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# Matching Topological and Frame Products of Modal Logics

**Abstract.** The simplest combination of unimodal logics $L_1$ and $L_2$ into a bimodal logic is their *fusion*, $L_1 \otimes L_2$, axiomatized by the theorems of $L_1$ for $\Box_1$ and of $L_2$ for $\Box_2$. Shehtman introduced combinations that are not only bimodal, but two-dimensional: he defined 2-d Cartesian products of 1-d Kripke frames, using these Cartesian products to define the *frame product* $L_1 \times L_2$ of $L_1$ and $L_2$. Van Benthem, Bezhanishvili, ten Cate and Sarenac generalized Shehtman's idea and introduced the *topological product* $L_1 \times_t L_2$, using Cartesian products of topological spaces rather than of Kripke frames. Frame products have been extensively studied, but much less is known about topological products. The goal of the current paper is to give necessary and sufficient conditions for the topological product to match the frame product, for Kripke complete extensions of S4: $L_1 \times_t L_2 = L_1 \times L_2$ iff $L_1 \supsetneq S5$ or $L_2 \supsetneq S5$ or $L_1, L_2 = S5$.

*Keywords*: Bimodal logic, Multimodal logic, Topological semantics, Topological product, Product space.

Let $\mathcal{L}$ be a propositional language with a set $PV$ of propositional variables; standard Boolean connectives $\wedge$, $\vee$ and $\neg$; and one modal operator, $\Box$. And let $\mathcal{L}_{12}$ be like $\mathcal{L}$, except with *two* modal operators, $\Box_1$ and $\Box_2$. We use standard definitions of $\supset$, $\equiv$, $\Diamond$, $\Diamond_1$ and $\Diamond_2$. A *normal unimodal* [*bimodal*] *logic* is any set L of formulas of $\mathcal{L}$ [$\mathcal{L}_{12}$] containing every propositional tautology and the formula $(\Box(p \supset q) \supset (\Box p \supset \Box q))$ [the formulas $(\Box_1(p \supset q) \supset (\Box_1 p \supset \Box_1 q))$ and $(\Box_2(p \supset q) \supset (\Box_2 p \supset \Box_2 q))$], and closed under modus ponens, necessitation for $\Box$ [for $\Box_1$ and $\Box_2$], and substitution – we will suppress the adjective 'normal'. A logic L is *consistent* iff L excludes some formula in the relevant language. L *extends* L$'$ iff $L' \subseteq L$. Given any logic L and any set $\Gamma$ of formulas, $L + \Gamma$ is the logic generated by closing $L \cup \Gamma$ under modus ponens, necessitation (for either $\Box$ or for each of $\Box_1$ and $\Box_2$, depending on the language) and substitution. If $\Gamma = \{A_1, \ldots, A_n\}$, then we write $L + A_1 + \cdots + A_n$ for $L + \Gamma$. K is the smallest unimodal logic. $S4 =_{df} K + (\Box p \supset p) + (\Box p \supset \Box\Box p)$, $S4.2 =_{df} S4 + (\Diamond\Box p \supset \Box\Diamond p)$, $S5 =_{df} S4 + (\Diamond p \supset \Box\Diamond p)$, $Triv =_{df} S4 + (p \supset \Box p)$, and $Verum =_{df} K + \Box p$.

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The simplest combination of two unimodal logics $\mathrm{L}_1$ and $\mathrm{L}_2$ into a bimodal logic is their bimodal *fusion*, $\mathrm{L}_1 \otimes \mathrm{L}_2$: let $\mathrm{L}_1'$ [$\mathrm{L}_2'$] be the set of formulas of $\mathcal{L}_{12}$ got by replacing each occurrence of $\Box$ in each formula in $\mathrm{L}_1$ [$\mathrm{L}_2$] by $\Box_1$ [$\Box_2$]; and let $\mathrm{L}_1 \otimes \mathrm{L}_2$ be the smallest set of formulas of $\mathcal{L}_{12}$ containing $\mathrm{L}_1' \cup \mathrm{L}_2'$ and closed under modus ponens, necessitation for $\Box_1$ and for $\Box_2$, and substitution.

Shehtman [13] introduces combinations that are not only bimodal, but two-dimensional: he defines a kind of birelational Kripke frame as a Cartesian product of two Kripke frames. The *frame product* $\mathrm{L}_1 \times \mathrm{L}_2$, is then the set of formulas in the language $\mathcal{L}_{12}$ validated by every product of a Kripke frame validating $\mathrm{L}_1$ with a Kripke frame validating $\mathrm{L}_2$.

For unimodal logics stronger than S4, the McKinsey-Tarski topological semantics [9–11] for the unimodal language $\mathcal{L}$ generalizes the Kripke semantics. Van Benthem et al. [15] generalize Shehtman's products of frames to products of topological spaces: they define a kind of bitopological space as a Cartesian product of two topological spaces. The *topological product* $\mathrm{L}_1 \times_t \mathrm{L}_2$, is then the set of formulas in the language $\mathcal{L}_{12}$ validated by every product of a topological space validating $\mathrm{L}_1$ with a topological space validating $\mathrm{L}_2$. Frame products have been extensively studied,[1] but much less is known about topological products. The main purpose of the current paper is to give necessary and sufficient conditions for the topological product to match the frame product, for Kripke complete extensions $\mathrm{L}_1$ and $\mathrm{L}_2$ of S4: $\mathrm{L}_1 \times_t \mathrm{L}_2 = \mathrm{L}_1 \times \mathrm{L}_2$ iff $\mathrm{L}_1 \supsetneq \mathrm{S5}$ or $\mathrm{L}_2 \supsetneq \mathrm{S5}$ or $\mathrm{L}_1 = \mathrm{L}_2 = \mathrm{S5}$.

## 1. Details

### 1.1. Kripke Semantics

Here are the gory, and typically routine, details. A *Kripke uniframe* [*biframe*] is an ordered pair [triple] $F = \langle W, R\rangle$ [$F = \langle W, R_1, R_2\rangle$] where $W$ is a nonempty set and $R$ is a binary relation on $W$ [$R_1$ and $R_2$ are binary relations on $W$]. We sometimes use the expression *frame* ambiguously for uniframes and biframes. If $F$ is a uniframe [biframe and $i \in \{1, 2\}$], then $F$ is *reflexive*, *transitive*, etc. [ $i$-reflexive, $i$-transitive, etc.], iff $R$ [$R_i$] is reflexive, transitive, etc. If $F$ is a biframe, then $F$ is simply *reflexive*, *transitive*, etc., iff $F$ is $i$-reflexive, $i$-transitive, etc., for each $i \in \{1, 2\}$.

[1] As noted in [15], a systematic study of multi-dimensional modal logics of products of Kripke frames can be found in [4], and an up-to-date account of the most important results in the field can be found in [5]. See also [7].

If $F$ is a uniframe [biframe and $i \in \{1,2\}$] and $S \subseteq W$, then the *interior*[*s*] of $S$ is [are] $\mathsf{Int}(S) =_{\rm df} \{w \in W : \forall w' \in W, wRw' \Rightarrow w' \in S\}$ [$\mathsf{Int}_i(S) =_{\rm df} \{w \in W : \forall w' \in W, wR_iw' \Rightarrow w' \in S\}$]. A *Kripke unimodel* [*bimodel*] is an ordered pair $M = \langle F, V\rangle$, where $F = \langle W, R\rangle$ [$F = \langle W, R_1, R_2\rangle$] is a uniframe [biframe] and $V : PV \rightarrow \mathcal{P}(W)$. The *valuation function* $V$ extends to all formulas in the language $\mathcal{L}$ [$\mathcal{L}_{12}$] as follows: $V(\neg A) = W - V(A)$, $V(A \wedge B) = V(A) \cap V(B)$, $V(A \vee B) = V(A) \cup V(B)$, and $V(\Box A) = \mathsf{Int}(V(A))$ [$V(\Box_i A) = \mathsf{Int}_i(V(A)), i = 1, 2$]. We say $w \Vdash A$ iff $w \in V(A)$. We say $M \Vdash A$ iff $V(A) = W$. We say $F \Vdash A$ iff $M \Vdash A$ for every model $M = \langle F, V\rangle$. If $\Gamma$ is a set of formulas, then we say that $F \Vdash \Gamma$ iff $F \Vdash A$ for every $A \in \Gamma$. If $\mathcal{F}$ is a class of frames, then we say that $\mathcal{F} \Vdash \Gamma$ iff $F \Vdash \Gamma$ for every $F \in \mathcal{F}$. $\mathsf{Fr}(\Gamma) =_{\rm df} \{F : F \Vdash \Gamma\}$. If $\mathcal{F}$ is a class of frames, then $\mathsf{Log}(\mathcal{F}) =_{\rm df} \{A : \forall F \in \mathcal{F}, F \Vdash A\}$: note that $\mathsf{Log}(\mathcal{F})$ is a normal modal logic. If $F$ is a frame, $\mathsf{Log}(F) =_{\rm df} \mathsf{Log}(\{F\})$. The following results are well-known in the unary case: $\mathsf{Fr}(\mathrm{K}) = \{F : F \text{ is a uniframe}\}$ and $\mathrm{K} = \mathsf{Log}(\mathsf{Fr}(\mathrm{K}))$; $\mathsf{Fr}(\mathrm{S4}) = \{F : F \text{ is a reflexive, transitive uniframe}\}$ and $\mathrm{S4} = \mathsf{Log}(\mathsf{Fr}(\mathrm{S4}))$; and $\mathsf{Fr}(\mathrm{S5}) = \{F : F \text{ is a reflexive, transitive, symmetric uniframe}\}$ and $\mathrm{S5} = \mathsf{Log}(\mathsf{Fr}(\mathrm{S5})) = \mathsf{Log}(\{\langle W, R\rangle : \forall w, w' \in W, wRw'\})$. A logic L is *Kripke complete* iff $\mathrm{L} = \mathsf{Log}(\mathcal{F})$ for some class $\mathcal{F}$ of frames; equivalently, iff $\mathrm{L} = \mathsf{Log}(\mathsf{Fr}(\mathrm{L}))$.

