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SUPERVALUATION FIXED-POINT LOGICS OF TRUTH

ABSTRACT. Michael Kremer defines *fixed-point logics of truth* based on Saul Kripke's fixed point semantics for languages expressing their own truth concepts. Kremer axiomatizes the strong Kleene fixed-point logic of truth and the weak Kleene fixed-point logic of truth, but leaves the axiomatizability question open for the supervaluation fixed-point logic of truth and its variants. We show that the principal supervaluation fixed point logic of truth, when thought of as consequence relation, is highly complex: it is not even analytic. We also consider variants, engendered by a stronger notion of 'fixed point', and by variant supervaluation schemes. A 'logic' is often thought of, not as a consequence relation, but as a set of sentences – the sentences true on each interpretation. We axiomatize the supervaluation fixed-point logics so conceived.

KEY WORDS: fixed point logics, languages, truth

1. Introduction

One reaction to the liar paradox and its kin is the thought that the offending sentences are neither true nor false. Kripke (1975) gives the first comprehensive semantic framework formalizing this idea. (See also Martin and Woodruff (1975).) Suppose that L is a language with a distinguished predicate T. Kripke considers models in which T is given a *partial* interpretation: T is assigned a mutually exclusive extension and anti-extension; there is no assumption that these are exhaustive, as there would be if T were interpreted classically.

Given such a model, a sentence can be true, false, or *neither*. The precise distribution of truth values (considering *neither* as a truth value) depends on the *scheme of evaluation* used to determine the truth value of a sentence in a model. Kripke explicitly considers the *weak Kleene* scheme, the *strong Kleene* scheme, and the *supervaluation* scheme with two variants. Given a scheme of evaluation, a *fixed point* is a model in which the true sentences are precisely those in the extension of T and the false sentences are precisely those in the antiextension of T. These are the models in which T can be understood as meaning 'true'.

For each valuation scheme, Kremer (1986, 1988) defines a *fixed-point logic of truth*. He axiomatizes the strong Kleene fixed-point logic of truth and the weak Kleene fixed-point logic of truth, but explicitly leaves open the question of whether the supervaluation fixed-point logics of truth are axiomatizable. (This question is also asked in Kremer and Kremer (2003).) In the current paper, we show that the principal supervaluation fixed point logic of truth, when thought of as a consequence relation, is highly complex: it is not even analytic. We also consider variants, engendered by a stronger notion of 'fixed point', and by variant supervaluation schemes. A 'logic' is often thought of, not as a consequence relation, but as a set of sentences – the sentences true on each interpretation. We axiomatize the supervaluation fixed-point logics so conceived. (These results refine a theorem of Kremer and Kremer (2003): the consequence relations that are shown in that paper to be nonaxiomatizable are in fact highly complex.)

2. Fixed-Point Semantics

If L is any first-order language, then a classical model for L is, as usual, an ordered pair $M=\langle D, I\rangle$, where D is a nonempty set and I is a function assigning a member of D to each name of L, an n-ary function on D to each n-ary function symbol of L, and a function from D^n to $\{\mathbf{t}, \mathbf{f}\}$ to each n-ary relation symbol of L. Suppose that $M=\langle D, I\rangle$ is a classical model and that s is an *assignment* (*of values to the variables*), i.e., a function s:Vbles→D, where Vbles is the set of variables of L. Then $\mathrm{CL}_{M, s}(A)$ is the classical truth value, either **t** or **f**, assigned to the formula A relative to the assignment s.

Suppose that L is a first-order language with a distinguished predicate T and with a quote name 'A' for every sentence A of L. We call such a first-order language a *truth language*. Let Sent(L)={A: A is a sentence of L}. The *T-free fragment of* L is the fragment of L with no occurrences of T, except in the scope of quotation marks. The *quote-name-free* fragment of L is the language just like L without quote names. A *ground model* for L is a classical model $M=\langle D, I\rangle$ for the T-free fragment of L, satisfying the following conditions:

- Sent(L)⊆D; and
- I('A')=A, for every $A\in$Sent(L).

If $M=\langle D, I\rangle$ is a ground model for L and h:$D\rightarrow\{\mathbf{t}, \mathbf{f}\}$, then we define $M+h$ to be the classical model for all of L, just like M, except that $M+h$ assigns the function h to the predicate T.

Suppose that $M=\langle D, I\rangle$ is a ground model for L, and that h:$D\rightarrow\{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$. Then we will consider *partial* models $M+h$ of all of L. Note that a classical model is a special case of a partial model. We say that $h\leq h'$ and that $M+h\leq M+h'$ iff $h(d)=\mathbf{t}[\mathbf{f}]\Rightarrow h'(d)=\mathbf{t}[\mathbf{f}]$ for each $d\in D$, and that h and $M+h$ are *classical* iff h:$D\rightarrow\{\mathbf{t}, \mathbf{f}\}$. Suppose that $M+h$ is a partial model and that s is an *assignment*. We define the truth value $\mathrm{SV}_{M+h,\,s}(A)$ assigned to the formula A (by the supervaluation scheme) as follows:

$$\mathrm{SV}_{M+h,\,s}(A) = \begin{cases} \mathbf{t}, \text{if } \mathrm{CL}_{M+h',\,s}(A) = \mathbf{t} \text{ for every classical } M + h' \geq M + h; \\ \mathbf{f}, \text{if } \mathrm{CL}_{M+h',\,s}(A) = \mathbf{f} \text{ for every classical } M + h' \geq M + h; \\ \mathbf{n}, \text{otherwise.} \end{cases}$$

If A is a sentence (i.e. a formula with no free variables) then we write $\mathrm{SV}_{M+h}(A)$ for $\mathrm{SV}_{M+h,\,s}(A)$ since s drops out as irrelevant.

A partial model $M+h$ (for L) is a *fixed point* (for L) iff $h(A)=\mathrm{SV}_{M+h}(A)$ for every $A\in\mathrm{Sent}(L)$. (Kripke further stipulates that, in a fixed point, $h(d)=\mathbf{f}$ for every $d\in D-\mathrm{Sent}(L)$. We follow M. Kremer in suppressing this stipulation: nonsentences can be false, true or neither.) The following theorem is the core of Kripke's fixed-point semantics for truth, as applied to the supervaluation scheme. (Kripke proves this theorem for a wide variety of valuation schemes.)

FIXED POINT THEOREM. (Kripke (1975)) If L is a truth language and M is a ground model for L, then there is a fixed point $M+h$ for L.

The now canonical proof of this theorem involves the construction of the *least* fixed point, a fixed point which many have taken to be definitive of Kripke's theory of truth. (See Grover (1977), Haack (1978), Davis (1979), Kroon (1984), Parsons (1984), Kirkham (1992), and Read (1994).) But M. Kremer (1988) argues convincingly that T means 'true', not just in the least fixed point, but in any fixed point: we refer the reader to his discussion. Accordingly, for formulas A and B, we define

$$A\vDash B \text{ iff for every fixed point } M + h \text{ and every assignment } s, \text{ if } \mathrm{SV}_{M+h,\,s}(A) = \mathbf{t} \text{ then } \mathrm{SV}_{M+h,\,s}(B) = \mathbf{t}.$$

(Kremer allows multiple antecedents, interpreted conjunctively; and multiple consequents, interpreted disjunctively. He also requires right-to-left falsehood preservation as well as left-to-right truth preservation. Our results would go through for $\vDash$ so conceived.)

There is a peculiarity here: whether $A\vDash B$ depends on the language that A and B are expressed in. In particular, suppose that we pay more careful attention to all the factors, and define

$$A\vDash_L B \text{ iff for every fixed point } M + h \text{ for } L \text{ and every assignment } s, \text{ if } \mathrm{SV}_{M+h,\,s}(A) = \mathbf{t} \text{ then } \mathrm{SV}_{M+h,\,s}(B) = \mathbf{t}.$$

Then there are languages L and L' and sentences A and B of both L and L' such that $A \not\models_L B$ and $A \models_{L'} B$. This pathological behaviour of the consequence relation $\models_L$ is caused by our requirement that the sentences of L be included in the universe of a ground model.

EXAMPLE. Let LA be the language of arithmetic, with the name 0 and function symbols s, + and ×, enriched with a predicate T and quote names for the sentences of LA. And let LA′ be the result of adding uncountably many new constants to LA, as well as the resulting new quote names. Let PA^{-} be the conjunction of the axioms of Peano Arithmetic except the induction axioms, and let Ind be the formula $T0 \ \& \ \forall x(Tx \supset Tssx) \supset \forall x T2x$. Then

(*) $PA^{-} \ \& \ Ind \not\models_{\mathrm{LA}} T`\exists x T2x' \vee T`\exists x \neg T2x'$, and
(**) $PA^{-} \ \& \ Ind \models_{\mathrm{LA}'} T`\exists x T2x' \vee T`\exists x \neg T2x'$.

To see (*), order Sent(LA) in an ω-sequence $A_0, A_1, \ldots, A_n, \ldots$, making sure that $A_{2n} = \neg Ts^{2n}0$ for each n, where $s^k 0$ is 0 preceded by k occurrences of s. Let D=Sent(LA) and define three functions S, ⊕ and ⊗ on D as follows: $S(A_n) = A_{n+1}$, and $A_m \oplus A_n = A_{m+n}$, and $A_m \otimes A_n = A_{m \times n}$. Finally let $M = \langle D, I \rangle$ be the ground model where $I(0) = A_0$, $I(s) = S$, $I(+) = \oplus$, $I(\times) = \otimes$, and $I(`A') = A$ for each $A \in$ Sent(LA). Let $M+h$ be the least fixed point based on M. Note that $M+h$, restricted to the T-free and quote-name-free fragment of LA, is isomorphic to the standard (classical) model of arithmetic so that $\mathrm{SV}_{M+h}(PA^{-}) = \mathbf{t}$ and $\mathrm{SV}_{M+h}(Ind) = \mathbf{t}$, since all forms of induction are satisfied in every classical $M+h' \geq M+h$. Also note that $h(A_{2n}) = \mathbf{n}$, for each $n \in \mathbb{N}$, since each A_{2n} is a liar sentence. So there are classical h' and $h'' \geq h$ such that, for each $n \in \mathbb{N}$, we have $h'(A_{2n}) = \mathbf{t}$ and $h''(A_{2n}) = \mathbf{f}$. So $\mathrm{CL}_{M+h'}(\exists x T2x) = \mathrm{CL}_{M+h''}(\exists x \neg T2x) = \mathbf{t}$ and $\mathrm{CL}_{M+h''}(\exists x T2x) = \mathrm{CL}_{M+h'}(\exists x \neg T2x) = \mathbf{f}$. So $\mathrm{SV}_{M+h}(\exists x T2x) = \mathrm{SV}_{M+h}(\exists x \neg T2x) = \mathbf{n}$. So $h(\exists x T2x) = h(\exists x \neg T2x) = \mathbf{n}$, since $M+h$ is a fixed point. So there are classical h', $h'' \geq h$ such that $h'(\exists x T2x) = h'(\exists x \neg T2x) = \mathbf{t}$ and $h''(\exists x T2x) = h''(\exists x \neg T2x) = \mathbf{f}$, so that $\mathrm{SV}_{M+h'}(T`\exists x T2x' \vee T`\exists x \neg T2x') = \mathbf{t}$ and $\mathrm{SV}_{M+h''}(T`\exists x T2x' \vee T`\exists x \neg T2x') = \mathbf{f}$. So $\mathrm{SV}_{M+h}(T`\exists x T2x' \vee T`\exists x \neg T2x') = \mathbf{n}$. So $M+h$ is a fixed point in which the premise is true but the conclusion is not.

To see (**), suppose that $M+h = \langle D, I \rangle + h$ is a fixed point for the language LA′ and that $\mathrm{SV}_{M+h}(PA^{-} \ \& \ Ind) = \mathbf{t}$ and $\mathrm{SV}_{M+h}(T`\exists x T2x' \vee T`\exists x \neg T2x') \neq \mathbf{t}$. Then $\mathrm{SV}_{M+h}(\exists x T2x) = \mathrm{SV}_{M+h}(T`\exists x T2x') \neq \mathbf{t}$ and $\mathrm{SV}_{M+h}(\exists x \neg T2x) = \mathrm{SV}_{M+h}(T`\exists x \neg T2x') \neq \mathbf{t}$. Say that $d \in D$ is *even* if we have $d = I(+)(d', d')$ for some $d' \in D$. Note that $h(d) = \mathbf{n}$ for every even $d \in D$, otherwise either $\mathrm{SV}_{M+h}(\exists x T2x) = \mathbf{t}$ or $\mathrm{SV}_{M+h}(\exists x \neg T2x) = \mathbf{t}$. We define the *standard* members of D as follows: $I(0)$ is standard, and if d is standard then so is $I(s)(d)$. Since $h(d) = \mathbf{n}$ for every even $d \in D$, there is a classical $h' \geq h$ so that $h'(d) = \mathbf{t}$ for every standard even $d \in D$, and $h'(d) = \mathbf{f}$ for every nonstandard even $d \in D$. Such nonstandard even $d \in D$ exist, because $M+h$ is an uncountable model in which PA^- is true. So $\mathrm{CL}_{M+h'}(Ind)=\mathbf{f}$. But this contradicts the fact that $\mathrm{SV}_{M+h}(PA^- \,\&\, Ind)=\mathbf{t}$. ⊣

Given the peculiarity indicated by (*) and (**), we redefine $\vDash$ in a way which is independent of the language L:

$A \vDash B$ iff for every truth language L of which A and B are both formulas, and for every fixed point $M+h$ for L and for every assignment s, if $\mathrm{SV}_{M+h,\,s}(A) = \mathbf{t}$ then $\mathrm{SV}_{M+h,\,s}(B) = \mathbf{t}$.

Thus, $PA^- \,\&\, Ind \vDash_{\mathrm{LA}'} T`\exists x T2x' \vee T`\exists x \neg T2x'$, but $PA^- \,\&\, Ind \nvDash T`\exists x T2x' \vee T`\exists x \neg T2x'$.

3. Complexity

Our main result is that $\vDash$ is highly complex. To state this result precisely, we need some definitions. First assume that the first order language L has countably many relational symbols of each arity, countably many non-quote names, a distinguished unary predicate T and a quote name 'A' for each $A \in \mathrm{Sent}(L)$. (We assume that L has no function symbols.) Let L^2 be the second-order language constructed out of L as follows: first, remove the quote names; second, treat each relational symbol of L (including T) as a relational variable of the same arity in L^2; and third, treat each individual constant (not including the quote names) in L as an individual variable in L^2. In L^2, there is nothing special about the predicate T, since T is treated in L^2 as an ordinary unary second-order relational variable. A second-order formula is Π^1_n (Σ^1_n) iff it is of the form $Q_1X_1,\ldots, Q_nX_n\, A$, where $X_1,\ldots, X_n$ are unary second-order relational variables, $Q_1,\ldots, Q_n$ is a string of alternating quantifiers with $Q_1=\forall$ ($\exists$), and A is a quote-name-free formula containing no second-order quantifiers – i.e., A is a quote-name-free formula of L. Given a nonempty set D, an assignment of values to the individual variables (of L^2) is a function s assigning a member of D to each individual variable (of L^2), and an assignment of values to the relational variables (of L^2) is a function S assigning a function from D^n to $\{\mathbf{t}, \mathbf{f}\}$ to each n-ary relational variable (of L^2). The truth of a formula A in D relative to s and S is defined in the standard way. SOL is the set of formulas that are true in each domain D relative to any assignments s and S. Π^2_2-SOL is the set of Π^1_2 formulas in SOL.

