; (ii) p stands for 'it is raining in Moscow'; (iii) Tracy is tall; and (iv) it is presently raining in

Moscow. If it stops raining in Moscow then the formula $(Fx \;\&\; p)$ stops being true of Tracy. Tracy thereby undergoes a "Cambridge" change, even though the end of the shower in Moscow has nothing to do with her, and, intuitively, she has not changed at all.[8] Geach presents other examples: Socrates becomes shorter than Theaetetus, as Theaetetus grows; Socrates becomes admired by a schoolboy; five ceases to be the number of someone's children.

One place to locate the unrealness of these changes in is the unrealness of the corresponding properties: $(Fx \;\&\; p)$; x is shorter than Theaetetus; x is admired by Johnny; and x is the number of Geach's children. This suggestion leads to an intuitive test for distinguishing formulas expressing Cambridge properties from formulas expressing what we will call "Geach" properties:

> A formula Ax expresses a Cambridge property (with respect to the variable x) iff, for some object (named by) c, Ac can change truth values without a corresponding change in c. Otherwise Ax expresses a Geach property.

The Geach–Cambridge distinction need not be understood as a distinction among *properties*. It can be understood as a distinction between Geach and Cambridge *formulas*. So the nominalist can make sense of it.

As given, Geach's distinction relies on intuitions concerning change. This has two limitations. First, we might want to wield the distinction in timeless contexts. We might want to distinguish between real and hokey properties of the number 5 or the set $\emptyset$, even when the hokey properties are themselves expressed in an atemporal mathematical vocabulary. Secondly, even in temporal contexts, there are some properties that a thing cannot lose, and so cannot lose without changing, but that we nonetheless count as Cambridge properties. Tracy cannot lose the following properties: being such that $2 + 2 = 4$; being such that at some future or past point in history, Napoleon will march or has marched across Europe. Yet these properties have as little to do with what Tracy is like as has *being such that it is raining in Moscow*.[9]

There are intuitive non-temporal ways of making the Geach/Cambridge distinction. Let a *theory* to be a set of sentences in first order logic; and consider the theories $T_1 = \{Fc, p\}$ and $T_2 = \{Fc, \neg p\}$, where Fx and p are interpreted as above, and where c names Tracy. T_1 and T_2 clearly disagree, but do they disagree *about Tracy*? Note that, where $Ax = (Fx \;\&\; p)$, we have $T_1 \vdash Ac$ and $T_2 \vdash \neg Ac$ on most accounts of logical consequence. So, according to a Cambridge criterion of disagreement, T_1 and T_2 disagree about Tracy. Nonetheless there is an intuition according to which this disagreement arises out of a disagreement regarding the weather in Moscow, and has nothing to do with a disagreement *about Tracy*.

This suggests a new intuitive test for distinguishing between Cambridge and Geach formulas:

> A formula Ax (of an interpreted language) is a Cambridge formula (with respect to variable x) iff, for some name c, there are theories T_1 and T_2 that disagree regarding Ac without disagreeing about (the object name by) c. Otherwise Ax is a Geach formula.

This new non-temporal intuitive test might be accused of explaining the obscure – Geach formulas – in terms of the more obscure – disagreement *about* (the object named by) c. Further, the putative explanation is circular: ultimately, it seems, *disagreement about* c will be cashed out in terms of disagreement about c's real properties. In a less obvious way the original temporal intuitive test is circular: one suspects that, ultimately, the notion of *real* change in c will be cashed out in terms of change in c's *real* properties.

Fair enough. But these tests are only meant to provide rough criteria for Cambridgeness, not definitions. They are meant to tie together and motivate various intuitions – in particular, to motivate weaker intuitions, regarding properties, in the light of stronger ones, regarding change or disagreement. If one accepts the *prima facie* plausibility of the intuition that there was no real change in Tracy corresponding to the change in the weather in Moscow, then one can accept the *prima facie* plausibility of the intuition that some formulas do not express real properties.

A different kind of problem arises when we confine our attention to classical logic and to a classical theory of consequence ($\vdash$). The new intuitive test still makes Geach formulas out of suspected impostors like $Ax = (Fx \,\&\, (p \rightarrow p))$. If we assume that Fx is a Geach formula, then for any name c and any classical first order theories T_1 and T_2 we have

	(T_1 and T_2 agree about c)	
$\Rightarrow$	($T_1 \vdash Fc$ iff $T_2 \vdash Fc$)	(since Fx is a Geach formula)
$\Rightarrow$	($T_1 \vdash Ac$ iff $T_2 \vdash Ac$)	(by classical logic)
$\Rightarrow$	(T_1 and T_2 agree regarding Ac).	

So T_1 and T_2 cannot disagree regarding Ac without disagreeing about c.[10] We nonetheless suspect $(Fx \,\&\, (p \rightarrow p))$ of being an impostor, since $(p \rightarrow p)$ seems to have as little to do with *what Tracy is like* as does p. Eventually, we will motivate Dunn's relevant predication as a formalisation of Geach's intuitions. As it will turn out, one can then take Fx to be a real property without taking $(Fx \,\&\, (p \rightarrow p))$ to be a real property. (See §13.6, below.)

5. CAN WE PROVIDE AN OBJECT LANGUAGE DEFINITION OF GEACH PREDICATION WITHIN THE CONFINES OF CLASSICAL LOGIC?

§4's remarks concerning the classical theory of consequence suggest that classical logic is not the right context in which to make the Geach/Cambridge distinction. Here we reinforce this suggestion, by considering the prospects of any object language account of Geach predication, within the classical context. Such an account should provide a definition of an object language formula Geach(A, x) = 'A expresses a Geach property (with respect to x)' = 'A is a Geach formula (with respect to x)' for each formula A and each individual variable x. We propose two minimal conditions that such a definition should satisfy.

Condition 1: *Uniformity.* We want a *general* notion of what it is for a formula to be a Geach formula, with respect to the variable x. So, for example, the formulas Geach(Fx, x) and Geach(Gx, x) should not differ, except that where there are occurrences of F in Geach(Fx, x), there should be occurrences of G in Geach(Gx, x). In general, the formula Geach(A, x) should result by a uniform substitution of A for Fx in the formula Geach(Fx, x).[11]

Condition 2: *Restraint.* Our discussion at the beginning of §4 motivated the claim that though Fx expresses a real property, (Fx & p) does not, even when p is true. Any account of Geach predication should be thus restrained: it should be possible that p be true and that the formula (Fx & p) not be a Geach predicate. So, at the very least, the following should be logically consistent: Geach(Fx, x) & p & $\neg$Geach(Fx & p, x).

THEOREM. No definition of 'A is a Geach formula (with respect to x)' given for standard first or higher order classical languages satisfies both uniformity and restraint. In particular, for any uniform definition of Geach(A, x), the following is a classical theorem: $\neg$(Geach(Fx, x) & p & $\neg$Geach(Fx & p, x)).[12]

Thus, no object language account of Geach predication given for a classical language can satisfy our minimal conditions. So we turn away from classical logic altogether.

