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ON THE COMPLEXITY OF PROPOSITIONAL QUANTIFICATION IN INTUITIONISTIC LOGIC

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Abstract. We define a propositionally quantified intuitionistic logic **H**π+ by a natural extension of Kripke's semantics for propositional intuitionistic logic. We then show that **H**π+ is recursively isomorphic to full second order classical logic. **H**π+ is the intuitionistic analogue of the modal systems **S5**π+, **S4**π+, **S4.2**π+, **K4**π+, **T**π+, **K**π+ and **B**π+, studied by Fine.

§1. Introduction. Kripke's [1963] semantics for propositional intuitionistic logic can be extended in a natural way to a language with propositional quantifiers. Kripke defines an *intuitionistic model structure* to be an ordered triple $(g, K, \leq)$ where K is a non-empty set, $g \in K$, and $\leq$ is a reflexive and transitive relation on K. We get a *model* by adding an assignment ϕ of a truth value to each propositional variable p at each point $h \in K$, with the following constraint: if $\phi(p, h) = T$ and $h \leq h'$ then $\phi(p, h') = T$. Kripke gives recursive clauses extending ϕ so that $\phi(A, h)$ is defined for every formula A of a propositional intuitionistic language. A formula A is *valid* iff, for every model $(g, K, \leq, \phi)$, $\phi(A, g) = T$. Kripke shows that a formula is valid iff it is a theorem of Heyting's intuitionistic logic, **H**.

Given a model, $M = (g, K, \leq, \phi)$, it is natural to take the *proposition* assigned to the formula A to be the subset of points $h \in K$ such that $\phi(A, h) = T$. Indeed, we could proceed by taking ϕ to be an assignment of *propositions* to the propositional variables, where a *proposition* is a subset P of K which is such that if $h \in P$ and $h \leq h'$ then $h' \in P$. Kripke's recursive clauses can be re-interpreted as clauses for determining the proposition, say $M(A)$, assigned to each formula A. Understood this way, A is valid iff, for every model $M = (g, K, \leq, \phi)$, $g \in M(A)$.

The present paper extends Kripke's semantics to propositionally quantified languages, by interpreting the quantifiers as ranging over the *propositions*, understood as above. Our main result is that the resulting set of *valid* formulas, which we call **H**π+, is recursively isomorphic to full second order classical logic (§3, Theorem 7, below).

It is important to distinguish **H**π+ from the propositionally quantified intuitionistic logics considered by Gabbay [1974] and [1981], Löb [1976], Sobolev [1977], Kreisel [1981], Scedrov [1984], Pitts [1992] and others. These logics are defined by extending **H** with new axioms or inference rules for the propositional quantifiers; Gabbay *et al* have shown various among these systems to be undecidable. Following Henkin [1950], we call the interpretations of the quantifiers resulting in these logics

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secondary interpretations; and we call the interpretation of the quantifiers resulting in **Hπ+** the *primary* or *principal* interpretation.

§2. **Related results.** Fine [1970] studies modal logics **S5π+**, **S4π+**, **S4.2π+**, **K4π+**, **Tπ+**, **Kπ+** and **Bπ+**, analogous to **Hπ+**. **S5π+** is also studied by Kaplan [1970]. And in Kremer [1993] I study an analogous relevance logic **RP+**. Kaplan and Fine show that **S5π+** is decidable, and Fine discusses proofs that second-order arithmetic can be recursively encoded in the other modal systems. And in Kremer [1993] I show that **RP+** is recursively isomorphic to full second classical order logic. My [1993] strategy can be extended to **S4π+**, **S4.2π+**, **K4π+**, **Tπ+**, **Kπ+** and **Bπ+**, to show that they too are recursively isomorphic to full second order classical logic. (Fine and Kripke inform me that they independently proved this result shortly after the publication of Fine [1970].

Until now, the problem of the complexity of **Hπ+** has remained open. Given the expressive weakness of intuitionistic logic, my [1993] strategy is not as straightforwardly extendable to **Hπ+** as it is to **S4π+** and all. In particular, we must proceed differently in the intuitionistic context than in the relevance context in showing that we can focus our attention on a class of models for which we can define a connective, $\neg$, with the following property: for each model M in the class and each formula A, $\neg A$ is true in M iff A is not true in M. (In the modal context, the class of all models will do the trick.) This "classical" negation connective is useful at various stages in the encoding of second order formulas in the intuitionistic language.

§3. **Precise statement of the main result.** We work with a propositional language with a countable set $PV = \{p_1, \dots, p_n, \dots\}$ of propositional variables; left and right parentheses; connectives &, $\vee$, $\sim$ and $\rightarrow$; and propositional quantifiers $\forall$ and $\exists$. We use p, q, ... as metalanguage variables over PV and A, B, ... as metalanguage variables over formulas.

DEFINITION 1. An (intuitionistic) *model structure* (ims) is an ordered triple $(g, K, \leq)$ such that K is a non-empty set, $g \in K$ and $\leq$ is a reflexive and transitive relation on K. Given an ims, a *proposition* is a set $P \subseteq K$ such that if $h \in P$ and $h \leq h'$ then $h' \in P$.

DEFINITION 2. An (intuitionistic) *model* is an ordered 4-tuple $M = (g, K, \leq, \phi)$ where $(g, K, \leq)$ is an ims and ϕ is a function on PV so that $\phi(p)$ is a proposition. $M = (g, K, \leq, \phi)$ is *based on* $(g, K, \leq)$. Given a model M, a proposition P and a propositional variable p, $M[P/p]$ is the model just like M except that it assigns the proposition P to the propositional variable p.

DEFINITION 3. Given a model $M = (g, K, \leq, \phi)$ and a formula A, we define $M(A)$, the *proposition assigned by M to A*, as follows:

$$M(p) = \phi(p), \text{ for propositional variables } p$$
$$M(A \,\&\, B) = M(A) \cap M(B)$$
$$M(A \vee B) = M(A) \cup M(B)$$
$$M(A \rightarrow B) = \{h \in K : \text{ for each } h' \in M(A), \text{ if } h \leq h' \text{ then } h' \in M(B)\}$$

$$M(\sim A) = \{h \in K \ : \ \text{for each } h' \in K, \text{ if } h \leq h' \text{ then } h' \notin M(A)\}$$
$$M(\forall pA) = \bigcap\{M[P/p](A) \ : \ P \text{ is a proposition}\}$$
$$M(\exists pA) = \bigcup\{M[P/p](A) \ : \ P \text{ is a proposition}\}.$$

LEMMA 4. *For every formula A, $M(A)$ is a proposition.*

PROOF. By induction on the complexity of A. ⊣

DEFINITION 5. *M validates A* ($M \models A$) iff $g \in M(A)$. A is *valid* ($\models A$) iff A is validated by every model.