COMPLEXITY THEOREM. Π^1_2-SOL is recursively encodable in both $\vDash$ and $\vDash_L$.

Before we prove this theorem, we stress just how complex $\prod_2^1$-SOL is. It is a familiar fact that Peano's axioms for the natural numbers, expressed as a universal $\left(\prod_1^1\right)$ formula of second-order logic, characterize the natural numbers up to isomorphism. Consequently, the set of all true sentences of first-order number theory is recursively embeddable in $\sum_1^1$-SOL, showing that $\sum_1^1$-SOL is not in the arithmetical hierarchy.

However, we can state much stronger results. The set of existential validities of second-order logic is so complex that no reasonable description of its complexity exists; thus it is essentially impossible to give an upper bound on the complexity of either $\vDash$ or $\vDash_L$. This follows from the work of Hintikka, Montague and others. Hintikka (1955) gives a recursive translation of formulas from the simple theory of types into $\sum_1^1$-SOL, that preserves validity in both directions. The basic idea behind the translation can be stated quite simply. Hintikka sets down a set of first-order axioms describing the type structure, and adds a $\prod_1^1$ sentence of second-order logic expressing the fact that the full comprehension axiom holds at each type level. The conjunction of these sentences can be written as a $\prod_1^1$ sentence $\forall X\, D$, so any sentence from the simple theory of types can be expressed as an equivalent $\sum_1^1$ second-order sentence of the form $\exists X(D \supset A)$. Consequently, the set of all true sentences expressed in the language of the simple theory of types, with the natural numbers as the ground type, is recursively embeddable in$\sum_1^1$-SOL, showing that $\vDash$ is not even analytic.

Richard Montague (1965) extends Hintikka's theorem to type theories where the type levels go far beyond the level ω of the simple theory of types. He shows that if all of the ordinals representing type levels in the language are describable in a precise sense, that the recursive embedding proved to exist by Hintikka can be generalized to this much more extended type-theoretical hierarchy. He remarks about the set of second-order logical truths:

> It is natural, however, to ask whether this set occurs in some natural extension of the Kleene arithmetical hierarchy, ... The next theorem, which was derived from Theorem 6 by Vaught and me, gives a negative answer; indeed, we have the stronger result that in a 'natural' higher-order language not even the set of Gödel numbers of *existential* second-order logical truths is definable [Montague 1965, 263].

The complexity theorem itself is a direct corollary of the next lemma. Before we state this lemma, we give a definition. Given a formula A of L and a unary predicate G not occurring in A, let A^G be the result of restricting all the first order quantification in A to G.

COMPLEXITY LEMMA. Suppose that $\forall X \exists TA$ is a $\prod_2^1$ formula, and that G is some unary predicate of L not occurring in A. Also suppose that $t_1, \ldots, t_n$ are all the terms occurring in A. (Note: the terms that can occur in A are either individual constants of L or variables of L. Every formula contains *some* term, so the list is not empty.) Then

(1) $\forall X \exists TA \in \prod_2^1$- SOL iff
(2) $\neg A^G$ & Gt_1 & ... & $Gt_n \models_L T$'$\exists x(Gx$ & $Tx)$' $\vee$ T'$\exists x(Gx$ & $\neg Tx)$' iff
(3) $\neg A^G$ & Gt_1 & ... & $Gt_n \models T$'$\exists x(Gx$ & $Tx)$' $\vee$ T'$\exists x(Gx$ & $\neg Tx)$'.

Before we prove this lemma, it might be helpful to explain the antecedent, $\neg A^G$ & Gt_1 & ... & Gt_n, and the consequent, T'$\exists x(Gx$ & $Tx)$' $\vee$ T'$\exists x(Gx$ & $\neg Tx)$', of (2) and (3). One way to understand these admittedly obscure formulas is to consider a different situation: In this new situation, we will define a consequence relation with multiple antecedents and multiple consequents, by considering *all* partial models rather than simply fixed points. Indeed we will generalize further: we will not assume that $M=\langle D, I\rangle$ is a ground model, but only a classical model for the T-free fragment of L. (Note that, in this generalized situation, quote names are not semantically different from nonquote names.) So, given a classical model $M=\langle D, I\rangle$ of the T-free fragment of L, and a function $h:D \rightarrow \{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$, we can consider the partial model $M+h$ of L. In this generalized situation, consider the following consequence relation, where Γ and Δ are sets of formulas of L:

$\Gamma \Vdash_L \Delta$ iff for every partial model $M+h$ for L (where M is a classical model for the T-free fragment of L) and for every assignment s, if $\mathrm{SV}_{M+h,\,s}(A) = \mathbf{t}$ for every $A \in \Gamma$, then $\mathrm{SV}_{M+h,\,s}(B) = \mathbf{t}$ for some $B \in \Delta$.

Note that, in the context of this new definition, the predicate T is no longer a plausible *truth* predicate: the only distinctive feature of T is that it is allowed a partial rather than a classical interpretation. Given this new consequence relation, we can prove the following, for any formula A of L that does not contain any quote names:

$$\forall X \exists TA \in \prod_2^1\text{-SOL iff} \neg A \Vdash_L \exists x Tx, \exists x \neg Tx.$$

Here is the proof. ($\Rightarrow$) Suppose that $\forall X \exists TA \in \prod_2^1$-SOL but that $\neg A \nVdash_L \exists x Tx, \exists x \neg Tx$. Then there is some partial model $M+h$ for L and some assignment s, with $\mathrm{SV}_{M+h,\,s}(\neg A)=\mathbf{t}$ and $\mathrm{SV}_{M+h,\,s}(\exists x Tx) \neq \mathbf{t}$ and $\mathrm{SV}_{M+h,\,s}(\exists x \neg Tx) \neq \mathbf{t}$. We will assume that $M=\langle D, I\rangle$. Given the last two inequalities, we know that $M+h$ assigns to T both an empty extension and an empty antiextension. In other words, $h(d)=\mathbf{n}$ for each $d \in D$. We now specify assignments s' and S of values to the individual and relational variables of L^2. Let $s'(x)=s(x)$ for every variable x of L occurring in A, and $s'(c)=I(c)$ for every nonquote name c of L occurring in A (these are variables in L^2), and $s'(v)=d$ for some arbitrarily chosen $d\in D$, for every other individual variable v of L^2. Let $S(T)(d)=\mathbf{f}$ for every $d\in D$ (the value assigned to T doesn't really matter); and let $S(Z)=I(Z)$ for every relational symbol Z of L other than T. Since $\forall X\exists TA \in \prod^1_2$-SOL, $\forall X\exists TA$ is true in D relative to s' and S. So $\exists TA$ is true in D relative to s' and S. So there is some $h':D\rightarrow\{\mathbf{t}, \mathbf{f}\}$ such that A is true in D relative to s' and S, where S' is just like S except that $S'(T)=h'$. Now consider the classical model $M+h'$. Clearly $h\leq h'$. So, since $SV_{M+h,\ s}(\neg A)=\mathbf{t}$ we have $SV_{M+h',\ s}(\neg A)=\mathbf{t}=CL_{M+h',\ s}(\neg A)$. But this contradicts our earlier claim that A is true in D relative to s' and S.

(⇐) Suppose that $\neg A \Vdash_L \exists xTx, \exists x\neg Tx$ but that $\forall X\exists TA \notin\prod^1_2$-SOL. Then there is a domain D, and assignments s and S such that $\exists TA$ is false in D relative to s and S. Define a classical model $M=\langle D, I\rangle$ for the language L as follows, where d_0 is some arbitrarily chosen member of D:

$I(R)\ =S(R)$, for each n-ary relation R of L
$I(c)\ =s(c)$, for each nonquote name c of L
$I(\text{‘}B\text{’})=d_0$, for each $B\in \text{Sent}(L)$

Define h as follows: $h(d)=\mathbf{n}$ for each $d\in D$. Notice that $SV_{M+h}(\exists xTx)= SV_{M+h}(\exists x\neg Tx)=\mathbf{n}$. Define the assignment s' of values to the individual variables of L as follows: $s'(x)=s(x)$, for each variable x of L. Thus $SV_{M+h,\ s'}(\exists xTx)=SV_{M+h,s'}(\exists x\neg Tx)=\mathbf{n}$. So, since $\neg A\Vdash_L \exists xTx, \exists x\neg Tx$, we have $SV_{M+h,\ s'}(\neg A)\neq\mathbf{t}$. So $SV_{M+h,\ s'}(A)\neq\mathbf{f}$. So there is some classical $h'\geq h$ such that $CL_{M+h',\ s'}(A)=\mathbf{t}$. Now we define an assignment S' of values to the second order variables of L^2: $S'(R)=S(R)$ for every $R\neq T$, and $S'(T)(d)=h'(d)$, for each $d\in D$. Since $\exists TA$ is false in D relative to s and S, A is false in D relative to s and to S'. So A, taken as a first-order formula of the quote-name-free fragment of L, is false in the classical model $M+h'$ relative to the assignment s'. But this contradicts $CL_{M+h',\ s'}(A)=\mathbf{t}$. ⊣

So, this is how a claim similar to the Complexity Lemma can be proved when we generalize our semantic definition of consequence in two ways: first, by not insisting that M be a ground model for L, but only a classical model for the T-free fragment L; second, by making no reference to fixed points $M+h$, but rather by considering all $M+h$. In this generalized situation, we have a very simple sequent, $\neg A\Vdash_L \exists xTx, \exists x\neg Tx$, instead of the more cumbersome $\neg A^G\ \&\ Gt_1\ \&\ ...\ \&\ Gt_n\vDash_L T\text{‘}\exists x(Gx\ \&\ Tx)\text{’} \vee T\text{‘}\exists x(Gx\ \&\ \neg Tx)$.

Moving to the specialized case at hand, where M is a *ground* model for L, and where the consequence relation is defined via only the fixed points, rather than all partial models $M+h$, presents special challenges. The main challenge is that $\exists xTx$ and $\exists x\neg Tx$ will be true in all fixed points: they won't ever take on the value **n**. This motivates the move to the sentences $\exists x(Gx \;\&\; Tx)$ and $\exists x(Gx \;\&\; \neg Tx)$: if the interpretation of G is constrained in the right way, then we can get the effect in the more specialized context that we got with the sentences $\exists xTx$ and $\exists x\neg Tx$ above. In fact, the interpretation of G will give us a useful subdomain of the domain D. Since we are only considering fixed points, we can replace the multiple consequent

$$\exists x(Gx \;\&\; Tx), \exists x(Gx \;\&\; \neg Tx)$$

with the single sentence T'$\exists x(Gx \;\&\; Tx)$' $\vee$ T'$\exists x(Gx \;\&\; \neg Tx)$': this sentence will take on the value **t** in a fixed point iff at least one of $\exists x(Gx \;\&\; Tx)$ or $\exists x(Gx \;\&\; \neg Tx)$ does. This explains the new consequent, T'$\exists x(Gx \;\&\; Tx)$' $\vee$ T'$\exists x(Gx \;\&\; \neg Tx)$'. As for the antecedent $\neg A^G \;\&\; Gt_1 \;\&\ldots\&\; Gt_n$, we note two things: (1) restricting the quantification in $\neg A$ to G allows us to restrict quantification to the subdomain determined by G; and (2) adding the conjuncts $Gt_1,\ldots, Gt_n$ ensures that anything picked out by relevant terms of L will be in the subdomain. Of course, to see how this really works, we move on to the proof.

PROOF OF THE COMPLEXITY LEMMA. (1)$\Rightarrow$(2): Suppose that $\neg A^G \;\&\; Gt_1 \;\&\; \ldots \;\&\; Gt_n \nvDash_L T$'$\exists x(Gx \;\&\; Tx)$' $\vee$ T'$\exists x(Gx\&\neg Tx)$'. Then for some fixed point $M+h=\langle D, I\rangle+h$ for L and some assignment s of values to the variables (of L) we have $SV_{M+h,\ s}(\neg A^G)=SV_{M+h,\ s}(Gt_i)=\mathbf{t}$ and SV_{M+h} (T'$\exists x(Gx \;\&\; Tx)$' $\vee$ T'$\exists x(Gx \;\&\; \neg Tx)$')$\neq\mathbf{t}$. Thus $SV_{M+h}(T$'$\exists x(Gx \;\&\; Tx)$'$)\neq\mathbf{t}$ and $SV_{M+h}(T$'$\exists x(Gx \;\&\; \neg Tx)$'$)\neq\mathbf{t}$. Thus $SV_{M+h}(\exists x(Gx \;\&\; Tx))\neq\mathbf{t}$ and SV_{M+h} $(\exists x(Gx \;\&\; \neg Tx))\neq\mathbf{t}$, since $M+h$ is a fixed point.

Let $D'=\{d\colon I(G)(d)=\mathbf{t}\}$, which is nonempty since $SV_{M+h,\ s}(Gt_1)=\mathbf{t}$. It suffices to find assignments s' and S of values to the individual and relational variables of L^2 so that $\exists TA$ is false in D' relative to s' and S. Let $s'(x)=s(x)$ for every variable x of L occurring in A, and $s'(c)=I(c)$ for every nonquote name c of L occurring in A (these are variables in L^2), and $s'(v)=d$ for some arbitrarily chosen $d\in D'$, for every other individual variable v of L^2. And for every relational symbol Z of L other than T, let $S(Z)=I(Z)|_{D'}$, i.e. the function $I(Z)$ restricted to the set D'; and let $S(T)(d)=\mathbf{f}$ for every $d\in D'$.

To see that $\exists TA$ is false in D' relative to s' and S, suppose not. Then there is some $h'\colon D'\rightarrow\{\mathbf{t}, \mathbf{f}\}$ such that A is true in D' relative to s' and S', where S' is just like S except that $S'(T)=h'$. Consider the following classical model for the quote-name-free fragment of L: $M'=\langle D', I'\rangle$ where $I'(c)=s'(c)$ for every nonquote name c of L, and $I'(Z)=S(Z)$ for every relational symbol Z of L. Clearly $CL_{M',\ s''}(A)=\mathbf{t}$ where s'' is s' restricted to the variables of L.