The definitions and results in this paragraph are from [13] and [4]. Given two uniframes $F_1 = \langle W_1, R_1\rangle$ and $F_2 = \langle W_2, R_2\rangle$, the biframe $F_1 \times F_2 =_{\rm df} \langle W_1 \times W_2, S_1, S_2\rangle$, where $\langle w_1, w_2\rangle S_1 \langle w_1', w_2'\rangle$ iff $w_1 R_1 w_1'$ and $w_2 = w_2'$; and where $\langle w_1, w_2\rangle S_2 \langle w_1', w_2'\rangle$ iff $w_1 = w_1'$ and $w_2 R_2 w_2'$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are classes of uniframes, then $\mathcal{F}_1 \times \mathcal{F}_2 =_{\rm df} \{F_1 \times F_2 : F_1 \in \mathcal{F}_1 \text{ and } F_2 \in \mathcal{F}_2\}$. If $\mathrm{L}_1$ and $\mathrm{L}_2$ are unimodal logics, then the *frame product* of $\mathrm{L}_1$ and $\mathrm{L}_2$ is the bimodal logic $\mathrm{L}_1 \times \mathrm{L}_2 =_{\rm df} \mathsf{Log}(\mathsf{Fr}(\mathrm{L}_1) \times \mathsf{Fr}(\mathrm{L}_2))$. Every product frame validates the following three formulas:

| | | |
|---|---|---|
| $com_{\supset}$ | (left commutativity) | $\Box_1\Box_2 p \supset \Box_2\Box_1 p$ |
| $com_{\subset}$ | (right commutativity) | $\Box_2\Box_1 p \supset \Box_1\Box_2 p$ |
| *chr* | (Church–Rosser) | $\Diamond_1\Box_2 p \supset \Box_2\Diamond_1 p$. |

The *commutator* of $\mathrm{L}_1$ and $\mathrm{L}_2$ is the bimodal logic $[\mathrm{L}_1, \mathrm{L}_2] =_{\rm df} \mathrm{L}_1 \otimes \mathrm{L}_2 + com_{\supset} + com_{\subset} + chr$. We always have $\mathrm{L}_1 \otimes \mathrm{L}_2 \subseteq \mathrm{L}_1 \times \mathrm{L}_2$ and almost always $\mathrm{L}_1 \otimes \mathrm{L}_2 \subsetneq \mathrm{L}_1 \times \mathrm{L}_2$;[2] For many popular modal logics, $\mathrm{L}_1 \times \mathrm{L}_2 = [\mathrm{L}_1, \mathrm{L}_2]$

[2]But not always. Suppose that one of $\mathrm{L}_1$ or $\mathrm{L}_2$ is either Triv or Verum, and that the other is Kripke complete. Then $\mathrm{L}_1 \otimes \mathrm{L}_2 = \mathrm{L}_1 \times \mathrm{L}_2$. Ditto, if either $\mathrm{L}_1$ or $\mathrm{L}_2$ is inconsistent, regardless of whether the other is Kripke complete.

(see [4, Theorem 7.12]), in particular when $L_1, L_2 \in \{S4, S5\}$. (Sometimes, however, this fails: see [4, Theorem 8.2].)

### 1.2. Topological Semantics

A *topological unispace* [*bispace*] is an ordered pair [triple] $T = \langle X, \tau \rangle$ [$T = \langle X, \tau_1, \tau_2 \rangle$] where $X$ is a nonempty set and $\tau$ is a topology on $X$ [$\tau_1$ and $\tau_2$ are topologies on $X$].[3] If $T$ is a unispace [bispace and $i \in \{1, 2\}$], then a set $Y \subseteq X$ is *open* [*i-open*] iff $Y \in \tau$ [$Y \in \tau_i$] and *closed* [*i-closed*] iff $X - Y \in \tau$ [$X - Y \in \tau_i$]. For unispaces [bispaces], the topology $\tau$ [each topology $\tau_i$] is associated with an interior operator $\mathsf{Int}$ [$\mathsf{Int}_i$] and a closure operator $\mathsf{Cl}$ [$\mathsf{Cl}_i$]. We sometimes use the expression *space* ambiguously for unispaces and bispaces. A *topological unimodel* [*bimodel*] is an ordered pair $M = \langle T, V \rangle$, where $T = \langle X, \tau \rangle$ [$T = \langle X, \tau_1, \tau_2 \rangle$] is a unispace [bispace] and $V : PV \to \mathcal{P}(W)$.The *valuation function* $V$ extends to all formulas in the language $\mathcal{L}$ [$\mathcal{L}_{12}$] as follows: $V(\neg A) = X - V(A)$, $V(A \wedge B) = V(A) \cap V(B)$, $V(A \vee B) = V(A) \cup V(B)$, and $V(\Box A) = \mathsf{Int}(V(A))$ [$V(\Box_i A) = \mathsf{Int}_i(V(A)), i = 1, 2$]. We say $x \Vdash A$ iff $x \in V(A)$. We say $M \Vdash A$ iff $V(A) = X$. We say $T \Vdash A$ iff $M \Vdash A$ for every model $M = \langle T, V \rangle$. If $\Gamma$ is a set of formulas, then we say that $T \Vdash \Gamma$ iff $T \Vdash A$ for every $A \in \Gamma$. If $\mathcal{T}$ is a class of spaces, then we say that $\mathcal{T} \Vdash \Gamma$ iff $T \Vdash \Gamma$ for every $T \in \mathcal{T}$. $\mathsf{Sp}(\Gamma) =_{\text{df}} \{T : T \Vdash \Gamma\}$. If $\mathcal{T}$ is a class of spaces, then $\mathsf{Log}(\mathcal{T}) =_{\text{df}} \{A : \forall T \in \mathcal{T}, T \Vdash A\}$: note that $\mathsf{Log}(\mathcal{T})$ is a normal extension of S4. If $T$ is a space, $\mathsf{Log}(T) =_{\text{df}} \mathsf{Log}(\{T\})$. A logic L is *topologically complete* iff $L = \mathsf{Log}(\mathcal{T})$ for some class $\mathcal{T}$ of spaces; equivalently, iff $L = \mathsf{Log}(\mathsf{Sp}(L))$.

A unispace [bispace] is *Alexandrov* [*i-Alexandrov*] iff any intersection of open [$i$-open] sets is open [$i$-open]. A bispace is simply *Alexandrov* iff it is $i$-Alexandrov for each $i \in \{1, 2\}$. Every reflexive, transitive uniframe $F = \langle W, R \rangle$ [biframe $F = \langle W, R_1, R_2 \rangle$] generates an Alexandrov unispace $T_F = \langle W, \tau \rangle$ [bispace $T_F = \langle W, \tau_1, \tau_2 \rangle$]: let $\tau = \{O \subseteq W : w \in O \text{ and } wRw' \Rightarrow w' \in O\}$ [$\tau_i = \{O \subseteq W : w \in O \text{ and } wR_iw' \Rightarrow w' \in O\}$]. Note that a space is Alexandrov iff it is generated in this way. Note also that the definition of $\mathsf{Int}(S)$ [$\mathsf{Int}_i(S)$] given for $S \subseteq W$ in Sect. 1.1 corresponds exactly to the topological interior associated with $\tau$ [$\tau_i$]. This last point implies that any valuation function $V : PV \to \mathcal{P}(W)$ extends in the same way when defined in terms of the Kripke model $\langle F, V \rangle$ or the topological model $\langle T_F, V \rangle$. We will treat reflexive, transitive frames as notational variants of Alexandrov spaces, identifying $F$ and $T_F$. Let *Alex* be the class of Alexandrov unispaces.