DEFINITION 6. $\mathbf{H\pi+} = \{A : \models A\}$.

THEOREM 7 (The Main Result). $\mathbf{H\pi+}$ *is recursively isomorphic to full second order classical logic.*

PROOF. See §§4–8, below. ⊣

NOTE. Dmitri Skvortsov has sent me a proof of a weaker result—that $\mathbf{H\pi+}$ is not r.e.—and he is preparing it for publication. I was aware of Skvortsov's result, but I had not seen the proof, when I discovered the current proof. The current, stronger, result requires quite a different proof from Skvorstov's.

§4. Proof of the main result: Preliminaries. It suffices to show that second order logic is 1-reducible to $\mathbf{H\pi+}$, since the fact that $\mathbf{H\pi+}$ is 1-reducible to second order logic can be shown by the methods of Kremer [1993, §I.5 and §II.7]. Our proof that second order logic is 1-reducible to $\mathbf{H\pi+}$ relies on a result that can be gleaned from Nerode and Shore [1980, §1]. Nerode and Shore reproduce unpublished considerations of Rabin and Scott, showing how to code arbitrary n-ary relations by sib (symmetric irreflexive binary) relations. The result is that second order logic is recursively isomorphic to second order logic with second order quantification restricted to sib relations. We will state a slightly different result, more useful for our purposes, that follows from these considerations (Theorem 9, below). To make things precise, we assume that we are working with a second order classical language with individual variables $x_1, \ldots, x_n, \ldots$; binary relational variables $X_1, \ldots, X_n, \ldots$; parentheses; connectives $\vee$ and $\neg$; identity, $=$; and first and second order universal quantifiers.

DEFINITION 8. A *classical second order SIB-model* (or simply *SIB-model*) is an ordered pair $M = (U, \phi)$ where U (the universe) is a non-empty set, and ϕ is a function mapping every individual variable to a member of U and every relational variable to a sib relation on U. The concepts of validity in $M (M \models A)$ and of validity ($\models A$) are defined in the standard ways, with the relational quantifiers ranging over all and only the sib relations. We define SIB^2 to be the set of formulas valid in every second order SIB-model. We also define a peculiar theory, 2-SIB^2, to be the set of formulas validated by every second order SIB-model whose universe has at least two elements.

THEOREM 9 (Nerode and Shore [1980], and Rabin and Scott). 2-SIB^2 *is recursively isomorphic to second order logic.*

REMARK. Nerode and Shore [1980] show how to encode binary relations as sib relations, and arbitrary n-ary relations as binary relations. The present paper's Appendix (§10) gives what we consider a simpler encoding of binary relations as sib relations. We refer the reader to Nerode and Shore [1980] for the encoding of n-ary relations as binary relations. We reserve this material for the Appendix due to its distance from our main topic, $\mathbf{H\pi+}$. ⊣

Given Theorem 9, it suffices, for our main result, to prove that 2-SIB^2 is 1-reducible to $\mathbf{H\pi+}$.

§5. Proof of the main result: more preliminaries.

DEFINITION 10. A *simple* ims is an ims $(g, K, \leq)$ such that $\leq$ is a partial ordering on K (if $h \leq h'$ and $h' \leq h$ then $h = h'$); and such that g is the $\leq$-least member of K (for each $h \in K$, $g \leq h$). For simple ims's, we write $h < h'$ for ($h \leq h'$ and $h \neq h'$). For each $h \in K$, we define the proposition $[h] = \{h' : h \leq h'\}$. The set of *classical points* is $CL = \{h : \text{ for no } h', h < h'\}$. A *simple* (intuitionistic) model is a model based on a simple ims.

THEOREM 11. $\mathbf{H\pi+} = \{A : $ *for every simple model* $M, M \models A\}$.

PROOF. Clearly $\mathbf{H\pi+} \subseteq \{A : $ for every simple model $M, M \models A\}$. Now suppose that $A \notin \mathbf{H\pi+}$. Then for some model $M = (g, K, \leq, \phi)$, $M \not\models A$. Define an equivalence relation $\approx$ on K: $h \approx h'$ iff $h \leq h'$ and $h' \leq h$. Let $\langle h \rangle = \{h' : h \approx h'\}$. And define a new model $M' = (\langle g \rangle, K', \leq', \phi')$, where $K' = \{\langle h \rangle : h \in K \,\&\, g \leq h\}$; $\langle h \rangle \leq' \langle h' \rangle$ iff $h \leq h'$; and $\langle h \rangle \in \phi'(p)$ iff $h \in \phi(p)$. Observe: (1) $(\langle g \rangle, K', \leq')$ is a *simple* ims; (2) ϕ' is well-defined; and (3) if $\langle h \rangle \leq' \langle h' \rangle$ and $\langle h \rangle \in \phi'(p)$ then $\langle h' \rangle \in \phi'(p)$. So M is a simple model. Also, the following can be shown by induction on the complexity of the formula B: $\langle h \rangle \in M'(B)$ iff $h \in M(B)$. So M' is a simple model such that $M' \not\models A$. ⊣

Given Theorem 11, we henceforth restrict our attention to simple ims's and simple models—in fact we will simply refer to these as *ims*'s and *models*, dropping the modifier "simple".

DEFINITION 12. An ims is $\leq$*n-tiered* iff there are no chains $h_1, \ldots, h_{n+1} \in K$ with $h_i < h_{i+1}$. An ims is *n-tiered* iff it is $\leq n$-tiered and is not $\leq(n-1)$-tiered. A model is $\leq$*n-tiered* (*n-tiered*) iff it is based on an $\leq n$-tiered (n-tiered) ims. We define the *tiers* of a $\leq n$-tiered ims or model as follows: $\text{tier}_1 = \{g\}$; $\text{tier}_{m+1} = \{h : \text{ for some } h' \in \text{tier}_m, h' < h \text{ and for no } h'', h' < h'' < h\}$. Figures 1 to 6 represent some sample 3-tiered ims's. Figures 4, 5 and 6 represent some 3-tiered ims's of a type that will be of special interest.