Define $h'':D\rightarrow\{\mathbf{t}, \mathbf{f}\}$ as follows:

$$h''(d) = h'(d) \text{ if } d \in D',\\ \mathbf{t} \text{ if } d \in D - D' \text{ and } h(d) = \boldsymbol{t},\\ \mathbf{f} \text{ if } d \in D - D' \text{ and } h(d) \neq \boldsymbol{t}.$$

We claim that $h\leq h''$. If $d\in D-D'$, then $h(d)=\mathbf{t}$ $[\mathbf{f}]\Rightarrow h''(d)=$ $\mathbf{t}$ $[\mathbf{f}]$, by definition of h''. For $d\in D'$, it suffices to show that $h(d)=\mathbf{n}$. Note that if $h(d)=\mathbf{t}$ and $d\in D'$, then we would have $SV_{M+h}(\exists x(Gx \ \&\ Tx))=\mathbf{t}$; and if $h(d)=\mathbf{f}$ and $d\in D'$, then we would have $SV_{M+h}(\exists x(Gx \ \&\ \neg Tx))=\mathbf{t}$. In either case we would contradict the above-noted fact that $SV_{M+h}(\exists x(Gx \ \&\ Tx))\neq\mathbf{t}$ and $SV_{M+h}(\exists x(Gx \ \&\ \neg Tx))\neq\mathbf{t}$. So $M+h''\geq M+h$. Also, $M+h''$ is a classical model for L.

So, since $SV_{M+h,\ s}(\neg A^G)=SV_{M+h,\ s}(Gt_i)=\mathbf{t}$, we have $CL_{M+h'',\ s}(A^G)=\mathbf{f}$. But notice that this gives us a contradiction, since, for any formula B whose names, free variables and relation symbols are the same as in A, we have $CL_{M',\ s''}(B)=CL_{M+h'',\ s}(B^G)$: the reason is that B^G is B restricted to G, and M' is that submodel of M (as far as the nonlogical constants in A are concerned) whose domain is the extension of G.

(2)⇒(3): This follows directly from the definitions.

(3)⇒(1): Suppose that $\forall X\exists TA \notin\Pi^1_2$-SOL. Then there is a domain D, and assignments s and S such that $\exists TA$ is false in D relative to s and S. We can assume that D contains no sentences of L. Define a ground model $M=\langle D', I\rangle$ for the language L as follows:

D'	=	$\text{Sent}(L)\cup D$
$I(R)$	=	$S(R)\cup\{\langle d_1, ..., d_n, \mathbf{f}\rangle$: some $d_i\in\text{Sent}(L)\}$, for each n-ary relation $R\neq G$ of L
$I(G)$	=	$(D\times\{\mathbf{t}\})\cup(\text{Sent}(L)\times\{\mathbf{f}\})$
$I(c)$	=	$s(c)$, for each nonquote name c of L
I(‘B’)	=	B, for each $B\in\text{Sent}(L)$

Let $M+h$ be any fixed point for L assigning $\mathbf{n}$ to each nonsentence, i.e. $h(d)=\mathbf{n}$ for each $d\in D=D'-\text{Sent}(L)$. The fixed point theorem can easily be strengthened to show that such a fixed point exists. Notice that SV_{M+h} $(\exists x$ $(Gx \ \&\ Tx))=SV_{M+h}(\exists x(Gx \ \&\ \neg Tx))=\mathbf{n}$. Thus $SV_{M+h}(T‘\exists x(Gx \ \&\ Tx)’)=$ $SV_{M+h}(T‘\exists x(Gx \ \&\ \neg Tx)’)=\mathbf{n}$. Thus $SV_{M+h}(T‘\exists x(Gx \ \&\ Tx)’\vee T‘\exists x(Gx \ \&\ \neg Tx)’)=\mathbf{n}$, as can be seen by considering any classical h' and h'' such that $h'(\exists x(Gx \ \&\ Tx))=h'(\exists x(Gx \ \&\ \neg Tx))=\mathbf{t}$ and $h''(\exists x(Gx \ \&\ Tx))=h''(\exists x(Gx \ \&\ \neg Tx))=\mathbf{f}$.

Since $SV_{M+h}(T‘\exists x(Gx \ \&\ Tx)’\vee T‘\exists x(Gx \ \&\ \neg Tx)’)=\mathbf{n}$, it suffices to specify an assignment s' of values to the variables of L, such that $SV_{M+h,\ s'}$

$(\neg A^G \,\&\, Gt_1 \,\&\, ... \,\&\, Gt_n)=\mathbf{t}$. Define s' as follows: $s'(x)=s(x)$, for each variable x of L. Clearly, $\mathrm{SV}_{M+h,\, s'}(Gt_i)=\mathbf{t}$, so it suffices to show that $\mathrm{SV}_{M+h,\, s'}(A^G)=\mathbf{f}$.

Suppose not. Then there is some classical $h''' \geq h$ such that $\mathrm{CL}_{M+h''',\, s'}(A^G)=\mathbf{t}$. Now we define an assignment S' of values to the second-order variables of L^2: $S'(R)=S(R)$ for every $R\neq T$, and $S'(T)(d)=h'''(d)$, for each $d\in D$. Since $\exists TA$ is false in D relative to s and S, A is false in D relative to s and to S'. So A, taken as a first-order formula of the quote-name-free fragment of L, is false in the classical model $M'=\langle D, I'\rangle$ relative to the assignment s'', where $I'(R)=S'(R)$ for each relational symbol R of L; $I'(c)=s(c)$ for each nonquote name c of L; and $s''(x)=s(x)$ for each variable x of L.

But notice that this gives us a contradiction, for the same reason that we got a contradiction at the end of the (1)$\Rightarrow$(2) proof above: for any formula B whose names, free variables and relation symbols are the same as in A, we have $\mathrm{CL}_{M',\, s''}(B)=\mathrm{CL}_{M+h''',\, s'}(B^G)$. ⊣

4. The Set of Sentences True in Every Model

One conception of the 'logic' generated by a semantics is the set of sentences true in every model. Thus, given a truth language L, we might want to axiomatize the set $\{A: A \text{ is a sentence of } L \text{ and } \vDash A\}$.

DEFINITION: THE SUPERVALUATION FIXED POINT LOGIC OF TRUTH. For each truth language L, Let SVFPLT_L (*the supervaluation fixed point logic of truth in the language* L) be the set of sentences axiomatized as follows:

Axioms:	'A' $\neq$ 'B', where A and B are distinct sentences of L.
Rules:	
Classical consequence:	From A, infer any classical consequence of A.
T rules:	From $A \supset B$, infer $A \supset T$'B' if A is in the T-free fragment of L.
	From $A \supset T$'B', infer $A \supset B$ if A is in the T-free fragment of L.
	From $A \supset \neg B$, infer $A \supset \neg T$'B' if A is in the T-free fragment of L.
	From $A \supset \neg T$'B', infer $A \supset \neg B$ if A is in the T-free fragment of L.

SOUNDNESS THEOREM. If $A\in\mathrm{SVFPLT}_L$ then both $\vDash A$ and $\vDash_L A$.

PROOF. Routine. ⊣

COMPLETENESS THEOREM.

(1) If $\vDash A$ then $A\in\mathrm{SVFPLT}_L$.
(2) If L is countable and $\vDash_L A$ then $A\in\mathrm{SVFPLT}_L$.

COROLLARY. If L is countable then $\vDash_L A$ iff $\vDash A$.

Completeness (1) follows from Completeness (2) by the following considerations. Suppose that L is some language (possibly uncountable) and that $A \notin \text{SVFPLT}_L$. Let L' be the sublanguage of L whose nonlogical vocabulary (other than T and quote names) consists of the vocabulary occurring in A, both in and not in the scope of quotation marks. L' is countable, and $A \notin \text{SVFPLT}_{L'}$. (This relies on the fact that SVFPLT_L is a conservative extension of $\text{SVFPLT}_{L'}$.) So, by Completeness (2), $\nvDash_L A$. Thus $\nvDash A$, as desired. So our remaining task is to prove Completeness (2).

As usual, we prove completeness by starting with a sentence $A \notin \text{SVFPLT}_L$, and then by building a theory Γ so that $A \notin \Gamma$ and so that a model can be constructed from Γ in such a way that A is not true in that model. Unfortunately, we have to be somewhat careful about our constructions. Before proving completeness, we state some definitions and lemmas.

DEFINITIONS. Given a set Γ of sentences, $\text{CN}(\Gamma)$ is the set of classical consequences of Γ and $\text{CN}_T(\Gamma)$ is the result of closing Γ under both classical consequence and the T rules. $\text{CN}_\omega(\Gamma)$ is the result of closing Γ under both classical consequence and the following ω-rule: From $A[t/x]$ for every term t to infer $\forall xAx$. A set Γ of sentences is a *theory* iff $\Gamma = \text{CN}(\Gamma)$. Γ is *T-closed* (ω-*closed*) iff it is closed under the T rules (ω-rule). A theory is *consistent* iff it is classically consistent. A theory Γ is *witnessing* iff for every sentence of the form $\forall xCx$, there is a sentence of the form $(Ct \supset \forall xCx) \in \Gamma$, where t is a closed term. Note: witnessing implies ω-closure, i.e. if a theory Γ is witnessing then Γ is ω-closed. A theory Γ is *almost complete* iff either $A \in \Gamma$ or $\neg A \in \Gamma$ for every sentence A in the T-free fragment of the language. A theory Γ is *complete* iff either $A \in \Gamma$ or $\neg A \in \Gamma$ for every sentence A in the language. Given a theory Γ and two closed terms (either quote names or other terms) t and t', we say that $t \equiv_\Gamma t'$ iff the sentence $(t = t') \in \Gamma$. $|t|_\Gamma =_{df} \{t' : t \equiv_\Gamma t'\}$. Note that, if A and B are distinct sentences, then 'A' $\not\equiv_\Gamma$ 'B' for every consistent theory $\Gamma \supseteq \text{SVFPLT}_L$.

MORE DEFINITIONS. Given a truth language L and a partial model $M+h$ for L, let $\text{Th}(M+h) = \{A$ is a sentence of L: $\text{SV}_{M+h}(A) = \mathbf{t}\}$. A ground model $M = \langle D, I \rangle$ is *explicit* iff for every $d \in D$ there is a name c (which can be a quote name) such that $I(c) = d$. A partial model $M+h$ is *explicit* iff the ground model M is.

MORE DEFINITIONS. Given an almost complete consistent witnessing theory $\Gamma \supseteq \{$'A'$\neq$'B': A and B are distinct sentences of $L\}$, define $M_\Gamma + h_\Gamma$, the *canonical partial model for* Γ, as follows. First let D=Sent

$(L) \cup \{|t|_\Gamma :\ t$ is a closed term of L and $t \not\equiv_\Gamma$ 'A' for any sentence $A\}$. For each closed term t, define

$$\langle t \rangle \quad = \quad |t|_\Gamma \text{ if } t \not\equiv_\Gamma \text{`}A\text{' for every sentence } A$$

$$A \text{ if } t \equiv_\Gamma \text{`}A\text{' for the sentence } A$$

Note that $\langle$'A'$\rangle$=A. Moreover, note that the well-definedness of $\langle$'$A\rangle$ depends on the fact that $\Gamma \supseteq \{$'A'$\neq$'B': A and B are distinct sentences of $L\}$. Now let

$M_\Gamma \quad = \quad \langle D_\Gamma, I_\Gamma\rangle$, where

$D_\Gamma = \text{Sent}(L) \cup \{\langle t\rangle$: t is a closed term of $L\}$,

$I_\Gamma(c) = \langle c\rangle$,

$I_\Gamma(f)(I(\langle t_1\rangle), ..., I(\langle t_n\rangle)) = \langle ft_1 ...t_n\rangle$ if f is an n-ary function symbol and $t_1, ..., t_n$ are closed terms, and

$I_\Gamma(R)(I(\langle t_1\rangle), ..., I(\langle t_n\rangle)) =$ **t** [**f**] if R is an n-ary relation symbol (other than T) and $t_1, ..., t_n$ are closed terms and $Rt_1 ...t_n \in \Gamma$ [$\neg Rt_1 ...t_n \in \Gamma$].

$h_\Gamma(\langle t\rangle) \quad =$ **t** if $Tt \in \Gamma$

f if $\neg Tt \in \Gamma$

n otherwise.

LEMMA 1. If Γ is a *complete* consistent witnessing theory and $\Gamma \supseteq \{$'A'$\neq$'B': A and B are distinct sentences of $L\}$ then $M_\Gamma + h_\Gamma$ is classical. Moreover, $\text{CL}_{M_\Gamma + h_\Gamma}(A) =$ **t** iff $A \in \Gamma$ for every sentence A of L.

PROOF. The classicalness of $M + h$ follows straight from the definition. The second part is a slight adaptation of the textbook theorem canonically stated when the objects in the model are equivalence classes of closed terms. ⊣

LEMMA 2. If Γ is an ω-closed theory and B is a sentence, then CN($\Gamma \cup \{B\}$) is ω-closed.

PROOF. Suppose that $A[t/x] \in \text{CN}(\Gamma \cup \{B\})$ for every closed term t. Then $(B \supset A[t/x]) \in \Gamma$ for every closed term t. So $\forall x(B \supset Ax) \in \Gamma$, since Γ is ω-closed. So $(B \supset \forall xAx) \in \Gamma$. So $\forall xAx \in \text{CN}(\Gamma \cup \{B\})$, as desired. ⊣

LEMMA 3. If Γ is an ω-closed theory such that $A \notin \Gamma$, and if Cx is a formula with one free variable, then there is a closed term t (which can be a quote name) such that $A \notin \mathrm{CN}_\omega(\Gamma \cup \{Ct \supset \forall x Cx\})$.

PROOF. Given Lemma 2, it suffices to show that there is a closed term t (which can be a quote name) such that $A \notin \mathrm{CN}(\Gamma \cup \{Ct \supset \forall x Cx\})$. Suppose not. Then, for every closed term t, the sentence $((Ct \supset \forall x Cx) \supset A) \in \Gamma$. So $\forall x((Cx \supset \forall x Cx) \supset A) \in \Gamma$. So $A \in \Gamma$, since A is a classical consequence of $\forall x((Cx \supset \forall x Cx) \supset A)$. ⊣

LEMMA 4. If the language L is countable and $M+h$ is an explicit partial model for L, then $\mathrm{Th}(M+h) = \mathrm{CN}_\omega(\{A : A$ is an atomic sentence and $\mathrm{SV}_{M+h}(A) = \mathbf{t}\} \cup \{\neg A : A$ is an atomic sentence and $\mathrm{SV}_{M+h}(A) = \mathbf{f}\})$.

PROOF. If $M+h$ is classical, then the result is standard. Suppose that $M+h$ is nonclassical. Let $\Gamma = \mathrm{CN}_\omega(\{A : A$ is an atomic sentence and $\mathrm{SV}_{M+h}(A) = \mathbf{t}\} \cup \{\neg A : A$ is an atomic sentence and $\mathrm{SV}_{M+h}(A) = \mathbf{f}\})$. Then $\Gamma \subseteq \mathrm{Th}(M+h)$, because

- $\{A$: A is an atomic sentence and $\mathrm{SV}_{M+h}(A)=\mathbf{t}\} \subseteq \mathrm{Th}(M+h)$,
- $\{\neg A$: A is an atomic sentence and $\mathrm{SV}_{M+h}(A)=\mathbf{f}\} \subseteq \mathrm{Th}(M+h)$, and
- $\mathrm{Th}(M+h)$ is closed under classical consequence and under the ω-rule (the latter, because $M+h$ is explicit).