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[3] We assume familiarity with the basics of point-set topology.

The following results are well-known, the first originally due to [10]: $\mathrm{S4} = \mathsf{Log}(\{T : T \text{ is a unispace}\}) = \mathsf{Log}(\mathbb{Q}) = \mathsf{Log}(\mathbb{R}) = \mathsf{Log}(\mathit{Alex})$, where $\mathbb{R}$ and $\mathbb{Q}$ are the reals and the rationals with the standard topologies; $\mathrm{S5} = \mathsf{Log}(\{T : T$ is an almost discrete unispace$\}) = \mathsf{Log}(\{T : T$ is a trivial unispace$\})$.[4] Also, $\mathsf{Sp}(\mathrm{S4}) = \{T : T$ is a unispace$\}$; and $\mathsf{Sp}(\mathrm{S5}) = \{T : T$ is an almost discrete unispace$\}$. Note that every almost discrete unispace is Alexandrov: thus, $\mathsf{Sp}(\mathrm{S5}) \subseteq \mathit{Alex}$. Indeed, if $\mathrm{L} \supseteq \mathrm{S5}$, then $\mathsf{Sp}(\mathrm{L}) \subseteq \mathit{Alex}$. Given the identification of reflexive, transitive Kripke frames with Alexandrov spaces, any Kripke complete extension of S4 is also topologically complete.

The definitions in this paragraph are from [15]. Given two unispaces $T_1 = \langle X_1, \tau_1 \rangle$ and $T_2 = \langle X_2, \tau_2 \rangle$, the bispace $T_1 \times T_2 =_{\mathrm{df}} \langle X_1 \times X_2, \sigma_1, \sigma_2 \rangle$, where $\sigma_1$ has as a basis the family $\{O \times \{x\} : O \in \tau_1$ and $x \in X_2\}$ and $\sigma_2$ has as a basis the family $\{\{x\} \times O : x \in X_1$ and $O \in \tau_2\}$.[5] If $\mathcal{T}_1$ and $\mathcal{T}_2$ are classes of unispaces, then $\mathcal{T}_1 \times \mathcal{T}_2 =_{\mathrm{df}} \{T_1 \times T_2 : T_1 \in \mathcal{T}_1$ and $T_2 \in \mathcal{T}_2\}$. If $\mathrm{L}_1$ and $\mathrm{L}_2$ are unimodal logics, then the *topological product* of $\mathrm{L}_1$ and $\mathrm{L}_2$ is the bimodal logic $\mathrm{L}_1 \times_t \mathrm{L}_2 =_{\mathrm{df}} \mathsf{Log}(\mathsf{Sp}(\mathrm{L}_1) \times \mathsf{Sp}(\mathrm{L}_2))$.

In general,

$$\mathrm{L}_1 \otimes \mathrm{L}_2 \;\subseteq\; \mathrm{L}_1 \times_t \mathrm{L}_2 \;\subseteq\; \mathrm{L}_1 \times \mathrm{L}_2.$$

The main result of [15] is that

$$\mathrm{S4} \otimes \mathrm{S4} \;=\; \mathrm{S4} \times_t \mathrm{S4} \;\subsetneq\; \mathrm{S4} \times \mathrm{S4}.$$

But going topological does not always have the same effect [6]:

$$\mathrm{S5} \otimes \mathrm{S5} \;\subsetneq\; \mathrm{S5} \times_t \mathrm{S5} \;=\; \mathrm{S5} \times \mathrm{S5}, \text{ and}$$

$$\mathrm{S4} \otimes \mathrm{S5} \;\subsetneq\; \mathrm{S4} \times_t \mathrm{S5} \;=\; \mathrm{S4} \otimes \mathrm{S5} + \mathit{com}_{\supset} + \mathit{chr} \;\subsetneq\; \mathrm{S4} \times \mathrm{S5}.$$

Given that $\mathrm{S4} \times_t \mathrm{S5} = \mathrm{S4} \otimes \mathrm{S5} + \mathit{com}_{\supset} + \mathit{chr}$, $\mathrm{S4} \times_t \mathrm{S5}$ is identical to the semiproduct (to use Shehtman's [14] expression) of S4 and S5: such logics are studied in [8].

Not much else is known about topological products. It is worth noting how different S4 and S5 are in this context: the topological product of S4

[4] A unispace $\langle X, \tau \rangle$ is *almost discrete* iff every open set is closed, and *trivial* iff $\tau = \{\emptyset, X\}$.

[5] Recall that every Kripke frame $F$ generates an Alexandrov space $T_F$: indeed, we are treating $F$ and $T_F$ as notational variants. Also note that, if $F_1$ and $F_2$ are Kripke frames, then $T_{F_1 \times F_2} = T_{F_1} \times T_{F_2}$. Thus the frame product of $F_1$ and $F_2$ is a notational variant of the topological product of $T_{F_1}$ and $T_{F_2}$, so we can be a bit sloppy about which product (Kripke or topological) we are using '$\times$' for when considering Kripke frames/Alexandrov spaces.

with itself matches the *fusion* of S4 with itself; by contrast, the topological product of S5 with itself matches the *frame product* of S5 with itself. This suggests two general questions: When does $\mathrm{L}_1 \times_t \mathrm{L}_2 = \mathrm{L}_1 \otimes \mathrm{L}_2$? And when does $\mathrm{L}_1 \times_t \mathrm{L}_2 = \mathrm{L}_1 \times \mathrm{L}_2$? As indicated above, the main purpose of this paper is to prove the following – and to answer the second question – for Kripke complete extensions $\mathrm{L}_1$ and $\mathrm{L}_2$ of S4:

THEOREM 1.1. $\mathrm{L}_1 \times_t \mathrm{L}_2 = \mathrm{L}_1 \times \mathrm{L}_2$ *iff* $\mathrm{L}_1 \supsetneq \mathrm{S5}$ *or* $\mathrm{L}_2 \supsetneq \mathrm{S5}$ *or* $\mathrm{L}_1 = \mathrm{L}_2 = \mathrm{S5}$.

Before we turn to the proof of Theorem 1.1, we note a fact about interiors and closures in product spaces. In particular, consider spaces $T_1 = \langle X_1, \tau_1 \rangle$ and $T_2 = \langle X_2, \tau_2 \rangle$, and their product $T_1 \times T_2 =_{\mathrm{df}} \langle X_1 \times X_2, \sigma_1, \sigma_2 \rangle$. We will write $\mathsf{Int}_{T_i}$ [$\mathsf{Cl}_{T_i}$] for the interior [closure] operator in the space $T_i$, and $\mathsf{Int}_i$ [$\mathsf{Cl}_i$] for the $i$th interior [closure] operator in the product space $T_1 \times T_2$. For any $x \in X_1$, we define a right projection operator $\mathsf{rt}_x : \mathcal{P}(X_1 \times X_2) \to \mathcal{P}(X_2)$ as follows: $\mathsf{rt}_x(S) = \{y \in X_2 : \langle x, y \rangle \in S\}$, for $S \subseteq X_1 \times X_2$. Similarly, for any $y \in X_2$, we define a left projection operator $\mathsf{lft}_y : \mathcal{P}(X_1 \times X_2) \to \mathcal{P}(X_1)$ as follows: $\mathsf{lft}_y(S) = \{x \in X_1 : \langle x, y \rangle \in S\}$. Note that $S = \bigcup_{x \in X_1} \{x\} \times \mathsf{rt}_x(S) = \bigcup_{y \in X_2} \mathsf{lft}_y(S) \times \{y\}$. As for interiors and closures:

LEMMA 1.2. *For any* $S \subseteq X_1 \times X_2$*:*

$$\mathsf{Int}_1(S) = \bigcup_{y \in X_2} \mathsf{Int}_{T_1}(\mathsf{lft}_y(S)) \times \{y\}$$

$$\mathsf{Cl}_1(S) = \bigcup_{y \in X_2} \mathsf{Cl}_{T_1}(\mathsf{lft}_y(S)) \times \{y\}$$

$$\mathsf{Int}_2(S) = \bigcup_{x \in X_1} \{x\} \times \mathsf{Int}_{T_2}(\mathsf{rt}_x(S))$$

$$\mathsf{Cl}_2(S) = \bigcup_{x \in X_1} \{x\} \times \mathsf{Cl}_{T_2}(\mathsf{rt}_x(S))$$