DEFINITION 13. (i) $\leq\mathbf{n\text{-}H\pi+} = \{A : $ for every $\leq n$-tiered model $M, M \models A\}$.
(ii) $\mathbf{n\text{-}H\pi+} = \{A : $ for every n-tiered model $M, M \models A\}$.

THEOREM 14. $\leq\mathbf{n\text{-}H\pi+} = \mathbf{n\text{-}H\pi+}$.

PROOF. Clearly $\leq\mathbf{n\text{-}H\pi+} \subseteq \mathbf{n\text{-}H\pi+}$. To see that $\mathbf{n\text{-}H\pi+} \subseteq \leq\mathbf{n\text{-}H\pi+}$, suppose $M = (g, K, \leq, \phi)$ is a $\leq n$-tiered model such that $M \not\models A$. So M is an m-tiered model, where $m \leq n$. Create an n-tiered model M' as follows: get K' by adding

FIGURE 1

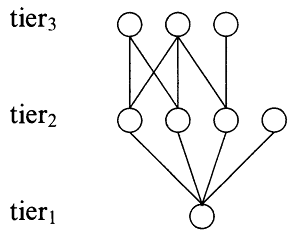


FIGURE 2

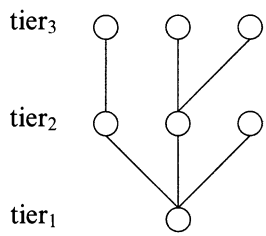


FIGURE 3

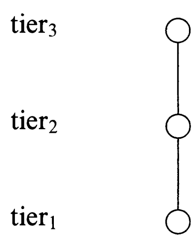


FIGURE 4

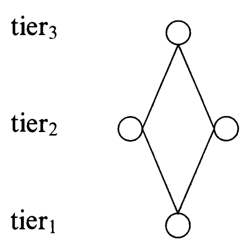


FIGURE 5

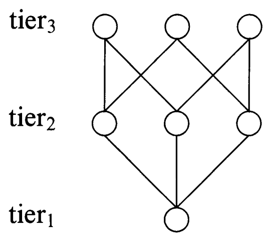
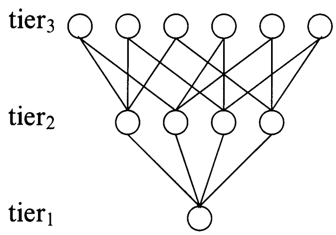


FIGURE 6



n-m new distinct members, $g_1, \dots, g_{n\text{-}m}$, to K; extend $\leq$ to $\leq'$ so that $g_i \leq' g_{i+1}$ and $g_{n\text{-}m} \leq' g$; and let $M' = (g_1, K', \leq', \phi)$. (Note that $g_1, \dots, g_{n\text{-}m} \notin \phi(p)$ for each propositional variable p.) Notice that, for each formula B, $M(B) = M'(B) \cap K$. So, since $M \not\models A$, $g \notin M'(A)$. So, by Lemma 4, $g_1 \notin M'(A)$. So $M' \not\models A$. ⊣

THEOREM 15. *Let C_1 be the formula $\forall p(p \vee {\sim}p)$. Given C_n, Let C_{n+1} be the formula $\forall p(p \vee (p \rightarrow C_n))$. Then, for each n, and for each model M, M is $\leq$n-tiered iff $M \models C_n$.*

PROOF. For any ims, let $CL_1 = CL$ as in Definition 10; $CL_{m+1} = \{h :$ for every h', if $h < h'$ then $h' \in CL_m\}$. First, we show by induction on n that, for each n, and for each model $M = (g, K, \leq, \phi)$, $M(C_n) = CL_n$.

Base case. To show $M(C_1) \subseteq CL_1$, suppose that $h \in M(C_1)$ and $h \notin CL_1$. Then there is some $h' > h$. Let $P = [h']$. Since $h \in M(\forall p(p \vee {\sim}p))$, and since $h \notin P$, $h \in M[P/p]({\sim}p)$. ($M[P/p]$ is defined in Definition 2.) So, for every $k > h$, $k \notin P = [h']$. So $h' \notin [h']$, which is clearly false. So $M(C_1) \subseteq CL_1$. On the other hand, to show that $CL_1 \subseteq M(C_1)$, suppose that $h \in CL_1$. To show that $h \in M(C_1) = M(\forall p(p \vee {\sim}p))$, fix a proposition P. We want to show that $h \in M[P/p](p \vee {\sim}p)$. Clearly this is true if $h \in P$. And if $h \notin P$, then, since there are no $k > h$, we have $h \in M[P/p]({\sim}p)$. So $h \in M[P/p](p \vee {\sim}p)$, as desired.

Inductive step. Suppose that, for each model M, $M(C_m) = CL_m$. Fix a model M. To show $M(C_{m+1}) \subseteq CL_{m+1}$, suppose that $h \in M(C_{m+1})$ and $h \notin CL_{m+1}$. So there is some $h' > h$ such that $h' \notin CL_m$. Let $P = [h']$. Since $h \in M(\forall p(p \vee (p \rightarrow C_m))$, and since $h \notin P$, $h \in M[P/p](p \rightarrow C_m)$. So, since $h' > h$ and $h' \in M[P/p](p) = P$, $h' \in M[P/p](C_m)$. But this contradicts $h' \notin CL_m$, given the inductive hypothesis. So $M(C_{m+1}) \subseteq CL_{m+1}$. On the other hand, to show that $CL_{m+1} \subseteq M(C_{m+1})$, suppose that $h \in CL_{m+1}$. To show that $h \in M(\forall p(p \vee (p \rightarrow C_m))$, fix a proposition P. We want to show that $h \in M[P/p](p \vee (p \rightarrow C_m))$. Clearly this is true if $h \in P$. Suppose $h \notin P$. Then, for each $k \geq h$,

$k \in M[P/p](p) \Rightarrow k \in P \Rightarrow k > h \Rightarrow k \in CL_m$ (since $h \in CL_{m+1}$) $\Rightarrow$ $k \in M[P/p](C_m)$, by the inductive hypothesis. So $h \in M[P/p](p \to C_m)$. So $h \in M[P/p](p \vee (p \to C_m))$, as desired.

Induction complete. Now fix a model $M = (g, K, \leq, \phi)$ and note: M is $\leq n$-tiered iff $CL_n = K$ iff $M(C_n) = K$ iff $g \in M(C_n)$ iff $M \models C_n$, as desired. ⊣

COROLLARY 16. **n-Hπ+** *is* 1*-reducible to* **Hπ+**.