Also note that, if A and B are distinct sentences, then 'A'≠'B'$\in \Gamma$, since M is a ground model.

To see that $\mathrm{Th}(M+h) \subseteq \Gamma$, suppose that $A \notin \Gamma$. List the formulas whose only free variable is x as follows: $C_1x, \ldots, C_nx, \ldots$ Build ω-closed theories $\Gamma_0 \subseteq \ldots \subseteq \Gamma_n \subseteq \ldots$ with $A \notin \Gamma_n$ as follows:

- $\Gamma_0 = \Gamma$
- Assume that Γ_n is an ω-closed theory and that $A \notin \Gamma_n$. Let $\Gamma_{n+1} = \mathrm{CN}_\omega(\Gamma \cup \{C_n t_n \supset \forall x C_n x\})$, where the closed term t_n is chosen so that $A \notin \Gamma_{n+1}$. This can be done by Lemma 3.

Let $\Gamma_\omega = \cup_n \Gamma_n$. Note that Γ_ω is witnessing. Now find, by standard methods, a *complete* theory $\Delta \supseteq \Gamma_\omega$ such that $A \notin \Delta$. Δ is a complete, consistent, witnessing theory and if C and B are distinct sentences, then 'C'≠'B'$\in \Delta$. So, by Lemma 1, $M_\Delta + h_\Delta$ is classical and $\mathrm{CL}_{M_\Delta + h_\Delta}(C) = \mathbf{t}$ iff $A \in \Delta$, for every sentence C. In particular, $\mathrm{CL}_{M_\Delta + h_\Delta}(A) = \mathbf{f}$.

Note that $M_\Delta + h_\Delta$ is isomorphic to $M+h'$ for some classical $h' \geq h$. So $\mathrm{CL}_{M+h'}(A) = \mathbf{f}$. So $\mathrm{SV}_{M+h}(A) \neq \mathbf{t}$, as desired. ⊣

Remark. This proof relies on the countability of the language in the construction of Γ_ω. At each step $(n+1)$, the fact that we can cite Lemma 3 in order to find an appropriate term t_n depends on the fact that Γ_n is ω-closed. If the language were uncountable, then the construction would have to be carried into the transfinite. But then we could not be sure that, at the limit steps, the relevant theories are ω-closed: a union of an increasing sequence of ω-closed theories, each excluding A, is guaranteed to be a theory excluding A, but is not guaranteed to be ω-closed.

MORE DEFINITIONS. Suppose that L is a truth language, and that $M = \langle D, I \rangle$ is a ground model for L. Given any nonempty $H \subseteq \{\mathbf{t}, \mathbf{f}, \mathbf{n}\}^D$, define $\lim H{:}D \rightarrow \{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$ as follows:

$$\lim H(d) = \mathbf{t}[\mathbf{f}] \text{ if } h(d) = \mathbf{t}[\mathbf{f}] \text{ for some } h \in H \text{ and } h(d) = \mathbf{f}[\mathbf{t}] \text{ for no } h \in H$$
$$\mathbf{n} \text{ otherwise}$$

Define the *supervaluation jump operator* $\sigma_M{:}\{\mathbf{t}, \mathbf{f}, \mathbf{n}\}^D \rightarrow \{\mathbf{t}, \mathbf{f}, \mathbf{n}\}^D$ as follows:

$$\begin{aligned}\sigma_M(h)(d) &= h(d) \text{ if } d \notin \text{Sent}(L) \\ &= \text{Val}_M(A) \text{ if } d = A \in \text{Sent}(L)\end{aligned}$$

For each ordinal α define σ_M^α as follows:

$$\begin{aligned}\sigma_M^0(h) &= h \\ \sigma_M^{\alpha+1}(h) &= \sigma_M\left(\sigma_M^\alpha(h)\right) \\ \sigma_M^\alpha(h) &= \lim\left\{\sigma_M^\alpha(h) : \beta<\alpha\right\} \text{for limit ordinals } \alpha\end{aligned}$$

LEMMA 5.

(1) If $h \leq h'$ then $\sigma_M(h) \leq \sigma_M(h')$
(2) If $h \leq \sigma_M(h)$ then for any ordinals α and β, $\sigma_M^\alpha(h) \leq \sigma_M^\beta(h)$
(3) If $h \leq \sigma_M(h)$ then there is some ordinal α such that $\sigma_M^{\alpha+1}(h) = \sigma_M^\beta(h)$.

Proof. These are standard results in the literature on fixed-point semantics. ⊣

PROOF OF COMPLETENESS (2). Assume that L is countable and that $A \notin \text{SVFPLT}_L$. We need to show that $\nvDash_L A$. We will assume that the variables of L are $x_0, \ldots x_n, \ldots$. Extend L to a language L' as follows: first add countably many new nonquote names $c_0, \ldots c_n, \ldots$; next add quote names to L' so that 'B' is a quote name of L' for every sentence B of L'. Thus L' will not only contain the new nonquote names $c_0, \ldots c_n, \ldots$ but also, for example, the quote names '$(c_1=c_2)\vee(c_3=c_4)$' and 'T'Tc_7''. List the formulas of L' with only the free variable x_0 as C_0,... C_n,.... For any such formula C and any term t, let Ct be the result of replacing the free occurrences of x_0 in C with t. We claim that $A\notin \mathrm{CN}_T(\mathrm{SVFPLT}_{L'}\cup\{C_nc_n\supset\forall x_0C_nx_0: n\in\mathbb{N}\})$.

To show this, we define a one-one translation Tr:Sent(L')→Sent(L). First, define the *depth* of a sentence B of L' as follows: if B has no quote names then depth(B)=0; if max{depth(C): 'C' occurs in B}=n, then depth (B)=$n+1$. Now, assuming that Tr(B) has been defined for every sentence B of L′ of depth<n, we will define Tr(B) for B of depth n. Beginning with B, let

$B'=B$, if B is of depth 0
= the result of replacing in B every quote name 'C' which does not itself occur in the scope of quotation marks with the quote name 'Tr(C)'.

Next, get B'' by replacing every bound variable x_k in B' that does not occur in the scope of quotation marks with the variable x_{2k}. If no new name c_m occurs in B'', then let Tr(B)=B''. Otherwise, choose the greatest $m\in\mathbb{N}$ such that c_m occurs in B''. Get the *formula* B''' by replacing each c_k in B'' by x_{2k+1}. And let Tr(B)=$\exists x_1\exists x_3...\exists x_{2m+1}B'''$.

Clearly Tr:Sent(L')→Sent(L) is one-one. Also, for every $B\in$Sent(L), notice that B and Tr(B) differ only in their bound variables, both in and out of the scope of quotation marks: for example, suppose that R is a binary relation symbol in L and that G is a unary predicate symbol of L and that B is the sentence $\forall x_7Rx_7$'$\exists x_3Gx_3$'; then Tr(B) is the sentence $\forall x_{14}Rx_{14}$'$\exists x_6Gx_6$'. Thus $B\in\mathrm{SVFPLT}_L$ iff Tr(B)$\in\mathrm{SVFPLT}_L$, for every $B\in$ Sent(L). So, in particular, Tr(A)$\notin\mathrm{SVFPLT}_L$.

We claim that, for every sentence B, if $B\in\mathrm{CN}_T(\mathrm{SVFPLT}_{L'}\cup\{C_nc_n\supset\forall x_0C_nx_0: n\in\mathbb{N}\})$ then Tr(B)$\in\mathrm{SVFPLT}_L$. We prove this by induction on the length of proof of B in $\mathrm{CN}_T(\mathrm{SVFPLT}_{L'}\cup\{C_nc_n\supset\forall x_0C_nx_0: n\in\mathbb{N}\})$. First, suppose that B is the sentence 'C'≠'D' where C and D are distinct. Then Tr(B) is sentence 'Tr(C)' ≠ 'Tr(D)' which is in SVFPLT_L, since Tr(C) and Tr (D) are distinct. Next, suppose that B is $C_nc_n\supset\forall x_0C_nx_0$. Then Tr($B$)=$\exists x_1\exists x_3...\ \exists x_{2m+1}(C'x_{2n+1}\supset\forall x_0C'x_0)$ for some formula C' and some $m\geq n$. In this case Tr(B)$\in\mathrm{SVFPLT}_L$ since Tr(B) is a classical theorem. Next suppose that Tr(B_1)$\in\mathrm{SVFPLT}_L$ and that B_2 is a classical consequence of B_1. Then Tr (B_2) is a classical consequence of Tr(B_1) so that Tr(B_2)$\in\mathrm{SVFPLT}_L$. Similarly, if Tr(B_1)$\in\mathrm{SVFPLT}_L$ and B_2 follows from B_1 by an application of one of the T rules, then Tr(B_2) follows from Tr(B_1) by the same T rule, so that Tr(B_2)$\in$ SVFPLT_L. Thus, as desired, if $B\in\mathrm{CN}_T(\mathrm{SVFPLT}_{L'}\cup\{C_nc_n\supset\forall x_0C_nx_0: n\in\mathbb{N}\})$ then Tr(B)$\in\mathrm{SVFPLT}_L$, for every sentence B of L'. Thus $A\notin \mathrm{CN}_T(\mathrm{SVFPLT}_{L'}\cup\{C_nc_n\supset\forall x_0C_nx_0: n\in\mathbb{N}\})$, since Tr($A$)$\notin\mathrm{SVFPLT}_L$.

Note that $\mathrm{CN}_T(\mathrm{SVFPLT}_{L'} \cup \{C_n c_n \supset \forall x_0 C_n x_0\colon n \in \mathbb{N}\})$ is among the theories Σ satisfying

(1) $\mathrm{SVFPLT}_{L'} \subseteq \Sigma$,
(2) For each formula Cx with one free variable, there is a name c such that $(Cc \supset \forall x Cx) \in \Sigma$,
(3) $A \notin \Sigma$, and
(4) Σ is T-closed.

Also notice that the union of any increasing chain of theories satisfying (1)–(4) is also a theory satisfying (1)–(4). So we can choose a *maximal* theory Γ satisfying (1)–(4). Note that Γ is a consistent witnessing theory and that the sentence 'C'$\neq$'B'$\in\Gamma$ for distinct sentences C and B of Γ. If we can prove that Γ is almost complete, then we can define $M_\Gamma + h_\Gamma$, the canonical partial model for Γ.

To see that Γ is almost complete, suppose that B is in the T-free fragment of L' and that neither B nor $\neg B \in \Gamma$. Now either $A \notin \mathrm{CN}(\Gamma \cup \{B\})$ or $A \notin \mathrm{CN}(\Gamma \cup \{\neg B\})$. Suppose that $A \notin \mathrm{CN}(\Gamma \cup \{B\})$ (the other case is treated similarly). We claim that $\mathrm{CN}(\Gamma \cup \{B\})$ is T-closed. To see this, suppose that $C \supset D \in \mathrm{CN}(\Gamma \cup \{B\})$, where C is in the T-free fragment of L'. Then $(B \supset (C \supset D)) \in \Gamma$ so that B & $C \supset D \in \Gamma$, so that B & $C \supset T$'D'$\in\Gamma$, since (B & C) is in the T-free fragment of L', and since Γ is T-closed. Thus $C \supset T$'D'$\in \mathrm{CN}(\Gamma \cup \{B\})$. Similarly $\mathrm{CN}(\Gamma \cup \{B\})$ is closed under the other T rules. But the T-closedness of $\mathrm{CN}(\Gamma \cup \{B\})$ contradicts the maximality of Γ with respect to properties (1)–(4). So Γ is almost complete.

So we can define $M_\Gamma + h_\Gamma$, the canonical partial model for Γ. Actually, we only need the ground model M_Γ. Let $h(d)=\mathbf{n}$ for each $d \in D_\Gamma$. For each ordinal α, define $h_\alpha = \sigma_M^\alpha(h)$. Note that $h \leq \sigma_{M_\Gamma}(h)$ so that, by Lemma 5 (2), if $\alpha \leq \beta$ then $h_\alpha \leq h_\beta$. Note also that $h_\alpha(d) = \mathbf{n}$ for each $d \in D_\Gamma - \mathrm{Sent}(L')$.

By induction on α we will prove that

(5) $\{B\colon B \in \mathrm{Sent}(L') \text{ and } h_\alpha(B) = \mathbf{t}\} \subseteq \Gamma$.

When $\alpha=0$, the result is given. Suppose that α is a limit ordinal and that $\{B\colon B \in \mathrm{Sent}(L') \text{ and } h_\beta(B) = \mathbf{t}\} \subseteq \Gamma$ for every $\beta<\alpha$. Now choose any sentence B and suppose that $h_\alpha(B) = \mathbf{t}$. Then $h_\beta(B) = \mathbf{t}$ for some $\beta<\alpha$, so that $B \in \Gamma$ by the inductive hypothesis. Finally, suppose that $\alpha = \beta + 1$ and that $\{B\colon B \in \mathrm{Sent}(L') \text{ and } h_\beta(B) = \mathbf{t}\} \subseteq \Gamma$. Note that $h_\alpha(B) = h_{\beta+1}(B) = \mathbf{t}$ iff $B \in \mathrm{Th}(M_\Gamma + h_\beta)$, for any $B \in \mathrm{Sent}(L')$. So in order to show that $\{B\colon B \in \mathrm{Sent}(L') \text{ and } h_\alpha(B) = \mathbf{t}\} \subseteq \Gamma$, it suffices to show that $\mathrm{Th}(M_\Gamma + h_\beta) \subseteq \Gamma$. By Lemma 4, $\mathrm{Th}(M_\Gamma + h_\beta) = \mathrm{CN}_\omega(\{B : B$ is an atomic sentence and

$\mathrm{SV}_{M_\Gamma+h_\beta}(B)=\mathbf{t}\} \cup \{\neg B : B$ is an atomic sentence and $\mathrm{SV}_{\mathrm{M*+h}_\beta}(\mathrm{B}) = \mathbf{f}\})$. Note that Γ is ω-closed, since Γ is witnessing. So it suffices to show that $\{B$: B is an atomic sentence and $\mathrm{SV}_{M_\Gamma+h_\beta}(B)=\mathbf{t}\} \cup \{\neg B$: B is an atomic sentence and $\mathrm{SV}_{M_\Gamma+h_\beta}(B)=\mathbf{f}\} \subseteq \Gamma$. Note that, if B is an atomic sentence in the T-free fragment of L' then $\mathrm{SV}_{M_\Gamma+h_\beta}(B)=\mathbf{t}[\mathbf{f}]$ iff $B \in \Gamma[\neg B \in \Gamma]$. So it suffices to show that $\{Tt$: t is a closed term and $\mathrm{SV}_{M_\Gamma+h_\beta}(Tt)=\mathbf{t}\} \cup \{\neg Tt$: Tt is a closed term and $\mathrm{SV}_{M_\Gamma+h_\beta}(Tt)=\mathbf{f}\} \subseteq \Gamma$. So suppose that t is a closed term and $\mathrm{SV}_{M_\Gamma+h_\beta}(Tt)=\mathbf{t}$. Then $h_\beta(\langle t\rangle)=\mathbf{t}$. So $\langle t\rangle=B$ for some $B\in$Sent(L'). So $B\in\Gamma$, by the inductive hypothesis. So T‘B’$\in\Gamma$, since Γ is T-closed. Also note that, since $\langle t\rangle=B$, the sentence (t=‘B’)$\in\Gamma$. So $Tt\in\Gamma$. Similarly, if t is a closed term and $\mathrm{SV}_{M_\Gamma+h_\beta}(Tt)=\mathbf{f}$ then $\neg Tt\in\Gamma$. And this suffices to complete the inductive proof of (5).