## 2. Proving Theorem 1.1

Theorem 1.1 follows directly from Corollaries 2.7 and 2.10, below. We begin by specifying some particular uniframes and unispaces. First our uniframes: $\substack{\circ\\\circ} =_{\mathrm{df}} \langle \{0, 1\}, \leq \rangle$ and, for each $n \geq 1$, $\circ_n =_{\mathrm{df}} \langle \{1, \ldots, n\}, \{1, \ldots, n\} \times \{1, \ldots, n\} \rangle$. Thus $\substack{\circ\\\circ}$ is a (or 'the', up to isomorphism) two-element reflexive chain, and $\circ_n$ is a (or 'the') $n$-element cluster. Using standard methods, it is easy to prove that $\mathsf{Log}(\substack{\circ\\\circ}) = \mathrm{S4.2} + (p \vee \Box(p \supset \Box p)) = \mathrm{S4} + (\Diamond\Box p \supset \Box\Diamond p) + (p \vee \Box(p \supset \Box p))$; that $\mathsf{Log}(\circ_1) = \mathrm{Triv}$; and that

$$\mathsf{Log}(\circ_n) = \mathrm{S5} + \left( \bigwedge_{i=1}^{n} \Diamond p_i \supset \left( \Box \bigvee_{i=1}^{n} p_i \vee \bigvee_{\substack{i,j=1 \\ i \neq j}}^{n} \Diamond (p_i \wedge p_j) \right) \right).$$

Next, our unispaces: The trivial space $\mathbb{N}^t =_{\mathrm{df}} \langle \mathbb{N}, \tau^t \rangle$, where $\tau^t =_{\mathrm{df}} \{\emptyset, \mathbb{N}\}$; and the *El'kin* space (in the terminology of [1]), $\mathbb{N}^e =_{\mathrm{df}} \langle \mathbb{N}, \tau^e \rangle$, where $\tau^e = U \cup \{\emptyset\}$ for some nonprincipal ultrafilter $U$ on $\mathbb{N}$. It follows from Theorem 4.7 in [1] that $\mathbb{N}^e \in \mathsf{Sp}(\mathsf{Log}(\overset{\circ}{\circ}))$. Also, clearly, $\mathbb{N}^t \in \mathsf{Sp}(\mathrm{S5})$. Given the identification in Sect. 1.2 of reflexive, transitive frames with Alexandrov spaces, $\overset{\circ}{\circ}$ is identified with the space $\{0, 1\}$ with three open sets, $\emptyset$, $\{0, 1\}$, and $\{1\}$, i.e., $\overset{\circ}{\circ}$ is a ('the', up to homeomorphism) Sierpinski space; and $\circ_n$ is identified with the space $\{1, \ldots, n\}$ with the trivial topology with only two open sets, $\emptyset$ and $\{1, \ldots, n\}$.

Now we consider $\mathrm{L} \times_t \mathsf{Log}(\circ_n)$, where $n \geq 1$. If $T = \langle X, \tau \rangle$ is a unispace, an open set $O \subseteq X$ is *trivial* iff $O$ has no open subsets other than $O$ and $\emptyset$.

LEMMA 2.1. *If $T \in \mathsf{Sp}(\mathsf{Log}(\circ_n))$, then $T$ is the disjoint union of trivial open sets, each of cardinality $\leq n$.*

COROLLARY 2.2. *For any class $\mathcal{T}$ of unispaces,* $\mathsf{Log}(\mathcal{T} \times \mathsf{Sp}(\mathsf{Log}(\circ_n))) = \mathsf{Log}(\mathcal{T} \times \{\circ_n\})$.

PROOF. Since $\circ_n \in \mathsf{Sp}(\mathsf{Log}(\circ_n))$, $\mathsf{Log}(\mathcal{T} \times \mathsf{Sp}(\mathsf{Log}(\circ_n))) \subseteq \mathsf{Log}(\mathcal{T} \times \{\circ_n\})$. So we need only show that $\mathsf{Log}(\mathcal{T} \times \{\circ_n\}) \subseteq \mathsf{Log}(\mathcal{T} \times \mathsf{Sp}(\mathsf{Log}(\circ_n)))$. So suppose that $A \notin \mathsf{Log}(\mathcal{T} \times \mathsf{Sp}(\mathsf{Log}(\circ_n)))$. Then there is a $T_1 = \langle X_1, \tau_1 \rangle \in \mathcal{T}$ and a $T_2 = \langle X_2, \tau_2 \rangle \in \mathsf{Sp}(\mathsf{Log}(\circ_n)))$, such that $T_1 \times T_2 \nVdash A$. Write $T_1 \times T_2 = \langle X_1 \times X_2, \sigma_1, \sigma_2 \rangle$. So, for some binary topological model $M = \langle T_1 \times T_2, V \rangle$ and for some $\langle x_1, x_2 \rangle \in X_1 \times X_2$, we have $\langle x_1, x_2 \rangle \notin V(A)$. By Lemma 2.1, there is a trivial open set $O \subseteq X_2$, with $m$ elements, such that $m \leq n$ and $x_2 \in O$.

Let $T = \langle O, \rho \rangle$, where $\rho = \{O \cap S : S \in \tau_2\}$. So $T$ is a subspace of $T_2$. Write $T_1 \times T = \langle X_1 \times O, \rho_1', \rho_2' \rangle$. Define $V' : PV \to \mathcal{P}(X_1 \times O)$ as follows: $V'(p) = V(p) \cap (X_1 \times O)$. By a standard inductive argument, for every formula $B$ of $\mathcal{L}_{12}$, $V'(B) = V(B) \cap (X_1 \times O)$. So $\langle x_1, x_2 \rangle \notin V'(A)$. So $T_1 \times T \nVdash A$. Note that $T$ is homeomorphic[6] to $\circ_m$. So $T_1 \times \circ_m \nVdash A$. If $m = n$

[6]Suppose that $T = \langle X, \tau \rangle$ [$T = \langle X, \tau_1, \tau_2 \rangle$] and $S = \langle Y, \rho \rangle$ [$T = \langle X, \rho_1, \rho_2 \rangle$] are unispaces [bispaces] and that $f : X \to Y$. Then $f$ is *continuous* [$i$-*continuous*] iff the inverse image of every open [$i$-open] subset of $Y$ is open [$i$-open]. And $f$ is *open* [$i$-*open*] iff the image of every open [$i$-open] subset of $X$ is open [$i$-open]. If $T$ and $S$ are bispaces, then we say that $f$ is *continuous* iff $f$ is 1- and 2-continuous, and that $f$ is *open* iff $f$ is

then we're done. Otherwise, we extend $\circ_m$ to $\circ_n$ by treating $m+1,\ldots,n$ as copies of 1, to get a valuation on $T_1 \times \circ_n$ that falsifies $A$. So $T_1 \times \circ_n \not\Vdash A$. So $A \notin \mathsf{Log}(\mathcal{T} \times \{\circ_n\})$, as desired. ■

Given any $n \geq 1$, associate with every propositional variable, $p \in PV$, $n$ propositional variables $p_1,\ldots,p_n$, in such a way that $PV = \{p_i : p \in PV\}$ and if $p,q \in PV$ with $p \neq q$ or $i \neq j$ then $p_i \neq q_j$. Next, given any $n \geq 1$, for each $i \in \{1,\ldots,n\}$ we define a translation $\mathsf{Tr}_i^n$ from formulas of $\mathcal{L}_{12}$ to formulas of $\mathcal{L}$:

$$\begin{aligned}
\mathsf{Tr}_i^n(p) &= p_i \\
\mathsf{Tr}_i^n(A \wedge B) &= \mathsf{Tr}_i^n(A) \wedge \mathsf{Tr}_i^n(B) \\
\mathsf{Tr}_i^n(A \vee B) &= \mathsf{Tr}_i^n(A) \vee \mathsf{Tr}_i^n(B) \\
\mathsf{Tr}_i^n(\neg A) &= \neg \mathsf{Tr}_i^n(A) \\
\mathsf{Tr}_i^n(\Box_1 A) &= \Box \mathsf{Tr}_i^n(A) \\
\mathsf{Tr}_i^n(\Box_2 A) &= \bigwedge_{j=1}^{n} \mathsf{Tr}_j^n(A)
\end{aligned}$$

LEMMA 2.3. *Fix $n \geq 1$. Suppose that $T = \langle X, \tau \rangle$ is a unispace, $M = \langle T, V \rangle$ is a topological unimodel, and $M' = \langle T \times \circ_n, V' \rangle$ is a topological bimodel with $\langle x, i \rangle \in V'(p)$ iff $x \in V(p_i)$. Then, for every formula $A$ of $\mathcal{L}_{12}$, every $x \in X$ and every $i \in \{1,\ldots,n\}$, we have $\langle x, i \rangle \in V'(A)$ iff $x \in V(\mathsf{Tr}_i^n(A))$.*