PROOF. Given Theorem 14, it suffices to show that **≤n-Hπ+** is 1-reducible to **Hπ+**. And for this, it suffices to show that, for each formula A, $A \in$ **≤n-Hπ+** iff $(C_n \to A) \in$ **Hπ+**, where C_n is as defined in Theorem 15. (⇒) Suppose that $(C_n \to A) \notin$ **Hπ+**. Then there is some model $M = (g, K, \leq, \phi)$ such that $M \not\models (C_n \to A)$. So, for some $h \in K$, $h \in M(C_n)$ and $h \notin M(A)$. Define $M' = (h, [h], \leq', \phi')$, where $\leq'$ is $\leq$ restricted to $[h]$, and where $\phi'(p) = \phi(p) \cap [h]$. For every formula B, $M'(B) = M(B) \cap [h]$. So $h \notin M'(A)$ and $h \in M'(C_n)$. So $M' \not\models A$, and $M' \models C_n$. So, by Theorem 15, M' is a $\leq n$-tiered model. So, since $M' \not\models A$, $A \notin$ **≤n-Hπ+**. (⇐) Suppose that $(C_n \to A) \in$ **Hπ+**. Note: M is a $\leq n$-tiered model $\Rightarrow M \models C_n \Rightarrow M \models A$. So $A \notin$ **≤n-Hπ+**. ⊣

Given Corollary 16 and Theorem 9, it suffices, for our main result, to show the following:

THEOREM 17. 2-SIB^2 *is* 1*-reducible to* **3-Hπ+**.

PROOF. See §§6–8, below. ⊣

§6. Proof of Theorem 17. The idea of the proof is to mimic the behaviour of a classical second order SIB-model (with two or more elements in the universe) with a special kind of 3-tiered intuitionistic model. We will construct particular 3-tiered models so that the points on tier$_2$ stand in for the *objects* in the universe of a SIB-model; and so that the points on tier$_3$ stand in for *unordered pairs of distinct objects* from the universe of a SIB-model. The *subsets* of the third tier can stand in for sib relations on the universe of a SIB-model, since a sib relation can be thought of as a set of unordered pairs of distinct objects. So, we will define a special kind of 3-tiered model, one that can mimic a SIB-model. Figures 4, 5 and 6, above, represent 3-tiered ims's that will represent SIB-models whose universes have, respectively, two, three and four members. Figures 1, 2 and 3 represent less well-behaved 3-tiered ims's.

DEFINITION 18. A 3-tiered ims is *SIB-like* iff (i) every pair of distinct points in tier$_2$ has a unique least upper bound; (ii) every point in tier$_3$ is the least upper bound of two distinct points in tier$_2$; and (iii) no three distinct points in tier$_2$ have an upper bound. A model is *SIB-like* iff it is based on a SIB-like ims. Notice that every 3-tiered SIB-like model will represent a classical second order SIB-model whose universe has two or more members.

DEFINITION 19. **SIB-Hπ+** $= \{A$: for every 3-tiered SIB-like intuitionistic model $M, M \models A\}$.

To prove Theorem 17, it suffices to show Theorems 20 and 21:

THEOREM 20. **SIB-Hπ+** *is* 1-*reducible to* **3-Hπ+**.

PROOF. See §7, below. ⊣

THEOREM 21. 2-SIB^2 *is* 1-*reducible to* **SIB-Hπ+**.

PROOF. See §8, below. ⊣

§7. Proof of Theorem 20: SIB-Hπ+ is 1-reducible to 3-Hπ+. The primary task of §7.1 and §7.2 is to define a sentence, *sib*, in the intuitionistic language, which is such that, for every 3-tiered model M, $M \models sib$ iff M is SIB-like (see Table 3, row 5).

7.1. Table 1: Object language connectives and their two meta-linguistic interpretations. Given a 3-tiered intuitionistic model $M = (g, K, \leq, \phi)$, we can think of a formula A as playing two rôles: (i) A *names* a subset of K, in particular, $M(A)$; and (ii) A *makes a claim* about the model. For example, $(p \to q)$ names the set $M(p \to q)$ and $(p \to q)$ *says that* $M(p) \subseteq M(q)$ since, for every 3-tiered model M, $M \models (p \to q)$ iff $M(p) \subseteq M(q)$. (I appeal to exactly the same considerations in Kremer [1993].) Table 1 lists formulas constructed with the primitive intuitionistic object language connectives, and indicates what the formulas *say* and what they *name*.

In reading Tables 1, 2 and 3, it is important to assume that the model M is a 3-tiered model. The blank entries in these tables are those of no particular interest.

TABLE 1

Formula	What the formula says: $M \models$ Formula iff	What the formula names: M(Formula) $=$
$(A \,\&\, B)$	$M \models A$ and $M \models B$	$M(A) \cap M(B)$
$(A \vee B)$	$M \models A$ or $M \models B$	$M(A) \cup M(B)$
$(A \to B)$	$M(A) \subseteq M(B)$	
$\sim A$	$M(A) = \emptyset$	
$\forall pA$	for each proposition P, $M[P/p] \models A$	
$\exists pA$	for some proposition P, $M[P/p] \models A$	

7.2. Tables 2 and 3: Defined object language connectives and formulas, and their metalinguistic interpretations. Tables 2 and 3 define some object language connectives and formulas. Table 2 is concerned with what the definienda *name* and Table 3 is concerned with what they *say*. Also, in Table 3, we define a one-place connective $\neg$ and a two place connective $\in$. If p is a propositional variable and if A and B are formulas, then $(\forall p \in A)B$ is an abbreviation of the formula $\forall p(\neg(p \in A) \vee B)$; and $(\exists p \in A)B$ is an abbreviation of the formula $\exists p((p \in A) \,\&\, B)$. Using $\in$, we will be able to mimic quantification over the *elements* of the partial order K by quantifying over propositions of the form $[h]$, for $h \in K$.