Now by Lemma 5, there is some ordinal α such that the partial model $M_\Gamma+h_\alpha$ is a fixed point, so that $\mathrm{SV}_{M_\Gamma+h_\alpha}(B)=h_\alpha(B)$ for each sentence B of L'. So, since $A\notin\Gamma$, we have $\mathrm{SV}_{M_\Gamma+h_\alpha}(A)\neq\mathbf{t}$, by (5) above. Now $M_\Gamma=\langle D_\Gamma, I_\Gamma\rangle$ is a ground model for the language L', but we can easily use it to define a ground model M for the language L as follows: $M=\langle D_\Gamma, I\rangle$ where I agrees with I_Γ on every constant, every relation symbol and every quote name in L. Clearly $\mathrm{SV}_{M+h_\alpha}(A)\neq\mathbf{t}$. So $\nvDash_L A$, as desired. ⊣

Remark. We have not closed the question of whether $\vDash_L A \Rightarrow A \in \mathrm{SVFPLT}_L$ for uncountable languages L, although we assert here that it just *must* be true!

Remark. Another conception of the ‘logic’ generated by a semantics is the set of sentences false in no model, i.e. the set $\{A : \neg A \vDash \bot\}$. Note that this ‘logic’ is not closed under classical logical consequence: both sentences b=‘¬Tb’⊃Tb and b=‘¬Tb’⊃¬Tb are in this set, but the sentence b≠‘¬Tb’ is not. We leave the complexity question open.

5. Variant Definitions of Consequence

So far, we have followed Kremer (1986) in taking a *fixed point* to be a partial model M+h where $h(A)=\mathrm{SV}_{M+h}(A)$ for every $A\in$Sent(L). As noted in Section 2 above, Kripke (1975) further requires that, in a fixed point, $h(d)=\mathbf{f}$ for each $d\in D-$Sent(L). We call such a fixed point a *Kripkean* fixed point, and define variants of $\vDash_L$ and $\vDash$:

$A\vDash'_L B$ iff	for every Kripkean fixed point M+h for L and every assignment s, if $\mathrm{SV}_{M+h,\ s}(A)=\mathbf{t}$ then $\mathrm{SV}_{M+h,\ s}(B)=\mathbf{t}$.
$A\vDash' B$ iff	for every language L (with a distinguished predicate T and quote names) of which A and B are both

formulas, and for every Kripkean fixed point $M+h$ for L and for every assignment s, if $\mathrm{SV}_{M+h,\ s}(A)=\mathbf{t}$ then $\mathrm{SV}_{M+h,\ s}(B)=\mathbf{t}$.

Analogously to Section 2, above, we have PA^{-} & $\mathrm{Ind}\vDash'_{\mathrm{LA}'}T$'$\exists xT2x$' $\vee$ T'$\exists x\neg T2x$' and PA^{-} & $Ind\nvDash' T$'$\exists xT2x$' $\vee$ T'$\exists xT2x$'. The proofs are identical. As for the relationship between these consequence relations and those of Section 2, it is clear that $A\vDash_L B \Rightarrow A\vDash'_L B$ and $A\vDash B \Rightarrow A\vDash' B$. But we have the following counterexample to the converses.

EXAMPLE. Consider the truth language L with no nonquote names, one binary function symbol p (think of a symbol for the *pairing* function), two unary predicates G and H in addition to the unary predicate T and quote names for all the sentences. Let A and B be the following sentences:

$$\begin{aligned} A=_{df}\ & \forall x(Hx\supset Tx)\ \&\ \forall x(Tx\supset Gx\vee Hx)\ \&\ \forall x\forall yGpxy\ \& \\ & \forall x\forall y\forall z\forall w(pxy=pzw\supset x=z\ \&\ y=w)\ \& \\ & \forall x\forall y\forall z(Tpxy\ \&\ Tpzy\supset x=z)\supset\exists y(Gy\ \&\ \forall x(Hx\supset\neg Tpxy)) \\ B=_{df}\ & T\text{‘}\exists x(Gx\ \&\ Tx)\text{’}\vee T\text{‘}\exists x(Gx\ \&\ \neg Tx)\text{’} \end{aligned}$$

Then we claim that,

(*) $A\vDash' B$ and $A\vDash'_L B$, but
(**) $A\nvDash B$ and $A\nvDash_L B$.

For (*), it suffices to show that $A\vDash' B$. So suppose not. Then for some language L' of which A and B are both sentences, and for some Kripkean fixed point $M+h=\langle D, I\rangle+h$ for L', we have $\mathrm{SV}_{M+h}(A)=\mathbf{t}$ and $\mathrm{SV}_{M+h}(B)\neq\mathbf{t}$. Define four subsets of D as follows:

G=$\{d\in D: I(G)(d)=\mathbf{t}\}$
H=$\{d\in D: I(H)(d)=\mathbf{t}\}$
T=$\{d\in D: h(d)=\mathbf{t}\}$
N=$\{d\in D: h(d)=\mathbf{n}\}$

Also define the function p:$D\times D\rightarrow D$ as $I(p)$.
Note:

(1) G$\subseteq$N, since $\mathrm{SV}_{M+h}(B)\neq\mathbf{t}$. To see this, note (1.1) $d\in$G and $h(d)=\mathbf{t}\Rightarrow \mathrm{SV}_{M+h}(\exists x(Gx\ \&\ Tx))=\mathbf{t}\Rightarrow \mathrm{SV}_{M+h}(T$‘$\exists x(Gx\ \&\ Tx)$’$)=\mathbf{t}$ (since $M+h$ is a fixed point) $\Rightarrow\mathrm{SV}_{M+h}(T$‘$\exists x(Gx\ \&\ Tx)$’$\vee T$‘$\exists x(Gx\ \&\ \neg Tx)$’$)=\mathbf{t}$; and (1.2) $d\in$G and $h(d)=\mathbf{f}\Rightarrow\mathrm{SV}_{M+h}(\exists x(Gx\ \&\ \neg Tx))=\mathbf{f}\Rightarrow\mathrm{SV}_{M+h}(T$‘$\exists x(Gx\ \&\ \neg Tx)$’$)=\mathbf{f}\Rightarrow\mathrm{SV}_{M+h}(T$‘$\exists x(Gx\ \&\ Tx)$’$\vee T$‘$\exists x(Gx\ \&\ \neg Tx)$’$)=\mathbf{t}$.
(2) H$\subseteq$T, since $\mathrm{SV}_{M+h}(\forall x(Hx\supset Tx))=\mathbf{t}$. If there were a $d\in$H$-$T, then there would be a classical extension $M+h'$ of $M+h$, such that $h'(d)=$

f, in which case $CL_{M+h'}(\forall x(Hx \supset Tx))=\mathbf{f}$, which would contradict $SV_{M+h}(\forall x(Hx \supset Tx))=\mathbf{t}$.

(3) $N \cup T \subseteq G \cup H$, since $SV_{M+h}(\forall x(Tx \supset Gx \vee Hx))=\mathbf{t}$. If there were a $d \in (N \cup T)-(G \cup H)$, then there would be a classical extension $M+h'$ of $M+h$, such that $h'(d)=\mathbf{t}$, in which case $CL_{M+h'}(\forall x(Tx \supset Gx \vee Hx))=\mathbf{f}$, which would contradict $SV_{M+h}(\forall x(Tx \supset Gx \vee Hx))=\mathbf{t}$.

(4) G=N and H=T. This follows from (1), (2) and (3) and the fact that $N \cap T=\varnothing$.

(5) $p(d, d') \notin T$. This follows from (4) and from the fact that $SV_{M+h}(\forall x \forall y Gpxy)=\mathbf{t}$.

(6) If $p(d_1, d_2)=p(d_3, d_4)$ then $d_1=d_3$ and $d_2=d_4$, since $SV_{M+h}(\forall x \forall y \forall z \forall w (pxy=pzw \supset x=z \;\&\; y=w))=\mathbf{t}$.

Since the model is Kripkean, we have $T \cup N \subseteq \text{Sent}(L')$. Also, clearly the cardinality of T is the same as the cardinality of Sent(L'), since $A \vee \neg A \in T$ for every $A \in \text{Sent}(L')$. So there is an onto function k:T→N. Let $K=\{p(d, d'): d'=k(d)\}$. Note:

(7) $p(d, d') \in K$ iff $d'=k(d)$. (Proof of ⇒): Suppose that $p(d, d') \in K$. Then, for some d'' and d''' $p(d, d')=p(d'', d''')$ and $d'''=k(d'')$. So $d' = d'''=k(d'')=k(d)$, by (6). This proves (⇒). The converse follows from the definition of K.

Since $K \subseteq N$, there is a classical $M+h' \geq M+h$ such that $T'=\{d: h'(d)=\mathbf{t}\}= T \cup K$. Note:

(8) If $p(d, d') \in T'$ and if $p(d, d'') \in T'$ then $d'=d''$. To see this, suppose that $p(d, d') \in T'$ and that $p(d, d'') \in T'$. Then $p(d, d') \in K$ and $p(d, d'') \in K$, by (5). So $d'=d''$ by (7).

So $CL_{M+h'}(\forall x \forall y \forall z(Tpxy \;\&\; Tpzy \supset x=z))=\mathbf{t}$. So $CL_{M+h'}(\exists y(Gy \;\&\; \forall x (Hx \supset \neg Tpxy)))=\mathbf{t}$. So we can choose a $d \in G=N$ such that, for every $d' \in H= T$, we have $p(d, d') \notin T'$, in which case $p(d, d') \notin K$, in which case $d' \neq k(d)$. But this contradicts the fact that k:T→N is onto. This suffices for the proof of (*).

For (**), it suffices to show that $A \not\models_L B$. So we want to construct a fixed point for L that makes A true, but that does not make B true. The idea is simple: we want $D=\text{Sent}(L) \cup E$, where E is some uncountable set; we want the truth values of all the nonsentences (i.e. elements of E) to be **n**; and we want the extension of H to be the set of sentences with the truth value **t**; and finally we want the extension of G all the sentences with truth value **n** together with the members of E. The actual construction is tricky, since we will build the extension of G and H as well as the interpretation of T up together.

First, choose some uncountable set E disjoint from Sent(L) and let D= Sent(L)∪E. Next, choose a one-one and onto (pairing) function p:$D\times D\rightarrow E$. Given G, H⊆D, we will use the notation G+H for the classical ground model $M=\langle D, I\rangle$ where p=$I(p)$, G={$d\in D$: $I(G)(d)$=**t**} and H= {$d\in D$: I(H)(d)=**t**}. And given non-overlapping T, F⊆D, we will use the notation T+F for the function from D to {**t**, **f**, **n**} where (T+F)(d)=**t** [**f**] iff $d\in$T [F]. Thus G+H+T+F is a partial model for L, where G, H, T, F⊆D and T∩F=∅.

We define a function π on partial models of the form G+H+T+F as follows. π(G+H+T+F)=G′+H′+T′+F′, where

G′=E∪{$C\in$Sent(L): $SV_{G+H+T+F}(C)$=**n**}
H′={$C\in$Sent(L): $SV_{G+H+T+F}(C)$=**t**}
T′=H′
F′={$C\in$Sent(L): $SV_{G+H+T+F}(C)$=**f**}

Later, we will define, using π, a sequence of partial models: this sequence will culminate in our desired partial model.

Recall the definition of the *depth* of a sentence, given in the completeness proof, above: if the sentence X has no quote names then depth (X)=0; if max{depth(C): 'C' occurs in X}=n, then depth(X)=n+1. Given any set $K\subseteq D$ and any $n\in\mathbb{N}$, let $K_{<n}=K\cap\{C\in \text{Sent}(L)\text{: depth}(C)<n\}$ and $K_{\geq n}=K\cap\{C\in \text{Sent}(L)\text{: depth}(C)n\}$. We say that a partial model G +H+T+F is *n-separated* iff

- G∪H∪F=D, and
- H=T, and
- G, H and F are pairwise disjoint
- E⊆G
- and $G_{\geq n}$, $H_{\geq n}$, $T_{\geq n}$ and $F_{\geq n}$ are all infinite.

We say that $G+H+T+F=_n G'+H'+T'+F'$ iff $G_{<n}=G'_{<n}$, $H_{<n}=H'_{<n}$, $T_{<n}=T'_{<n}$ and $F_{<n}=F'_{<n}$.

Given any two partial models $M+h=\langle D, I\rangle+h$ and $M'+h'=\langle D', I'\rangle+h'$ for L and any $n\in\mathbb{N}$, we say that a function Ψ:$D\rightarrow D'$ is an *n-isomorphism* from $M+h$ to $M'+h'$ iff

- Ψ is one-one and onto,
- Ψ ($I(p)(d, d')$)=$I'(p)$(Ψ (d), Ψ (d')), for every d, $d'\in D$;
- $I'(G)$(Ψ (d))=$I(G)(d)$, for every $d\in D$;
- $I'(H)$(Ψ (d))=$I(H)(d)$, for every $d\in D$;
- h'(Ψ (d))=$h(d)$, for every $d\in D$; and
- Ψ(C)=C for every sentence C of depth <n.

We say that $M+h$ and $M'+h'$ are *n-isomorphic* iff there is an n-isomorphism between them.

Now we have the tools to prove the following:

(#) Suppose that G+H+T+F and G′+H′+T′+F′ are n-separated and that $\mathrm{G}+\mathrm{H}+\mathrm{T}+\mathrm{F} =_n \mathrm{G}'+\mathrm{H}'+\mathrm{T}'+\mathrm{F}'$. Then $\pi(\mathrm{G}+\mathrm{H}+\mathrm{T}+\mathrm{F}) =_{n+1} \pi(\mathrm{G}'+\mathrm{H}'+\mathrm{T}'+\mathrm{F}')$.