PROOF. By a straightforward induction on construction of the formula $A$. We consider two cases in the inductive step: $A = \Box_1 B$ and $A = \Box_2 B$. Recall that $T \times \circ_n = \langle X \times \{1,\ldots,n\}, \sigma_1, \sigma_2 \rangle$, where $\{O \times \{i\} : O \in \tau \text{ and } 1 \leq i \leq n\}$ is a basis for $\sigma_1$, and where $\{\{x\} \times \{1,\ldots n\} : x \in X\}$ is a basis for $\sigma_2$.
Case 1: $A = \Box_1 B$. Note: $\langle x, i \rangle \in V'(A)$
  iff, for some $O \in \tau$, $\langle x, i \rangle \in O \times \{i\} \subseteq V'(B)$,
  iff, for some $O \in \tau$, $x \in O \subseteq V(\mathsf{Tr}_i^n(B))$, by the inductive hypothesis,
  iff $x \in V(\Box \mathsf{Tr}_i^n(B))$,
  iff $x \in V(\mathsf{Tr}_i^n(\Box_1 B)) = V(\mathsf{Tr}_i^n(A))$.
Case 2: $A = \Box_2 B$. Note: $\langle x, i \rangle \in V'(A)$
  iff, for every $j \in \{1,\ldots,n\}$, $\langle x, j \rangle \in V'(B)$,

Footnote 6 continued
1- and 2-open. A *homeomorphism* from $T$ to $S$ is any continuous open bijection. And we say that $T$ and $S$ are *homeomorphic* iff there is a homeomorphism from $T$ onto $S$. It is clear that if $T$ and $S$ are homeomorphic unispaces and $U$ is some other unispace, then $U \times T$ is homeomorphic to $U \times S$. It is also clear that if $T$ and $S$ are homeomorphic unispaces [bispaces] and $A$ is a formula of $\mathcal{L}$ [$\mathcal{L}_{12}$], then $T \Vdash A$ iff $S \Vdash A$.

iff, for every $j \in \{1, \dots, n\}$, $x \in V(\mathsf{Tr}_j^n(B))$, by the inductive hypothesis,
iff $x \in V(\bigwedge_{j=1}^n \mathsf{Tr}_j^n(B))$
iff $x \in V(\mathsf{Tr}_i^n(\Box_2 B)) = V(\mathsf{Tr}_i^n(A))$. ■

COROLLARY 2.4. *Fix $n \geq 1$. Suppose that $T = \langle X, \tau \rangle$ is a unispace. Then, for every formula $A$ of $\mathcal{L}_{12}$ and every $i \in \{1, \dots, n\}$, $T \times \circ_n \Vdash A$ iff $T \Vdash \mathsf{Tr}_i^n(A)$.*

COROLLARY 2.5. *Fix $n \geq 1$. Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are nonempty classes of unispaces such that $\mathsf{Log}(\mathcal{T}) = \mathsf{Log}(\mathcal{T}')$. Then $\mathsf{Log}(\mathcal{T} \times \{\circ_n\}) = \mathsf{Log}(\mathcal{T}' \times \{\circ_n\})$.*

PROOF. Suppose that $A \notin \mathsf{Log}(\mathcal{T} \times \{\circ_n\})$. Then $A \notin \mathsf{Log}(T \times \circ_n)$, for some $T \in \mathcal{T}$. So $\mathsf{Tr}_i^n(A) \notin \mathsf{Log}(T)$, by Corollary 2.4. So $\mathsf{Tr}_i^n(A) \notin \mathsf{Log}(\mathcal{T})$. So $\mathsf{Tr}_i^n(A) \notin \mathsf{Log}(\mathcal{T}')$. So $\mathsf{Tr}_i^n(A) \notin \mathsf{Log}(T')$, for some $T' \in \mathcal{T}'$. So $A \notin \mathsf{Log}(T' \times \circ_n)$. So $A \notin \mathsf{Log}(\mathcal{T}' \times \{\circ_n\})$.

Thus $\mathsf{Log}(\mathcal{T}' \times \{\circ_n\}) \subseteq \mathsf{Log}(\mathcal{T} \times \{\circ_n\})$. Similarly, $\mathsf{Log}(\mathcal{T} \times \{\circ_n\}) \subseteq \mathsf{Log}(\mathcal{T}' \times \{\circ_n\})$. ■

COROLLARY 2.6. *If* L *is a Kripke complete extension of* S4*, then* $\mathrm{L} \times_t \mathsf{Log}(\circ_n) = \mathrm{L} \times \mathsf{Log}(\circ_n)$.

PROOF. Since L is Kripke complete, $\mathrm{L} = \mathsf{Log}(\mathsf{Fr}(\mathrm{L}))$. Since $\mathsf{Fr}(\mathrm{L})$ is a class of reflexive transitive uniframes, we are treating it also as a class of Alexandrov unispaces. So $\mathsf{Fr}(\mathrm{L}) \subseteq \mathsf{Sp}(\mathrm{L})$. So $\mathrm{L} \subseteq \mathsf{Log}(\mathsf{Sp}(\mathrm{L})) \subseteq \mathsf{Log}(\mathsf{Fr}(\mathrm{L})) = \mathrm{L}$. So $\mathsf{Log}(\mathsf{Fr}(\mathrm{L})) = \mathsf{Log}(\mathsf{Sp}(\mathrm{L}))$. So $\mathsf{Log}(\mathsf{Fr}(\mathrm{L}) \times \{\circ_n\}) = \mathsf{Log}(\mathsf{Sp}(\mathrm{L}) \times \{\circ_n\})$, by Corollary 2.5. So $\mathsf{Log}(\mathsf{Fr}(\mathrm{L}) \times \mathsf{Sp}(\mathsf{Log}(\circ_n))) = \mathsf{Log}(\mathsf{Sp}(\mathrm{L}) \times \mathsf{Sp}(\mathsf{Log}(\circ_n)))$, by Corollary 2.2.

Also notice that every unispace $T$ with $T \Vdash \mathsf{Log}(\circ_n)$ is Alexandrov. Thus, every such $T$ can be identified with a uniframe. Thus $\mathsf{Fr}(\mathsf{Log}(\circ_n)) = \mathsf{Sp}(\mathsf{Log}(\circ_n))$. So $\mathsf{Log}(\mathsf{Fr}(\mathrm{L}) \times \mathsf{Fr}(\mathsf{Log}(\circ_n))) = \mathsf{Log}(\mathsf{Fr}(\mathrm{L}) \times \mathsf{Sp}(\mathsf{Log}(\circ_n))) = \mathsf{Log}(\mathsf{Sp}(\mathrm{L}) \times \mathsf{Sp}(\mathsf{Log}(\circ_n)))$. So $\mathrm{L} \times \mathsf{Log}(\circ_n) = \mathrm{L} \times_t \mathsf{Log}(\circ_n)$. ■

So we get the right-to-left direction of the biconditional in Theorem 1.1:

COROLLARY 2.7. *Suppose that* $\mathrm{L}_1$ *and* $\mathrm{L}_2$ *are Kripke complete extensions of* S4*. Then if either* $\mathrm{L}_1 \supsetneq \mathrm{S5}$ *or* $\mathrm{L}_2 \supsetneq \mathrm{S5}$ *or* $\mathrm{L}_1 = \mathrm{L}_2 = \mathrm{S5}$*, then* $\mathrm{L}_1 \times_t \mathrm{L}_2 = \mathrm{L}_1 \times \mathrm{L}_2$.