6. RELEVANCE LOGIC: A SUGGESTION CONCERNING IMPLICATION AND IDENTITY

There are several ways to express the thought that the end of a shower in Moscow does not bring about a real change in Tracy: the end of the shower has nothing to do with Tracy; it is unrelated to her; it has no connection to her. These locutions suggest that the right logical context for an object language account of Geach predication is one designed to formalise a notion of relatedness or connection. And Anderson and Belnap's *relevance logic* enterprise provides a context in which this notion plays a central role.[13] The *loci classici* for this enterprise are Anderson and Belnap 1975 and Anderson *et al.* 1992, henceforth Ent1 and Ent2. This enterprise's animating intuition is that a conditional $(A \rightarrow B)$ is false unless A is relevant to B. Thus neither $((p \mathbin{\&} {\sim}p) \rightarrow q)$ nor $(p \rightarrow (q \rightarrow p))$, for example, are theorems of standard relevance logics.

Can the connectedness implicit in the claim that *being tall* is a Geach property of Tracy be assimilated to the connectedness implicit in a conditional claim? Here is one approach. The fact that Tracy is tall is connected to the way Tracy *is*, or to the fact that Tracy is as she is, while the fact that it is raining in Moscow is not. Put another way, the proposition that Tracy is tall is *implicit* in the proposition that Tracy is as Tracy is, while the proposition that it is raining in Moscow is not. Taking

(14) the proposition that B is implicit in the proposition that A

to be rendered formally by the expression $(A \rightarrow B)$, our suggestion leads to the following: 'Tallness is a Geach property of Tracy' can be expressed by the conditional

(15) Tracy is as Tracy is $\rightarrow$ Tracy is tall.

How should we express the claim that Tracy is as Tracy is? To say of a and b that a is as b is, is to say that a has the properties b has, or that what is true of a is true of b. These locutions suggest that 'a is as b is' can be formally expressed by '$a = b$'. This suggestion leads to the following object language rendering of 'F is a Geach property of c':

(16) $(c = c \rightarrow Fc)$.

This rendering needs fleshing out. We have said little about the relevant implication connective '$\rightarrow$' and nothing about which relevance logic to use. Furthermore, once we settle on the relevance logic **R**, we must provide a coherent interpretation of identity in **R**. We have made a stab here: '$a = b$' says that a is as b is. But this needs sharpening: though axioms for first order **R** have been clearly spelled out, there is neither a generally accepted axiomatisation of identity in **R** nor a generally accepted philosophical interpretation of identity in **R**. To bring matters into focus, we begin with the relevance logic enterprise, and the interpretation of '$\rightarrow$'.

7. THE RELEVANCE LOGIC ENTERPRISE

7.1. **E** *and* **R**

The hero of Ent1 and Ent2 is the logic **E**, which formalises the notion of *entailment* or *purely logical implication*. For 'A entails B' to be true, B must *follow* from A: not only must A be *relevant* to B; the relevant connection must be a *necessary* one. An early gloss on such necessary relevant connections is that they must be connections of *meaning*.[14] **E** is defined in a propositional language with truth-functional &, $\vee$, and $\sim$, and the non-truth-functional entailment connective, $\rightarrow$. The heart of **E** is its arrow fragment $\mathbf{E}_{\rightarrow}$, "the pure calculus of entailment".

To define $\mathbf{E}_{\rightarrow}$, Entl introduces a Fitch-style natural deduction system, in which we derive formulas on the basis of hypothetical assumptions. Ent1 satisfies our *relevance* intuitions with a device for keeping track of which assumptions were used to derive which conclusions: to infer $(A \rightarrow B)$ from a subderivation of B on assumption A, we must first make sure that A was *used* in deriving B. Ent1 satisfies our *necessity* intuitions by restricting which lines of a derivation can be reiterated into subderivations: to show that A *entails* B, it does not suffice to show that A can be used in deriving B. We must not use collateral information, unless it comes in the form of a necessary entailment claim. So only formulas of the form $(C \rightarrow D)$ are reiterable into subderivations.

Ent1 notes that, formally, we can separate the relevance- and necessity-motivated proof-theoretic devices. Keeping the necessity-motivated restrictions but ignoring relevance yields the strict implication fragment of **S4**. Suppose we ignore the necessity-motivated restrictions, but keep track of which assumptions were used in deriving which conclusions. Ent1 defines $\mathbf{R}_{\rightarrow}$ to be the resulting implicational system. Ent1 defines the full logic **R**, with connectives &, $\vee$, $\sim$ and $\rightarrow$, by extending $\mathbf{R}_{\rightarrow}$ in the same way that $\mathbf{E}_{\rightarrow}$ is extended to **E**.

Ent1's motivation for **E** is in terms of a family of related pre-formal notions: logical consequence, entailment, meaning-connections. But the original motivation for **R** was formal: **R** is the result of weakening the constraints on an **E**-derivation. Regarding **R**, Ent1 offers only preliminary suggestions along interpretive lines:

> **E** lacks a relevant, *contingent* sense of 'if-then'; accordingly, **R** promises to have applications wherever what is wanted is a conditional the meaning of which, while non-logical, involves relevance of antecedent to consequent. (p. 351)[15]

As an object of formal study, **R** now rivals **E** in importance. But not much more of philosophical interest has been said about **R** before or since –

particularly about the relevance character of *relevant implication* and how it differs from the *entailment* of **E**.[16]

Despite this, the contingent yet relevant character of **R**'s arrow is well-suited to our application. Tracy's connection to her tallness is real but contingent. So we want the conditional $(c = c \rightarrow Fc)$ to be true, but contingently so. So we choose **R** rather than **E** as the logic in which to formalise the Geach-Cambridge distinction. Given the lack of interpretive work on **R**, we should say something along interpretive lines.

First, some preliminary remarks.

Though **R**'s motivation was mainly formal, one can see it as a formalisation of certain pre-formal intuitions about ordinary contingent 'if then'. Typically, 'if p then q' is not equivalent to $(\sim p \vee q)$ even when contingent: when we say 'if it is sunny outside then we will have a picnic' we mean to draw a connection between the antecedent and consequent. The connection is not a connection of meaning, but one that holds because this is in fact the way that the world is – a fact caused, perhaps, by a decision we made about lunch next Sunday.

A qualification: there are many intuitions, more or less informed by theory, concerning ordinary contingent implication. Formalising implication with an eye on some of these intuitions – for example *relevance* intuitions – might run counter to others.[17] The ultimate significance of a given formalisation lies in its connections to other ideas, and in particular in its applications.

In what follows, we will not give a full specification of what kind of connection must exist between A and B for $(A \rightarrow B)$ to be true. We will try to clarify the pertinent notion of relevance in two ways: firstly, by arguing against one strategy for interpreting relevant implication (§7.2); and secondly by considering some of the consequences of the particular formalisation of implication embodied in **R** (§7.3).