TABLE 2

Definiendum	Definiens	What the definiendum names: M(Definiendum) =
C_1	$\forall p(p \vee \sim p)$	CL (see Theorem 15)
C_2	$\forall p(p \vee (p \rightarrow C_1))$	CL_2 (see Theorem 15) $= \text{tier}_2 \cup \text{tier}_3$ (since M is 3-tiered)

TABLE 3

Definiendum	Definiens	What the definiendum says: $M \models$ Definiendum iff
$(A \leftrightarrow B)$	$(A \rightarrow B) \& (B \rightarrow A)$	$M(A) = M(B)$
$\diamond A$	$(\sim(A \& C_1) \rightarrow C_2)$	$M(A) \neq \emptyset$
$\neg A$	$\diamond{\sim}A$	$M \not\models A$
$(A \in B)$	$\diamond A \& (A \rightarrow B) \& [A \vee \forall p (\neg(p \rightarrow C_1) \vee \sim(p \& A) \vee (A \rightarrow p)) \vee ((A \rightarrow C_2) \& \neg(A \rightarrow C_1) \& \forall p ((p \& A \rightarrow C_1) \vee (A \rightarrow p)))]$	for some $h \in M(B)$, $M(A) = [h]$
sib	$(\forall p \in C_1)(\exists q \in C_2)(\exists r \in C_2)(\neg(q \in C_1) \& \neg(r \in C_1) \& \neg(q \leftrightarrow r) \& (q \& r \leftrightarrow p)) \& (\forall p \in C_2)(\forall q \in C_2)((p \leftrightarrow q) \vee (p \rightarrow C_1) \vee (q \rightarrow C_1) \vee ((p \& q) \in C_1)) \& (\forall p \in C_2)(\forall q \in C_2)(\forall r \in C_2)((p \rightarrow C_1) \vee (q \rightarrow C_1) \vee (r \rightarrow C_1) \vee (p \leftrightarrow q) \vee (q \leftrightarrow r) \vee (p \leftrightarrow r) \vee \sim(p \& q \& r))$	M is SIB-like

The entries in the third columns of Tables 2 and 3 express non-obvious claims. Here we verify the last four rows in Table 3.

TABLE 3 ROW 2. Suppose that M is 3-tiered. Then $M \models \diamond A$ iff $M(A) \neq \emptyset$.

PROOF. First notice that $M(A \& C_1) = \emptyset$ iff $M \models \sim(A \& C_1)$ (by Table 1 row 5) iff $M(\sim(A \& C_1)) = K$ iff $M(\sim(A \& C_1)) \not\subseteq \text{tier}_2 \cup \text{tier}_3$ (since M is 3-tiered) iff $M{\sim}(A \& C_1)) \not\subseteq M(C_2)$ (by Table 2 row 2) iff $M \not\models (\sim(A \& C_1) \rightarrow C_2)$ (by Table 1 row 3). So it suffices to show that $M(A) = \emptyset$ iff $M(A \& C_1) = \emptyset$. ($\Rightarrow$) is obvious. To show ($\Leftarrow$) suppose that $M(A) \neq \emptyset$. Also, $M(A)$ is closed upwards. So, since M is finitely-tiered, there is an $h \in M(A)$ such that, for every $h' \in K$, $h \not< h'$. So $h \in CL_1 = M(C_1)$ (Table 2 row 2). So $M(A \& C_1) = M(A) \cap M(C_1) \neq \emptyset$. ⊣

TABLE 3 ROW 3. Suppose that M is 3-tiered. Then $M \models \neg A$ iff $M \not\models A$.

PROOF. $M \not\models \neg A$ iff $M \not\models \diamond{\sim}A$ iff $M({\sim}A) = \emptyset$ iff $M(A) = K$ iff $M \models A$. ⊣

TABLE 3 ROW 4. Suppose that M is 3-tiered. Then $M \models (A \in B)$ iff for some $h \in M(B)$, $M(A) = [h]$.

PROOF. $M \models (A \in B)$ iff both
(1) $M(A)$ is a non-empty subset of $M(B)$; and
(2) either
(2.1) $M \models A$, in which case $M(A) = K = [g]$; or
(2.2) for every proposition $P \subseteq CL_1$, either $P \cap M(A) = \emptyset$ or $M(A) \subseteq P$; or
(2.3) $M(A) \subseteq CL_2$ and $M(A) \not\subseteq CL_1$ and, for every proposition P, either $P \cap M(A) \subseteq CL_1$ or $M(A) \subseteq P$.

So $M \models (A \in B)$ iff both
(1) $M(A)$ is a non-empty subset of $M(B)$; and
(2) either
(2.1) $M(A) = [g]$; or
(2.2) $M(A) = [h]$ where $h \in CL_1$; or
(2.3) $M(A) = [h]$ where $h \in CL_2 - CL_1$.

So $M \models (A \in B)$ iff for some $h \in M(B)$, $M(A) = [h]$. ⊣

TABLE 3 ROW 5. Suppose that M is 3-tiered. Then $M \models sib$ iff M is SIB-like.

PROOF. $M \models sib$ iff
(1) for every $h \in CL_1$, there are distinct h', $h'' \in CL_2 - CL_1$ such that $[h'] \cap [h''] = [h] = \{h\}$ (note that $[h] = \{h\}$ for each $h \in CL_1$); and
(2) for any distinct $h, h' \in CL_2 - CL_1$, $[h] \cap [h'] = [h'']$ where $h'' \in CL_1$, so that $[h] \cap [h'] = \{h''\}$; and
(3) for any distinct $h, h', h'' \in CL_2 - CL_1$, $[h] \cap [h'] \cap [h''] = \emptyset$.

So $M \models sib$ iff
(1) every point in CL_1 is the unique least upper bound of two distinct points in $CL_2 - CL_1$; and
(2) any two distinct points in $CL_2 - CL_1$ have a unique upper bound; and
(3) no three distinct points in $CL_2 - CL_1$ have an upper bound.

Now note that, if (1), (2) and (3) hold, then $CL_1 = \text{tier}_3$ and $CL_2 - CL_1 = \text{tier}_2$. Further, if M is SIB-like, then $CL_1 = \text{tier}_3$ and $CL_2 - CL_1 = \text{tier}_2$. So $M \models sib$ iff
(0) $CL_1 = \text{tier}_3$ and $CL_2 - CL_1 = \text{tier}_2$; and
(1) every point in tier_3 is the unique least upper bound of two distinct points in tier_2; and
(2) any two distinct points in tier_2 have a unique upper bound; and
(3) no three distinct points in tier_2 have an upper bound.

So $M \models sib$ iff M is SIB-like. ⊣

Our goal in this section is to establish Theorem 20 (§6). Theorem 20 follows from Theorem 22:

THEOREM 22. $A \in$ **SIB-H**π+ *iff* $(\neg\, sib \vee A) \in$ **3-H**π+.