PROOF. Suppose $\mathrm{L}_2 \supsetneq \mathrm{S5}$, i.e., $\mathrm{L}_2$ is a strict extension of S5. Then either $\mathrm{L}_2$ is inconsistent or $\mathrm{L}_2 = \mathsf{Log}(\circ_n)$ for some $n$, a classic result of [12]. So $\mathrm{L}_1 \times_t \mathrm{L}_2 = \mathrm{L}_1 \times \mathrm{L}_2$, either trivially if $\mathrm{L}_2$ is inconsistent or by Corollary 2.6

if $\mathrm{L}_2$ is consistent. Similarly, if $\mathrm{L}_1 \supsetneq \mathrm{S5}$, then $\mathrm{L}_1 \times_t \mathrm{L}_2 = \mathrm{L}_1 \times \mathrm{L}_2$. Finally, consider $\mathrm{S5} \times_t \mathrm{S5}$. Note that every member of $\mathsf{Sp}(\mathrm{S5})$ is Alexandrov, and so is identified with some Kripke frame. So $\mathrm{S5} \times_t \mathrm{S5} = \mathsf{Log}(\mathsf{Sp}(\mathrm{S5}) \times \mathsf{Sp}(\mathrm{S5})) = \mathsf{Log}(\mathsf{Fr}(\mathrm{S5}) \times \mathsf{Fr}(\mathrm{S5})) = \mathrm{S5} \times \mathrm{S5}$. ■

Remark 2.8. Before we address the left-to-right direction of the biconditional in Theorem 1.1, some notation and a few remarks. We will be interested in the two unispaces, $\mathbb{N}^e$ and $\mathbb{N}^t$, defined above; and the two product spaces, $\mathbb{N}^e \times \mathbb{N}^t$ and $\mathbb{N}^e \times \mathbb{N}^e$. We will write $\mathsf{Int}_{\mathbb{N}^e}$ [$\mathsf{Int}_{\mathbb{N}^t}$] for the interior operator in the space $\mathbb{N}^e$ [$\mathbb{N}^t$] and similarly for $\mathsf{Cl}_{\mathbb{N}^e}$ [$\mathsf{Cl}_{\mathbb{N}^t}$]. We will write $\mathsf{Int}_i$ [$\mathsf{Cl}_i$] for the $i$th interior [closure] operator in the product space $\mathbb{N}^e \times \mathbb{N}^t$ and $\mathsf{Int}'_i$ [$\mathsf{Cl}'_i$] for the $i$th interior [closure] operator in the product space $\mathbb{N}^e \times \mathbb{N}^e$. Note that $\mathsf{Int}'_1 = \mathsf{Int}_1$ but that $\mathsf{Int}'_2 \neq \mathsf{Int}_2$. We will also use the notation introduced at the end of Sect. 1.2: for any $S \subseteq \mathbb{N} \times \mathbb{N}$, and any $n \in \mathbb{N}$, $\mathsf{rt}_n(S) =_{\mathrm{df}} \{m \in \mathbb{N} : \langle n, m\rangle \in S\}$ and $\mathsf{lft}_n(S) =_{\mathrm{df}} \{m \in \mathbb{N} : \langle m, n\rangle \in S\}$. For each $n \in \mathbb{N}$, define the $n$th row and the $n$th column in $\mathbb{N} \times \mathbb{N}$ as follows: $\mathrm{R}_n =_{\mathrm{df}} \mathbb{N} \times \{n\}$ and $\mathrm{C}_n =_{\mathrm{df}} \{n\} \times \mathbb{N}$. Note that, for any $S \subseteq \mathbb{N} \times \mathbb{N}$, $S \cap \mathrm{R}_n = \mathsf{lft}_n(S) \times \{n\}$ and $S \cap \mathrm{C}_n = \{n\} \times \mathsf{rt}_n(S)$.

Lemma 2.9. (1) $com_{\subset} \notin \mathsf{Log}(\overset{\circ}{\underset{\circ}{|}}) \times_t \mathrm{S5}$.

(2) $com_{\supset} \notin \mathrm{S5} \times_t \mathsf{Log}(\overset{\circ}{\underset{\circ}{|}})$.

(3) $com_{\subset} \notin \mathsf{Log}(\overset{\circ}{\underset{\circ}{|}}) \times_t \mathsf{Log}(\overset{\circ}{\underset{\circ}{|}})$.

(4) $com_{\supset} \notin \mathsf{Log}(\overset{\circ}{\underset{\circ}{|}}) \times_t \mathsf{Log}(\overset{\circ}{\underset{\circ}{|}})$.

(5) $chr \notin \mathsf{Log}(\overset{\circ}{\underset{\circ}{|}}) \times_t \mathsf{Log}(\overset{\circ}{\underset{\circ}{|}})$.

Proof. We only prove (1), (3) and (5), since the proofs of (2) and (4) are symmetric to the proofs of (1) and (3) respectively. Recall that $\mathbb{N}^e \in \mathsf{Sp}(\mathsf{Log}(\overset{\circ}{\underset{\circ}{|}}))$ and $\mathbb{N}^t \in \mathsf{Sp}(\mathrm{S5})$. So to prove (1) it suffices to show that $\mathbb{N}^e \times \mathbb{N}^t \not\Vdash com_{\subset}$; to prove (3) it suffices to show that $\mathbb{N}^e \times \mathbb{N}^e \not\Vdash com_{\subset}$; and to prove (5) it suffices to show that $\mathbb{N}^e \times \mathbb{N}^e \not\Vdash chr$.

*Proof of* (1) *and* (3). It suffices to specify a set $P \subseteq \mathbb{N} \times \mathbb{N}$ such that both $\mathsf{Int}_2\mathsf{Int}_1(P) - \mathsf{Int}_1\mathsf{Int}_2(P) \neq \emptyset$ and $\mathsf{Int}'_2\mathsf{Int}'_1(P) - \mathsf{Int}'_1\mathsf{Int}'_2(P) \neq \emptyset$. Let $P =_{\mathrm{df}} \{\langle m, n\rangle : m = 0 \text{ or } n = 0 \text{ or } 0 < n < m\}$. Figure 1 represents $P$: the bullets represent the ordered pairs in $P$ and the open circles represent the ordered pairs not in $P$. Figure 1 also indicates the rows and columns, $\mathrm{R}_n$ and $\mathrm{C}_n$.

**Calculating both $\mathsf{Int}_1(P)$ and $\mathsf{Int}'_1(P)$.** Note that $\mathsf{lft}_n(P) \times \{n\} = P \cap \mathrm{R}_n$ is cofinite in $\mathrm{R}_n$ for every $n \in \mathbb{N}$. So $\mathsf{lft}_n(P)$ is cofinite in $\mathbb{N}$ for every $n \in \mathbb{N}$.

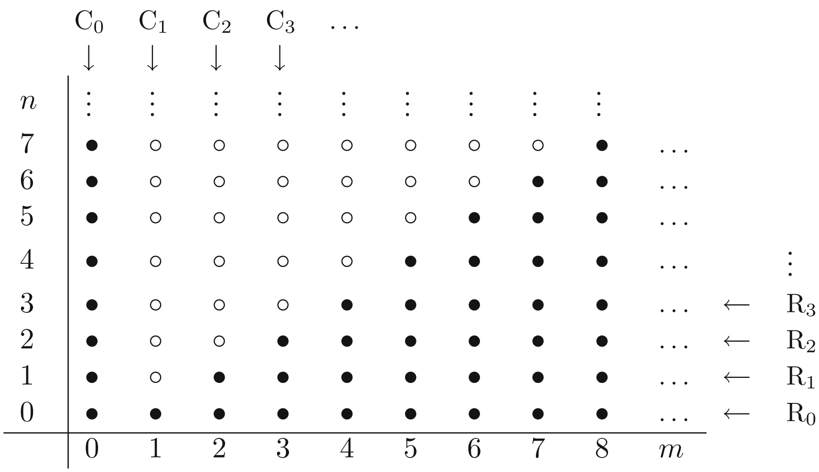


Figure 1. The set $P \subseteq \mathbb{N} \times \mathbb{N}$

So $\mathsf{lft}_n(P)$ is open in $\mathbb{N}^e$, since every cofinite subset of $\mathbb{N}$ is open in $\mathbb{N}^e$. So $\mathsf{Int}_{\mathbb{N}^e}(\mathsf{lft}_n(P)) = \mathsf{lft}_n(P)$. So

$$\begin{aligned}\mathsf{Int}_1(P) &= \bigcup_{n\in\mathbb{N}} \mathsf{Int}_{\mathbb{N}^e}(\mathsf{lft}_n(P)) \times \{n\}, \text{ by Lemma 1.2}\\ &= \bigcup_{n\in\mathbb{N}} \mathsf{lft}_n(P) \times \{n\} = P.\end{aligned}$$

As noted in Remark 2.8, the operators $\mathsf{Int}_1$ and $\mathsf{Int}'_1$ are identical. Thus we also have $\mathsf{Int}'_1(P) = P$.