7.2. *Proof-Theoretic Explications of the Truth of Relevant Conditionals*

It might be illuminating to explicate the truth of true entailments, formalised with the '$\rightarrow$' of **E**, in terms of meaning-connections. But true relevant conditionals, formalised with the '$\rightarrow$' of **R**, are often contingent. So the meaning-connection strategy is inappropriate for explicating their truth. A common alternative is to explicate this truth proof-theoretically, in terms of the derivability of the consequent from the antecedent in a way that *uses* the antecedent. Consider:

> ... the consequent of a relevant implication is supposed to depend on the antecedent in a somewhat technical sense, but one that intuitively means that the antecedent can be used in deriving the consequent. (Dunn 1990a, p. 181)

To gain perspective on such a strategy, we do not consider what it is for a relevant conditional to be true, but rather what is involved in postulating a relevant conditional as an axiom of a theory. For example, many of the axioms of Meyer's 197+ Relevant Peano Arithmetic, RPA, are relevant conditionals. (See Ent2, §82.) Granted, the main *consequence* of postulating $(A \rightarrow B)$ can be spelled out in proof-theoretic terms: once $(A \rightarrow B)$ has been postulated, B can be inferred from A in the course of a derivation. But in general the *justification* for postulating $(A \rightarrow B)$ is not proof-theoretic: if the justification were a derivation of B from A, then there would be no point in postulating $(A \rightarrow B)$ in the first place, since one could just derive it.

The justification is typically metaphysical, based on intuitions about connections among the items in the universe of discourse. Consider Ent2 on the axiom $(x = y \rightarrow x' = y')$ of RPA: "To say that [this axiom] holds is or might be to say ... something true because of arithmetic and not just because of logic. [This axiom] says that the generation of the integers by the successor function is relevant." (Ent2, p. 431). The point here is that more is involved in the truth of a relevant conditional than can be accounted for by the proof theory.[18] (We return to this point at the end of §7.3, in §8 and in §13.2.)

7.3. *Interpreting Relevant Implication*

Our current interpretive strategy for **R** is to take it as a formalisation of certain intuitions about ordinary contingent implication. One way to pursue this strategy is to consider some of its consequences. In particular consider the following theorems of **R**: $p \rightarrow ((p \rightarrow q) \rightarrow q)$ and $p \rightarrow ((p \rightarrow p) \rightarrow p)$. (These are not, by the way, theorems of **E**.) If p is contingently true, then so are $((p \rightarrow q) \rightarrow q)$ and $((p \rightarrow p) \rightarrow p)$. What factual connection exists between the tautologous $(p \rightarrow p)$ and the contingently true p? Our pre-formal intuitions are simply unclear regarding contingent relevant conditionals whose antecedents are conditionals.

There *is* an intuitive locution for expressing conditionals with complex antecedents. Furthermore, this locution allows true conditionals whose antecedents are necessary, and whose consequents are contingent. Consider (17):

(17) all bachelors are men.

Though (17) is necessary, we might consider its *upshot* in some contingent circumstances. If there is a bachelor in the room, it is natural to say that part of the *upshot* of (17) is (18):

(18) there is a man in the room.

On the other hand, if the room contains men but no bachelors, then (18) is true, but not part of the *upshot* of (17). Note that the upshot of a necessary claim can be contingent. Our main suggestion here is that the locution 'B is part of the upshot of A' is another way of expressing 'if A then B', in our relevant sense of 'if . . . then . . . '.

Of course, (18) might also be part of the upshot of a contingently true claim such as (19):

(19) all former Presidents are men,

In interpreting '(19) $\rightarrow$ (18)', we can think of the hypothetical postulation of (19) by partial analogy with the postulation of an axiom of a theory (see §7.3). In this conditional, (19) is only hypothetically postulated, and demands for its justification do not arise. But the question of its hypothetical consequences does. (19) licenses inferences to (18) from (20):

(20) there is a former President in the room.

So if (20) is true, then the inference licence (19) is relevantly connected to (18): (18) can be derived from (20) using the licence. This is why, when (20) is true, ((19) $\rightarrow$ (18)) is also true. But if (20) becomes false and (18) remains true, ((19) $\rightarrow$ (18)) becomes false: (19) loses its connection to (18), which it once had through (20).

A similar analysis can be given to conditionals with conditional antecedents, such as $((p \rightarrow q) \rightarrow q)$. Here $(p \rightarrow q)$ plays the role of (19) and q of (18). Whether the inference licence $(p \rightarrow q)$ is relevantly connected to q – whether q is part of the upshot of $(p \rightarrow q)$ – can depend on whether p is true. $((p \rightarrow p) \rightarrow p)$ is a limiting case. If p is contingently true, then p is part of the upshot of $(p \rightarrow p)$: using the inference licence we can obtain p. If p is false then this is no longer the case. Note that the hypothetical postulation of the theorem $(p \rightarrow p)$ as the antecedent of $((p \rightarrow p) \rightarrow p)$, is not an empty gesture. Its consequences are all true, but many truths are not among its consequences. The consequences of $(p \rightarrow p)$ are just those truths, contingent and necessary, involving p in some intimate way.

Adding the moral of §7.2 to the considerations of the current section, teaches us a combined lesson. Proof-theoretic ideas are not enough to explicate, in general, the truth of true relevant conditionals. As we saw, in RPA the truth of $(x = y \rightarrow x' = y')$ is best understood in metaphysical terms. The proof theory, however, can help us see what other factual connections exist once certain factual connections – even certain non-conditional claims – have been established on non-proof-theoretic grounds. In particular, once p has been established on whatever grounds, we can see that there is a factual connection between $(p \to q)$ and q. Or, to consider a different example, once $(p \to (q \to r))$ and q have been established on whatever grounds, one can then use the proof theory to establish the factual connection expressed by the conditional, $(p \to r)$.

8. INTERPRETING $(c = c \to Fc)$

§6 suggested rendering 'F is a Geach property of c' as (16):

(16) $(c = c \to Fc)$,

where $c = c$ formalises 'c is as c is', and '$\to$' is the contingent relevant implication of **R**. In light of §7.2 and §7.3, we can begin to flesh this suggestion out.

One way to understand the contingent truth of $(c = c \to Fc)$ is by analogy with other contingent truths with necessary antecedents. Recall our discussion in §7.3 of $((p \to p) \to p)$, which is true if p is true. Analogously to that case, the hypothetical postulation of the theorem $c = c$ in the conditional $(c = c \to Fc)$ is not an empty gesture. Its consequences are those truths involving c in some intimate way.

In addition to the analogy between $(c = c \to Fc)$ and $((p \to p) \to p)$, there is an important disanalogy. We know that $((p \to p) \to p)$ is true if p is true on proof-theoretic grounds: $p \to ((p \to p) \to p)$ is a theorem of **R**. But, without a theory of relevant identity, it is not clear whether $Fc \to (c = c \to Fc)$ is a theorem of **R**. In §13.2 we will suggest, in effect, that it is not. $Fc \to (c = c \to Fc)$, it will turn out, may be true, but that is the sort of thing to be established on metaphysical grounds, not proof-theoretic ones. See §12 and §13.2 for a further elaboration.

9. RELEVANT PREDICATION, AGAIN

Rather than considering the upshot of $c = c$, we could consider more generally the upshot of $x = c$. Suppose that F is a property of c, intimately connected to the way c is. The hypothesis that x is as c is seems to license the inference to the claim that F is intimately connected with x as well as with c. So if F is a Geach property of c not only is $(c = c \to Fc)$ plausible, but so is (21):

(21) $\forall x(x = c \to Fx)$.

Furthermore, (21) should be true only when F is a Geach property of c.

We now have two rival formalisations of 'F is a Geach property of c', (21) and (16):

(16) $(c = c \rightarrow Fc)$.