PROOF. (⇒) Suppose that $(\neg\, sib \vee A) \notin$ **3-H**π+. Then there is some 3-tiered model M such that $M \not\models (\neg\, sib \vee A)$. So $M \not\models A$ and $M \not\models \neg\, sib$. So $M \models sib$, by

Table 3 row 3. So M is SIB-like, by Table 3 row 5. Since M is SIB-like and $M \not\models A$, $A \notin$ **SIB-Hπ+**. ($\Leftarrow$) Suppose that $A \notin$ **SIB-Hπ+**. Then for some SIB-like 3-tiered model M, $M \not\models A$. Since M is SIB-like, $M \models$ *sib*, by Table 3 row 5; in which case $M \not\models \neg\, sib$, by Table 3 row 3. So M is a 3-tiered model such that $M \not\models (\neg\, sib \vee A)$.⊣

§8. Proof of Theorem 21: 2-SIB^2 is 1-reducible to SIB-Hπ+. This section will define, in stages, a recursive function f, from second order formulas to intuitionistic formulas, such that a second order formula $A \in$ 2-SIB^2 iff $f(A) \in$ **SIB-Hπ+** (see Corollary 30, below). This will suffice for Theorem 21.

Definitions 23 and 25, below, will formalise our considerations at the top of §6. We will represent a typical classical second order SIB-model (whose universe has at least two members) by an intuitionistic SIB-like model.

Quantification over the *elements* of the classical second order SIB-model will become quantification over the propositions of the form $[h]$, where $h \in CL_2 - CL_1 =$ tier$_2$. Quantification over such propositions itself plays the role of quantification over the *elements* of tier$_2$. Recall from §6 that the elements of tier$_2$ in the intuitionistic model represent the *objects* in the universe of the classical second order SIB-model.

Quantification over the sib relations becomes quantification over the propositions that are subsets of $CL_1 =$ tier$_3$. Recall from §6 that these propositions represent the *sib relations* of a classical second order SIB-model.

We will use propositional variables with odd indices to stand in for relational variables of the second order language, and we will use propositional variables with even indices to stand in for individual variables of the second order language.

DEFINITION 23. Suppose that the set U has at least two members; that $\emptyset \notin U$; and that for $u, v \in U \cup \{\emptyset\}$, we have $u \notin v$. We then say that U is *representable*. If U is representable, then the *representative* of U is the ims REP$(U) = (g, K, \leq)$ where $g = \emptyset$; $K = \{\emptyset\} \cup U \cup \{\{u, v\} : u, v \in U$ and $u \neq v\}$; and, for $h, k \in K$, $h \leq k$ iff $h = \emptyset$ or $h = k$ or $h \in U \cap k$. Given a classical second order SIB-model $M = (U, \phi)$, the *representative* of M is the intuitionistic model REP$(M) = (g, K, \leq, \phi')$ where $(g, K, \leq) =$ REP(U) and where ϕ' is defined as follows: $\phi'(p_{2i}) = [\phi(x_i)]$ and $\phi'(p_{2i-1}) = \{\{u, v\} : (u, v) \in \phi(X_i)\}$.

LEMMA 24. *If U is representable and if $M = (U, \phi)$ is a classical second order* SIB-*model, then* REP(U) *is a* SIB-*like ims and* REP(M) *is a* SIB-*like intuitionistic model. Further, in* REP(M), $CL_2 - CL_1 =$ tier$_2 = U$ *and* $CL_1 =$ tier$_3 = \{\{u, v\} : u, v \in U$ *and* $u \neq v\}$.

DEFINITION 25. We define a recursive 1-1 function, f_1, from second order formulas to intuitionistic formulas:

$$f_1(x_i = x_j) = (p_{2i} \leftrightarrow p_{2j})$$
$$f_1(X_i x_j x_k) = (p_{2j} \,\&\, p_{2k} \rightarrow p_{2i-1})$$
$$f_1(\neg A) = \neg f_1(A)$$
$$f_1(A \vee B) = f_1(A) \vee f_1(B)$$
$$f_1(\forall x_i A) = \forall p_{2i}(\neg(p_{2i} \in C_2) \vee (p_{2i} \in C_1) \vee f_1(A))$$
$$f_1(\forall X_i A) = \forall p_{2i-1}(\neg(p_{2i-1} \rightarrow C_1) \vee f_1(A)).$$

LEMMA 26. *For each second order formula A we have the following: if U is a representable set (Definition 23) and if $M = (U, \phi)$ is a classical second order* SIB-*model, then $M \models A$ iff* $\text{REP}(M) \models f_1(A)$.

PROOF. By induction on A. Here we skip the atomic cases and the inductive steps for $\neg$ and $\vee$, and concentrate on the inductive step for $\forall x_i$. ($\forall X_i$ is treated similarly.) So suppose that $A = \forall x_i B$ for some formula B. Then $f_1(A) = \forall p_{2i}(\neg(p_{2i} \in C_2) \vee (p_{2i} \in C_1) \vee f_1(B))$. Now suppose that U is representable and that $M = (U, \phi)$. We want to show that $M \models A$ iff $\text{REP}(M) \models f_1(A)$. ($\Rightarrow$) Suppose that $\text{REP}(M) \not\models f_1(A)$. Then, for some proposition P, $\text{REP}(M)[P/p_{2i}] \not\models \neg(p_{2i} \in C_2)$ and $\text{REP}(M)[P/p_{2i}] \not\models (p_{2i} \in C_1)$ and $\text{REP}(M)[P/p_{2i}] \not\models f_1(B)$. So, for some proposition P, $P = [h]$ where $h \in CL_2$; and $P \neq [h]$ for any $h \in CL_1$; and $\text{REP}(M)[P/p_{2i}] \not\models f_1(B)$. So, for some $h \in \text{tier}_2 = U$, $\text{REP}(M)[[h]/p_{2i}] \not\models f_1(B)$. Note that $\text{REP}(M)[[h]/p_{2i}] = \text{REP}(M')$ where $M' = M[h/x_i]$ is the classical second order SIB-model just like M except that it assigns the object $h \in U = \text{tier}_2$ to the individual variable x_i. So $\text{REP}(M') \not\models f_1(B)$. So, by the inductive hypothesis, $M' \not\models B$. So $M \not\models \forall x_i B$. So $M \not\models A$.