To prove (#), suppose that G+H+T+F and G′+H′+T′+F′ are n-separated and that $\mathrm{G}+\mathrm{H}+\mathrm{T}+\mathrm{F} =_n \mathrm{G}'+\mathrm{H}'+\mathrm{T}'+\mathrm{F}'$. Given the definition of π, it suffices to define an n-isomorphism Ψ from G+H+T+F to G′+H′+T′+F′. Before we can construct Ψ, we distinguish some subsets of E. Let $E_0=\{\mathrm{p}(X, Y): X, Y\in \mathrm{Sent}(L)\}$ and let $E_{n+1}=\{\mathrm{p}(d, d'): d\in E_n\}\cup E_n$. First, construct a function $\Phi:\mathrm{Sent}(L)\rightarrow \mathrm{Sent}(L)$ by pasting together,

- the identity function on $\{C\in \mathrm{Sent}(L): \mathrm{depth}(C)<n\}$
- a one-one onto function from $\mathrm{G}_{\geq n}$ to $\mathrm{G}'_{\geq n}$.
- a one-one onto function from $\mathrm{H}_{\geq n} = \mathrm{T}_{\geq n}$ to $\mathrm{H}'_{\geq n} = \mathrm{T}'_{\geq n}$.
- a one-one onto function from $\mathrm{F}_{\geq n}$ to $\mathrm{F}'_{\geq n}$.

Construct Ψ by setting $\Psi(C)=\Phi(C)$ for $C\in \mathrm{Sent}(L)$, and by setting

- $\Psi(\mathrm{p}(X, Y))=\mathrm{p}(\Psi(X), \Psi(Y))$ for $X, Y\in \mathrm{Sent}(L)$;
- $\Psi(\mathrm{p}(d, d'))=\mathrm{p}(\Psi(d), \Psi(d'))$ for $d, d'\in E_0$;
- $\Psi(\mathrm{p}(d, d'))=\mathrm{p}(\Psi(d), \Psi(d'))$ for $d, d'\in E_{n+1} - E_n$;
- $\Psi(d)=d$, for $d \in E - \cup_{n\in\mathbb{N}} E_n$.

Note that Ψ is, as desired, an n-isomorphism. Thus (#) is proved.

Next, we define a sequence of partial models $\mathrm{G}_n+\mathrm{H}_n+\mathrm{T}_n+\mathrm{F}_n$ as follows.

$$\begin{aligned}
&\mathrm{G}_0 = D\\
&\mathrm{H}_0 = \mathrm{T}_0 = \mathrm{F}_0 = \varnothing.\\
&\mathrm{G}_{n+1}+\mathrm{H}_{n+1}+\mathrm{T}_{n+1}+\mathrm{F}_{n+1} = \pi(\mathrm{G}_n+\mathrm{H}_n+\mathrm{T}_n+\mathrm{F}_n).
\end{aligned}$$

First we claim that, if $n\geq 1$, then $\mathrm{G}_n+\mathrm{H}_n+\mathrm{T}_n+\mathrm{F}_n$ is m-separated for each $m\leq n$. Clearly the first four conditions for m-separatedness are satisfied. As for the fifth condition, first note that each of $(\mathrm{G}_n)_{\geq m}$, $(\mathrm{H}_n)_{\geq m}$, $(\mathrm{T}_n)_{\geq m}$, and $(\mathrm{F}_n)_{\geq m}$ is closed under double negation, so it suffices to show that they are not empty. For any $k\in\mathbb{N}$ and sentence C, define the sentence T^kC as follows: $T^0C=C$ and $T^{k+1}C=T‘T^kC’$. Then notice that $T^m\forall x(x=x)\vee\neg T^m\forall x(x=x)\in(\mathrm{H}_n)_{\geq m}=(\mathrm{T}_n)_{\geq m}$, and $T^m\forall x(x=x)\ \&\ \neg T^m\forall x(x=x)\in(\mathrm{F}_n)_{\geq m}$. Also, an easy inductive argument shows that $\forall x(Gx\supset Tx)\in \mathrm{G}_k$ for each k. So $((T^m\forall x(x=x)\ \&\ \neg T^m\forall x(x=x))\vee\forall x(Gx\supset Tx))\in(\mathrm{G}_n)_{\geq m}$. Thus, as desired, if $n\geq 1$, then $\mathrm{G}_n+\mathrm{H}_n+\mathrm{T}_n+\mathrm{F}_n$ is m-separated for each $m\leq n$.

By induction on n, we will prove

(##)$G_{n+1} + H_{n+1} + T_{n+1} + F_{n+1} =_n G_{n+2} + H_{n+2} + T_{n+2} + F_{n+2}$,
for each $n \in \mathbb{N}$.

This is vacuously true for n=0. For the inductive step, assume that $G_{n+1} + H_{n+1} + T_{n+1} + F_{n+1} =_n G_{n+2} + H_{n+2} + T_{n+2} + F_{n+2}$. The fact that $G_{n+2} + H_{n+2} + T_{n+2} + F_{n+2} =_{n+1} G_{n+3} + H_{n+3} + T_{n+3} + F_{n+3}$ follows from (#) and from the n-separatedness of the partial models $G_{n+2} + H_{n+2} + T_{n+2} + F_{n+2}$ and $G_{n+3} + H_{n+3} + T_{n+3} + F_{n+3}$.

Now define the partial model G+H+T+F as follows:

$G = \{d : (\exists m \in \mathbb{N})(\forall n \geq m)(d \in G_n)\} = \cup_m \cap_{n \geq m} G_n$
$H = \cup_m \cap_{n \geq m} H_n$
$T = \cup_m \cap_{n \geq m} T_n$
$F = \cup_m \cap_{n \geq m} F_n$

We will show that $\pi(G + H + T + F) = G + H + T + F$. It suffices to show that $\pi(G + H + T + F) =_m G + H + T + F$, for each m. Given (##), if C is a sentence of depth m, then either $(\forall n \geq m + 2)(C \in G_n)$ or $(\forall n \geq m + 2)(C \notin G_n)$, and similarly for H, T and F in place of G. So $G + H + T + F =_m G_{m+2} + H_{m+2} + T_{m+2} + F_{m+2}$. Thus, by (#), $\pi(G + H + T + F) =_{m+1} G_{m+3} + H_{m+3} + T_{m+3} + F_{m+3}$. But by (##), $G_{m+2} + H_{m+2} + T_{m+2} + F_{m+2} =_{m+1} G_{m+3} + H_{m+3} + T_{m+3} + F_{m+3}$. So $\pi(G + H + T + F) =_m G + H + T + F$. So $\pi(G + H + T + F) = G + H + T + F$. But in this case, the partial model G+H+T+F is a fixed point: $SV_{G+H+T+F}(C) = \mathbf{t}[\mathbf{f}]$ iff $C \in T[F]$, for each sentence C. Note that this fixed point was carefully constructed so $SV_{G+H+T+F}(A) = \mathbf{t}$ and $SV_{G+H+T+F}(B) = \mathbf{n}$. ⊣

Remark. The above example and the example from Section 2 show that $\vDash$, $\vDash_L$, $\vDash'$ and $\vDash'_L$ are all distinct.

As for the issue of complexity, let the first and second order languages L and L^2 be as in Section 3.

COMPLEXITY THEOREM.

(1) $\prod^1_2$-SOL is recursively encodable in $\vDash'$.
(2) $\vDash'_L$ is $\prod^1_2$-hard.

Here is what we mean when we say that $\vDash'_L$ is Π^1_2-hard. First treat $\vDash'_L$ as a set of ordered pairs, in particular $\{\langle A, B\rangle : A \vDash'_L B\}$. To say that $\vDash'_L$ is Π^1_2-hard is to say that for any Π^1_2 set X of natural numbers, there is a one-one recursive function $f : \mathbb{N} \to \mathrm{Sent}(L) \times \mathrm{Sent}(L)$ such that, for every $n \in \mathbb{N}$, we have $n \in X$ iff $f(n) \in \vDash'_L$.

Part (1) of the Complexity Theorem is a corollary to a lemma analogous to the Complexity Lemma of Section 3.

COMPLEXITY LEMMA. Suppose that $\forall X \exists TA$ is a Π^1_2 formula, and that G is some unary predicate of L not occurring in A. Also suppose that $t_1,\ldots, t_n$ are all the terms occurring in A. (Note: the terms that can occur in A are either individual constants of L or variables of L. Every formula contains *some* term, so the list is not empty.) Then

$$\forall X \exists TA \in \Pi^1_2\text{-SOL iff } \neg A^G \;\&\; Gt_1 \;\&\; \ldots \;\&\; Gt_n \vDash' T\text{‘}\exists x(Gx \;\&\; Tx)\text{’} \vee T\text{‘}\exists x(Gx \& \neg Tx)\text{’}.$$

Proof. ($\Rightarrow$) Suppose that $\neg A^G$ & Gt_1 & ... & $Gt_n \nvDash' T$‘$\exists x(Gx$ & $Tx)$’$\vee$ T‘$\exists x(Gx \neg Tx)$’. Then for some language L' which includes all the vocabulary on the right and left-hand sides of the turnstile, we have $\neg A^G$ & Gt_1 &... & $Gt_n \nvDash'_{L'} T$‘$\exists x(Gx$ & $Tx)$’ $\vee$ T‘$\exists x(Gx$ & $\neg Tx)$’. At this point, the proof follows the proof of (1)$\Rightarrow$(2) in Section 3, with straightforward adjustments added to deal with the fact that we're working in L' instead of L.

($\Leftarrow$) This proof is somewhat similar to the proof of (3)$\Rightarrow$(1) in Section 3, except that, in the construction of the fixed point, we have to make sure that we construct a Kripkean fixed point, unlike in Section 3. So suppose that $\forall X \exists TA \notin \Pi^1_2$-SOL. Then there is a domain D, and assignments s and S such that $\exists TA$ is false in D relative to s and S. Let κ be the cardinality of D, and expand L to a language L' by adding κ many new constants $\{c_\alpha : \alpha < \kappa\}$ to L, and also adding the new quote names. We might as well assume that the domain D is precisely the following set of sentences of L':$\{\neg Tc_\alpha : \alpha < \kappa\}$, since only the cardinality of D is really at issue in falsifying $\forall X \exists TA$.

Define a ground model $M = \langle D', I\rangle$ for the language L as follows:

D' = Sent(L') (note that $D \subseteq D'$)

$I(R)$ = $S(R) \cup \{\langle d_1,\ldots, d_n, \mathbf{f}\rangle$: some $d_i \notin D\}$, for each n-ary relation $R \neq G$ of L

$I(G)$ = $(D \times \{\mathbf{t}\}) \cup ((D' - D) \times \{\mathbf{f}\})$

$I(c)$ = $s(c)$, for each nonquote name c of L

$I(c_\alpha) = \neg Tc_\alpha$, for the new nonquote names c_α
$I(\text{'}A\text{'}) = A$, for each $A \in \mathrm{Sent}(L')$

Let $M+h$ be any fixed point for L'. M will be Kripkean since there are no nonsentences in D. Note that $h(d)=\mathbf{n}$ for each $d \in D$, since the members of D are liar sentences. Notice that $\mathrm{SV}_{M+h}(\exists x(Gx \,\&\, Tx))=\mathrm{SV}_{M+h}(\exists x(Gx \,\&\, \neg Tx))=\mathbf{n}$. Thus $\mathrm{SV}_{M+h}(T\text{'}\exists x(Gx \,\&\, Tx)\text{'})=\mathrm{SV}_{M+h}(T\text{'}\exists x(Gx \,\&\, \neg Tx)\text{'})=\mathbf{n}$. Thus $\mathrm{SV}_{M+h}(T\text{'}\exists x(Gx \,\&\, Tx)\text{'} \vee T\text{'}\exists x(Gx \,\&\, \neg Tx)\text{'})=\mathbf{n}$, as can be seen by considering any classical h' and h'' such that $h'(\exists x(Gx \,\&\, Tx))=h'(\exists x(Gx \,\&\, \neg Tx))=\mathbf{t}$ and $h''(\exists x(Gx \,\&\, Tx))=h''(\exists x(Gx \,\&\, \neg Tx))=\mathbf{f}$. Since $\mathrm{SV}_{M+h}(T\text{'}\exists x(Gx \,\&\, Tx)\text{'} \vee T\text{'}\exists x(Gx \,\&\, \neg Tx)\text{'})=\mathbf{n}$, it suffices to specify an assignment s' of values to the variables of L, such that $\mathrm{SV}_{M+h,\, s'}(\neg A^G \,\&\, Gt_1 \,\&\ldots\&\, Gt_n)=\mathbf{t}$. The proof that this can be done is exactly as in the proof of (3)⇒(1) in Section 3. ⊣

Part (2) of the Complexity Theorem is weaker than Part (1) for the following reason: given that the language L is countable, we cannot get the effect of full second-order quantification but only of second-order quantification over countable sets. In particular, define ω-SOL as the set of second-order formulas true in each *countable* domain (relative to any assignments s and S') and define $\prod_2^1$-ω-SOL as the set of $\prod_2^1$ formulas in ω-SOL. Then we can prove a weaker complexity lemma, analogous to the above lemma:

$$\forall X \exists T A \in \prod\nolimits_2^1\text{-}\,\omega\,\text{-}\,\mathrm{SOL} \text{ iff } \neg A^G \,\&\, Gt_1 \,\&\, \ldots \,\&\, Gt_n \vDash_L' T\text{'}\exists x(Gx \,\&\, Tx)\text{'} \vee T\text{'}\exists x(Gx \,\&\, \neg Tx)\text{'}.$$

The proof of (⇒) runs as in the proof of (1)⇒(2) in Section 3: the countability of the new domain $D'=\{d\colon I(G)(d)=\mathbf{t}\}$ is guaranteed since $I(G)(d)=\mathbf{t} \Rightarrow h(d)=\mathbf{n}$, in which case d must be a sentence, if we are taking the fixed point to be a Kripkean fixed point. The proof of (⇐) runs as in the proof of (⇐) above, with the following exceptions: we know that D is countable; rather than add *new* constants to L, we can just assume that $D=\{\neg Tc\colon c \in C\}$ for some countable set C of constants already in L. Given both directions of the biconditional, we can infer that $\prod_2^1$-ω-SOL is encodable in $\vDash_L'$. And this suffices for Part (2) of the Complexity Theorem, given the following lemma:

LEMMA. $\prod_2^1$-ω-SOL is $\prod_2^1$-hard.