Both link c's identity to the property F. (21) is a stronger claim than (16). So (21) draws a tighter connection between *the way* c is and Fc than does (16). In light of §7, what is wanted in expressing the claim that 'F is a Geach property of c' is a formula that draws a tight, but possibly contingent, connection between c and F. So we opt for (21) as a refinement of 'F is a Geach property of c', which we first formalised as (16).[19]

If we replace Fx with an arbitrary formula Ax, (21) becomes the following formalisation of 'Ax is a Geach property of c':

(22) $\forall x(x = c \rightarrow Ax)$.

Observe: (22) is the definiens of Dunn's definition of 'Ax is a relevant property of c'!

10. A PROBLEM AND A SOLUTION

We have motivated (22) as a formalisation of Geach predication. But there is a problem: our formalisation seems to diverge from the original notion of Geach predication in the case of predicates of the form $(Fx \vee p)$.

To see this, suppose that Fx stands for 'x is tall'; p stands for 'it is raining in Moscow'; and suppose further that it is raining in Moscow. Also suppose that Tracy's friend Sergio, whom we will call c, is short. If it stops raining in Moscow, then c will lose the property $(Fx \vee p)$ without really changing. So $(Fx \vee p)$ is, as expected, a Cambridge property and not a Geach property.

Unfortunately, on our definition of 'Ax is a Geach property of c', (23) is a theorem of **R**:

(23) If Fx is a Geach property of c, then $(Fx \vee p)$ is a Geach property of c.
Formally: $\forall x(x = c \rightarrow Fx) \rightarrow \forall x(x = c \rightarrow Fx \vee p)$.

So $(Fx \vee p)$ is, on our formal definition, a Geach property of c. This seems to conflict with the conclusion of the preceding paragraph, that $(Fx \vee p)$ is a Cambridge property. So our attempted formal definition of Geach predication seems to fail.

This is only an appearance. Our formal notion of Geach predication is an object-specific notion, 'Ax is a Geach property of c'. But our informal notion of Geach predication is a non-object-specific general notion, 'Ax is a Geach property'. The informal notion was articulated in §4 with intuitive tests for whether a formula Ax is Geach *tout court*, i.e. not relative to some particular object c. An intuitive test, in Geach's spirit, for an informal *object-specific* notion of Geach predication would be as follows:

> Ax expresses a Cambridge property of c iff Ac is true and Ac can become false without a corresponding change in c. Ax expresses a Geach property of c iff Ac is true and Ac cannot become false without a corresponding change in c.

We will shortly show that (23) holds for our informal object-specific notion of Geach predication, just as it holds for our formally defined object-specific notion. Thus our formalisation of Geach predication (of c) is not, after all, at odds with the appropriate informal considerations. (23) should not be confused with (24):

(24) If Fx is a Geach property *tout court*, then $(Fx \vee p)$ is a Geach property *tout court*.

(24) has no formal meaning yet. But it fails for the informal notion, as shown above. Below, we formalise Geach predication tout court. We expect (24) not to be a theorem of first order **R** with identity. See §13.6, below.

In our argument for (23), we will use the notation '$t : A$' for 'A is true at time t'. And we will let Gx abbreviate $(Fx \vee p)$. To begin, assume that $t : (Fx$ is a Geach property of $c)$. So $t : Fc$. So $t : Gc$. At time t, is Gx a Geach or a Cambridge property of c? Applying the intuitive test just given, we suppose that $t' : {\sim}Gc$ at time $t' > t$. So $t' : {\sim}Fc$. So, since t: (Fx is a Geach property of c), the passage from t to t' corresponds to a real change in c: c has lost a Geach property. So, somewhat to our surprise, Gx is a Geach property of c, since c cannot lose this property without really changing. So (23), interpreted informally, is established as desired. (No parallel argument works for (Fx & p).) Thus the theoremhood of the formal reading of (23) is unproblematic.

11. GEACH PROPERTIES TOUT COURT

We began in §4 with the intuitive notion of a Geach property *tout court* rather than the notion of a Geach property of c. And in some sense, the *tout court* notion is more fundamental. Can we formalise the *tout court* notion in first order **R** with identity? We begin by noting that the argument for (23) can be turned into an argument for

(25) F is a Geach property *tout court* $\rightarrow$ [$Fc \rightarrow$ (Gx is a Geach property of c)].

Proof of (25): Assume that F is a Geach property *tout court*. Also assume that $t : Fc$. Then $t : Gc$. Suppose that c loses the property Gx at time $t' > t$. Then $t' : {\sim}Gc$. So $t' : {\sim}Fc$. So, since F is Geach property *tout court*, c has changed in the period of time from t to t'. Thus, Gc cannot become false without a corresponding change in c. So Gx is a Geach property of c. QED.

Why can't Gc become false at $t' > t$ without a corresponding change in c? Because $t : Fc$. If all we had was the weaker assumption that $t : Gc$, we could not infer that Gx is a Geach property of c. So, though we can argue that [$Fc \rightarrow$ (Gx is a Geach property of c)], we cannot argue that [$Gc \rightarrow$ (Gx is a Geach property of c)]. It is not the general features of G that make it a Geach property of c. Rather, it is the general features of F.

What is required for G to be a Geach property *tout court*? G's general features must be what make G unlosable by an object without that object changing. This says *more* than that G is a Geach property of c for every c that has it. Rather, this says that, for every c, there is a *connection* between c's being G, and c's not being able to lose G without changing. So we postulate the following connection between being a Geach property *tout court* and being a Geach property of c, where the arrow represents relevant implication:

(26) G is a Geach property *tout court* iff
$\forall y(Gy \rightarrow$ (y cannot lose G without changing)), i.e.
$\forall y(Gy \rightarrow$ (G is a Geach property of y)).

Recall that at the end of §9 we formalised 'Ax is a Geach property of c' as

(22) $\forall x(x = c \rightarrow Ax)$.

If we combine this formalisation with our new connection between being a Geach property *tout court* and being a Geach property of c, then we obtain the following formalisation of 'Ax is a Geach property *tout court*':

(27) $\forall y(Ay \rightarrow \forall x(x = y \rightarrow Ax))$

Observe: (27) is the definiens in Dunn's definition of 'Ax is a relevant properties *tout court*'!

As promised, we have re-motivated Dunn's two central definitions: of (i) 'Ax is a relevant property of c', which we have re-expressed as 'Ax is a Geach property of c'; and of (ii) 'Ax is a relevant property *tout court*', which we have re-expressed as 'Ax is a Geach property *tout court*'.

12. WHAT IS A GOOD THING TO MEAN BY '$x = y$'?

In re-motivating Dunn's definitions, we have been relying on a an intuitive interpretation of $x = y$: x is as y is. Before we sharpen this interpretation, a recap.

In addition to our intuitive interpretation of $x = y$, we have been relying on the intuition that p, even if true, generally has nothing to do with *the way x is* (or the way y is). So, in general, $x = y$ should not relevantly imply p. On the other hand, some claims *are* implied by $x = y$: $x = y$ implies Fx whenever F is a real property and Fy is true. §10 worried about the following: if F is a real property then if Fc, then $\forall x(x = c \rightarrow Fx \vee p)$ (see (23)). Our gloss: Fc might be the reason that $\forall x(x = c \rightarrow Fx)$; but $(Fc \vee p)$ is *not* the reason that $\forall x(x = c \rightarrow Fx \vee p)$. Thus we motivated, in §11, the definition of 'Ax is a Geach property *tout court*': $\forall y(Ay \rightarrow \forall x(x = y \rightarrow Ax))$.