($\Leftarrow$) Suppose $M \not\models A$. Then, for some $u \in U$, $M[u/x_i] \not\models B$. So by the inductive hypothesis, $\text{REP}(M[u/x_i]) \not\models f_1(B)$. Note that, in the model $\text{REP}(M[u/x_i])$, $u \in \text{tier}_2 = CL_2 - CL_1$. Further $\text{REP}(M[u/x_i])(p_{2i}) = [u]$, by Definition of REP (Definition 23). So $\text{REP}(M[u/x_i]) \models (p_{2i} \in C_2)$ and $\text{REP}(M[u/x_i]) \not\models (p_{2i} \in C_1)$. So $\text{REP}(M[u/x_i]) \not\models (\neg(p_{2i} \in C_2) \vee (p_{2i} \in C_1) \vee f_1(B))$. Also note that $\text{REP}(M[u/x_i]) = \text{REP}(M)[[u]/p_{2i}]$. So $\text{REP}(M) \not\models \forall p(\neg(p_{2i} \in C_2) \vee (p_{2i} \in C_1) \vee f_1(B))$. So $\text{REP}(M) \not\models A$, as desired. ⊣

DEFINITION 27. Let f_2 be the following recursive 1-1 function from second order formulas to second order formulas. Suppose that A is a second order formula and that n is the greatest number such that x_n or X_n appears in A. Let $f_2(A) = \forall x_1 \cdots \forall x_n \forall X_1 \ldots \forall X_n A$.

LEMMA 28. *For any second order formula A, $A \in$ 2-SIB^2 iff $f_2(A) \in$ 2-SIB^2.*

LEMMA 29. *For any closed second order formula A, $A \in$ 2-SIB^2 iff $f_1(A) \in$* **SIB-Hπ+**.

PROOF. Let A be a closed second order formula. ($\Rightarrow$) Suppose that $f_1(A) \notin$ **SIB-Hπ+**. Then there is an SIB-like intuitionistic model $M = (g, K, \leq, \phi)$ such that $M \not\models f_1(A)$. We can assume that M has the following two properties: (1) the set tier_2 is representable (Definition 23) and (2) for each i, $\phi(p_{2i})$ has a least member, which is in tier_2, and $\phi(p_{2i-1}) \subseteq CL_1 = \text{tier}_3$. We can assume (1) since, if tier_2 is not representable, then there is some other SIB-like intuitionistic model M such that $M \not\models f_1(A)$ *and* such that tier_2 is representable. And we can assume (2), since $f_1(A)$ is a closed formula, and the values that ϕ takes on are irrelevant to whether or not a particular model validates a particular closed formula.

Let $U = \text{tier}_2$. Note that, since M is SIB-like, U has at least two members. Let M' be the classical second order SIB-model (U, ϕ'), where $\phi'(x_i) =$ the least member of $\phi(p_{2i})$, and where $\phi'(X_i) = \{(u, v) : u, v \in \text{tier}_2$ and the least upper bound of u and v is in $\phi(p_{2i-1})\}$. Note: M' is representable, since U is representable. Further, the intuitionistic model $\text{REP}(M')$ is isomorphic to the intuitionistic model

M. (The notion of isomorphism between intuitionistic models is defined in the obvious way.) So, since $M \not\models f_1(A)$, we also have REP$(M') \not\models f_1(A)$ (by the obvious isomorphism theorem). So, by Lemma 26, $M' \not\models A$. So $A \notin$ 2-SIB^2.

($\Leftarrow$) Suppose that $A \notin$ 2-SIB^2. Then there is a classical second order SIB-model $M = (U, \phi)$ such that $M \not\models A$. We can assume that U is representable. So, by Lemma 26, REP$(M) \not\models f_1(A)$. So $f_1(A) \notin$ **SIB-Hπ+**. ⊣

COROLLARY 30. *Let $f = f_1 f_2$. For any second order formula A,*

$$A \in \text{2-SIB}^2 \textit{ iff } f(A) \in \textbf{SIB-H}\boldsymbol{\pi}+.$$

PROOF. By Lemmas 28 and 29. ⊣

Theorem 21 follows from Corollary 30 together with the fact that f is a recursive function. Theorem 21 and Theorem 20 (§6) together yield Theorem 17 (§5), which, given Corollary 16 (§5), suffices for our main result (Theorem 7, §3).

§9. Concluding remarks. So the logic **Hπ+** is recursively isomorphic to full second order classical logic. This result does not bode well for the **Hπ+**. A well-behaved extension of intuitionistic logic should not be so non-constructive, so beyond our computational grasp. This seems to speak against any primary interpretation of intuitionistic propositional quantifiers, and in favour of the axiomatisable propositionally quantified extensions of **H**, studied by Gabbay and others.

There is, however, a different primary interpretation of intuitionistic propositional quantifiers that is of some interest. Though Kripke [1963] allows for a large variety of intuitionistic model structures, both he and the subsequent literature pay special attention to *trees*. We could define a new logic **Hπ++** by insisting that the only ims's taken into consideration be trees. **Hπ++** is strictly stronger than **Hπ+**: $(\forall p(\sim p \vee \sim\sim p) \rightarrow \forall p \forall q((p \rightarrow q) \vee (q \rightarrow p))) \in$ **Hπ++** but $\notin$ **Hπ+**. The current strategy for establishing the complexity of **Hπ+** will not work for **Hπ++**, since our SIB-like models are not trees, and so would no longer count as models on the new thinking. If **Hπ++** is axiomatisable—we believe that the question is open—then it is a plausible second order propositional intuitionistic logic, based on a primary interpretation of the propositional quantifiers.

§10. Appendix: encoding binary relations as sib relations. This appendix will show how to encode arbitrary binary relations as sib relations—our encoding is quite different from but, we believe, simpler than that of Nerode and Shore [1980]. This will show that second order logic with only binary relational variables, interpreted as ranging over *all* binary relations, is 1-reducible to 2-SIB^2. As mentioned in the remark after Theorem 9, we refer the reader to Nerode and Shore [1980] for the encoding of n-ary relations as binary relations. Recall that our logic SIB^2 is defined for a second order language with individual variables $x_1, \ldots, x_n, \ldots$; binary relational variables $X_1, \ldots, X_n, \ldots$; parentheses; connectives $\vee$ and $\neg$; identity, $=$; and first and second order universal quantifiers. We define BIN^2 to be standard second order logic, restricted to this language. So we define BIN^2 so that the X_i range over all binary relations.

Consider a non-empty set U. We will represent binary relations on U as sib relations on a larger set U', where U' is equipped with a privileged sib relation R.