Proof. If our second-order language L^2 had a name **o** ('zero') and an unary function symbols **s**, and binary function symbols + and ×, then we could proceed as follows with our proof. Suppose that X is a $\prod_2^1$ set of natural numbers. Let *Arith* be the conjunction of the second-order axioms of Peano Arithmetic: *Arith* characterizes, up to isomorphism, the standard model of arithmetic. So, since X is $\prod_2^1$, there is a $\prod_2^1$ formula $B(x)= \forall X\exists TA(X, T, x)$ with a free individual variable x and no other free variables so that, for each $n \in \mathbb{N}$ we have $n\in$X iff $B(\mathbf{s}^n\mathbf{o})$ is true in the standard model of arithmetic, where $\mathbf{s}^n\mathbf{o}$ is **o** prefixed by n occurrences of **s**. Thus, for each $n \in \mathbb{N}$ we have $n\in$X iff $(Arith\supset B(\mathbf{s}^n\mathbf{o}))\in\omega$-SOL. Note that, since *Arith* is a $\prod_1^1$ formula and $B(\mathbf{s}^n\mathbf{o})$ is a $\prod_2^1$ formula, there is a $\prod_2^1$ formula C_n which is equivalent to $(Arith\supset B(\mathbf{s}^n\mathbf{o}))$, i.e. $(C_n\equiv(Arith\supset B(\mathbf{s}^n\mathbf{o})))\in$SOL: moreover, the function from the $B(\mathbf{s}^n\mathbf{o})$'*s* to the C_n'*s* is recursive. Thus, for each $n \in \mathbb{N}$ we have $n \in$ X iff $C_n \in \prod_2^1$- ω- SOL. And note that the map from $\mathbb{N}$ to the sentences of L^2, taking n to C_n, is recursive.

Unfortunately this proof does not quite work, since our second-order language L^2 has no names and no function symbols – L^2 does not even have functional variables. So the above proof has to be modified by letting some specified individual variable v stand in for the constant **o**, and some binary relational variable stand in for **s**, and some ternary relational variables stand in for + and ×. ⊣

Analogously to our Soundness and Completeness Theorems in Section 4, we have the following.

SOUNDNESS THEOREM. If $A\in\mathrm{SVFPLT}_L$ then $\vDash' A$ and $\vDash'_L A$.

COMPLETENESS THEOREM.

(1) If $\vDash' A$ then $A\in\mathrm{SVFPLT}_L$.
(2) If L is countable and $\vDash'_L A$ then $A\in\mathrm{SVFPLT}_L$.

COROLLARY.

(1) $\vDash' A$ iff $\vDash A$.
(2) If L is countable then $\vDash_L A$ iff $\vDash A$ iff $\vDash'_L A$ iff $\vDash' A$.

As before, soundness is routine and Completeness (1) follows from Completeness (2).

Before we proceed to the proof of Completeness (2), we need some new notions. Recall the definition of the *depth* of a sentence, given in the proof of Completeness (1) in Section 4, and repeated in the Example in Section 5. Fix any truth language L. Suppose that $M + h = \langle D, I\rangle + h$

and $M' + h' = \langle D', I' \rangle + h'$ are two partial models of L. We say that $M + h \equiv_n M' + h'$ iff $\mathrm{SV}_{M+h}(B) = \mathrm{SV}_{M'+h'}(B)$ for every sentence B of depth $\leq n$. Also, we generalize the definition of an *n-isomorphism* between two models: a function Ψ:$D \rightarrow D'$ is an *n-isomorphism* from M+ h to M'+h' iff

- Ψ is one-one and onto,
- $\Psi(C)=C$ for every sentence C of depth < n.
- $I'(c)=\Psi(I(c))$ for every nonquote name c
- $\Psi(I(f)(d_1, ..., d_n))=I'(f)(\Psi(d_1), ..., \Psi(d_n))$ for every n-ary function symbol f and every $d_1, ..., d_n \in D$
- $I(R)(d_1, ..., d_n)=I'(R)(\Psi(d_1), ..., \Psi(d_n))$ for every n-ary relation symbol f and every $d_1, ..., d_n \in D$
- $h(d)=h'(\Psi(d))$, for every $d \in D$

We say that M+h and M'+h' are *n-isomorphic* iff there is an n-isomorphism between them. Note that if M+h and M'+h' are n-isomorphic then $M + h \equiv_n M' + h'$.

Suppose that Φ:Sent(L)$\rightarrow$Sent(L). Then we say that $M + h \equiv_{\Phi\,,\,n} M'$+ h' iff $\mathrm{SV}_{M+h}(B)$= $\mathrm{SV}_{M'+h'}(\Phi(B))$ for every sentence B of depth $\leq n$.

If $M + h = \langle D, I \rangle + h$ is a partial model for L and Ψ:$D \rightarrow$Sent(L) is one-one and onto, then $M_\Psi + h_\Psi = \langle \mathrm{Sent}(L), I_\Psi \rangle + h_\Psi$ is the partial model satisfying the following:

- $I_\Psi(\text{‘}E\text{’})$ $= E$, for each sentence E
- $I_\Psi(c)$ $= \Psi(I(c))$, for each nonquote name c
- $I_\Psi(f)(E_1, \ldots, E_m) = \Psi\big(\mathrm{I}(f)\big(\Psi^{-1}(E_1), \ldots, \Psi^{-1}(E_m)\big)\big)$, for each n-ary function symbol f and for E_1,..., $E_m \in$Sent(L)
- $I_\Psi(R)(E_1, \ldots, E_m) = I(R)\big(\Psi^{-1}(E_1), \ldots, \Psi^{-1}(E_m)\big)$, for each n-ary relation symbol R and for E_1,..., $E_m \in$Sent(L)
- $h_\Psi(E)$ $= h\big(\Psi^{-1}(E)\big)$ for $E \in$Sent(L).

Note that Ψ is a 0-isomorphism between M+h and $M_\Psi + h_\Psi$, so that $M_\Psi + h_\Psi \equiv_0 M + h$. We also define the sets $\mathrm{T}_\Psi = \{\mathrm{E} : \mathrm{SV}_{\mathrm{M}_\Psi + \mathrm{h}_\Psi}(\mathrm{E}) = \mathbf{t}\}$, $\mathrm{N}_\Psi = \{E : \mathrm{SV}_{M_\Psi + h_\Psi}(E) = \mathbf{n}\}$, and $\mathrm{F}_\Psi = \{E : \mathrm{SV}_{M_\Psi + h_\Psi}(E) = \mathbf{f}\}$. And we define the set $D_{\Psi,n} = \{d \in D - \mathrm{Sent}(L) : \Psi(d) \text{is of depth} > n\}$.

Finally, we say that a sentence B is a *constituent* of a formula C iff ‘B’ occurs in C, possibly in the scope of quotation marks. A *constituent occurrence* of B in C is an occurrence of B in the quote name ‘B’.

PROOF (SKETCH) OF COMPLETENESS (2). Suppose that L is a countable truth language and that $A \notin \mathrm{SVFPLT}_L$. Recall the construction, in Section 4, of a fixed point $M + h = \langle D, I \rangle + h$, in which A is not true.

Define $D'=D-\text{Sent}(L)$, and define $\text{T}=\{d\in D\text{: } h(d)=\mathbf{t}\}$, $\text{F}=\{d\in D\text{: } h(d)=\mathbf{f}\}$ and $\text{N}=\{d\in D\text{: } h(d)=\mathbf{n}\}$. An inspection of the construction in Section 4 reveals that $M+h$ satisfies the following:

- D is countably infinite.
- $D'\cap\text{T}$ is infinite.
- $D'\cap\text{F}$ is infinite.
- If $\text{Sent}(L)\cap\text{N}$ is nonempty, then $D'\cap\text{N}$ is infinite.
- If $\text{Sent}(L)\cap\text{N}$ is empty, then $D'\cap\text{N}$ is either empty or infinite.

At this point, we consider three cases.

(Case 1) M is nonclassical and $\text{Sent}(L)\cap\text{N}$ is nonempty. It suffices to construct a one-one onto function $\Psi{:}D\rightarrow\text{Sent}(L)$ so that

(1) $\Psi(A)=A$,
(2) $M+h\equiv_{\Psi,n} M_\Psi+h_\Psi$ for each n, and
(3) $M_\Psi+h_\Psi$ is a fixed point.

$M_\Psi+h_\Psi$ will be a *Kripkean* fixed point, since the domain of the ground model $M_\Psi=\text{Sent}(L)$. Also, we will have the following, where n is the depth of A: $\text{SV}_{M+h}(A)=\text{SV}_{M_\Psi+h_\Psi}(\Psi(A))$ by (2). Thus $\text{SV}_{M+h}(A)=\text{SV}_{M_\Psi+h_\Psi}(A)$, by (1). Thus $\text{SV}_{M_\Psi+h_\Psi}(A)\neq\mathbf{t}$. Thus $\nvDash'_{\text{L}}A$ as desired.

Our function Ψ will be the *limit* in some sense of a sequence $\{\Phi_n{:}D\rightarrow\text{Sent}(L)\}$ of functions, which we will construct by induction. But first, choose $B\in\text{Sent}(L)\cap\text{T}$ and $C\in\text{Sent}(L)\cap\text{N}$, both of depth 0, making sure that

- neither B nor C is a constituent nor a subformula of A or of each other, and
- for each n and for each (quote or nonquote) name c occurring in A (including those in the scope of quotation marks), $I(C)\neq\neg^nB$ and $I(C)\neq\neg^nC$.

Note that $\neg^{2n}B\in\text{Sent}(L)\cap\text{T}$, and $\neg^{2n+1}B\in\text{Sent}(L)\cap\text{F}$, and $\neg^{2n}C\in\text{Sent}(L)\cap\text{N}$, for each $n\in\mathbb{N}$, since $M+h$ is a fixed point.

Now define a function $\Phi{:}\text{Sent}(L)\rightarrow\text{Sent}(L)$ as follows:

$\Phi(\neg^{2n}\pm B) = \neg^{4n}\pm B$
$\Phi(\neg^{2n}C) = \neg^{4n}C$
$\Phi(E)$ = the result of replacing constituent occurrences of $\neg^{2n}\pm B$ [$\neg^{2n}C$] with $\neg^{4n}\pm B$ [$\neg^{4n}C$]. Here, we are assuming that all the occurrences of $\neg$ are indicated. Thus, for example, Φ(G‘$\neg\neg C$’ ∨ T‘T‘$\neg\neg\neg B$’’)=G‘$\neg\neg\neg\neg C$’ ∨ T‘T‘$\neg\neg\neg\neg\neg\neg B$’’.

Let $U=\text{range}\Phi$ and $V=\text{Sent}(L)-\text{range}\Phi$. And let $V_{>n}=\{E\in V:\text{depth}(E)>n\}$ and $V_{\leq n}=\{E\in V:\text{depth}(E)\leq n\}$. Note that the depth of $\Phi(E)$ is the same as the depth of E. Also note that $\Phi(A)=A$, since neither $\pm B$ nor C is a constituent of A.

Now we define our sequence of functions $\{\Phi_n{:}D\rightarrow \mathrm{Sent}(L)\}$. Define $\Phi_0{:}D\rightarrow \mathrm{Sent}(L)$ by patching together

- $\Phi{:}\mathrm{Sent}(L)\rightarrow \mathrm{Sent}(L)$
- A one-one function from $D'\cap \mathrm{T}$ to $V\cap \mathrm{T}$
- A one-one function from $D'\cap \mathrm{N}$ to $V\cap \mathrm{N}$
- A one-one function from $D'\cap \mathrm{F}$ to $V\cap \mathrm{F}$

Note that $M+h\equiv_{\Phi,0} M_{\Phi_0}+h_{\Phi_0}$, since $M+h\equiv_0 M_{\Phi_0}+h_{\Phi_0}$ and since $\Phi(E)$ is logically equivalent to E, for sentences E of depth 0. Note also that each of the sets $D_{\Phi_0,0}\cap \mathrm{T}$, $D_{\Phi_0,0}\cap \mathrm{N}$, and $D_{\Phi_0,0}\cap \mathrm{F}$ is infinite, as is each of $V_{>0}\cap \mathrm{T}_{\Phi_0}$, $V_{>0}\cap \mathrm{N}_{\Phi_0}$, and $V_{>0}\cap \mathrm{F}_{\Phi_0}$.

Now suppose that we have defined the one-one onto function $\Phi_n{:}D\rightarrow \mathrm{Sent}(L)$, so that each of the sets $D_{\Phi_n,n}\cap \mathrm{T}$, $D_{\Phi_n,n}\cap \mathrm{N}$, and $D_{\Phi_n,n}\cap \mathrm{F}$ is infinite. Also suppose that $\Phi_n(E)=\Phi(E)$, for each $E\in \mathrm{Sent}(L)$. Choose $\Phi_{n+1}{:}D\rightarrow \mathrm{Sent}(L)$ by insuring that,

- $\Phi_{n+1}(E)=\Phi(E)$, for each $E\in \mathrm{Sent}(L)$,
- $\Phi_{n+1}(d)=\Phi_n(d)$, if $d\in D'$ and $\Phi_n(d)$ is of depth $\leq n$,
- Φ_{n+1} maps $D_{\Phi_n,n}\cap \mathrm{T}$ one-one and onto $V_{>n}\cap \mathrm{T}_{\Phi_n}$
- Φ_{n+1} maps $D_{\Phi_n,n}\cap \mathrm{N}$ one-one and onto $V_{>n}\cap \mathrm{N}_{\Phi_n}$
- Φ_{n+1} maps $D_{\Phi_n,n}\cap \mathrm{F}$ one-one and onto $V_{>n}\cap \mathrm{F}_{\Phi_n}$.

We can construct this mapping, since each of the sets $V_{>n}\cap \mathrm{T}_{\Phi_n}$, $V_{>n}\cap \mathrm{N}_{\Phi_n}$, and $V_{>n}\cap \mathrm{F}_{\Phi_n}$ is infinite. To see this, note that, for each $m>n$, we have $T^m\neg\neg B\vee \neg T^m\neg\neg B\in V_{>n}\cap \mathrm{T}_{\Phi_n}$, $T^m\neg\neg B\ \&\ \neg T^m\neg\neg B\in V_{>n}\cap \mathrm{F}_{\Phi_n}$, and $(T^m\neg\neg B\vee\neg T^m\neg\neg B)\ \&\ C\in V_{>n}\cap \mathrm{N}_{\Phi_n}$.

We make a number of observations about the Φ_n. We will not prove the claims whose proofs are straightforward.

(4) $\Phi_n(E)=\Phi(E)$, for each $E\in \mathrm{Sent}(L)$.
(5) Each of the sets $D_{\Phi i_n,n}\cap \mathrm{T}$, $D_{\Phi_n,n}\cap \mathrm{N}$, and $D_{\Phi_n,n}\cap \mathrm{F}$ is infinite.
(6) $\Phi_n^{-1}(E)=\Phi_k^{-1}(E)$ for each sentence E of depth $\leq k\leq n$.
(7) $M_{\Phi_n}+h_{\Phi_n}\equiv_k M_{\Phi_k}+h_{\Phi_k}$, for each $k\leq n$. This is proved by induction on k. The base case is trivial. For the inductive step, it suffices to show that $M_{\Phi_{n+1}}+h_{\Phi_{n+1}}\equiv_n M_{\Phi_n}+h_{\Phi_n}$. And for this it suffices to note that the function $\Phi_{n+1}\circ\Phi_n^{-1}$ is an n-isomorphism from $M_{\Phi_n}+h_{\Phi_n}$ to $M_{\Phi_{n+1}}+h_{\Phi_{n+1}}$.