So if Ax expresses a real property *tout court*, then $x = y$ licenses inferences from Ax to Ay, since $x = y$ relevantly implies $(Ax \rightarrow Ay)$. Even when F is a relevant property, this leaves the following inferences unlicensed by $x = y$, though they may be licensed on other grounds: from p to p; from $(Fx \vee p)$ to $(Fy \vee p)$; from $(Fx \& (p \rightarrow p))$ to $(Fy \& (p \rightarrow p))$. Soon we attribute to $x = y$ a very minimal inference licensing content. Given this interpretation, $(Fx \vee p)$ is not a relevant property, as expected. And neither is $(Fx \& (p \rightarrow p))$. In light of the remarks at the end of §4, relevant predication turns out to be a refinement of the notion of Geach properties.

Since identity is symmetric, $x = y$ licenses inferences from Ay to Ax, as well as from Ax to Ay, if Ax expresses a real property. We suggest interpreting identity so that this exhausts the meaning of $x = y$: for x to be as y is is for x and y to have the same real properties. This amounts to taking $x = y$ to be equivalent to the infinite conjunction of formulas of the form $(Ax \leftrightarrow Ay)$ where A ranges over real properties. Or, if we allow second order constructions, this amounts to taking $x = y$ to be equivalent to $\forall F(Fx \leftrightarrow Fy)$ where F ranges over real properties. And this, we believe, is a good thing to mean by $x = y$.

13. CONSEQUENCES OF OUR INTERPRETATION OF IDENTITY

Our interpretation of identity draws a tight circle. Real predication is defined in terms of identity, which is in turn interpreted in terms of real predication. We can think of this circle as simply a making explicit of the relationships among real predication, relevant implication and identity. Now that we have made this relationship explicit, we can draw out some non-trivial consequences of our interpretation of identity. The non-trivial consequences show that our circle, though tight, is not uninteresting.

Before we consider these consequences, we note that the interpretation can be taken in two ways. On one hand, it can be thought of as a reductive analysis, reducing the notion of identity to the notion of real properties. On this line, $(u = x \leftrightarrow u = y)$ is not among the infinitely many biconditionals whose conjunction we are using to interpret $x = y$. On the other hand, if we are thinking of ourselves as simply making explicit the relationships among real predication, relevant implication and identity, we can think of our interpretation as an *explication* of identity rather than a reductive analysis of it. Thus, '=' can appear in the infinite conjunction of biconditionals by which we are interpreting $x = y$. This is in keeping with the point made in §13.3, below: on our interpretation, the property of being identical to u, expressed by the formula $x = u$, is of a kind to determine a relevant property with respect to x.

The second explicative line has the advantage that it does not commit us to the identity of indiscernibles in any objectionable sense. One objection to such a principle is that we can conceive of *numerically* distinct individuals x and y having all of their properties in common. But if *being identical* to x is among the properties of x, then we can longer conceive of such numerically distinct individuals.

Our remarks below, on the consequences of our interpretation of identity, do not depend on whether we take the interpretation of identity as a reductive analysis or as an explication.

13.1. *The Fundamental Notion is that of a Real Property Tout Court*

Our dialectic has proceeded by first defining 'Ax is a real property of c' (§9), and then defining 'Ax is a real property tout court' in terms of it (§11). The definition of 'Ax is a real property of c' involves identity in a central way, and identity is in turn interpreted (§12) in terms of real properties *tout court*. So the *fundamental* notion is that of a real property *tout court*, not the notion of a real property of c.

13.2. *The Weakness of the Logic*

The logic suggested by our interpretation of identity is very weak, since it is hardly ever the case that Ax is a real property on logical grounds alone, unless Ax is entirely built up from logical vocabulary. For example, for no nonlogical first order constant F would we expect $x = y \rightarrow (Fx \rightarrow Fy)$ to be a theorem. For, independently of any interpretation, we have no reason to expect that F expresses a real property.

A worry: nothing makes '=' an *identity* relation rather than just an *equivalence* relation. What is distinctive of identity ought to be the way it interacts with a language's nonlogical vocabulary. It seems, however, that '=' as here interpreted does not so interact.

This appearance is misleading. It is based on the false assumption that formulas of the form $x = y$ are conferred content from the *theorems* in which they are involved. One place in which such formulas are also conferred content is within the context of a *theory*. For example, suppose that a particular theory stipulates that the nonlogical predicate constant F is relevant. Then '=' is conferred the inference licensing content implicit in $\forall x \forall y(x = y \rightarrow (Fx \leftrightarrow Fy))$.

Identity claims are provided with *content* by way of just this kind of stipulation. Further, the motivation for such a stipulation cannot in general be proof-theoretic. It must rely on frankly metaphysical intuitions about *real* properties and real connections among the objects in our universe of discourse. This is an instance of a general point made in §7.2, above.

Despite its weakness, **R** itself can help us understand relevant predication in the absence of any particular theory. It helps us see what the consequences are of relevant property claims of various kinds. In particular, if we provisionally assume that such and such atomic formulas are relevant in their variables, **R** gives us insight into which *other* formulas are relevant predicates. See Kremer 1989, where we show that the resulting set of relevant predicates can be given natural and independently motivated grammatical characterisations.

13.3. *Nested Transitivity*

Identity respects the axiom of nested transitivity: $x = y \rightarrow (y = z \rightarrow x = z)$, since identities are being treated as conjunctions of biconditionals or as second order universal closures of biconditionals, and since *these* satisfy nested transitivity. As a result, the property of being identical to u, expressed by the formula $x = u$, is of a kind to determine a relevant property with respect to x. We point this out because Dunn 1990a worries about transitivity, and considers the possibility of replacing nested transitivity with the strictly weaker "conjoined" transitivity: $((x = y \,\&\, y = z) \rightarrow x = z)$.

13.4. *The Axiom of Substitution*

On our interpretation of identity, it is hard to motivate the axiom of substitution,

(28) $\quad x = y \,\&\, A[x/u] \rightarrow A[y/u]$.

Firstly, if there are irrelevant atomic predicates like Fx = 'x is admired by Johnny', we might not want to allow the importation that $((x = y \& Fx) \rightarrow Fy)$ suggests. Further, suppose that there is only one nonlogical predicate F and that F is relevant. Then there is good reason to think that the meaning of $x = y$ should be captured by these four axioms: $x = x$; $x = y \rightarrow y = x$; $x = y \rightarrow (y = z \rightarrow x = z)$; and $x = y \rightarrow (Fx \leftrightarrow Fy)$. And substitution simply does not follow from these. A fuller story on substitution is given in Kremer 1994.