FIGURE 7

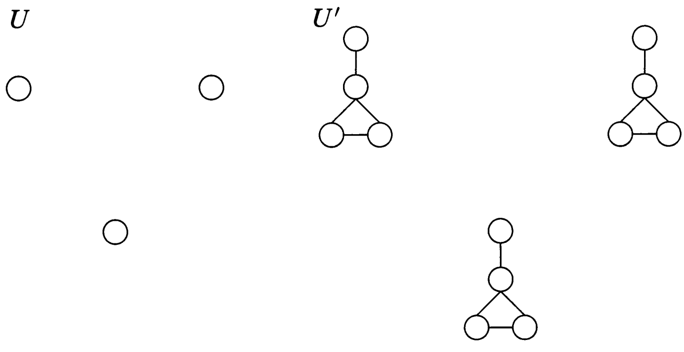
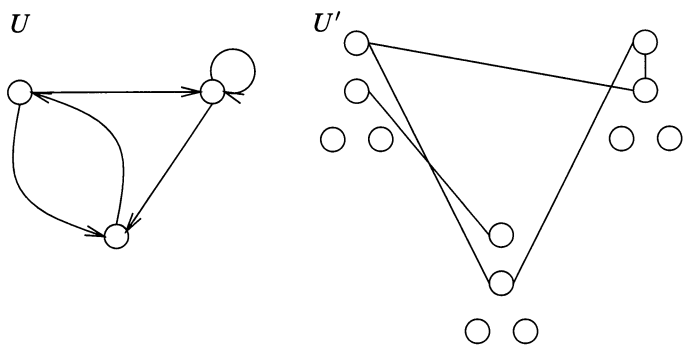


FIGURE 8



For each point $u \in U$, add three new points, and define R so each point in U is represented by a *constellation* in U', as in Figure 7. The lines connecting the points in U' represent the relation R.

Figure 8 shows how a particular binary relation, say B, on U, is represented by a sib relation B' on U'. The directed lines connecting the points in U represent B, and the lines connecting the points in U' represent B'. Note that the top point in each constellation in U' represents a point acting as the left member of an ordered pair in $U \times U$; the middle point in each constellation represents a point acting as a right member of an ordered pair; and the other two points serve to distinguish the "left" and the "right" points in each constellation.

Formally, we will define a recursive 1-1 map, g, from second order formulas to second order formulas (where the only second order variables in each case are binary) so that $A \in \text{BIN}^2$ iff $g(A) \in \text{2-SIB}^2$. This will establish that BIN^2 is 1-reducible to 2-SIB^2. This, coupled with the obvious fact that 2-SIB^2 is 1-reducible to BIN^2, shows that BIN^2 is recursively isomorphic to 2-SIB^2. We will define g in stages. But first, some preliminary definitions.

DEFINITION 31. The formula *constellation* is the following (where & and $\equiv$ have their standard definitions in terms of $\neg$ and $\vee$): $\forall x \exists y((x = y \vee X_1xy) \,\&\, \exists a \exists b \exists c(a \neq b \,\&\, a \neq c \,\&\, X_1bc \,\&\, \neg X_1ac \,\&\, \neg X_1ab \,\&\, \forall z(X_1yz \equiv (z = a \vee z = b \vee z = c)))$. This formula says that the sib relation X_1 splits the world up into *constellations* as in the world U' in Figure 7.

DEFINITION 32. Suppose that the sib-model M satisfies *constellation* and that U is M's universe of discourse and that $R = M(X_1)$. Then a point $u \in U$ is a *left point* iff u is R-related to exactly one point. And u is a *right point* iff u is R-related to exactly three points. A sib relation S is a *representative* (of some binary relation) iff, for every $(u, v) \in S$, either u is a left point and v is a right point, or u is a right point and v is a left point.

DEFINITION 33. For each individual variable x the formulas *left*(x) and *right*(x) are $\exists y \forall z(X_1 xz \equiv z = y)$ and $\exists a \exists b \exists c(a \neq b \,\&\, a \neq c \,\&\, b \neq c \,\&\, \forall z(X_1 xz \equiv (z = a \vee z = b \vee z = c))$, respectively. In those models that satisfy *constellation*, *left*(x) says that x is a left point, and *right*(x) says that x is a right point.

DEFINITION 34. For each binary relational variable X, the formula *representative*(X) is the following:

$$\forall x \forall y(\neg Xxy \vee ((\mathit{left}(x) \,\&\, \mathit{right}(y)) \vee (\mathit{left}(y) \,\&\, \mathit{right}(x)))).$$

In those models that satisfy *constellation*, *representative*(X) says that the sib relation X represents an arbitrary binary relation.

DEFINITION 35. We define a map g_1 from second order formulas to second order formulas. Those individual variables indexed by odd numbers will be restricted to left points, and those individual variables indexed by even numbers will be restricted to right points. Further, the relational variables will be restricted to representatives. We increase the index of the relational variables because of the special purpose served by X_1.

$$\begin{aligned}
g_1(X_n x_i x_j) &= X_{n+1} x_{2i-1} x_{2j} \\
g_1(x_i = x_j) &= X_1 x_{2i-1} x_{2j} \\
g_1(\neg A) &= \neg g_1(A) \\
g_1(A \vee B) &= g_1(A) \vee g_1(B) \\
g_1(\forall x_i A) &= \forall x_{2i-1} \forall x_{2i}(\neg\, \mathit{left}(x_{2i-1}) \vee \mathit{right}(x_{2i}) \vee \neg X_1 x_{2i-1} x_{2i} \vee g_1(A)) \\
g_1(\forall X_i A) &= \forall X_{i+1}(\neg\, \mathit{representative}(X_{i+1}) \vee g_1(A)).
\end{aligned}$$

DEFINITION 36. Let f_2 be defined as in Definition 27. Let $g(A) = \neg\, \mathit{constellation}$ $\vee g_1 f_2(A)$.

LEMMA 37. *For any closed second order formula A,* $A \in \mathrm{BIN}^2$ *iff* $\neg\, \mathit{constellation}$ $\vee g_1(A) \in$ 2-SIB^2.

PROOF. This is similar to the proof of Lemma 29 (§8). Analogous preliminary definitions and lemmas are useful. ⊣

COROLLARY 38. *For any second order formula A,* $A \in \mathrm{BIN}^2$ *iff* $g(A) \in$ 2-SIB^2.

PROOF. This is proved by Lemma 28 (§8) and Corollary 34. ⊣

COROLLARY 39. BIN^2 *is recursively isomorphic to* 2-SIB^2.

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