Given (6), above, we can define a function $\Psi{:}D\rightarrow \mathrm{Sent}(L)$:

$\Psi(d)=E$ if for some k, for every $n\geq k$, $\Phi_n(d)=E$.

Note that $\Psi^{-1}(E) = \Phi_n^{-1}(E)$ for each sentence E of depth n. As noted above, we will be done if we can show,

(1) $\Psi(A)=A$,
(2) $M + h \equiv_{\Psi,n} M_\Psi + h_\Psi$ for each n, and
(3) $M_\Psi\Psi + h_\Psi\Psi$ is a fixed point.

Re (1). Note that $\Phi_n(E)=\Phi(E)$, for each $E\in\text{Sent}(L)$ and each n. So $\Psi(E)=\Phi(E)$, for each $E\in\text{Sent}(L)$. Also $\Phi(A)=A$. So $\Psi(A)=A$, as desired.

The proofs of (2) and (3) still need to be filled in. But the proof of (2) will rely on the fact that Φ's mapping of, for example, $\neg\neg B$ to $\neg\neg\neg\neg B$ will be compensated for by mapping G'$\neg\neg B$' to G'$\neg\neg\neg\neg B$'. And the proof of (3) will rely on the claim that $M_{\Phi_0} + h_{\Phi_0}$ is, in a sense, a fixed point relative to sentences of depth $< n$: if E is of depth $< n$, then E will be **t**[**f**,**n**] iff $h_{\Phi_0}(E) = \mathbf{t}[\mathbf{f}, \mathbf{n}]$. This will complete Case 1.

(Case 2) $M+h$ is classical. Thus $D'\cap\text{N}$ is empty. Then the construction in Case 1 can be repeated, with a slight simplification: all mention of C and of N can be excised.

(Case 3) $M+h$ is nonclassical and $\text{Sent}(L)\cap\text{N}$ is empty. Choose any classical partial model $M + h'\geq M + h$. Note that, for any sentence E, $\text{SV}_{M+h}(E)=\mathbf{t}$ or **f**, since $\text{Sent}(L)\cap\text{N}$ is empty. Also, for any sentence E, $\text{SV}_{M+h}(E)=\mathbf{t}\ [\mathbf{f}]\Rightarrow\text{SV}_{M+h'}(E)=\mathbf{t}\ [\mathbf{f}]$, since $M + h'\geq M + h$. Thus, for any sentence E, $\text{SV}_{M+h'}(E)=\text{SV}_{M+h}(E)$. Also, for any sentence E, $h'(E)=h(E)$, since $\text{Sent}(L)\cap\text{N}$ is empty. So $M+h'$ is a classical fixed point, making the same sentence true [false] as $M+h$. In particular $\text{SV}_{M+h}(A)=$ **f**. Note that this case now reduces to Case 2. ⊣

6. Variant Supervaluation Schemes

As mentioned above, Kripke defines two variants of the supervaluation scheme. Fix a truth language L. Suppose that $M=\langle D, I\rangle$ is a ground model. Thus $\text{Sent}(L)\subseteq D$. We say that $h{:}D\rightarrow\{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$ is *weakly consistent* iff the set $\{A\in\text{Sent}(L){:}\ h(A)=\mathbf{t}\}$ is consistent. We say that h is *strongly consistent* iff the set $\{A\in\text{Sent}(L){:}\ h(A)=\mathbf{t}\}\cup\{\neg A\in\text{Sent}(L){:}\ h(A)=\mathbf{f}\}$ is consistent. Note that if $h(A)=\mathbf{t}$ or **f** for every $A\in\text{Sent}(L)$, then h is strongly consistent iff the set $\{A\in\text{Sent}(L){:}\ h(A)=\mathbf{t}\}$ is a complete consistent theory. If h is weakly consistent we define, for each assignment s and formula A,

$$\mathrm{SV1}_{M+h,\,s}(A) = \begin{cases} \mathbf{t}, & \text{if } \mathrm{CL}_{M+h',\,s}(A) = \mathbf{t} \text{ for every classical } M+h' \geq M+h, \\ & \text{where } h' \text{ is weakly consistent;} \\ \mathbf{f}, & \text{if } \mathrm{CL}_{M+h',\,s}(A) = \mathbf{f} \text{ for every classical } M+h' \geq M+h, \\ & \text{where } h' \text{ is weakly consistent;} \\ \mathbf{n}, & \text{otherwise.} \end{cases}$$

If h is strongly consistent, the definition of $\mathrm{SV2}_{M+h,\,s}(A)$ is analogous, with the word ‘weakly’ replaced by the word ‘strongly’.

This produces two new evaluation schemes, at least for partial models $M+h$, where h is weakly (for SV1) or strongly (for SV2) consistent. A partial model $M+h$ (for L) is an SV1-*fixed point* (for L) iff $h(A) = \mathrm{SV1}_{M+h}(A)$ for every $A \in \mathrm{Sent}(L)$. Similarly for SV2-*fixed point*. The fixed point theorem holds for SV1- and SV2-fixed points. An SV1-fixed point [SV2-fixed point] is *Kripkean* iff $h(d)=\mathbf{f}$ for every $d \in D - \mathrm{Sent}(L)$. Analogously to the definitions in Section 3 and Section 5, we define consequence relations:

$A \vDash_{1L} B$ iff	for every SV1-fixed point $M+h$ for L and every assignment s, if $\mathrm{SV1}_{M+h,\,s}(A)=\mathbf{t}$ then $\mathrm{SV1}_{M+h,\,s}(B)=\mathbf{t}$.
$A \vDash_{1} B$ iff	for every truth language L of which A and B are both formulas, and for every SV1-fixed point $M+h$ for L and for every assignment s, if $\mathrm{SV1}_{M+h,\,s}(A)=\mathbf{t}$ then $\mathrm{SV1}_{M+h,\,s}(B)=\mathbf{t}$.
$A \vDash'_{1L} B$ iff	for every Kripkean SV1-fixed point $M+h$ for L and every assignment s, if $\mathrm{SV1}_{M+h,\,s}(A)=\mathbf{t}$ then $\mathrm{SV1}_{M+h,\,s}(B)=\mathbf{t}$.
$A \vDash'_{1} B$ iff	for every language L (with a distinguished predicate T and quote names) of which A and B are both formulas, and for every Kripkean SV1-fixed point $M+h$ for L and for every assignment s, if $\mathrm{SV1}_{M+h,\,s}(A)=\mathbf{t}$ then $\mathrm{SV1}_{M+h,\,s}(B)=\mathbf{t}$.

The consequence relations $\vDash_{2L}$, $\vDash_{2}$, $\vDash'_{2L}$ and $\vDash'_{2}$ are defined analogously.

Analogously to Section 2, Section 3 and Section 5, we have the following results (1)–(6) for $\vDash_{1L}$, $\vDash_{1}$, $\vDash'_{1L}$ and $\vDash'_{1}$. For (1) and (2), LA and LA′ are the languages in the example in Section 2; for (3)–(5), L is the first-order language specified in Section 3; and for (6), L is the language of the counterexample in Section 5.

(1) PA^{-} & $Ind \nvDash_{1\mathrm{LA}} T$'$\exists xT2x$' $\vee$ T'$\exists x \neg T2x$' (thus PA^{-}& $Ind \nvDash_{1} T$'$\exists xT2x$' $\vee$ $T\exists x\neg T2x$)
(2) PA^{-} & $Ind \nvDash_{1\mathrm{LA}'} T$'$\exists xT2x$' $\vee$ T'$\exists xT2x$'.
(3) $\prod_{2}^{1}$- SOL is recursively encodable in both $\vDash_{1}$ and $\vDash_{1L}$.
(4) $\prod_{2}^{1}$- SOL is recursively encodable in $\vDash_{1}'$.
(5) $\vDash_{1L}'$ is $\prod_{2}^{1}$- hard.
(6) $A \vDash_{1}' B$ and $A \vDash_{1L}' B$, but $A \vDash_{1} B$ and $A \nvDash_{1L} B$, where

$$A =_{df} \quad \forall x(Hx \supset Tx) \& \forall x(Tx \supset Gx \vee Hx) \ \& \ \forall x \forall y Gpxy \ \&$$
$$\forall x \forall y \forall z \forall w(pxy = pzw \supset x = z \& y = w) \ \&$$
$$\forall x \forall y \forall z(Tpxy \ \& \ Tpzy \supset x = z) \supset \exists y(Gy \ \& \ \forall x(Hx \supset \neg Tpxy)), \text{and}$$
$$B =_{df} \quad T\text{`}\exists x(Gx \ \& \ Tx)\text{'} \vee T\text{`}\exists x(Gx \ \& \ \neg Tx)\text{'}.$$

The arguments for (1) to (6) are almost exactly the same as the arguments for the analogous claims in Section 2, Section 3 and Section 5, except that a few more considerations must be adduced. Consider, for example, (1) and (2). First, in the proofs, the occurrences of 'SV' must be replaced by 'SV1'. To prove (1) we have to alter the proof of (*) in the example in Section 2, from the middle, as follows, with the additional considerations in italics: "Also note that $h(A_{2n})$=**n**, for each $n \in \mathbb{N}$, since each A_{2n} is a liar sentence. *Clearly there is a weakly consistent classical* $h'' \geq h$ *such that, for each* $n \in \mathbb{N}$, *we have* $h''(A_{2n})$=**f**. *Moreover there is a weakly consistent classical* $h' \geq h$ *such that, for each* $n \in \mathbb{N}$, *we have* $h'(A_{2n})$=**t**, *because the set of* A_{2k}*'s is logically consistent with* Th(M+h). So $\mathrm{CL}_{M+h'}$ ($\exists xT2x$)= $\mathrm{CL}_{M+h''}$($\exists x\neg T2x$)=**t** and $\mathrm{CL}_{M+h''}$($\exists xT2x$)=$\mathrm{CL}_{M+h'}$($\exists x\neg T2x$)=**f**. So $\mathrm{SV1}_{M+h}$ ($\exists xT2x$)=$\mathrm{SV1}_{M+h}$($\exists x\neg T2x$)=**n**. So h($\exists xT2x$)=h($\exists x\neg T2x$)=**n**, since M+h is a fixed point. So there are *weakly consistent* classical h', $h'' \geq h$ such that h'($\exists xT2x$)=h'($\exists x\neg T2x$)=**t** (*since* $\exists xT2x$ *and* $\exists x\neg T2x$ *are consistent with* Th(M+h)) and h''($\exists xT2x$)=h''($\exists x\neg T2x$)=**f**, so that $\mathrm{SV1}_{M+h'}$ (T'$\exists xT2x$' $\vee$ T'$\exists x\neg T2x$')=**t** and $\mathrm{SV1}_{M+h''}$(T'$\exists xT2x$' $\vee$ T'$\exists x\neg T2x$')=**f**. So $\mathrm{SV1}_{M+h}$(T'$\exists xT2x$' $\vee$ T'$\exists x\neg T2x$')=**n**. So M+h is a fixed point in which the premise is true but the conclusion is not."

To prove (2) we have to alter the proof of (**) in the example in Section 2, by adding to the last few lines as follows, with the additional considerations in italics: "We define the *standard* members of D as follows: I(0) is standard, and if d is standard then so is I(s)(d). *Since* $\mathrm{SV1}_{M+h}$

$(\exists x\neg T2x)\neq\mathbf{t}$, *there is a weakly consistent* $h''\geq h$ *such that* $h''(d)=\mathbf{t}$ *every even* $d\in D$. *So the set* $\{A\in\text{Sent(LA}')$: A is even$\}\cup\text{Th}(M+h)$ *is consistent. So the set* $\{A\in\text{Sent(LA}')$: A is even and standard$\}\cup\text{Th}(M)$ *is consistent. So there is a weakly consistent classical* $h'\geq h$ *so that* $h'(d)=\mathbf{t}$ for every standard even $d\in D$, and $h'(d)=\mathbf{f}$ for every nonstandard even $d\in D$. Such nonstandard even $d\in D$ exist, because $M+h$ is an uncountable model in which PA^{-} is true. So $\text{CL}_{M+h'}(Ind)=\mathbf{f}$. But this contradicts the fact that $\text{SV1}_{M+h}(PA^{-}\ \&\ Ind)$-$\mathbf{t}$."

Similar amendments to the proofs of the complexity lemmas yield (3), (4) and (5), above; and to the proofs in the counterexample of Section 5 yield (6). As for the axiomatizability of "the set of true sentences", we have the following.

SOUNDNESS AND COMPLETENESS (SV1). (1) $\vDash_1 A$ iff $\vDash_1' A$ iff $A \in \text{SV1FPLT}_L$, where SV1FPLT_L is defined by the following axiom, in addition to the axioms and rules of SVFPLT_L.

Weak consistency axiom: $\neg(T\text{ `}A_1\text{'}\ \&\ldots\&\ T\text{ `}A_n\text{'})$ is an axiom, where $\neg(A_1\ \&\ldots\&\ A_n)$ is a classical theorem.

(2) If L is countable, then $\vDash_{1L} A$ iff $\vDash_{1L}' A$ iff $\vDash_1 A$ iff $\vDash_1' A$ iff $A \in \text{SV1FPLT}_L$.

PROOF. This is proved by straightforwardly amending the proofs of the soundness and completeness theorems for $\vDash_L$, $\vDash_L'$, $\vDash$, and $\vDash'$. ⊣

As for SV2, we have the following:

SOUNDNESS AND COMPLETENESS (SV2). (1) $\vDash_2 A$ iff $\vDash_2' A$ iff $A \in \text{SV2FPLT}_L$, where SV2FPLT_L is defined by the following axioms, in addition to the axioms and rules of SVFPLT_L.

Strong consistency axioms : $T\text{`}A\supset B\text{'}\ \&\ T\text{ `}A\text{'}\supset T\text{ `}B\text{'}$

$T\text{ `}\neg A\text{'}\equiv\neg T\text{ `}A\text{'}$

(2) If L is countable, then $\vDash_{2L} A$ iff $\vDash_{2L}' A$ iff $\vDash_2 A$ iff $\vDash_2' A$ iff $A \in \text{SV2FPLT}_L$.

PROOF. Again, this is proved by straightforwardly amending the proofs of the soundness and completeness theorems for $\vDash_L$, $\vDash_L'$, $\vDash$, and $\vDash'$. ⊣

Unfortunately, we have found proofs neither for the analogues of the complexity theorems, nor of the distinctness of the consequence relations $\vDash_{2L}$, $\vDash_2$, $\vDash'_{2L}$ and $\vDash'_2$. So we conclude with two open questions:

OPEN QUESTIONS.

(1) Are $\vDash_{2L}$, $\vDash_2$, $\vDash'_{2L}$ and $\vDash'_2$ distinct?
(2) What is the complexity of $\vDash_{2L}$, $\vDash_2$, $\vDash'_{2L}$ and $\vDash'_2$?

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