13.5. $x = y$ *and* $\Box x = y$

Ent1 defines a system $\mathbf{R}^{\Box}$, by extending **R** with an **S4**-style necessity operator, $\Box$. We might define a quantified modal relevance logic with identity, by extending $\mathbf{R}^{\forall\exists x=}$ with an $\mathbf{R}^{\Box}$-style necessity operator. Some of Dunn's 1990b discussion of essential predication takes place in the context of such a logic. Dunn 1 990b provisionally assumes as an axiom, $x = y \rightarrow \Box x = y$. On our interpretation of identity, however, such an axiom is problematic. The technical reason is that a finite conjunction of relevant biconditionals logically does not, as a rule, imply its own necessity. So we must be careful about postulating that the infinite conjunction of relevant biconditionals, represented by $x = y$, logically implies its own necessity. This issue is further discussed in Kremer 1994.

13.6. *Uniformity and Restraint*

In §5 we proposed two conditions on a good object language definition of Geach predication: uniformity and restraint. Our formal definition of Geach predication tout court clearly satisfies uniformity. Moreover, suppose that we add '=' to $\mathbf{R}^{\forall\exists x}$ with the axioms suggested by our interpretation of identity.[20] Then our definition satisfies stronger forms of restraint than that proposed in §5, since the formulas (29)–(34) listed below are all consistent.[21] Here, the formula Geach(Ax, x) is $\forall x(Ax \rightarrow \forall y(x = y \rightarrow Ay))$, where Ay is the result of replacing every free occurrence of x in Ax with a free occurrence of y.

(29) $\quad$ Geach$(Fx, x) \,\&\, p \,\&\, {\sim}$Geach$(Fx \,\&\, p, x)$

(30) $\text{Geach}(Fx, x)\ \&\ p\ \&\ {\sim}\text{Geach}(Fx \vee p, x)$

(31) $\text{Geach}(Fx, x)\ \&\ {\sim}p\ \&\ {\sim}\text{Geach}(Fx\ \&\ p, x)$

(32) $\text{Geach}(Fx, x)\ \&\ {\sim}p\ \&\ {\sim}\text{Geach}(Fx \vee p, x)$

(33) $\text{Geach}(Fx, x)\ \&\ p\ \&\ {\sim}\text{Geach}(Fx\ \&\ (p \to p), x)$

(34) $\text{Geach}(Fx, x)\ \&\ {\sim}p\ \&\ {\sim}\text{Geach}(Fx\ \&\ (p \to p), x)$.[22]

Incidentally, the consistency of (30) shows that (24) in §10 is not a theorem of first order **R** with identity, as expected.

14. CONCLUDING REMARKS

We have presented Dunn's relevant predication as a formalisation of Geach predication. We believe that it can be used to formalise any notion of real predication, where what is wanted is that *real* properties be caught up with the identity of their terms. Dunn 1990a suggests that this is the case for Moore's 1919 internal properties and for St. Thomas's real, as opposed to merely logical, relations.

There is, however, no immediate application of relevant predication to the hornet's nest of issues surrounding confirmation theory and Goodman's 1955 new riddle of induction. For example, we cannot identify projectible properties with the relevant ones, since there are many relevant properties that are not projectible: $x = c$; ${\sim}Fx$; $(Fx \vee Gx)$. So there is more to a property's projectibility than the fact that it is relevant. Dunn's account might be of some help in these matters, but it must be supplemented by other considerations.

Nonetheless, it is interesting to see an account that so explicitly links a thing's identity to its real properties. And within the context of relevance as opposed to classical logic, we can make sure that things stay clean: all sorts of irrelevancies do not sneak in to our notion of what it is for a thing to be as it is.

We end this paper with a qualification. So far our dialectic has proceeded according to a suggestion that contingent relevant connections are, as Dunn 1987 puts it, "part of the objective ontological furniture of the world" (p. 446), and perhaps even irreducibly so (see §7.4, above). As Dunn acknowledges, however, what counts as relevant to what might be relative to human concerns. Indeed, it might even vary from inquiry to inquiry, and from application to application. If this is true, then we have advanced an application-relative notion of real properties and an application-relative notion of identity that parallel the application-relative notion of relevance.

ACKNOWLEDGEMENTS

For useful comments concerning content and exposition I thank Nuel Belnap, Mike Dunn, Bob Brandom, Jamie Tappenden, Ken Manders, Bob Daley, Sergio Tenenbaum and an anonymous referee at *Erkenntnis*.

NOTES

[1] We often suppress quotation marks surrounding mentioned formal expressions.
[2] See Note 3, below, for an elaboration of this point.
[3] Above, we noted that Dunn's project trades talk of real properties in for talk of relevant predication. So, though some of the motivation is put in terms of properties, the project need not offend nominalistic scruples. Indeed, Dunn's object language formula 'A is of a kind to determine relevant properties' does not even mention or quantify over *predicates*, let alone properties.
[4] Anderson and Belnap's 1962a variable sharing condition on a successful relevance logic – i.e., for every theorem $(A \rightarrow B)$, A and B must share some propositional variable – can be construed as a typographical requirement.
[5] It is tricky to express such a principle for a language containing individual variables but no individual constants, since '$x = y$' has no truth value. But there is a sense in which '$x = y$' can be taken to be true or false: it is true or false relative to an assignment of values to the individual variables. Similarly 'x' and 'y' can be taken to have referents, relative to an assignment of values to the variables. As we proceed, we informally treat variables as referring expressions and open formulas as bearers of truth or falsity.
[6] In conversation, Nuel Belnap pointed out the pertinence of the use-mention distinction here.
[7] Dunn 1987 mentions Geach's distinction in a list of possible applications of relevant predication, though he does not offer any details there. Our dialectical position is quite different from Dunn's: rather than reinforce an already defined notion of relevant predication with an after-the-fact application, we want to use intuitions surrounding Geach's distinction to give primary motivations leading up to the definitions of relevant predication. Dunn 199+, which I first saw after writing an early draft of the present paper, contains a long discussion of Cambridge change. Therein, he cites my dissertation (Kremer 1994) which contains an earlier version of the dialectic presented here. The ideas and emphases are different, but obviously related, having grown up together.
[8] Geach does not consider Cambridge properties expressed by formulas of this conjunctive form. Robert Brandom suggested the idea to me. Dunn has directed me to another property expressed by such a formula, Fodor's 1987 property of being a fridgeon: x is a *fridgeon* at t iff x is a particle at t and my fridge is on at t.
[9] Dunn 1990b considers a similar problem: that of necessarily existing objects that have all of their intrinsic properties essentially.
[10] Implicit in this four line argument is the intuition that, for a formula A and two theories

T_1 and T_2, T_1 and T_2 agree regarding A iff ($T_1 \vdash A$ iff $T_2 \vdash A$). But consider $T_1 = \{\neg p\}$ and $T_2 = \emptyset$. Though $T_1 \vdash p$ iff $T_2 \vdash p$, we might take it that T_1 and T_2 disagree regarding p since they disagree regarding the truth of p. The condition for theory-agreement (regarding a formula A) might then be strengthened to: T_1 and T_2 agree regarding A iff $((T_1 \vdash A$ iff $T_2 \vdash A)$ and $(T_1 \vdash \neg A$ iff $T_2 \vdash \neg A))$. Assuming the stronger condition for theory-agreement, we can still infer that $Ax = (Fx \;\&\; (p \rightarrow p))$ is a real predicate from the assumption that Fx is.