**Calculating both $\mathsf{Int}_2(P)$ and $\mathsf{Int}'_2(P)$.** Note that $\mathsf{rt}_0(P) = \mathbb{N}$ and $\mathsf{rt}_n(P)$ is finite for every $n \geq 1$. So $\mathsf{Int}_{\mathbb{N}^t}(\mathsf{rt}_0(P)) = \mathsf{Int}_{\mathbb{N}^e}(\mathsf{rt}_0(P)) = \mathbb{N}$; also $\mathsf{Int}_{\mathbb{N}^t}(\mathsf{rt}_n(P)) = \mathsf{Int}_{\mathbb{N}^e}(\mathsf{rt}_n(P)) = \emptyset$ for every $n \geq 1$, since $\emptyset$ is the only open finite set in $\mathbb{N}^t$ and the only open finite set in $\mathbb{N}^e$. So we get two very similar calculations:

$$\mathsf{Int}_2(P) = \bigcup_{n\in\mathbb{N}} \{n\} \times \mathsf{Int}_{\mathbb{N}^t}(\mathsf{rt}_n(P)) \text{ (by Lemma 1.2) } = \{0\} \times \mathbb{N} = \mathrm{C}_0.$$

$$\mathsf{Int}'_2(P) = \bigcup_{n\in\mathbb{N}} \{n\} \times \mathsf{Int}_{\mathbb{N}^e}(\mathsf{rt}_n(P)) \text{ (by Lemma 1.2) } = \{0\} \times \mathbb{N} = \mathrm{C}_0.$$

Thus $\mathsf{Int}_2\mathsf{Int}_1(P) = \mathsf{Int}_2(P) = \mathrm{C}_0 = \mathsf{Int}'_2(P) = \mathsf{Int}'_2\mathsf{Int}'_1(P)$: see Figure 2.

**Wrapping up.** Note that $\mathsf{Int}_{\mathbb{N}^e}(S) = \emptyset$ for any finite $S \subseteq \mathbb{N}$. So $\mathsf{Int}_1(\mathrm{C}_0) = \mathsf{Int}'_1(\mathrm{C}_0) = \cup_{n\in\mathbb{N}}(\mathsf{Int}_{N^e}(\{0\}) \times \{n\}$ (by Lemma 1.2) $= \emptyset$. So $\mathsf{Int}_1\mathsf{Int}_2(P) =$

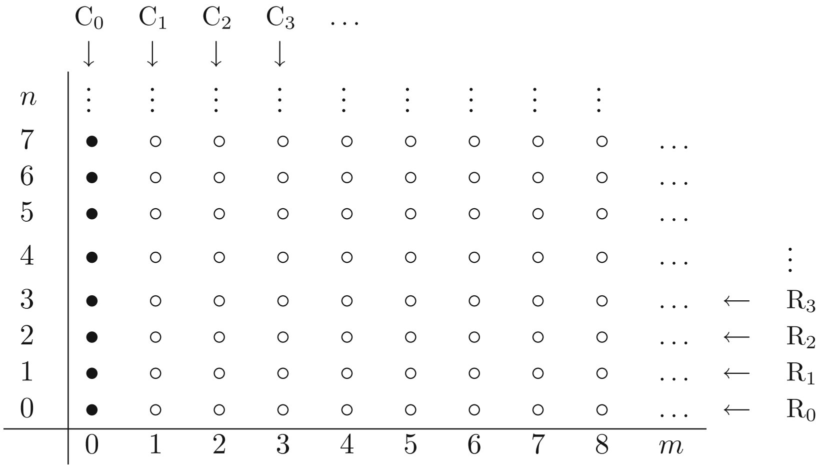


Figure 2. $\mathsf{Int}_2(P) = \mathsf{Int}_2\mathsf{Int}_1(P) = \mathrm{C}_0 = \mathsf{Int}'_2(P) = \mathsf{Int}'_2\mathsf{Int}'_1(P)$

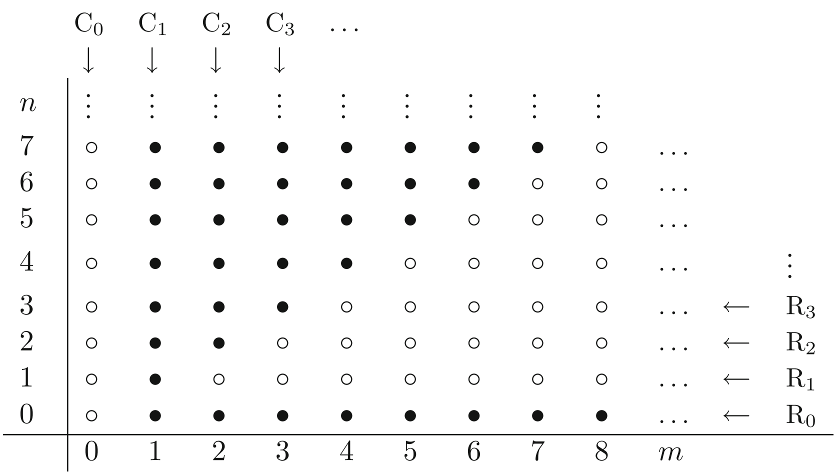


Figure 3. The set $Q \subseteq \mathbb{N} \times \mathbb{N}$

$\mathsf{Int}'_1\mathsf{Int}'_2(P) = \emptyset$. So $\mathsf{Int}_2\mathsf{Int}_1(P) - \mathsf{Int}_1\mathsf{Int}_2(P) = \mathrm{C}_0 \neq \emptyset$ and $\mathsf{Int}'_2\mathsf{Int}'_1(P) - \mathsf{Int}'_1\mathsf{Int}'_2(P) = \mathrm{C}_0 \neq \emptyset$, as desired.

*Proof of* (5). It suffices to specify a set $Q \subseteq \mathbb{N} \times \mathbb{N}$ such that $\mathsf{Cl}'_1\mathsf{Int}'_2(Q) - \mathsf{Int}'_2\mathsf{Cl}'_1(Q) \neq \emptyset$. Let $Q =_{\mathrm{df}} \{\langle m, n\rangle : m \neq 0$ and either $n = 0$ or $m \leq n\}$, as represented in Figure 3.

**Calculating** $\mathsf{Int}'_2(Q)$. Note that $\{n\} \times \mathsf{rt}_n(Q) = Q \cap \mathrm{C}_n$ is either empty or cofinite in $\mathrm{C}_n$ for every $n \in \mathbb{N}$. So $\mathsf{rt}_n(Q)$ is either empty or cofinite in $\mathbb{N}$ for every $n \in \mathbb{N}$. So $\mathsf{rt}_n(Q)$ is open in $\mathbb{N}^e$, since every cofinite subset of $\mathbb{N}$ is

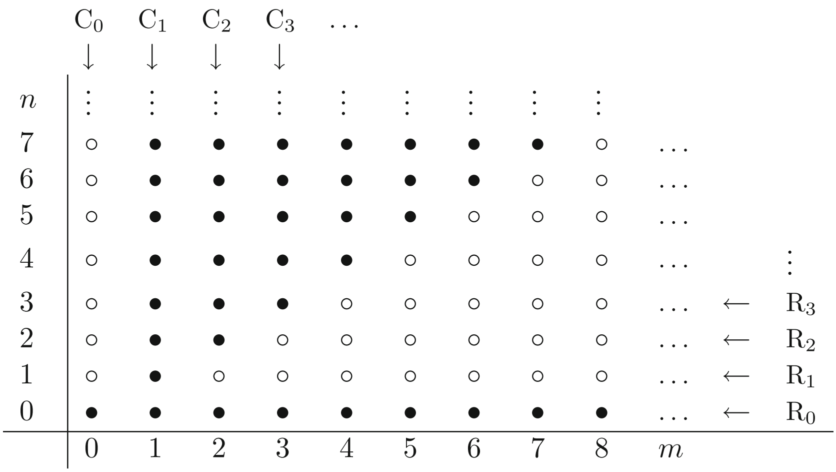


Figure 4. $\mathsf{Cl}'_1\mathsf{Int}'_2(Q) = \mathsf{Cl}'_1(Q) = \{\langle 0,0\rangle\} \cup Q$

open in $\mathbb{N}^e$. So $\mathsf{Int}_{\mathbb{N}^e}(\mathsf{rt}_n(Q)) = \mathsf{rt}_n(Q)$. So

$$\mathsf{Int}'_2(Q) = \bigcup_{n\in\mathbb{N}} \{n\} \times \mathsf{Int}_{\mathbb{N}^e}(\mathsf{rt}_n(Q)) \text{ (by Lemma 1.2) } = \bigcup_{n\in\mathbb{N}} \{n\} \times \mathsf{rt}_n(Q) = Q.$$