[11] Church 1956 precisely defines the *uniform substitution* of a formula Ax for an atomic formula Fx in another formula B, in first and higher order languages. The idea is to replace every occurrence of Fx with Ax, and every occurrence of Fy with Ay, and so on, making sure that things do not go wrong with bound variables.

[12] This theorem is corollary to the following lemma. Suppose that **L** is an interpreted first or higher order extensional classical language. If **L** is second or higher order, the higher order quantifiers can receive either their secondary or their principle interpretations, in the terminology of Henkin 1950. Suppose that Geach(A, x) is a map from formulas and variables to formulas such that, for each formula A and variable x, the formula Geach(A, x) results by a uniform substitution of A for Fx in the formula Geach(Fx, x). Then the following is true in **L**: (Geach(Fx, x) & p) $\supset$ Geach(Fx & p, x). For a proof, suppose that (Geach(Fx, x) & p) is true in **L**. Then $\forall y(Fy \equiv (Fy \;\&\; p))$ is also true in **L**. So Fx can be uniformly replaced by $(Fx \;\&\; p)$ salve veritate, in any formula. So, by uniformity, Geach(Fx & p, x) is true in **L**.

[13] Dunn chooses this context for the same reason, but he does not demonstrate in the same way the inappropriateness of the classical context.

[14] See for example Belnap 1960, who refers us to Nelson 1930 ("[implication] is a necessary connection of meaning"); Duncan-Jones 1934–35, (A entails B only when B "arises out of the meaning of" A); and Baylis 1931 (A entails B only when "the intensional meaning of B is identical with a part of the intensional meaning of A"). See also Anderson and Belnap 1962 and 1962a.

[15] The authors "hasten to add that these remarks are programmatic", and that they "have given them little thought".

[16] We have sought additional insight in Belnap 1960, Anderson and Belnap 1962 and 1962a, Prawitz 1964, Hockney and Wilson 1965, Woods 1967, Meyer 1968, Hockney 1968, Woods 1969, Köningsveld 1970, Bacon 1971, Meyer 1971, Anderson 1972, Curley 1972, Urquhart 1972, Köningsveld 1973, Anderson 1974, Garderförs 1978, Wolf 1978, Epstein 1979, Iseminger 1980, Epstein 1981, Diaz 1981, Palmer 1982, Burgess 1983, Burgess 1984, Copeland 1984, Dunn 1986, Rice 1986, Read 1988, Lewis 1988, van Dijk 1989, Parks-Clifford 1989, Woods 1989, Angell 1989, Myhill 1989, Urquhart 1989, Freeman 1989, Barker 1989, Priest and Crosthwaite 1989, and Ent2.

[17] The most notoriously counter-intuitive feature of most relevance logics is the rejection of the disjunctive syllogism. See Ent1, pp. 165–167 and 296–300, Ent2 pp.4 90 and 498–502, Barker 1975, Belnap 1977, Belnap and Dunn 1981, Routley 1984 and Lavers 1988.

[18] This point should be no surprise. Ater all, we do not expect the proof theory of, say, **S4** to tell us which claims of the form $\Box A$ are true. The proof theory's job is to formalise such claims' inferential connections to other claims and such.

[19] We could consider both formalisations, taking (21) to capture a stronger notion of Geach predication than (16). At this point in the dialectic, however, it is best to keep fine distinctions to a minimum.

[20] These axioms are $x = x$ (reflexivity), $x = y \rightarrow y = x$ (symmetry), and $x = y \rightarrow (y = z \rightarrow x = z)$ (nested transitivity).

[21] A formula A is consistent iff $\neg A$ is not a theorem of $\mathbf{R}^{\forall\exists x=}$, where $\mathbf{R}^{\forall\exists x=}$ results by enriching $\mathbf{R}^{\forall\exists x}$ with the axioms for identity listed in note 20.
[22] These are consistent even if we add the following identity axiom schemes to $\mathbf{R}^{\forall\exists x=}$: (1) substitution, $(x = y \,\&\, Ax) \rightarrow Ay$; and (2) an axiom scheme stating that all atomic formulas with x free express Geach properties with respect to x: $\forall x(Ax \rightarrow \forall y(x = y \rightarrow Ay))$, where Ax is atomic and x is free in Ax.

REFERENCES

Anderson, A. R.: 1972, 'An Intensional Interpretation of Truth Values', *Mind* **81**, 348–371.

Anderson, A. R.: 1974, 'Meaning and Implication', *Idealistic Studies* **4**, 79–88.

Anderson, A. R. and Belnap, N. D.: 1962, 'Tautological Entailments', *Philosophical Studies* **13**, 9–24.

Anderson, A. R. and Belnap, N. D.: 1962a, 'The Pure Calculus of Entailment', *Journal of Symbolic Logic* **27**, 19–52.

Anderson, A. R. and Belnap, N. D.: 1975, *Entailment: The Logic of Relevance and Necessity*, Volume I, Princeton University Press.

Anderson, A. R., Belnap, N. D., and Dunn, J. M.: 1992, *Entailment: The Logic of Relevance and Necessity*, Volume II, Princeton University Press.

Angell, R. B.: 1989, 'Deducibility, Entailment and Analytic Containment', Norman and Sylvan 1989, 119–143.

Bacon, J.: 1971, 'The Subjunctive Conditional as Relevant Implication', *Philosophia* **1**, 61–80.

Barker, J. A.: 1975, 'Relevance Logic, Classical Logic and Disjunctive Syllogism', *Philosophical Studies* **27**, 361–376.

Barker, J. A.: 1989, 'Relevance Logic and Inferential Knowledge', Norman and Sylvan 1989, 305–325.

Baylis, C. A.: 1931, 'Implication and Subsumption', *The Monist* **41**, 392–399.

Bealer, G.: 1982, *Quality and Concept*, Clarendon Press, Oxford.

Belnap, N. D.: 1960, 'Entailment and Relevance', *The Journal of Symbolic Logic* **25**, 144–146.

Belnap, N. D.: 1977, 'A Useful Four-Valued Logic', in J. M. Dunn and G. Epstein (eds.), *Modern Uses of Multiple-Valued Logic*, D. Reidel Publishing Co., Dordrecht, pp. 8–37.

Belnap, N. D. and Dunn, J. M.: 1981, 'Entailment and the Disjunctive Syllogism', in G. Fløistad and G. H. von Wright, (eds.), *Philosophy of Language/Philosophical Logic*, Martinus Nijhoff, The Hague, pp. 337–366.

Boolos, G.: 1971, 'The Iterative Conception of a Set', *Journal of philosophy* **68**, 215–232.

Bradley, F. H.: 1893, *Appearance and Reality: A Metaphysical Essay*, Swann Sonnenschein & Co. Ltd, London. Second edition, revised, 1897.

Burgess, J. P.: 1983, 'Common Sense and Relevance', *Notre Dame Journal of Formal Logic* **24**, 41–53.

Burgess, J. P.: 1984, 'Read on Relevance: a Rejoinder', *Notre Dame Journal of Formal Logic* **25**, 217–223.

Church, A.: 1956, *Introduction to Mathematical Logic*, Princeton University Press, Princeton.

Copeland, B. J.: 1984, 'Horseshoe, Hook and Relevance', *Theoria* **50**, 148–164.

Curley, E. M.: 1972, 'Lewis and Entailment', *Philosophical Studies* **23**, 198–204.

Diaz, M. R.: 1981, *Topics in the Logic of Relevance*, Analytica, Berlin and New York.

Duncan-Jones, A. E.: 1935, 'Is Strict Implication the Same as Entailment?', *Analysis* **2**, 70–78.

Dunn, J. M.: 1986, 'Relevant Logic and Entailment', in D. Gabbay and F. Guenthner, (eds.), *Handbook of Philosophical Logic, Volume III: Alternatives to Classical Logic*, D. Reidel Publishing Co., Dordrecht, pp. 117–229.

Dunn, J. M.: 1987, 'Relevant Predication 1: The Formal Theory', *Journal of Philosophical Logic* **16**, 347–381. Reprinted in Anderson, Belnap and Dunn 1992, §74, pp. 445–471. Page and section references are to this reprinting.

Dunn, J. M.: 1990, 'The Frame Problem and Relevant Predication', in H. Kyburg, R. Loui and C. Carlson, (eds.), *Knowledge, Representation and Defeasible Reasoning*, Kluwer Academic Publishers, Dordrecht, pp. 108–116.

Dunn, J. M.: 1990a, 'Relevant Predication 2: Intrinsic Properties and Internal Relations', *Philosophical Studies* **60**, 177–206.

Dunn, J. M.: 1990b, 'Relevant Predication 3: Essential Properties', in J. M. Dunn and A. Gupta, (eds.), *Truth or Consequences: Essays in Honor of Nuel Belnap*, Kluwer Academic Publishers, Dordrecht, pp. 77–95.

Dunn, J. M.: 199+, 'A Logical Framework for the Notion of Natural Property', manuscript.

Dunn, J. M.: 199+a, 'Is Existence a (Relevant) Predicate?', manuscript.

Epstein, R. L.: 1979, 'Relatedness and Implication', *Philosophical Studies* **36**, 137–173.

Epstein, R. L.: 1981, 'Relatedness and Dependence in Propositional Logics', *The Journal of Symbolic Logic* **46**, 202.

Fitch, F. B.: 1952, *Symbolic Logic: An Introduction*, The Ronald Press Co., New York.

Fodor, J. A.: 1987, 'Modules, Frames, Fridgeons, Sleeping Dogs, and the Music of the Spheres', in Z. W. Plyshyn, (ed.), *The Robot's Dilemma*, Ablex Publishing Corporation, Norwood, NI., pp. 139–149.

Freeman, J. B.: 1989, 'Quantification, Identity, and Opacity in Relevance Logic', in Norman and Sylvan 1989, pp. 305–316.

Gardenfors, P.: 1978, 'On the Logic of Relevance', *Synthese* **37**, 251–367.

Geach, P.: 1969, *God and the Soul*, Routledge and Kegan Paul Ltd., London.

Goodman, N.: 1955, *Fact, Fiction, and Forecast*, Harvard University Press, Cambridge MA. Fourth edition, revised, The Bobbs-Merrill Co., Inc., Indianapolis 1983.

Hamilton, E. and Cairns, H. (eds.): 1961, *The Collected Dialogues of Plato*, Bollingen Foundation, New York.

Henkin, L.: 1950, 'Completeness in the Theory of Types', *Journal of Symbolic Logic* **15**, 81–91.

Hockney, D. J.: 1968, 'A Vindication of System E', *Logique et Analyse* **11**, 480–491.

Hockney, D. J. and Wilson, J. C.: 1965, 'In Defense of a Relevance Condition', *Logique et Analyse* **8**, 211–220.

Iseminger, G.: 1980, 'Is Relevance Necessary for Validity?', *Mind* **89**, 73–88.

Köningsveld, H.: 1973, 'What Does '*p* entails *c*' Mean?', *Mind* **82**, 118–122.

Kremer, P.: 1989, 'Relevant Predication: Grammatical Characterisations', *Journal of Philosophical Logic* **18**, 349–382.

Kremer, P.: 1994, *Real Properties, Relevance Logic and Identity*, Doctoral Dissertation, University of Pittsburgh, University Microfilms, Ann Arbor.

Lavers, P.: 1988. 'Relevance and Disjunctive Syllogism', *Notre Dame Journal of Formal Logic* **29**, 34–44.

Lewis, D. K.: 1988, 'Relevant Implication', *Theoria* **54**, 162–174.

Meyer, R. K.: 1968, 'Entailment and Relevant Implication', *Logique et Analyse* **11**, 472–479.

Meyer, R. K.: 1971, 'Entailment', *Journal of Philosophy* **68**, 808–818.

Meyer, R. K.: 197+, 'Arithmetic Relevantly Formulated', unpublished manuscript.

Moore, G. E.: 1919, 'External and Internal Relations', *Proceedings of the Aristotelian Society, 1919–20.* Reprinted in *Philosophical Studies*, Harcourt, Brace and Co. Inc., New York.

Myhill, J.: 1989, 'Real Implication', in Norman and Sylvan 1989, pp. 157–165.

Nelson, E. J.: 1930, 'Intensional Relations', *Mind* **39**, 440–453.

Norman, J. and Sylvan, R. (eds.): 1989, *Directions in Relevant Logic*, Kluwer Academic Publishers, Dordrecht.

Palmer, H.: 1982, 'Revisionary Ifs', *Erkenntnis* **17**, 249–261.

Parks-Clifford, J. E.: 1989, 'Literal Relevance', in Norman and Sylvan 1989, pp. 59–75.

Pegis, A. C. (trans): 1945, *The Basic Writings of St. Thomas Aquinas* (two volumes), Random House, New York.

Prawitz, D.: 1964, 'Normal Deductions', *Journal of Symbolic Logic* **29**, 152.

Priest, G. and Crosthwaite, J.: 1989, 'Relevance, Truth and Meaning', in Norman and Sylvan 1989, pp. 377–397.

Read, S.: 1988, *Relevant Logic*, Basil Blackwell, Oxford.

Rice, H.: 1986, 'Entailment', *Mind* **95**, 345–360.

Routley, R.: 1984, 'Relevantism, Material Detachment, and the Disjunctive Syllogism Argument', *Canadian Journal of Philosophy* **14**, 167–188.

Urquhart, A.: 1972, 'A General Theory of Implication', *The Journal of Symbolic Logic* **37**, 443.

Urquhart, A.: 1989, 'What is Relevant Implication?', in Norman and Sylvan 1989, pp. 167–174.

van Dijk, T. A.: 1989, ' "Relevance" in Logic and Grammar', in Norman and Sylvan 1989, pp. 25–57.

Wolf, R. G.: 1978, 'Are Relevant Logics Deviant?', *Philosophia* **7**, 327–340.

Woods, J.: 1967, 'Is There a Relation of Intensional Conjunction?', *Mind* **76**, 357–368.

Woods, J.: 1969, 'Intensional Relations', *Studia Logica* **25**, 61–73.

Manuscript submitted November 28, 1995
Final version received December 24, 1996

Dept. of Philosophy
Yale University
P.O. Box 208306
New Haven, CT 0652-8306
E-mail: philip.kremer@yale.edu