**Calculating** $\mathsf{Cl}'_1\mathsf{Int}'_2(Q) = \mathsf{Cl}'_1(Q)$. Note that $\mathsf{lft}_n(Q)\times\{n\} = Q\cap \mathrm{R}_n$ is infinite for $n = 0$ and finite for $n \geq 1$. Thus $\mathsf{lft}_n(Q)$ is infinite for $n = 0$ and finite for $n \geq 1$. Now, every finite subset of $\mathbb{N}$ is closed in $\mathbb{N}^e$, while the closure of any infinite set is simply $\mathbb{N}$. So $\mathsf{Cl}_{\mathbb{N}^e}(\mathsf{lft}_0(Q)) = \mathbb{N}$ and $\mathsf{Cl}_{\mathbb{N}^e}(\mathsf{lft}_n(Q)) = \mathsf{lft}_n(Q)$, for $n \geq 1$. So,

$$\begin{aligned}
\mathsf{Cl}'_1(Q) &= \bigcup_{n\in\mathbb{N}} \mathsf{Cl}_{\mathbb{N}^e}(\mathsf{lft}_n(Q)) \times \{n\}, \text{ by Lemma 1.2} \\
&= \mathbb{N} \times \{0\} \cup \bigcup_{n\geq 1} \mathsf{lft}_n(Q) \times \{n\} \\
&= \{\langle 0,0\rangle\} \cup (Q \cap \mathrm{R}_0) \cup \bigcup_{n\geq 1} \mathsf{lft}_n(Q) \times \{n\} \\
&= \{\langle 0,0\rangle\} \cup (\mathsf{lft}_0(Q) \times \{0\}) \cup \bigcup_{n\geq 1} \mathsf{lft}_n(Q) \times \{n\} \\
&= \{\langle 0,0\rangle\} \cup \bigcup_{n\in\mathbb{N}} \mathsf{lft}_n(Q) \times \{n\} \\
&= \{\langle 0,0\rangle\} \cup Q
\end{aligned}$$

$\mathsf{Cl}'_1\mathsf{Int}'_2(Q) = \mathsf{Cl}'_1(Q)$ is represented by Figure 4.

**Calculating** $\mathsf{Int}_2'\mathsf{Cl}_1'(Q)$. Let $Q' = \mathsf{Cl}_1'(Q) = \{\langle 0,0\rangle\} \cup Q$. First note that $\mathsf{rt}_n(Q') = \mathsf{rt}_n(Q)$ for $n \geq 1$. Note that $\{n\} \times \mathsf{rt}_n(Q') = Q' \cap \mathrm{C}_n$ is finite if $n = 0$ and is cofinite in $\mathrm{C}_n$ if $n \geq 1$. Thus $\mathsf{rt}_n(Q')$ is finite if $n = 0$ and is cofinite in $\mathbb{N}$ if $n \geq 1$. The only finite subset of $\mathbb{N}$ which is open in $\mathbb{N}^e$ is $\emptyset$ and every cofinite subset of $\mathbb{N}$ is open in $\mathbb{N}^e$. So $\mathsf{Int}_{\mathbb{N}^e}(\mathsf{rt}_n(Q')) = \emptyset$ and $\mathsf{Int}_{\mathbb{N}^e}(\mathsf{rt}_n(Q')) = \mathsf{rt}_n(Q') = \mathsf{rt}_n(Q)$ if $n \geq 1$. So,

$$\begin{aligned}\mathsf{Int}_2'\mathsf{Cl}_1'(Q) &= \mathsf{Int}_2'(Q')\\ &= \bigcup_{n\in\mathbb{N}} \{n\} \times \mathsf{Int}_{\mathbb{N}^e}(\mathsf{rt}_n(Q')), \text{ by Lemma 1.2}\\ &= \bigcup_{n\geq 1} \{n\} \times \mathsf{rt}_n(Q)\\ &= \bigcup_{n\in\mathbb{N}} \{n\} \times \mathsf{rt}_n(Q), \text{ since } \mathsf{rt}_0(Q) = \emptyset\\ &= Q\end{aligned}$$

**Wrapping up**. $\mathsf{Cl}_1'\mathsf{Int}_2'(Q) - \mathsf{Int}_2'\mathsf{Cl}_1'(Q) = (\{\langle 0,0\rangle\} \cup Q) - Q = \{\langle 0,0\rangle\} \neq \emptyset$. ■

The following corollary is equivalent to the left-to-right direction of the biconditional in Theorem 1.1.

Corollary 2.10. *Suppose that* $\mathrm{L}_1$ *and* $\mathrm{L}_2$ *are Kripke complete extensions of* S4. *Then if*

(1) $\mathrm{L}_1, \mathrm{L}_2 \not\supseteq \mathrm{S5}$ *or*

(2) $\mathrm{L}_1 \not\supseteq \mathrm{S5}$ *and* $\mathrm{L}_2 = \mathrm{S5}$ *or*

(3) $\mathrm{L}_2 \not\supseteq \mathrm{S5}$ *and* $\mathrm{L}_1 = \mathrm{S5}$.

*then* $\mathrm{L}_1 \times_t \mathrm{L}_2 \neq \mathrm{L}_1 \times \mathrm{L}_2$.

Proof. First note that by the structure of extensions of S4, if L is any extension of S4, then either $\mathrm{L} \supseteq \mathrm{S5}$ or $\mathrm{L} \subseteq \mathsf{Log}(\between)$. This was originally proved in [2]. See also [3]. We proceed by considering only two cases, (1) $\mathrm{L}_1, \mathrm{L}_2 \not\supseteq \mathrm{S5}$ and (2) $\mathrm{L}_1 \not\supseteq \mathrm{S5}$ and $\mathrm{L}_2 = \mathrm{S5}$, since the third case (3) $\mathrm{L}_2 \not\supseteq \mathrm{S5}$ and $\mathrm{L}_1 = \mathrm{S5}$, is symmetric to (2).

Case 1: $\mathrm{L}_1, \mathrm{L}_2 \not\supseteq \mathrm{S5}$. Then $\mathrm{L}_1, \mathrm{L}_2 \subseteq \mathsf{Log}(\between)$. So $\mathrm{L}_1 \times_t \mathrm{L}_2 \subseteq \mathsf{Log}(\between) \times_t \mathsf{Log}(\between)$. So, by Lemma 2.9, $com_{\subset}, com_{\supset}, chr \notin \mathrm{L}_1 \times_t \mathrm{L}_2$. But $com_{\subset}$, $com_{\supset}$, $chr \in [\mathrm{L}_1, \mathrm{L}_2] \subseteq \mathrm{L}_1 \times \mathrm{L}_2$. So $\mathrm{L}_1 \times_t \mathrm{L}_2 \neq \mathrm{L}_1 \times \mathrm{L}_2$.

Case 2: $\mathrm{L}_1 \not\supseteq \mathrm{S5}$ and $\mathrm{L}_2 = \mathrm{S5}$. Then $\mathrm{L}_1 \subseteq \mathsf{Log}(\between)$. So $\mathrm{L}_1 \times_t \mathrm{S5} \subseteq \mathsf{Log}(\between) \times_t \mathrm{S5}$. So, by Lemma 2.9, $com_{\subset} \notin \mathrm{L}_1 \times_t \mathrm{L}_2$. So, as in Case 1, $\mathrm{L}_1 \times_t \mathrm{L}_2 \neq \mathrm{L}_1 \times \mathrm{L}_2$. ■

## 3. Concluding Remarks

We have given necessary and sufficient conditions for the topological product of Kripke complete extensions of S4 to match their frame product. In the most basic case, the topological product matches not the frame product but the fusion: $\mathrm{S4} \times_t \mathrm{S4} = \mathrm{S4} \otimes \mathrm{S4} \subsetneq \mathrm{S4} \times \mathrm{S4}$. Given this, there are nine easy examples of $\mathrm{L}_1 \times_t \mathrm{L}_2 = \mathrm{L}_1 \otimes \mathrm{L}_2$: when each of $\mathrm{L}_1$ and $\mathrm{L}_2$ is either S4, Triv, or inconsistent. We know of no other examples; nor of any counterexamples, except in cases where $\mathrm{L}_1 \supseteq \mathrm{S5}$ or $\mathrm{L}_2 \supseteq \mathrm{S5}$ – e.g., $\mathrm{S4} \times_t \mathrm{S5} \neq \mathrm{S4} \otimes \mathrm{S5}$, as noted in Sect. 1.2. This suggests three related projects, the third much more ambitious than the first two:

1. find other examples of $\mathrm{L}_1 \times_t \mathrm{L}_2 = \mathrm{L}_1 \otimes \mathrm{L}_2$, or show there aren't any;
2. find counterexamples to $\mathrm{L}_1 \times_t \mathrm{L}_2 = \mathrm{L}_1 \otimes \mathrm{L}_2$, where $\mathrm{L}_1$ and $\mathrm{L}_2$ are topologically complete and $\mathrm{L}_1, \mathrm{L}_2 \not\supseteq \mathrm{S5}$, or show there aren't any;
3. find nontrivial necessary and sufficient conditions for $\mathrm{L}_1 \times_t \mathrm{L}_2 = \mathrm{L}_1 \otimes \mathrm{L}_2$.

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